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Higher-Order Numeric Solutions for Nonlinear Systems Based on the Modified Decomposition Method

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ABSTRACT

Higher-order numeric solutions for nonlinear differential equations based on the Rach-Adomian-Meyers modified decomposition method are designed in this work. The presented one-step numeric algorithm has a high efficiency due to the new, efficient algorithms of the Adomian polynomials, and it enables us to easily generate a higher-order numeric scheme such as a 10th-order scheme, while for the Runge-Kutta method, there is no general procedure to generate higher-order numeric solutions. Finally, the method is demonstrated by using the Duffing equation and the pendulum equation.

KEYWORDS

Adomian Polynomials; Modified Decomposition Method; Adomian-Rach Theorem; Nonlinear Differential Equations; Numeric Solution

1. Introduction

The Adomian decomposition method (ADM) [1-4] is a practical technology for solving linear or nonlinear ordinary differential equations, partial differential equations, integral equations, etc. The ADM provides an efficient analytic approximate solution of nonlinear equations, which model real-world applications in engineering and the applied sciences. The Adomian decomposition series has been shown to be equivalent to a Banach-space analog of the Taylor series expansion about the initial solution component function, instead of the classic Taylor series expansion about a constant that is the initial point [5].

The ADM decomposes the pre-existent, unique, analytic solution into a series

$$u=\sum_{n=0}^{\infty}u_n, \qquad (1)$$

and decomposes the nonlinearity $Nu = f(u)$ into the series of the Adomian polynomials

$$f(u)=\sum_{n=0}^{\infty}A_n, \qquad (2)$$

where the Adomian polynomials A_n, depend on the solution component functions $u_0, u_1, \cdots, u_n$, and are defined by the formula [3]

$$A_n=\frac{1}{n!}\frac{d^n}{d\lambda^n}f(\sum_{k=0}^{\infty}u_k\lambda^k)\Big|_{\lambda=0}, n\ge 0. \qquad (3)$$

For convenient reference, we list the first five Adomian polynomials

$$A_0=f(u_0).$$

$$A_1=f'(u_0)u_1.$$

$$A_2=f'(u_0)u_2+f''(u_0)\frac{u_1^2}{2!}.$$

$$A_3 = f'(u_0)u_3 + f''(u_0)u_1u_2 + f'''(u_0)\frac{u_1^3}{3!}.$$

$$A_4 = f'(u_0)u_4 + f''(u_0)(\frac{u_2^2}{2!} + u_1u_3) + f'''(u_0)\frac{u_1^2u_2}{2!} + f^{(4)}(u_0)\frac{u_1^4}{4!}.$$

Some algorithms for symbolic programming have since been devised to efficiently generate the Adomian polynomials quickly to high orders, such as in [5-10]. Rach's Rule for the Adomian polynomials reads

$$A_n = \sum_{k=1}^{n} f^{(k)}(u_0)C_n^k,\ n \ge 1. \tag{4}$$

where the coefficients C_n^k are the sums of all possible products of k components from $u_1, u_2, \cdots, u_{n-k+1}$, whose subscripts sum to n, divided by the factorial of the number of repeated subscripts [6].

New, more efficient algorithms and subroutines in MATHEMATICA for fast generation of the one-variable and multi-variable Adomian polynomials to high orders have been provided in [8-10]. Here we list Corollary 3 algorithm [10] for the one-variable Adomian polynomials.

Corollary 3 algorithm [10]:

For $n \ge 1$,

$$C_n^1 = u_n. \tag{5}$$

For $2 \le k \le n$,

$$C_n^k = \frac{1}{n}\sum_{j=0}^{n-k}(j+1)u_{j+1}C_{n-1-j}^{k-1}. \tag{6}$$

Then the Adomian polynomials are given by the formula

$$A_n = \sum_{k=1}^{n} f^{(k)}(u_0)C_n^k.$$

The recurrence procedure for C_n^k does not involve the differentiation operator, only requires the operations of addition and multiplication, which is eminently convenient for computer algebra systems.

In 1992, Rach, Adomian and Meyers [11] proposed a modified decomposition method based on the nonlinear transformation of series by the Adomian-Rach theorem [12,13]:

If $u(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$,

then

$$f(u(x)) = \sum_{n=0}^{\infty} A_n (x - x_0)^n, \tag{7}$$

where $A_n = A_n(a_0, a_1, \cdots, a_n)$ are the Adomian polynomials in terms of the solution coefficients.

The Rach-Adomian-Meyers modified decomposition method [11] combines the power series solution and the Adomian-Rach theorem [12,13], and has been efficiently applied to solve various nonlinear models [2,14-16].

