A Gauge Transformation between Ragnisco-Tu Hierarchy and a Related Lattice Hierarchy

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Abstract

A new lattice hierarchy related to Ragnisco-Tu equation is proposed and its gauge equivalence to Ragnisco-Tu equation is proven. As an application of gauge transformation, we construct Darboux transformation (DT) of this new equation through DT of Ragnisco-Tu equation. An explicit exact solution is presented as an example.

Keywords

Ragnisco-Tu Hierarchy, Gauge Transformation, Transfer Operator, Darboux Transformation

1. Introduction

Nonlinear integrable equations usually have some marvellous properties such as Hamilton structure and infinitely many conservation laws. There are close connections between many of these equations. For instance, the cerebrated KdV equation, modified KdV equation, and nonlinear Schrödinger equation are reduction of AKNS system. Boussinesq equation and derivative nonlinear Schrödinger equation are linked to the constraint of KP equation (c.f. [1]). Jaulent-Miodek equation, Kaup-Newell equation, Levi equation and Heisenberg equation were found to be equivalent to AKNS equation [1]-[6]. The relation between two equations plays an important role, which makes one tackle with relatively unfamiliar equations through relatively familiar equations. However, in a general survey, there is comparatively less research on relatedness of lattice soliton equations than that of continuous soliton equations. One of the reasons behind this actuality is the lack of related example. In this paper we put forward a pair of nonlinear integrable lattice equations and investigate some relations such as gauge

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equivalence relation and Darboux transformation between them. Utilizing the relation which has been found, we will obtain an exact solution of equation.

Ragnisco-Tu equation [7] [8]

$$Q_{nt} = EQ_n - R_n Q_n^2, \quad R_{nt} = -E^{-1}R_n + R_n^2 Q_n, \tag{1.1}$$

is an integrable lattice soliton equation. Ref. [8] discussed its Hamilton structure, and proved that its continuous limit may result in AKNS system. Ref. [9] obtained its inverse scattering transformation and exact solution. Ref. [10]-[12] researched more general problems, and studied Hamilton structure and Darboux transformation and geometric algerba solutions. Ragnisco-Tu equation has spectral problem [7]

$$E\varphi_n = M\varphi_n, \quad M = \begin{pmatrix} \lambda + R_n Q_n & Q_n \\ R_n & 1 \end{pmatrix}, \tag{1.2}$$

where $\varphi_n = (\varphi_{1n}, \varphi_{2n})^{\mathrm{T}}$, E means a shift of space variable n, subscript t denotes partial derivative with respect to time t, λ is spectral parameter and Q_n, R_n are potential functions. The derivation of equation hierarchy will be given in Section 2. This spectral problem can be generalized to

$$E\psi_n = M'\psi_n, \quad \psi_n = (\psi_{1n}, \psi_{2n})^{\mathrm{T}}, \quad M' = \begin{pmatrix} \lambda + (\lambda+1)q_n r_n & (\lambda+1)q_n \\ r_n & 1 \end{pmatrix}. \tag{1.3}$$

This method for constructing new lattice equation was first used in modified Toda equation [13] [14]. Surprisingly, equations obtained via these two spectral equations are equivalent, but this feature does not appear on Toda equation. On this basis, we further discuss Darboux transformation of them. With the help of gauge transformation and Darboux transformation of Ragnisco-Tu equation, we get a Darboux transformation of new equation, which is complex and difficult to construct directly.

This paper is organized as follows: in Section 2 and Section 3, we deduce the general hierarchies of Ragnisco-Tu and related generalized lattice hierarchy respectively. In Section 4, we derive a gauge transformation and transfer operator of two hierarchies. Section 5 will contribute to the Darboux transformation of two equations. Finally, in Section 6, a conclusion is presented.

2. The Derivation of Ragnisco-Tu Hierarchy

The derivation of Ragnisco-Tu hierarchy can be referred to [8] [9], but for completeness we still give a concise version.

Consider time evolution corresponding to (1.2)

$$\varphi_{n,t} = N\varphi_n, \quad N = \begin{pmatrix} A_n & B_n \\ C_n & D_n \end{pmatrix}, \tag{2.1}$$

discrete zero curvature equation results in following equalities directly

$$\lambda_t + (R_n Q_n)_t = (\lambda + R_n Q_n)(E-1)A_n + R_n EB_n - Q_n C_n, \tag{2.2}$$

$$Q_{n,t} = Q_n(EA_n - D_n) + EB_n - (\lambda + R_n Q_n)B_n, \tag{2.3}$$

$$R_{n,t} = (\lambda + R_n Q_n)EC_n - C_n + R_n(ED_n - A_n), \tag{2.4}$$

$$0 = Q_n EC_n - R_n B_n + (E-1)D_n. \tag{2.5}$$

From these equations we draw out relations between related quantities

$$D = (E-1)^{-1}(R_n B_n - Q_n EC_n) + d_0, \tag{2.6}$$

$$A_n + D_n = n\lambda^{-1}\lambda_t + a_0, \tag{2.7}$$

where $(E-1)^{-1}$ defined as $(E-1)^{-1}f_n = \dfrac{1}{2}\left(-\sum_{k=n}^{\infty} + \sum_{k=-\infty}^{n-1}\right)f_k$, and d_0, a_0 are constants independent of variable n. Let

$$\begin{pmatrix} EC_n \\ B_n \end{pmatrix} = \sum_{j=0}^{k} \begin{pmatrix} Ec_{nj} \\ b_{nj} \end{pmatrix} \lambda^{k-j},$$

the relations (2.2)-(2.5) can be written as

$$\begin{pmatrix} Q_n \\ R_n \end{pmatrix}_t = \sum_{j=0}^{k} L_1 \begin{pmatrix} Ec_{nj} \\ b_{nj} \end{pmatrix} \lambda^{k-j} + \begin{pmatrix} (a_0 - 2d_0)Q_n \\ -(a_0 - 2d_0)R_n \end{pmatrix} - \sum_{j=-1}^{k-1} L_2 \begin{pmatrix} Ec_{n,j+1} \\ b_{n,j+1} \end{pmatrix} \lambda^{k-j} + \begin{pmatrix} (n+1)Q_n\lambda^{-1}\lambda_t \\ -nR_n\lambda^{-1}\lambda_t \end{pmatrix}, \tag{2.8}$$

where operators $L_i, i = 1, 2$ are defined as

$$L_1 = \begin{pmatrix} Q_n\Delta Q_n & -Q_n\Delta R_n + E - R_n Q_n \\ R_n Q_n - E^{-1} - R_n\Delta Q_n & R_n\Delta R_n \end{pmatrix},$$

$$L_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \Delta = (E+1)(E-1)^{-1}. \tag{2.9}$$

