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Finite Time Domain Dynamics of Vector Fields: Stueckelberg Lagrangian

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Abstract

We study the finite time domain dynamics of vector fields under the Stueckelberg lagrangian. We consider the relevant Hamiltonian and expand in term of creation and annihilation operators. We integrate via path integral methods within the holomorphic representation and extract the corresponding Green functions in various dimensions. Further, we study the photonic massless case. Finally, we consider the possible generation of vector fields and in particular of photons from currents.

Keywords

Vector Fields, Photons, Finite Time Interval, Functional Techniques, Currents

1. Introduction

The quantum theory of field constitutes a large area of physics with various applications [1]-[5]. It studies particles and fields and their possible structure. Here, we consider finite time path integral methods to study the possible evolution of fields in contrast to possible numerical simulations [6] [7] or the use of lattice quantum field theory [5]. On considering initial experimental preparations, these methods can give predictions on finite time effects, opposed to the outcomes of asymptotic in/out ($T \to \infty$) scattering theories. Such finite time expressions apply to the case of the possible generation of particles or after the removal of trapping fields or configurations.

In previous papers, we have considered Dirac, scalar and massive vector fields [8]-[10]. Here we study the dynamics of finite time domain vector fields within the standard gauge invariant Stueckelberg lagrangian formulation with an auxiliary scalar field [1] [2]. Such a description can be used in the study of systems such as massive vector Bosons such as the W and Z particles of the electroweak theory and further photons [11]-[13]. We approach them via path integral methods. We expand the system's vector field in term of annihilation and creation operators and evaluate the corresponding Hamiltonian in term of such operators. We integrate it within the holomorphic representation and extract the Green functions in various dimensions. These Green functions describe fully the system and give the whole dynamical information of possible evolution. In fact, here we use the derived Green functions in the possible generation of fields by currents in spacetime dimensions three and four.

The present paper proceeds as follows. In section 2, we give the present system's Stueckelberg lagrangian and derive the equations that the fields satisfy. We expand in annihilation and creation operators and integrate to obtain the transition amplitude between vacuum states. Then, in section 3, we use that result to derive the Green functions for certain spacetime dimensions and give possible numerical forms. In section 4, we use these Green functions to obtain the potentials of vector fields generated by conserved currents and further we consider the vector potential that a charge and more particularly a stationary charge can generate. Moreover, in section 5 we give our conclusions. Finally, in **Appendix A**, we give the integrals that appear in section 3.

Here we set $c = \hbar = 1$. Moreover, if d is the spacetime dimension, then we assume Latin indices to range from 1 to $d-1$ and Greek ones to range from 0 to $d-1$.

2. System and Path Integration

Here we quantize a vector field. We proceed in real time with the metric $g_{\mu\nu} = diag\left(1, \underbrace{-1, \cdots, -1}_{d-1}\right)$. The first component corresponds to time. We denote the spacetime with $\{x_0 \equiv t, \boldsymbol{x}\}$, $\boldsymbol{x} \in R^{d-1}$, and the vector field with $A_\mu = \{A_0, \boldsymbol{A}\}$. Let $F_{\mu\nu}(t, \boldsymbol{x})$ be the field strength tensor. Then the Stueckelberg lagrangian of the vector field couple with a real current J_μ has the form

$$\begin{aligned} L = &-\frac{1}{4} F_{\mu\nu}(t,\boldsymbol{x}) F^{\mu\nu}(t,\boldsymbol{x}) + \frac{1}{2} m^2 A_\mu(t,\boldsymbol{x}) A^\mu(t,\boldsymbol{x}) \\ &-\frac{1}{2}\lambda\left(\partial_\mu A^\mu(t,\boldsymbol{x})\right)^2 - J_\mu(t,\boldsymbol{x}) A^\mu(t,\boldsymbol{x}) \end{aligned} \tag{1}$$

where m is the mass and

$$F^{\mu\nu}(t,\boldsymbol{x}) = \partial^\mu A^\nu(t,\boldsymbol{x}) - \partial^\nu A^\mu(t,\boldsymbol{x}) \tag{2}$$

We notice that the present model corresponds to the standard gauge invariant Stueckelberg formulation with an auxiliary scalar field.