In this work, higher-order numeric one-step methods are designed for solving nonlinear differential equations based on the Rach-Adomian-Meyers modified decomposition method [11] and the previous research [17].

In the next section, we develop the numeric solution based on the modified decomposition method for nonlinear second-order differential equations, and demonstrate its application.

2. Higher-Order Numeric Solutions Based on the Modified Decomposition Method

We consider the IVP for the second-order ODE

$$\frac{d^2u}{dt^2} + \alpha(t)\frac{du}{dt} + \beta(t)u(t) + \gamma(t)f(u) = g(t), \tag{8}$$

$$u(t_0) = C_0,\ u'(t_0) = C_1, \tag{9}$$

where $t_0 \le t \le T$, $\alpha(t), \beta(t), \gamma(t), g(t)$ are specified bounded, analytic functions, and f is an analytic nonlinear operator.

The modified decomposition method supposes an analytic solution

$$u(t) = \sum_{m=0}^{\infty} a_m (t - t_0)^m. \tag{10}$$

Then the functions $\alpha(t), \beta(t), \gamma(t), g(t)$ are decomposed into the Taylor expansions

$$\alpha(t) = \sum_{m=0}^{\infty} \alpha_m (t - t_0)^m, \tag{11}$$

$$\beta(t) = \sum_{m=0}^{\infty} \beta_m (t - t_0)^m, \tag{12}$$

$$\gamma(t) = \sum_{m=0}^{\infty} \gamma_m (t - t_0)^m, \tag{13}$$

$$g(t) = \sum_{m=0}^{\infty} g_m (t - t_0)^m, \tag{14}$$

and the analytic nonlinearity $f(u)$ is decomposed into the Taylor series

$$f(u) = \sum_{m=0}^{\infty} A_m (t - t_0)^m, \tag{15}$$

where the coefficients $A_n = A_n(a_0, a_1, \cdots, a_n)$ are the Adomian polynomials in terms of the solution coefficients a_k due to the Adomian-Rach theorem [12,13].

Substituting Equations (10)-(15) in Equations (8), regrouping terms, equating the coefficients of like powers of $(t - t_0)$, and using the initial condition we obtain the recurrence scheme for the solution coefficients $a_0 = C_0$, $a_1 = C_1$,

$$a_{m+2} = \frac{1}{(m+1)(m+2)}[g_m - \sum_{l=0}^{m}((m+1-l)\alpha_l a_{m+1-l} + \beta_l a_{m-l} + \gamma_l A_{m-l})], \tag{16}$$

where $m \ge 0$ and the $A_n = A_n(a_0, a_1, \cdots, a_n)$ are the Adomian polynomials in terms of the coefficients a_k for the nonlinear function $f(u)$.

In particular, if $\alpha(t) = \alpha$, $\beta(t) = \beta$ and $\gamma(t) = \gamma$ are constants, then the recurrence formula becomes $a_0 = C_0$, $a_1 = C_1$,

$$a_{m+2} = \frac{1}{(m+1)(m+2)}[g_m - (m+1)\alpha a_{m+1} - \beta a_m - \gamma A_m], \; m \ge 0. \tag{17}$$

Further if $g(t) = g$ is also a constant, then the recurrence formula becomes

$$a_0 = C_0, \; a_1 = C_1,$$

$$a_2 = \frac{1}{2}[g - \alpha a_1 - \beta a_0 - \gamma A_0],$$

$$a_{m+2} = \frac{-1}{(m+1)(m+2)}[(m+1)\alpha a_{m+1} + \beta a_m + \gamma A_m], m \ge 1. \tag{18}$$

We denote the (n + 1)-term approximation of the solution as

$$\phi_{n+1}(t, t_0, C_0, C_1) = \sum_{m=0}^{n} a_m (t - t_0)^m. \tag{19}$$

We regard t_0, C_0, C_1 as three parameters, and generate the numeric solutions by using the (n + 1)-term approximation ϕ_{n+1}.