Giving boundary condition

$$\begin{pmatrix} A_n & B_n \\ C_n & D_n \end{pmatrix}\Bigg|_{Q_n=R_n=0} = \begin{pmatrix} \dfrac{1}{2}\lambda^{k+1} & 0 \\ 0 & -\dfrac{1}{2}\lambda^{k+1} \end{pmatrix} \tag{2.10}$$

and taking $\lambda_t = 0, a_0 - 2d_0 = \lambda^{k+1}$ we may deduce the iso-spectral hierarchy as follows:

$$\begin{pmatrix} Q_n \\ R_n \end{pmatrix}_t = \left(L_1 L_2^{-1}\right)^{k+1} \begin{pmatrix} Q_n \\ -R_n \end{pmatrix}. \tag{2.11}$$

The case of $k = 0$ just gives Ragnisco-Tu Equation (1.1).

If the boundary condition is given as

$$\begin{pmatrix} A_n & B_n \\ C_n & D_n \end{pmatrix}\Bigg|_{Q_n=R_n=0} = \begin{pmatrix} \dfrac{n(\lambda+1)\lambda^k}{2} & 0 \\ 0 & -\dfrac{n(\lambda+1)\lambda^k}{2} \end{pmatrix} \tag{2.12}$$

and $\lambda_t = (\lambda+1)\lambda^{k+1}, a_0 - 2d_0 = 0$ we get non-iso-spectral Ragnisco-Tu hierarchy

$$\begin{pmatrix} Q_n \\ R_n \end{pmatrix}_t = \left(L_1 L_2^{-1} + I\right)\left(L_1 L_2^{-1}\right)^k \begin{pmatrix} (n+1)Q_n \\ -nR_n \end{pmatrix}, \tag{2.13}$$

where I is an identity operator. In more general case, Ragnisco-Tu hierarchy is expressed by

Lemma 2.1. If $\omega(\lambda), \rho(\lambda)$ are the polynomials of λ with degree $k, k+1$ respectively, $\lambda_t = (\lambda+1)\lambda\omega(\lambda)$, and the boundary condition is as follows

$$\begin{pmatrix} A_n & B_n \\ C_n & D_n \end{pmatrix}\Bigg|_{Q_n=R_n=0} = \begin{pmatrix} \dfrac{n(\lambda+1)}{2}\omega(\lambda) + \dfrac{1}{2}\rho(\lambda) & 0 \\ 0 & -\dfrac{n(\lambda+1)}{2}\omega(\lambda) - \dfrac{1}{2}\rho(\lambda) \end{pmatrix}, \tag{2.14}$$

then general Ragnisco-Tu hierarchy adopts the from

$$\begin{pmatrix} Q_n \\ R_n \end{pmatrix}_t = (L+I)\omega(L)\begin{pmatrix} (n+1)Q_n \\ -nR_n \end{pmatrix} + \rho(L)\begin{pmatrix} Q_n \\ -R_n \end{pmatrix}, \quad L = L_1 L_2^{-1}. \tag{2.15}$$

3. A New Lattice Hierarchy Related to the Ragnisco-Tu Hierarchy

With regard to generalized spectral problem (1.3), introduce the time evolution

$$\psi_{n,t} = N'\psi_n, \quad N' = \begin{pmatrix} A'_n & (\lambda+1)B'_n \\ C'_n & D'_n \end{pmatrix}. \tag{3.1}$$

Then from discrete zero curvature equation, we have

$$(\lambda+1)(q_n r_n)_t = \left[\lambda+(\lambda+1)q_n r_n\right](E-1)A'_n + (\lambda+1)(r_n EB'_n - q_n C'_n) - (1+q_n r_n)\lambda_t, \tag{3.2a}$$

$$(\lambda+1)q_{n,t} = (\lambda+1)q_n(EA'_n - D'_n) + (\lambda+1)\left[EB'_n - \lambda B'_n - (\lambda+1)q_n r_n B'_n\right] - q_n \lambda_t, \tag{3.2b}$$

$$r_{n,t} = r(ED'_n - A'_n) - C'_n + \left[\lambda+(\lambda+1)q_n r_n\right]EC'_n, \tag{3.2c}$$

$$0 = (\lambda+1)(q_n EC'_n - r_n B'_n) + (E-1)D'_n. \tag{3.2d}$$

It is ease to know that there only have three independent equations, for instance, (3.2b), (3.2c), (3.2d). Now, from them we work out

$$A'_n + D'_n = n\lambda^{-1}\lambda_t + a'_0, \tag{3.3}$$

and

$$D'_n = -(\lambda+1)(E-1)^{-1}(q_n, -r_n)\begin{pmatrix} EC'_n \\ B'_n \end{pmatrix} + d'_0, \tag{3.4}$$

where a'_0, d'_0 are independent of *n*. Introducing two operators L'_1, L'_2

$$L'_1 = \begin{pmatrix} q_n \Delta q_n & -q_n \Delta r_n - q_n r_n + E \\ -r_n \Delta q_n + q_n r_n - E^{-1} & r_n \Delta r_n \end{pmatrix}, \tag{3.5}$$

$$L'_2 = \begin{pmatrix} -q_n \Delta q_n & q_n \Delta r_n + 1 + q_n r_n \\ r_n \Delta q_n - 1 - q_n r_n & -r_n \Delta r_n \end{pmatrix}, \tag{3.6}$$

we get matrix form

$$\begin{pmatrix} q_n \\ r_n \end{pmatrix}_t = L'_1\begin{pmatrix} EC'_n \\ B'_n \end{pmatrix} - \lambda L'_2\begin{pmatrix} EC'_n \\ B'_n \end{pmatrix} - \begin{pmatrix} q_n \\ 0 \end{pmatrix}\frac{\lambda_t}{\lambda+1} + (a'_0 - 2d'_0)\begin{pmatrix} q_n \\ -r_n \end{pmatrix} + \lambda^{-1}\lambda_t\begin{pmatrix} (n+1)q_n \\ -nr_n \end{pmatrix}. \tag{3.7}$$

Set

$$\begin{pmatrix} EC'_n \\ B'_n \end{pmatrix} = \sum_{j=0}^{k}\begin{pmatrix} Ec'_{n,j} \\ b'_{n,j} \end{pmatrix}\lambda^{k-j},$$

the general lattice hierarchy (called generalized Ragnisco-Tu hierarchy) is deduced in