According to variational techniques the field obeys the equation:

$$\left(\square + m^2\right) A^\nu - (1-\lambda)\partial^\nu\left(\partial_\mu A^\mu\right) = J^\nu \tag{3}$$

where $\square$ is the d'Alembertian. On taking the d-divergence of the above equation, we get

$$\lambda\left[\Box+\frac{m^2}{\lambda}\right]\partial_\mu A^\mu=\partial_\mu J^\mu \tag{4}$$

Within the present approach of a transverse/scalar decomposition (see Equation (9) below) we have to assume a conserved current. So, for conserved currents (see for instance the example in Equation (78) and the discussion there) we get

$$\lambda\left[\Box+\frac{m^2}{\lambda}\right]\partial_\mu A^\mu=0 \tag{5}$$

For $\lambda\neq 0$, $\partial_\mu A^\mu$ is a scalar field obeying a Klein-Gordon equation with square mass

$$M^2=\frac{m^2}{\lambda} \tag{6}$$

We assume λ to be positive in order M^2 not to be negative. We define the field

$$A_\nu^{\mathrm{T}}=A_\nu+\frac{1}{M^2}\partial_\nu\left(\partial_\mu A^\mu\right)=A_\nu+\frac{\lambda}{m^2}\partial_\nu\left(\partial_\mu A^\mu\right) \tag{7}$$

Due to Equation (5) A_ν^{T} is divergenceless

$$\partial^\nu A_\nu^{\mathrm{T}}=0 \tag{8}$$

Correspondingly A splits into a transverse (spin 1) and a scalar part. *i.e.*

$$A_\nu=A_\nu^{\mathrm{T}}-\frac{\lambda}{m^2}\partial_\nu\left(\partial_\mu A^\mu\right) \tag{9}$$

The conjugate momenta have the form

$$\pi^\nu\left(t,\boldsymbol{x}\right)=-F^{0\nu}\left(t,\boldsymbol{x}\right)-\lambda g^{0\nu}\left(\partial_\mu A^\mu\right) \tag{10}$$

Then we have $\pi^0=-\lambda\partial_\rho A^\rho$ while and the electric field is given as $E^i=\pi^i=-F^{0i}=-\partial^0A^i+\partial^iA^0$.

So, we can get the following expression for the Hamiltonian

$$\begin{aligned}H\left(E,A\right)&=\pi^\mu\partial_0A_\mu-L=\pi^i\left(-\pi_i+\partial_iA_0\right)-\lambda\partial_\rho A^\rho\partial_0A_0-L\\&=\frac{1}{2}\boldsymbol{E}^2+\frac{1}{4}F^{ij}F_{ij}-\left(\partial_i\pi^i\right)A^0+\frac{m^2}{2}\boldsymbol{A}^2-\frac{m^2}{2}\left(A^0\right)^2\\&\quad+\frac{1}{2}\lambda\left(\partial_\rho A^\rho\right)^2-\lambda\partial_\rho A^\rho\partial_0A^0+J^0A^0-\boldsymbol{J}\cdot\boldsymbol{A}\\&=\frac{1}{2}\boldsymbol{E}^2+\frac{1}{4}F^{ij}F_{ij}-\left(\nabla\cdot\boldsymbol{E}\right)A^0+\frac{m^2}{2}\boldsymbol{A}^2-\frac{m^2}{2}\left(A^0\right)^2\\&\quad-\frac{1}{2}\lambda\left[\left(\partial_0A^0\right)^2-\left(\partial_iA^i\right)^2\right]+J^0A^0-\boldsymbol{J}\cdot\boldsymbol{A}\end{aligned} \tag{11}$$

We have used $\partial^0A^i=F^{0i}+\partial^iA^0=-\pi^i+\partial^iA^0$ and integrated by parts to make the replacement $\pi^i\partial_iA_0\to\left(\partial_i\pi^i\right)A^0$. Further $\partial_i\pi^i=\nabla\cdot\boldsymbol{E}$.