Partition the interval $[t_0, t_N]$ into $t_0 < t_1 < \cdots < t_N$. Here we consider an equal step-size partition with $h = t_k - t_{k-1}$. The numeric solution generated by ϕ_{n+1} is of order n. We denote the nth-order numeric solution by $u_k^{<n>}$, k = 0, 1, …, N. The one-step recurrence scheme is as follows:

$$u_0^{<n>} = C_0, \; \dot{u}_0 = C_1,$$

$$u_k^{<n>} = \phi_{n+1}(t_k, t_{k-1}, u_{k-1}^{<n>}, \dot{u}_{k-1})$$
$$= u_{k-1}^{<n>} + \dot{u}_{k-1}h + \sum_{m=2}^{n} a_m^{(k-1)} h^m,$$
$$\dot{u}_k = \frac{d}{dt}\phi_{n+1}(t, t_{k-1}, u_{k-1}^{<n>}, \dot{u}_{k-1})\Big|_{t=t_k}$$
$$= \dot{u}_{k-1} + \sum_{m=2}^{n} m a_m^{(k-1)} h^{m-1},$$
$$k = 1, 2, \cdots, N,$$

where $a_m^{(0)}$ are the a_m in (16), and for $k = 2, \cdots, N$, $a_m^{(k-1)}$, $m = 2, 3, \cdots, n$, are determined by a recursion similar to (16) with $a_0^{(k-1)} = u_{k-1}^{<n>}$ and $a_1^{(k-1)} = \dot{u}_{k-1}$.

Example 1. Consider the IVP for the Duffing equation

$$\frac{d^2u}{dt^2} - u + 2u^3 = -\sin t \sin 2t,$$
$$u(0) = 1, u'(0) = 0.$$

The IVP has the exact solution $u^*(t) = \cos t$. The Adomian polynomials for the nonlinearity $f(t) = u^3$ are

$$A_0 = a_0^3,$$
$$A_1 = 3a_0^2 a_1,$$
$$A_2 = 3a_0 a_1^2 + 3a_0^2 a_2,$$
$$A_3 = a_1^3 + 6a_0 a_1 a_2 + 3a_0^2 a_3,$$
$$A_4 = 3a_1^2 a_2 + 3a_0 a_2^2 + 6a_0 a_1 a_3 + 3a_0^2 a_4,$$
$$\cdots.$$

The 8th-order numeric solutions on the interval [0,45] are plotted in **Figure 1** with the step-size h = 0.5. The numeric solution is suitable for a larger domain as the order increases.

Example 2. Consider the IVP for the pendulum equation

$$\frac{d^2u}{dt^2} + 25\sin u = 0, u(0) = 0, u'(0) = 9.$$

The exact solution can be expressed in terms of a Jacobi elliptic function as

$$u^*(t) = 2\arcsin(\frac{9}{10}\mathrm{sn}(5t, \frac{81}{100})).$$

The Adomian polynomials in terms of the decomposition coefficients a_k for the sinusoidal nonlinearity $\sin u$ are

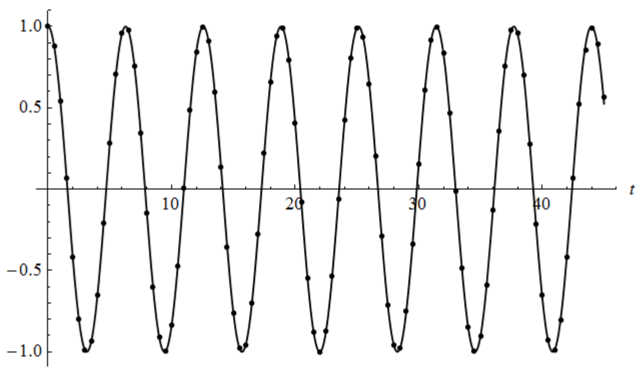

Figure 1. The exact solution (solid line) and the 8th-order numeric solution on [0,45] with h = 0.5 (dots).

$$A_0 = \sin a_0,$$
$$A_1 = a_1 \cos a_0,$$
$$A_2 = a_2 \cos a_0 - \frac{a_1^2}{2}\sin a_0,$$
$$A_3 = -\frac{a_1^3}{6}\cos a_0 - a_1 a_2 \sin a_0 - a_3 \cos a_0,$$
$$\cdots.$$

Using the initial conditions, the coefficients of solution series are calculated to be

$$a_0 = 0, a_1 = 9, a_2 = 0, a_3 = -75/2,$$
$$a_4 = 0, a_5 = 795/4, \ldots.$$

The 5-term, 10-term and 20-term approximations $\phi_5(t,0,0,9)$, $\phi_{10}(t,0,0,9)$, $\phi_{20}(t,0,0,9)$ are plotted in **Figure 2**. It is shown that the decomposition solution has a radius of convergence of more than 0.2.