Lemma 3.1. Let $\tilde{\omega}(\lambda)$ and $\tilde{\rho}(\lambda)$ be the polynomials of λ with degree k and $(k+1)$ respectively, take $\lambda_t = \lambda(\lambda+1)\tilde{\omega}(\lambda), a'_0 - 2d'_0 = \tilde{\rho}(\lambda)$, then under boundary condition

$$\begin{pmatrix} A'_n & (\lambda+1)B'_n \\ C'_n & D'_n \end{pmatrix}\Bigg|_{q_n=r_n=0} = \begin{pmatrix} \dfrac{n(\lambda+1)}{2}\tilde{\omega}(\lambda) + \dfrac{1}{2}\tilde{\rho}(\lambda) & 0 \\ 0 & -\dfrac{n(\lambda+1)}{2}\tilde{\omega}(\lambda) - \dfrac{1}{2}\tilde{\rho}(\lambda) \end{pmatrix}, \tag{3.8}$$

the generalized Ragnisco-Tu hierarchy is

$$\begin{pmatrix} q_n \\ r_n \end{pmatrix}_t = L'\tilde{\omega}(L')\begin{pmatrix} nq_n \\ -nr_n \end{pmatrix} + \tilde{\omega}(L')\begin{pmatrix} (n+1)q_n \\ -nr_n \end{pmatrix} + \tilde{\rho}(L')\begin{pmatrix} q_n \\ -r_n \end{pmatrix}, \quad L' = L'_1(L'^{-1}_2). \tag{3.9}$$

Especially when $\tilde{\omega}(\lambda) = 0$ and $\tilde{\rho}(\lambda) = \lambda^{k+1}$, it is the iso-spectral hierarchy

$$\begin{pmatrix} q_n \\ r_n \end{pmatrix}_t = L'^{k+1}\begin{pmatrix} q_n \\ -r_n \end{pmatrix}. \tag{3.10}$$

The first one ($k = 0$) is

$$\begin{pmatrix} q_n \\ r_n \end{pmatrix}_t = \begin{pmatrix} E\dfrac{q_n}{1+q_nr_n} - \dfrac{q_n^2 r_n}{1+q_nr_n} \\ -E^{-1}\dfrac{r_n}{1+q_nr_n} + \dfrac{q_nr_n^2}{1+q_nr_n} \end{pmatrix}. \tag{3.11}$$

If we take $\tilde{\omega}(\lambda) = \lambda^k, \tilde{\rho}(\lambda) = 0$, then we get the non-iso-spectral hierarchy as follows

$$\begin{pmatrix} q_n \\ r_n \end{pmatrix}_t = (L')^{k+1}\begin{pmatrix} nq_n \\ -nr_n \end{pmatrix} + (L')^k\begin{pmatrix} (n+1)q_n \\ -nr_n \end{pmatrix}. \tag{3.12}$$

The first one ($k = 0$) is

$$\begin{pmatrix} q_n \\ r_n \end{pmatrix} = \begin{pmatrix} (n+1)E\dfrac{q_n}{1+q_nr_n} + \dfrac{(n+1)q_n^2 r_n}{1+q_nr_n} \\ -(n-1)E^{-1}\dfrac{r_n}{1+q_nr_n} - \dfrac{nr_n}{1+q_nr_n} \end{pmatrix}. \tag{3.13}$$

Proof. Expanding (3.7) we have

$$\begin{pmatrix} q_n \\ r_n \end{pmatrix}_t = \sum_{j=0}^{k} L_1'\begin{pmatrix} Ec_{n,j}' \\ b_{n,j}' \end{pmatrix}\lambda^{k-j} - \sum_{j=-1}^{k-1} L_2'\begin{pmatrix} Ec_{n,j+1}' \\ b_{n,j+1}' \end{pmatrix}\lambda^{k-j} + \sum_{j=-1}^{k-1}\tilde{\omega}_{j+1}\begin{pmatrix} nq_n \\ -nr_n \end{pmatrix}\lambda^{k-j}$$

$$+ \sum_{j=0}^{k}\tilde{\omega}_j\begin{pmatrix} (n+1)q_n \\ -nr_n \end{pmatrix}\lambda^{k-j} + \sum_{j=-1}^{k}\tilde{\rho}_{j+1}\begin{pmatrix} q_n \\ r_n \end{pmatrix}\lambda^{k-j}.$$

Equating the coefficients of power of λ leads to

$$\begin{pmatrix} q_{n,t} \\ r_{n,t} \end{pmatrix} = L_1'\begin{pmatrix} Ec_{n,k}' \\ b_{n,k}' \end{pmatrix} + \tilde{\omega}_k\begin{pmatrix} (n+1)q_n \\ -nr_n \end{pmatrix} + \tilde{\rho}_{k+1}\begin{pmatrix} q_n \\ r_n \end{pmatrix},$$

$$L_2'\begin{pmatrix} Ec_{n,j+1}' \\ b_{n,j+1}' \end{pmatrix} = L_1'\begin{pmatrix} Ec_{n,j}' \\ b_{n,j}' \end{pmatrix} + \tilde{\omega}_{j+1}\begin{pmatrix} nq_n \\ -nr_n \end{pmatrix} + \tilde{\omega}_j\begin{pmatrix} (n+1)q_n \\ -nr_n \end{pmatrix} + \tilde{\rho}_{j+1}\begin{pmatrix} q_n \\ -r_n \end{pmatrix}, j = 0,1,\cdots,k-1,$$

$$L_2'\begin{pmatrix} Ec_{n,0}' \\ b_{n,0}' \end{pmatrix} = \tilde{\omega}_0\begin{pmatrix} nq_n \\ -nr_n \end{pmatrix} + \tilde{\rho}_0\begin{pmatrix} q_n \\ -r_n \end{pmatrix}.$$

Through mathematical induction we get the recursion relation

$$\begin{pmatrix} Ec_{n,j}' \\ b_{n,j}' \end{pmatrix} = \left((L_2')^{-1}L_1'\right)^j\begin{pmatrix} Ec_{n,0}' \\ b_{n,0}' \end{pmatrix} + \sum_{s=0}^{j-1}\left((L_2')^{-1}L_1'\right)^s (L_2')^{-1}\tilde{\rho}_{j-s}\begin{pmatrix} q_n \\ -r_n \end{pmatrix}$$

$$+ \sum_{s=0}^{j-1}\tilde{\omega}_{j-s}\left((L_2')^{-1}L_1'\right)^s (L_2')^{-1}\begin{pmatrix} nq_n \\ -nr_n \end{pmatrix} + \sum_{s=0}^{j-1}\tilde{\omega}_{j-1-s}\left((L_2')^{-1}L_1'\right)^s (L_2')^{-1}\begin{pmatrix} (n+1)q_n \\ -nr_n \end{pmatrix}.$$

From it the conclusion of Lemma 3.1 is got.