Moreover, the following equal time canonical commutation relations must be valid

$$\left[A^\mu\left(t,\boldsymbol{x}\right),\pi^\nu\left(t,\boldsymbol{y}\right)\right]=ig^{\mu\nu}\delta^{(d-1)}\left(\boldsymbol{x}-\boldsymbol{y}\right) \tag{12}$$

$$\left[\pi^{\mu}(t,\boldsymbol{x}),\pi^{\nu}(t,\boldsymbol{y})\right]=\left[A^{\mu}(t,\boldsymbol{x}),A^{\nu}(t,\boldsymbol{y})\right]=0 \tag{13}$$

$$\left[\dot{A}^{0}(t,\boldsymbol{x}),\dot{A}^{0}(t,\boldsymbol{y})\right]=\left[\dot{A}^{i}(t,\boldsymbol{x}),\dot{A}^{j}(t,\boldsymbol{y})\right]=0 \tag{14}$$

$$\left[\dot{A}^{\mu}(t,\boldsymbol{x}),A^{\nu}(t,\boldsymbol{y})\right]=ig^{\mu\nu}\left[1+\left(\frac{1}{\lambda}-1\right)g^{\mu 0}\right]\delta^{(d-1)}(\boldsymbol{x}-\boldsymbol{y}) \tag{15}$$

$$\left[\dot{A}^{0}(t,\boldsymbol{x}),\dot{A}_{j}(t,\boldsymbol{y})\right]=i\left(1-\frac{1}{\lambda}\right)\partial_{j}^{x}\delta^{(d-1)}(\boldsymbol{x}-\boldsymbol{y}) \tag{16}$$

Now we expand the field in terms of creation and annihilation operators. So

$$\begin{aligned}A^{\mu}(t,\boldsymbol{x})=&\int\frac{\mathrm{d}^{d-1}k}{(2\pi)^{d-1}2\omega_{\boldsymbol{k}}^{m}}\sum_{\lambda=1}^{d-1}\left[a^{(\lambda)+}(\boldsymbol{k})\varepsilon^{\mu*}(\boldsymbol{k},\lambda)\mathrm{e}^{-i\boldsymbol{k}\cdot\boldsymbol{x}}+a^{(\lambda)}(\boldsymbol{k})\varepsilon^{\mu}(\boldsymbol{k},\lambda)\mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{x}}\right]\\&+\int\frac{\mathrm{d}^{d-1}k}{(2\pi)^{d-1}2\omega_{\boldsymbol{k}}^{M}}\frac{k^{\mu}}{m}\left[a^{(0)+}(\boldsymbol{k})\mathrm{e}^{-i\boldsymbol{k}\cdot\boldsymbol{x}}+a^{(0)}(\boldsymbol{k})\mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{x}}\right]\end{aligned} \tag{17}$$

where

$$\omega_{\boldsymbol{k}}^{m}=\sqrt{m^{2}+\boldsymbol{k}^{2}} \tag{18}$$

and

$$\omega_{\boldsymbol{k}}^{M}=\sqrt{M^{2}+\boldsymbol{k}^{2}} \tag{19}$$

From the dynamic equations we can derive the following equations that the free creation and annihilation operators obey

$$a^{(\lambda)+}(t,\boldsymbol{k})=a^{(\lambda)+}(0,\boldsymbol{k})\mathrm{e}^{i\omega_{\boldsymbol{k}}^{m}t} \tag{20}$$

$$a^{(0)+}(t,\boldsymbol{k})=a^{(0)+}(0,\boldsymbol{k})\mathrm{e}^{i\omega_{\boldsymbol{k}}^{M}t} \tag{21}$$

$$a^{(\lambda)}(t,\boldsymbol{k})=a^{(\lambda)}(0,\boldsymbol{k})\mathrm{e}^{-i\omega_{\boldsymbol{k}}^{m}t} \tag{22}$$

$$a^{(0)}(t,\boldsymbol{k})=a^{(0)}(0,\boldsymbol{k})\mathrm{e}^{-i\omega_{\boldsymbol{k}}^{M}t} \tag{23}$$