The 5-term approximation $\phi_5(t,t_0,C_0,C_1)$ under the general initial conditions are calculated to be

$$\phi_5(t,t_0,C_0,C_1) = C_0 + C_1(t-t_0) - \frac{1}{2}\gamma(t-t_0)^2 \sin(C_0) - \frac{1}{6}\gamma C_1(t-t_0)^3 \cos(C_0)$$
$$+\frac{1}{24}\gamma(t-t_0)^4 \sin(C_0)(\gamma\cos(C_0) + C_1^2).$$

The 4th-order and 5th-order numeric solutions on the interval [0,6] with h = 0.1 are plotted in **Figures 3** and **4**, respectively. The 9th-order numeric solutions on the interval [0,10] with $h = 0.2$ are plotted in **Figure 5**. We observe that the higher-order numeric solutions permit a larger step-size, and enlarge the effective region.

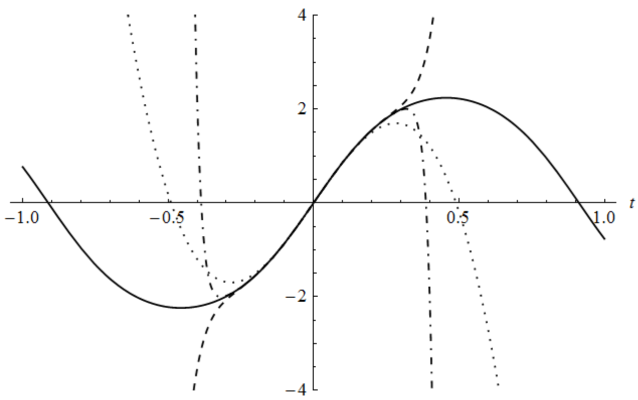

Figure 2. The exact solution $u^*(t)$ (solid line), the 5-term approximation $\phi_5(t,0,0,9)$ (dot line), the 10-term approximation $\phi_{10}(t,0,0,9)$ (dash line) and the 20-term approximation $\phi_{20}(t,0,0,9)$ (dot-dash line).

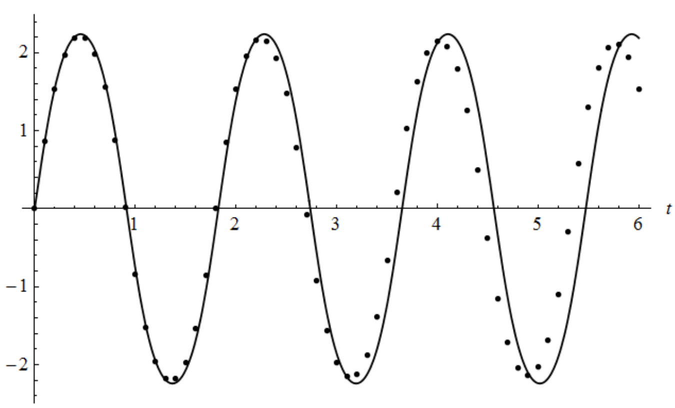

Figure 3. The exact solution (solid line) and the 4th-order numeric solution on [0,6] with h = 0.1 (dots).

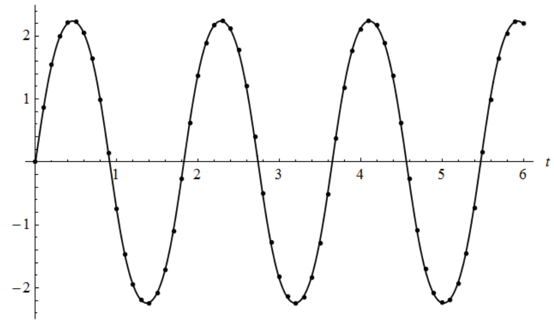

Figure 4. **The exact solution (solid line) and the 5th-order numeric solution on [0,6] with h = 0.1 (dots).**

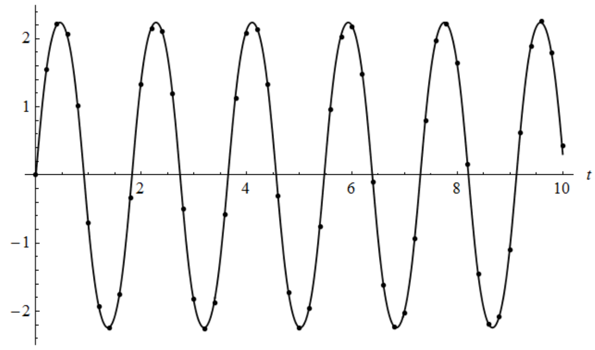

Figure 5. **The exact solution (solid line) and the 9th-order numeric solution on [0,10] with h = 0.2 (dots).**

3. Conclusions

We have developed higher-order numeric solutions for nonlinear differential equations based on the Rach-Adomian-Meyers modified decomposition method. Due to the new, efficient algorithms of the Adomian polynomials, the one-step numeric algorithm has a high efficiency, and permits us to easily generate a higher-order numeric scheme such as a 10th-order scheme, while for the Runge-Kutta method, there is no general procedure to generate higher-order numeric solutions. We demonstrated the presented numeric method by two nonlinear physical models.