4. A Gauge Transformation and Transfer Operator between the Ragnisco-Tu Hierarchy and Generalized Ragnisco-Tu Hierarchy

In this section we will give the conclusion about gauge transformation and transfer operator between the Ragnisco-Tu hierarchy and generalized Ragnisco-Tu hierarchy.

Theorem 4.1. There exists a gauge transformation changing Lax pair of generalized Ragnisco-Tu hierarchy (1.3), (3.1) into Lax pair of Ragnisco-Tu hierarchy:

$$\varphi_n = T\psi_n, \quad T = \begin{pmatrix} E^{-1}\mu_n & \mu_n q_n \\ 0 & E^{-1}\mu_n^{-1} \end{pmatrix}, \quad \mu_n = \prod_{j=n+1}^{\infty} (1+q_j r_j). \tag{4.1}$$

Further, potentials in (1.2) and those in (1.3) have the relations

$$Q_n = \mu_n^2 q_n + \mu_n E(\mu_n q_n), \quad R_n = \frac{r_n}{\mu_n E^{-1}\mu_n}. \tag{4.2}$$

When $\tilde{\omega}(\lambda) = \omega(\lambda)$, $\tilde{\rho}(\lambda) = \rho(\lambda)$, hierarchy (2.15) and hierarchy (3.9) satisfy

$$L_2^{-1}\left[\begin{pmatrix} Q_n \\ R_n \end{pmatrix}_t - (L+I)\omega(L)\begin{pmatrix} (n+1)Q_n \\ -nR_n \end{pmatrix} + \rho(L)\begin{pmatrix} Q_n \\ -R_n \end{pmatrix}\right]$$
$$= \Omega(L_2')^{-1}\left[\begin{pmatrix} q_n \\ r_n \end{pmatrix}_t - L'\omega(L')\begin{pmatrix} nq_n \\ -nr_n \end{pmatrix} + \omega(L')\begin{pmatrix} (n+1)q_n \\ -nr_n \end{pmatrix} + \rho(L')\begin{pmatrix} q_n \\ -r_n \end{pmatrix}\right], \tag{4.3}$$

where Ω is transfer operator defined as

$$\Omega = \begin{pmatrix} \mu_n^{-2} & 0 \\ \mu_n^2 q_n^2 & \mu_n^2(E+1) \end{pmatrix}.$$

Proof. As gauge transformation, T should satisfy

$$(ET)M' = MT. \tag{4.4}$$

Set

$$T = \begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix}, \tag{4.5}$$

the entries of it must meet the following equations

$$\lambda(1+q_n r_n)ET_1 + q_n r_n ET_1 + r_n ET_2 = (\lambda + Q_n R_n)T_1 + Q_n T_3,$$

$$ET_2 + (\lambda+1)q_n ET_1 = (\lambda + R_n Q_n)T_2 + Q_n T_4,$$

$$\lambda ET_3 + (\lambda+1)q_n r_n ET_3 + r_n ET_4 = R_n T_1 + T_3,$$

$$(\lambda+1)q_n ET_3 + ET_4 = R_n T_2 + T_4.$$

Notice that T is independent of λ, its entries are determined easily.

Transformation matrix T also changes time evolution (3.1) into (2.1). To justify this assertion, for a newly defined $\tilde{N} = (T_t + TN')T^{-1}$ or $\tilde{N} = S^{-1}(N'S - S_t)$, where $S = T^{-1}$, we need to prove that equality $\tilde{N} = N$ is hold. A simple calculation shows us

$$\tilde{N} = \begin{pmatrix} \tilde{N}_{11} & \tilde{N}_{12} \\ \tilde{N}_{21} & \tilde{N}_{22} \end{pmatrix}, \tag{4.6}$$

where

$$\tilde{N}_{11} = A_n' + E^{-1}(\ln\mu_n)_t + C_n' q_n/(1+q_n r_n),$$

$$\tilde{N}_{12} = (D_n' - A_n')q_n\mu_n E^{-1}\mu_n + (\lambda+1)B_n' E^{-1}\mu_n^2 - C_n' q_n^2 \mu_n^2$$
$$+ q_{n,t}\mu_n E^{-1}\mu_n + q_n\mu_n E^{-1}\mu_n(1-E^{-1})(\ln\mu_n)_t,$$

$$\tilde{N}_{21} = C_n'/E^{-1}\mu_n^2,$$

$$\tilde{N}_{22} = D_n' - E^{-1}(\ln\mu_n)_t - C_n' q_n/(1+q_n r_n).$$

It is evident that $\tilde{N}_{11}$, $\tilde{N}_{12}$, $\tilde{N}_{21}$, $\tilde{N}_{22}$ are the polynomial of λ with degree $k+1$, k, k, $k+1$ respec-

tively, which are the same as that of elements of N. In the meantime, the condition $q_n = r_n = 0$ is equivalent to $Q_n = R_n = 0$ and thus if permitting $\tilde{\omega}(\lambda) = \omega(\lambda)$, $\tilde{\rho}(\lambda) = \rho(\lambda)$ then $\tilde{N}\big|_{q_n=r_n=0} = N\big|_{Q_n=R_n=0}$ is valid. On the other hand, we can verify directly that $M_t = (E\tilde{N})M - M\tilde{N}$. Combining all discussed above we conclude that $\tilde{N} = N$. That is, the gauge transformation (4.1) conveys time evolution of (3.1) into that of (2.1).

Now we deduce transfer operator of two hierarchies. A dull calculation simplifies the expression of $\tilde{N}_{12}$ as

$$\tilde{N}_{12} = \mu_n^2 \left(EC_n' \right) q_n^2 + \mu_n^2 \left(E + 1 \right) B_n' + \frac{\lambda^{-1}\lambda_t}{\lambda + 1} \mu_n^2 q_n.$$

Thus we have

$$\begin{pmatrix} Ec_{nj} \\ b_{nj} \end{pmatrix} = \begin{pmatrix} \mu_n^{-2} & 0 \\ \mu_n^2 q_n^2 & \mu_n^2 (E+1) \end{pmatrix} \begin{pmatrix} Ec_{nj}' \\ b_{nj}' \end{pmatrix}, \tag{4.7}$$

where $j = 0, 1, \cdots, k-1$ for iso-spectral and $j = 1, 2, \cdots, k-1$ for non-iso-spectral.