The $d-1$ vectors $\varepsilon(\boldsymbol{k},\lambda)$ give the $d-1$ polarization directions. $d-2$ of those polarization directions are supposed to be around the direction of motion and another spacelike one is supposed to have momentum in the direction of motion, so that $k\cdot\varepsilon(\boldsymbol{k},\lambda)=0$. In the particular case of a massive vector boson moving along the $d-1$ direction we have $k^{\mu}=\left(\omega_{\boldsymbol{k}}^{m},\cdots,|\boldsymbol{k}|\right)$. On setting $\boldsymbol{\varepsilon}(\boldsymbol{k},\lambda)=\left(\varepsilon^{1}(\boldsymbol{k},\lambda),\cdots,\varepsilon^{d-1}(\boldsymbol{k},\lambda)\right)$ we get $\varepsilon^{\mu}(\boldsymbol{k},\lambda)=\left(0,\boldsymbol{\varepsilon}(\boldsymbol{k},\lambda)\right)$ for $\lambda=1,\cdots,d-2$ and $\varepsilon^{\mu}(\boldsymbol{k},d-1)=\left(\frac{|\boldsymbol{k}|}{m},\cdots,\frac{\omega_{\boldsymbol{k}}^{m}}{m}\right)$.

The electric field has the form

$$\begin{aligned}E^{i}(t,\boldsymbol{x})=&-i\int\frac{\mathrm{d}^{d-1}k}{(2\pi)^{d-1}2\omega_{\boldsymbol{k}}^{m}}\sum_{\lambda=1}^{d-1}\left[\omega_{\boldsymbol{k}}^{m}a^{(\lambda)+}(\boldsymbol{k})\tilde{\varepsilon}^{i*}(\boldsymbol{k},\lambda)\mathrm{e}^{-i\boldsymbol{k}\cdot\boldsymbol{x}}-\omega_{\boldsymbol{k}}^{m}a^{(\lambda)}(\boldsymbol{k})\tilde{\varepsilon}^{i}(\boldsymbol{k},\lambda)\mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{x}}\right]\\&-i\int\frac{\mathrm{d}^{d-1}k}{(2\pi)^{d-1}\omega_{\boldsymbol{k}}^{M}}\frac{k^{i}}{m}\left[\omega_{\boldsymbol{k}}^{M}a^{(0)+}(\boldsymbol{k})\mathrm{e}^{-i\boldsymbol{k}\cdot\boldsymbol{x}}-\omega_{\boldsymbol{k}}^{M}a^{(0)}(\boldsymbol{k})\mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{x}}\right]\end{aligned} \tag{24}$$

From the definition of the electric field upon setting

$\tilde{\boldsymbol{\varepsilon}}(\boldsymbol{k},\lambda)=\left(\tilde{\varepsilon}^{1}(\boldsymbol{k},\lambda),\cdots,\tilde{\varepsilon}^{d-1}(\boldsymbol{k},\lambda)\right)$ we obtain the relation [2]

$$\tilde{\boldsymbol{\varepsilon}}(\boldsymbol{k},\lambda)=\boldsymbol{\varepsilon}(\boldsymbol{k},\lambda)-\frac{\boldsymbol{k}}{\omega_{\boldsymbol{k}}^{m}}\varepsilon^{0}(\boldsymbol{k},\lambda)=\boldsymbol{\varepsilon}(\boldsymbol{k},\lambda)-\boldsymbol{k}\frac{\boldsymbol{k}\cdot\boldsymbol{\varepsilon}(\boldsymbol{k},\lambda)}{\left(\omega_{\boldsymbol{k}}^{m}\right)^{2}} \tag{25}$$

From the relations $k\cdot\varepsilon(\boldsymbol{k},\lambda)=\omega_{\boldsymbol{k}}^{m}\varepsilon^{0}(\boldsymbol{k},\lambda)-\boldsymbol{k}\cdot\boldsymbol{\varepsilon}(\boldsymbol{k},\lambda)=0$ we have obtained

$$\varepsilon^{0}(\boldsymbol{k},\lambda)=\frac{\boldsymbol{k}\cdot\boldsymbol{\varepsilon}(\boldsymbol{k},\lambda)}{\omega_{\boldsymbol{k}}^{m}} \tag{26}$$