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REFERENCES

[1] G. Adomian, “Nonlinear Stochastic Operator Equations,” Academic, Orlando, 1986.

[2] G. Adomian, “Solving Frontier Problems of Physics: The Decomposition Method,” Kluwer Academic, Dordrecht, 1994. http://dx.doi.org/10.1007/978-94-015-8289-6

[3] G. Adomian and R. Rach, “Inversion of Nonlinear Stochastic Operators,” *Journal of Mathematical Analysis and Applications*, Vol. 91, 1983, pp. 39-46. http://dx.doi.org/10.1016/0022-247X(83)90090-2

[4] A. M. Wazwaz, “Partial Differential Equations and Solitary Waves Theory,” Higher Education and Springer, Beijing and Berlin, 2009. http://dx.doi.org/10.1007/978-3-642-00251-9

[5] R. Rach, “A New Definition of the Adomian Polynomials,” *Kybernetes*, Vol. 37, 2008, pp. 910-955. http://dx.doi.org/10.1108/03684920810884342

[6] R. Rach, “A Convenient Computational Form for the Adomian Polynomials,” *Journal of Mathematical Analysis and Applications*, Vol. 102, 1984, pp. 415-419. http://dx.doi.org/10.1016/0022-247X(84)90181-1

[7] A. M. Wazwaz, “A New Algorithm for Calculating Adomian Polynomials for Nonlinear Operators,” *Applied Mathematics and Computation*, Vol. 111, 2000, pp. 53-69. http://dx.doi.org/10.1016/S0096-3003(99)00063-6

[8] J. S. Duan, “Recurrence Triangle for Adomian Polynomials,” *Applied Mathematics and Computation*, Vol. 216, 2010, pp. 1235-1241. http://dx.doi.org/10.1016/j.amc.2010.02.015

[9] J. S. Duan, “An Efficient Algorithm for the Multivariable Adomian Polynomials,” *Applied Mathematics and Computation*, Vol. 217, 2010, pp. 2456-2467. http://dx.doi.org/10.1016/j.amc.2010.07.046

[10] J. S. Duan, “Convenient Analytic Recurrence Algorithms for the Adomian Polynomials,” *Applied Mathematics and Computation*, Vol. 217, 2011, pp. 6337-6348. http://dx.doi.org/10.1016/j.amc.2011.01.007

[11] R. Rach, G. Adomian and R. E. Meyers, “A modified Decomposition,” *Computers & Mathematics with Applications*, Vol. 23, 1992, pp. 17-23. http://dx.doi.org/10.1016/0898-1221(92)90076-T

[12] G. Adomian and R. Rach, “Transformation of Series,” *Applied Mathematics Letters*, Vol. 4, 1991, pp. 69-71. http://dx.doi.org/10.1016/0893-9659(91)90058-4

[13] G. Adomian and R. Rach, “Nonlinear Transformation of Series—Part II,” *Computers & Mathematics with Applications*, Vol. 23, 1992, pp. 79-83. http://dx.doi.org/10.1016/0898-1221(92)90058-P

[14] G. Adomian and R. Rach, “Modified Decomposition Solution of Linear and Nonlinear Boundary-Value Problems,” *Foundations of Physics Letters*, Vol. 23, 1994, pp. 615-619. http://dx.doi.org/10.1016/0362-546X(94)90240-2

[15] G. Adomian, R. Rach and N. T. Shawagfeh, “On the Analytic Solution of the Lane-Emden Equation,” *Foundations of Physics Letters*, Vol. 8, 1995, pp. 161-181. http://dx.doi.org/10.1007/BF02187585

[16] J. C. Hsu, H. Y. Lai and C. K. Chen, “An Innovative Eigenvalue Problem Solver for Free Vibration of Uniform Timoshenko Beams by Using the Adomian Modified Decomposition Method,” *Journal of Sound Vibration*, Vol. 325, 2009, pp. 451-470. http://dx.doi.org/10.1016/j.jsv.2009.03.015

[17] G. Adomian, R. C. Rach and R. E. Meyers, “Numerical Integration, Analytic Continuation, and Decomposition,” *Applied Mathematics and Computation*, Vol. 88, 1997, pp. 95-116. http://dx.doi.org/10.1016/S0096-3003(96)00052-5