Because of in the case of iso-spectral and non-iso-spectral, the following recursion formula always holds

$$\begin{pmatrix} Ec_{n,j+1} \\ b_{n,j+1} \end{pmatrix} = L_2^{-1} L_1 \begin{pmatrix} Ec_{n,j} \\ b_{n,j} \end{pmatrix}, \ j = 1, 2, \cdots, k-1,$$

(In the case of iso-spectral, $j = 0, 1, \cdots, k-1$),

$$\begin{pmatrix} Ec_{n,j+1}' \\ b_{n,j+1}' \end{pmatrix} = \left(L_2' \right)^{-1} L_1' \begin{pmatrix} Ec_{n,j}' \\ b_{n,j}' \end{pmatrix}, \ j = 1, 2, \cdots, k-1,$$

(In the case of iso-spectral, $j = 0, 1, \cdots, k-1$), we can deduce transformation relation by substituting the above relations into (4.7)

$$L_2^{-1} L_1 \Omega = \Omega \left(\left(L_2' \right)^{-1} L_1' \right), \tag{4.8}$$

where Ω is the transfer operator of two hierarchies. Comparing the coefficients of λ^0 in $B_n = \tilde{N}_{12}, C_n = \tilde{N}_{21}$ we have

$$\begin{pmatrix} Ec_{nk} \\ b_{nk} \end{pmatrix} = \Omega \begin{pmatrix} Ec_{nk}' \\ b_{nk}' \end{pmatrix}.$$

According to the derivation expressions of iso-spectral and non-iso-spectral equation we arrive at the relation of two hierarchies immediately

$$L_2^{-1} \begin{pmatrix} Q_n \\ R_n \end{pmatrix}_t = \Omega \left(L_2' \right)^{-1} \begin{pmatrix} q_n \\ r_n \end{pmatrix}. \tag{4.9}$$

When we focus our attention on the iso-spectral case, (4.7) holds for $j = 0$. That is

$$\begin{pmatrix} R_n \\ Q_n \end{pmatrix} = \Omega \begin{pmatrix} \dfrac{r_n}{1 + q_n r_n} \\ \dfrac{q_n}{1 + q_n r_n} \end{pmatrix},$$

which can be verified readily. When we concern about the non-iso-spectral case, (4.7) holds for $j = 1$, the verification is not so easy. To get the equation

$$\begin{pmatrix} Ec_{n1} \\ b_{n1} \end{pmatrix} = \Omega \begin{pmatrix} Ec_{n1}' \\ b_{n1}' \end{pmatrix} \tag{4.10}$$

we first prove

$$\Omega \left(L_2' \right)^{-1} \begin{pmatrix} (n+1)q_n \\ -nr_n \end{pmatrix} = L_2^{-1} L_1 \begin{pmatrix} 0 \\ \mu_n^2 q_n \end{pmatrix} + L_2^{-1} \begin{pmatrix} (n+1)Q_n \\ -nR_n \end{pmatrix}. \tag{4.11}$$

Denote

$$l.h.s = \begin{pmatrix} leftline1 \\ leftline2 \end{pmatrix}, \quad r.h.s = \begin{pmatrix} rightline1 \\ rightline2 \end{pmatrix}.$$

It is ease to know " $leftline1 = rightline1$ ". As for " $line2$ " we have

$$leftline2 = n\mu_n^2 q_n + \mu_n E(\mu_n q_n) + \frac{\mu_n^2 q_n}{1 + q_n r_n} + \mu_n(E\mu_n)(Enq_n)$$

$$- \mu_n^2 E\left(\frac{q_n}{1 + q_n r_n} \Delta \frac{q_n r_n}{1 + q_n r_n}\right) - \mu_n^2 q_n \Delta \frac{q_n r_n}{1 + q_n r_n}$$

and

$$rightline2 = n\mu_n^2 q_n + E(\mu_n^2 q_n) + \frac{\mu_n^2 q_n}{1 + q_n r_n} - \mu_n E(\mu_n q_n)\frac{q_n r_n}{1 + q_n r_n}$$

$$+ \mu_n(E\mu_n)(Enq_n) - \left(\mu_n^2 q_n + \mu_n E(\mu_n q_n)\right)\Delta \frac{q_n r_n}{1 + q_n r_n}.$$

Their difference is

$$\mu_n E(\mu_n q_n) - E(\mu_n^2 q_n) + \mu_n E(\mu_n q_n)\frac{q_n r_n}{1 + q_n r_n} - \mu_n^2 E\left(\frac{q_n}{1 + q_n r_n}\Delta\frac{q_n r_n}{1 + q_n r_n}\right) + \mu_n E(\mu_n q_n)\Delta\frac{q_n r_n}{1 + q_n r_n}$$

$$= \mu_n E(\mu_n q_n) - E(\mu_n^2 q_n) + \mu_n E(\mu_n q_n)\frac{q_n r_n}{1 + q_n r_n} - \mu_n E(\mu_n q_n)(E+1)\frac{q_n r_n}{1 + q_n r_n} = 0.$$

The Equation (4.11) is proved.

On the other hand, through comparing the coefficients of λ^k in $B_n = \tilde{N}_{12}, C_n = \tilde{N}_{21}$ we have

$$\begin{pmatrix} nR_n \\ (n+1)Q_n \end{pmatrix} = \Omega \begin{pmatrix} \dfrac{nr_n}{1 + q_n r_n} \\ \dfrac{nq_n}{1 + q_n r_n} \end{pmatrix} + \begin{pmatrix} 0 \\ \mu_n^2 q_n \end{pmatrix}. \tag{4.12}$$

That is

$$L_2^{-1}\begin{pmatrix} (n+1)Q_n \\ -nR_n \end{pmatrix} = \Omega(L_2')^{-1}\begin{pmatrix} nq_n \\ -nr_n \end{pmatrix} + \begin{pmatrix} 0 \\ \mu_n^2 q_n \end{pmatrix}. \tag{4.13}$$

Using the recursion relations of $\begin{pmatrix} Ec_{nj} \\ b_{nj} \end{pmatrix}$ and $\begin{pmatrix} Ec'_{nj} \\ b'_{nj} \end{pmatrix}$ together with (4.11), (4.12) and the formula of transformation operator (4.8) we finally obtain the recursion relation of non-iso-spectral (4.10).