By direct evaluation F^{ij} has the form

$$F^{ij}(t,\boldsymbol{x})=-i\int\frac{\mathrm{d}^{d-1}k}{(2\pi)^{d-1}2\omega_{\boldsymbol{k}}^{m}}\sum_{\lambda=1}^{d-1}\Big\{a^{(\lambda)+}(\boldsymbol{k})\left[k^{i}\varepsilon^{j*}(\boldsymbol{k},\lambda)-k^{j}\varepsilon^{i*}(\boldsymbol{k},\lambda)\right]\mathrm{e}^{-i\boldsymbol{k}\cdot\boldsymbol{x}} \\ -a^{(\lambda)}(\boldsymbol{k})\left[k^{i}\varepsilon^{j}(\boldsymbol{k},\lambda)-k^{j}\varepsilon^{i}(\boldsymbol{k},\lambda)\right]\mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{x}}\Big\} \tag{27}$$

In order the above equations to be consistent the following commutation rules must be valid

$$\left[a^{(\lambda)}(\boldsymbol{k}),a^{(\lambda')+}(\boldsymbol{k}')\right]=\delta_{\lambda\lambda'}2\omega_{\boldsymbol{k}}^{m}(2\pi)^{d-1}\delta^{(d-1)}(\boldsymbol{k}-\boldsymbol{k}')\quad 1\le\lambda,\lambda'\le d-1 \tag{28}$$

$$\left[a^{(0)}(\boldsymbol{k}),a^{(0)+}(\boldsymbol{k}')\right]=-2\omega_{\boldsymbol{k}}^{M}(2\pi)^{d-1}\delta^{(d-1)}(\boldsymbol{k}-\boldsymbol{k}') \tag{29}$$

All the other commutators vanish.

The minus sign in the last commutant is a signal of the indefinite metric that we introduce to face the gauge dependence of A_{μ}. So, we introduce an indefinite norm Fock space to preserve the locality properties although only vectors in a physical positive norm subspace receive physical interpretation.

Eventually, we get the following diagonal Hamiltonian in normal order form

$$I\left(a^{(\lambda)}(\boldsymbol{k}),a^{(0)}(\boldsymbol{k}),a^{(\lambda)+}(\boldsymbol{k}),a^{(0)+}(\boldsymbol{k}),t\right)=\int\mathrm{d}^{d-1}xH(E,A) \\ =\int\frac{\mathrm{d}^{d-1}k}{(2\pi)^{d-1}2\omega_{\boldsymbol{k}}^{m}}\sum_{\lambda=1}^{d-1}\left[\omega_{\boldsymbol{k}}^{m}a^{(\lambda)+}(\boldsymbol{k})a^{(\lambda)}(\boldsymbol{k})+j^{(\lambda)*}(t,\boldsymbol{k})a^{(\lambda)}(\boldsymbol{k})+j^{(\lambda)}(t,\boldsymbol{k})a^{(\lambda)+}(\boldsymbol{k})\right] \\ +\int\frac{\mathrm{d}^{d-1}k}{(2\pi)^{d-1}2\omega_{\boldsymbol{k}}^{M}}\left[-\omega_{\boldsymbol{k}}^{M}a^{(0)+}(\boldsymbol{k})a^{(0)}(\boldsymbol{k})+j^{(0)*}(t,\boldsymbol{k})a^{(0)}(\boldsymbol{k})+j^{(0)}(t,\boldsymbol{k})a^{(0)+}(\boldsymbol{k})\right] \tag{30}$$

where

$$j^{(\lambda)}(t,\boldsymbol{k})=\varepsilon^{\mu*}(\boldsymbol{k},\lambda)J_{\mu}(t,\boldsymbol{k})\quad 1\le\lambda\le d-1 \tag{31}$$

$$j^{(0)}(t,\boldsymbol{k})=\frac{k^{\mu}}{m}J_{\mu}(t,\boldsymbol{k}) \tag{32}$$

$$J_{\mu}(t,\boldsymbol{k})=\int\mathrm{d}^{d-1}xJ_{\mu}(t,\boldsymbol{x})\mathrm{e}^{-i\boldsymbol{k}\cdot\boldsymbol{x}} \tag{33}$$