Finally, we consider relevancy of two hierarchy. The time part of (4.3) has given in (4.9). The following equation is deduced according to (4.13) and (4.8)

$$L_2^{-1}L\begin{pmatrix} (n+1)Q_n \\ -nR_n \end{pmatrix} = \Omega(L_2')^{-1}\begin{pmatrix} nq_n \\ -nr_n \end{pmatrix} + \begin{pmatrix} 0 \\ \mu_n^2 q_n \end{pmatrix}.$$

Equation (4.11) together with above expression yields

$$L_2^{-1}(L+I)\begin{pmatrix} (n+1)Q_n \\ -nR_n \end{pmatrix} = \Omega(L_2')^{-1}\left[L'\begin{pmatrix} nq_n \\ -nr_n \end{pmatrix} + \begin{pmatrix} (n+1)q_n \\ -nr_n \end{pmatrix}\right].$$

Through mathematical induction we can prove the part of ω in (4.3). The proof of part of ρ is similar and it is much simpler. Thus we finish the theorem.

5. Darboux Transformation to Generalized Ragnisco-Tu Equation

Darboux transformation is a very useful tool to obtain exact solutions of nonlinear integral equation. It plays role in every type of equations such as lattice equation, discrete equation and high dimensional integral equation [15]-[22]. However, the construction of DT of complex system may still encounter difficulty. Here, we will have the aid of gauge transformation to consider DT of generalized Ragnisco-Tu equation.

5.1. Darboux Transformation to Ragnisco-Tu Equation

Consider transformation

$$\bar{\varphi} = H\varphi, \; H = \begin{pmatrix} \lambda + h_1 & h_2 \\ h_3 & \lambda + h_4 \end{pmatrix}, \tag{5.1}$$

where $h_j, j = 1, 2, 3, 4$ are independent of λ. We can see that it is a DT of Ragnisco-Tu hierarchy:

Lemma 5.1. (see also [12]) Suppose $h_2, h_3 \to 0$ as $|n| \to \infty$, then aforementioned transformation (5.1) is DT of Ragnisco-Tu hierarchy expressed by Lemma 2.1 with $\omega(\lambda) = 0$, $\rho(\lambda) = \lambda^{k+1}$. If $\bar{R}_n$, $\bar{Q}_n$ and R_n, Q_n are used to denote potentials of spectral problem (1.2) and that of spectral problem transformed through formula (5.1) respectively, then the relations between them are formulized as follows:

$$\bar{Q}_n = Q_n - h_2, \; \bar{R}_n = R_n + Eh_3, \tag{5.2}$$

where $h_j, j = 1, 2, 3, 4$ are determined by

$$h_j = \frac{\Delta_j}{\Delta}, \; \Delta = \theta_n^{(2)} - \theta_n^{(1)}, \; \Delta_1 = \lambda_2 \theta_n^{(1)} - \lambda_1 \theta_n^{(2)}, \; \Delta_2 = \lambda_1 - \lambda_2,$$

$$\Delta_3 = (\lambda_2 - \lambda_1) \theta_n^{(1)} \theta_n^{(2)}, \; \Delta_4 = \lambda_1 \theta_n^{(1)} - \lambda_2 \theta_n^{(2)}, \tag{5.3}$$

where $\theta_n^{(j)} = \theta_n(\lambda_j)$, $j = 1, 2$ and $\theta_n(\lambda)$ is solution of Riccati equation

$$E\theta_n = \frac{R + \theta_n}{\lambda + R_n Q_n + Q_n \theta_n}, \; \theta_{nt} = C_n + (D_n - A_n)\theta_n - B\theta_n^2. \tag{5.4}$$

Proof. Transformation T as DT must solve the following equation

$$\bar{M}H = (EH)M, \; \bar{M} = \begin{pmatrix} \lambda + \bar{R}_n \bar{Q}_n & \bar{Q}_n \\ \bar{R}_n & 1 \end{pmatrix}. \tag{5.5}$$

Comparing coefficients of λ^j, $j = 0, 1$ in entries of both side yields

$$\bar{R}_n \bar{Q}_n - R_n Q_n = (E-1)h_1, \; \bar{Q}_n (h_1 \bar{R}_n + h_3) = (Q_n Eh_1 + Eh_2)R_n, \tag{5.6}$$

$$\bar{Q}_n = Q_n - h_2, \; \bar{Q}_n (\bar{R}_n h_2 + h_4) = R_n (Q_n Eh_3 + Eh_4), \tag{5.7}$$

$$\bar{R}_n = R_n + Eh_3, \; \bar{R}_n Eh_1 + Eh_3 = R_n (Q_n Eh_3 + Eh_4), \tag{5.8}$$

$$\bar{R}_n h_2 - Q_n Eh_3 = (E-1)h_4. \tag{5.9}$$

Suppose $\lambda_j, j = 1, 2$ are two zeros of $det(H)$, then components of $H\varphi$ are linear dependent, which means $H\varphi = 0$. Thus h_j can be determined according to formulas (5.3).

Transformation (5.1) also change (2.1) to time evolution which matches to $\bar{M}$. To justify this assertion, we first consider

$$V = (H_t + HN)H^*, \; V = \begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix},$$

where H^* means adjoint matrix of H. The expressions of $V_{ij}, i, j = 1, 2$ are as follows

$$V_{11} = (\lambda + h_4)h_{1t} - h_{2t}h_3 + (\lambda + h_1)(\lambda + h_4)A_n + h_2(\lambda + h_4)C_n - (\lambda + h_1)h_3 B_n - h_2 h_3 D_n,$$

$$V_{12} = (\lambda + h_1)h_{2t} - h_{1t}h_2 - (\lambda + h_1)h_2 A_n - h_2^2 C_n + (\lambda + h_1)^2 B_n + (\lambda + h_1)h_2 D_n,$$

$$V_{21} = (\lambda + h_4) h_{3t} - h_3 h_{4t} + (\lambda + h_4) h_3 A_n + (\lambda + h_4)^2 C_n - h_3^2 B_n - h_3 (\lambda + h_4) D_n,$$

$$V_{22} = -h_2 h_{3t} + (\lambda + h_1) h_{4t} - h_2 h_3 A_n - h_2 (\lambda + h_4) C_n + (\lambda + h_1) h_3 B_n + (\lambda + h_1)(\lambda + h_4) D_n.$$

We will prove that λ_j, $j = 1, 2$ are zeros of V_{ij}, $i, j = 1, 2$ and V_{11}, V_{22} and V_{12}, V_{21} are polynomials of λ with degree $k + 3$ and $k + 2$ respectively. This assures entries of matrix $\tilde{N} = (H_t + HN)H^{-1}$ have the same degrees as that of N. Assume asymptotic condition

$$N = \begin{pmatrix} A_n & B_n \\ C_n & D_n \end{pmatrix} \rightarrow \begin{pmatrix} \dfrac{\lambda^{k+1}}{2} & 0 \\ 0 & -\dfrac{\lambda^{k+1}}{2} \end{pmatrix}, \text{ when } |n| \rightarrow \infty,$$

and $h_2, h_3 \rightarrow 0$ when $|n| \rightarrow \infty$ we will find that $\tilde{N} = (H_t + HN)H^{-1}$ has the same asymptotic behavior. Therefore, because of $N, \tilde{N}$ also satisfy the same different equation, we say $\tilde{N} = \bar{N}$, *i.e.*, transformation (5.1) change (2.1) into time evolution matching to $\bar{M}$.

Now we deal with V_{11} as an example. First of all, referring to the fact that A_n, D_n and B_n, C_n are polynomials of λ with degree $k + 1$ and k, it is ease to know V_{11} is polynomial of λ with degree $k + 3$. Secondly, according to the definition of h_j, $j = 1, 2, 3, 4$ we work out

$$h_{1t} = \frac{h_2 \left(\theta_n^{(1)} \theta_{nt}^{(2)} - \theta_n^{(2)} \theta_{nt}^{(1)} \right)}{\theta_n^{(2)} - \theta_n^{(1)}}, \; h_{2t} = \frac{h_2 \left(\theta_{nt}^{(1)} - \theta_{nt}^{(2)} \right)}{\theta_n^{(2)} - \theta_n^{(1)}}.$$

Substituting them and (5.4) into V_{11} gives rise to

$$\begin{aligned}
V_{11}(\lambda_j) = \frac{1}{\theta_n^{(2)} - \theta_n^{(1)}} &\Big[h_2 (\lambda_j - \lambda_1) \theta_n^{(1)} \left(C_n^{(2)} - 2A_n^{(2)} \theta_n^{(2)} - B_n^{(2)} \theta_n^{(2)2} \right) \\
&- h_2 (\lambda_j - \lambda_2) \theta_n^{(2)} \left(C_n^{(1)} - 2A_n^{(1)} \theta_n^{(1)} - B_n^{(1)} \theta_n^{(1)2} \right) + 2(\lambda_1 - \lambda_2) h_3 A_n^{(j)} \\
&+ (\lambda_1 - \lambda_2)(\lambda_j + h_4) C_n^{(j)} + (\lambda_1 - \lambda_2)(\lambda_j + h_1) \theta_n^{(1)} \theta_n^{(2)} B_n^{(j)} \Big], \; j = 1, 2.
\end{aligned}$$

It is not difficult to check that $V_{11}(\lambda_j) = 0$, $j = 1, 2$. Noticing that $det(H) = (\lambda - \lambda_1)(\lambda - \lambda_2)$, we may say that $\tilde{A}_n$ in

$$\tilde{N} = \begin{pmatrix} \tilde{A}_n & \tilde{B}_n \\ \tilde{C}_n & \tilde{D}_n \end{pmatrix}$$

is $k + 1$ power polynomial of λ. As for asymptotic behavior of V_{11}, obviously, $|\lambda_1| > 1 > |\lambda_2|$ make $h_2, h_3 \rightarrow 0$ as $|n| \rightarrow \infty$ hold. In the mean time, h_1, h_4 tend to $-\lambda_1$, $-\lambda_2$. These results assure $\tilde{A}_n \rightarrow \dfrac{\lambda^{k+1}}{2}$. When it comes to V_{12}, V_{21}, V_{22}, the proof is similar, we do not repeat it. Now we finish the proof that (5.1) is a Darboux transformation of Ragnisco-Tu hierarchy.

As an application we present a exact solution to Ragnisco-Tu Equation (1.1). Starting from seed solution $R = Q = 0$, we first obtain a solution to Riccati Equations (5.4)

$$\theta_n = \lambda^{-n} e^{-\lambda^k t}. \tag{5.10}$$

Then according to Lemma 5.1, a solution to Ragnisco-Tu can be calculated out as follows

$$\bar{Q}_n = -\frac{\lambda_1 - \lambda_2}{\lambda_2^{-n} e^{-\lambda_2 t} - \lambda_1^{-n} e^{-\lambda_1 t}}, \; \bar{R}_n = -\frac{(\lambda_1 - \lambda_2)(\lambda_1 \lambda_2)^{-n-1} e^{-(\lambda_1 + \lambda_2)t}}{\lambda_2^{-n} e^{-\lambda_2 t} - \lambda_1^{-n} e^{-\lambda_1 t}}, \tag{5.11}$$

5.2. Darboux Transformation to Generalized Ragnisco-Tu Equation

From gauge transformation $\bar{\varphi} = T\bar{\psi}$, $\varphi = T\psi$ and DT of Ragnisco-Tu equation $\bar{\varphi} = H\varphi$, we find relation between $\bar{\psi}$ and ψ:

$$\overline{\psi} = \overline{T}^{-1} H T \psi, \tag{5.12}$$

which forms DT of generalized Ragnisco-Tu equation. Matrix $\overline{T} H T^{-1}$ admits the following form

$$\overline{T} H T^{-1} = \begin{pmatrix} (\lambda + h_1) E^{-1} \dfrac{\mu_n}{\overline{\mu}_n} - \overline{\mu}_n \overline{q}_n h_3 E^{-1} \mu_n & (\lambda + h_1) \dfrac{\mu_n q_n}{E^{-1} \overline{\mu}_n} - (\lambda + h_4) \dfrac{\overline{q}_n \overline{\mu}_n}{E^{-1} \mu_n} + \dfrac{h_2}{E^{-1} \mu_n \overline{\mu}_n} - h_3 q_n \overline{q}_n \mu_n \overline{\mu}_n \\[2ex] h_3 E^1 \mu_n \overline{\mu}_n & (\lambda + h_4) E^{-1} \dfrac{\overline{\mu}_n}{\mu_n} + h_3 q_n \mu_n E^{-1} \overline{\mu}_n \end{pmatrix}. \tag{5.13}$$

We can adopt simple notation to write it

$$\overline{\psi} = P \psi, \quad P = \begin{pmatrix} p_{11}^{(1)} \lambda + p_{11}^{(0)} & p_{12}^{(1)} \lambda + p_{12}^{(0)} \\ p_{21}^{(0)} & p_{22}^{(1)} \lambda + p_{22}^{(0)} \end{pmatrix}, \tag{5.14}$$

where

$$p_{11}^{(1)} = E^{-1} \frac{\mu_n}{\overline{\mu}_n}, \; p_{11}^{(0)} = h_1 E^{-1} \frac{\mu_n}{\overline{\mu}_n} - \overline{\mu}_n \overline{q}_n h_3 E^{-1} \mu_n, \; p_{12}^{(1)} = \frac{\mu_n q_n}{E^{-1} \overline{\mu}_n} - \frac{\overline{q}_n \overline{\mu}_n}{E^{-1} \mu_n},$$

$$p_{12}^{(0)} = h_1 \frac{\mu_n q_n}{E^{-1} \overline{\mu}_n} - h_4 \frac{\overline{q}_n \overline{\mu}_n}{E^{-1} \mu_n} + \frac{h_2}{E^{-1} \mu_n \overline{\mu}_n} - h_3 q_n \overline{q}_n \mu_n \overline{\mu}_n,$$

$$p_{21}^{(0)} = h_3 E^1 \mu_n \overline{\mu}_n, \; p_{22}^{(1)} = E^{-1} \frac{\overline{\mu}_n}{\mu_n}, \; p_{22}^{(0)} = h_4 E^{-1} \frac{\overline{\mu}_n}{\mu_n} + h_3 q_n \mu_n E^{-1} \overline{\mu}_n,$$

As a DT of generalized Ragnisco-Tu equation, P should satisfy

$$\overline{M}' P = (EP) M', \tag{5.15}$$

where

$$\overline{M}' = \begin{pmatrix} \lambda (1 + \overline{q}_n \overline{r}_n) + \overline{q}_n \overline{r}_n & (\lambda + 1) \overline{q}_n \\ \overline{r}_n & 1 \end{pmatrix}.$$

From this expression, we can draw the following equalities

$$(1 + \overline{q}_n \overline{r}_n) p_{11}^{(1)} = (1 + q_n r_n) E p_{11}^{(1)}, \tag{5.16a}$$

$$p_{11}^{(0)} + \overline{q}_n \overline{r}_n p_{11}^{(1)} + \overline{q}_n \left(\overline{r}_n p_{11}^{(0)} + p_{21}^{(0)} \right) = E p_{11}^{(0)} + q_n r_n E p_{11}^{(1)} + r_n \left(q_n E p_{11}^{(1)} + E p_{12}^{(1)} \right), \tag{5.16b}$$

$$\overline{q}_n \left(\overline{r}_n p_{11}^{(0)} + p_{21}^{(0)} \right) = r_n \left(q_n E p_{11}^{(0)} + E p_{12}^{(0)} \right), \tag{5.16c}$$

$$p_{12}^{(1)} + \overline{q}_n \left(\overline{r}_n p_{12}^{(1)} + p_{22}^{(1)} \right) = q_n E p_{12}^{(1)}, \tag{5.17a}$$

$$p_{12}^{(0)} + \overline{q}_n \left(\overline{r}_n p_{12}^{(1)} + p_{22}^{(1)} \right) + \overline{q}_n \left(\overline{r}_n p_{12}^{(0)} + p_{22}^{(0)} \right) = q_n \left(E p_{11}^{(0)} + E p_{11}^{(1)} \right) + E p_{12}^{(1)}, \tag{5.17b}$$

$$\overline{q}_n \left(\overline{r}_n p_{12}^{(0)} + p_{22}^{(0)} \right) = q_n E p_{11}^{(0)} + E p_{12}^{(0)}, \tag{5.17c}$$

$$\overline{r}_n p_{11}^{(1)} = (1 + q_n r_n) E p_{21}^{(0)} + r_n E p_{22}^{(1)}, \tag{5.18a}$$

$$\overline{r}_n p_{11}^{(0)} + p_{21}^{(0)} = r_n \left(q_n E p_{21}^{(0)} + E p_{22}^{(0)} \right), \tag{5.18b}$$

$$\overline{r}_n p_{12}^{(1)} + p_{22}^{(1)} = q_n E p_{21}^{(0)} + E p_{22}^{(1)}, \tag{5.19a}$$

$$\overline{r}_n p_{12}^{(0)} + p_{22}^{(0)} = q_n E p_{21}^{(0)} + E p_{22}^{(0)}. \tag{5.19b}$$

The acquisition of solution of them must be combined with relation exhibited in Darboux matrix (5.13). Here we do not consider general formula of solution but present a special solution related to $q_n = r_n = 0$.

Notice that $det(H) = (\lambda - \lambda_1)(\lambda - \lambda_2)$ and the definitions of $h_j, j = 1,2,3,4$ (5.3), we can find some simple relations easily

$$h_1 Eh_4 = \lambda_1 \lambda_2, \; h_3 Eh_4 = -\lambda_1 \lambda_2 Eh_3, \; p_{11}^{(0)} p_{22}^{(0)} - p_{12}^{(0)} p_{21}^{(0)} = \lambda_1 \lambda_2.$$

When seed solution $q_n = r_n = 0$ is substituted into (5.16a)-(5.19b) we find following relations

$$p_{11}^{(0)} = p_{11}^{(1)} + 1, \; p_{12}^{(0)} = p_{12}^{(1)}.$$

These equalities produce

$$E^{-1}\overline{\mu}_n = h_1 - 1 - \frac{h_3}{Eh_3}\overline{q}_n \overline{r}_n \tag{5.20}$$

and

$$\overline{q}_n \overline{r}_n = \frac{h_2 Eh_3}{h_4 - 1}. \tag{5.21}$$

Thus we get

$$E^{-1}\overline{\mu}_n = \frac{(\lambda_1 + 1)(\lambda_2 + 1)}{h_4 - 1},$$

and form (5.17c), we obtain

$$\overline{q}_n = \frac{Eh_4 - 1}{(\lambda_1 + 1)(\lambda_2 + 1)\Delta}.$$

Substituting it into (5.21), $\overline{r}_n$ is figured out

$$\overline{r}_n = \frac{(\lambda_1 + 1)(\lambda_2 + 1)(h_2 Eh_3)\Delta}{(h_4 - 1)(Eh_4 - 1)}. \tag{5.22}$$

6. Conclusion

We propose a lattice equation hierarchy related to Rangnisco-Tu hierarchy (generalized RT equation) and prove that it is equivalent to Rangnisco-Tu hierarchy itself. The transfer operator of two hierarchies is obtained. As an application of gauge transformation, we obtain a Darboux transformation of generalized RT equation and acquire an exact solution of this equation.

Acknowledgements

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