The space-like orthonormalized vectors $\varepsilon^{\mu}(\boldsymbol{k},\lambda)$ are orthogonal to the time-like vector k^{μ} as well, and therefore if we assume them real

$$\varepsilon(\boldsymbol{k},\lambda)\cdot\varepsilon(\boldsymbol{k},\lambda')=-\delta_{\lambda\lambda'} \tag{34}$$

and

$$\sum_{\lambda}\varepsilon^{\mu}(\boldsymbol{k},\lambda)\varepsilon^{\nu}(\boldsymbol{k},\lambda)=-\left(g^{\mu\nu}-\frac{k^{\mu}k^{\nu}}{m^{2}}\right) \tag{35}$$

Now we study the Hamiltonian

$$H_0 = \omega a^+ a + j^*(t)a + j(t)a^+ \tag{36}$$

Hamiltonians of such a form appear in Equation (30).

We intend to construct a path integral representation of the evolution operator

$$U_0(T) = \mathrm{e}^{-i\int_0^T H_0 \mathrm{d}\tau} \tag{37}$$

We work within the holomorphic representation. We introduce the complex variables α^* and proceed via path integral methods. We represent the operators a^+, a with the operators α^* and $\frac{\partial}{\partial\alpha^*}$ respectively. They act on functions of α^* obeying the same commutation relations. Then the Hamiltonian H_0 has the representation

$$H_0 = \omega\alpha^* \frac{\partial}{\partial\alpha^*} + j^*(t)\frac{\partial}{\partial\alpha^*} + j(t)\alpha^* \tag{38}$$

and for a small time t we obtain

$$\begin{aligned} \left\langle \alpha^* \middle| U_0(t) \middle| \alpha' \right\rangle &= \left[1 - it\left(\omega\alpha^* \frac{\partial}{\partial\alpha^*} + j^*(t)\frac{\partial}{\partial\alpha^*} + j(t)\alpha^*\right) + O(t^2)\right]\mathrm{e}^{\alpha^*\alpha'} \\ &= \left[1 - it\left(\omega\alpha^*\alpha' + j^*(t)\alpha' + j(t)\alpha^*\right) + O(t^2)\right]\mathrm{e}^{\alpha^*\alpha'} \\ &= \mathrm{e}^{\alpha^*\alpha'(1-it\omega) - it\left(\alpha^* j(t) + j^*(t)\alpha'\right)} + O(t^2) \end{aligned} \tag{39}$$

Moreover, according to the group property

$$\left\langle \alpha^* \middle| U_2 U_1 \middle| \alpha' \right\rangle = \int \frac{\mathrm{d}\alpha'' \mathrm{d}\alpha''^*}{2\pi i} \left\langle \alpha^* \middle| U_2 \middle| \alpha'' \right\rangle \mathrm{e}^{-\alpha''^*\alpha''} \left\langle \alpha''^* \middle| U_1 \middle| \alpha' \right\rangle \tag{40}$$

After multiple application of the Equations (39)-(40) we obtain the evolution operator at finite time in the following path integral representation

$$\left\langle \alpha_f^* \middle| U_0(T) \middle| \alpha_i \right\rangle = \lim_{N\to\infty} \int \prod_{p=1}^{N-1} \frac{\mathrm{d}\alpha_p^* \mathrm{d}\alpha_p}{2\pi i} \exp\left[iS_0\left(\alpha_p^*, \alpha_p\right)\right] \tag{41}$$

where

$$S_0\left(\alpha_p^*, \alpha_p\right) = i\sum_{p=1}^{N-1} \alpha_p^*\left(\alpha_p - \alpha_{p-1}\right) - i\alpha_N^*\alpha_{N-1} - \omega\varepsilon\sum_{p=1}^{N} \alpha_p^*\alpha_{p-1} - \varepsilon\sum_{p=1}^{N}\left(j_p\alpha_p^* + j_{p-1}^*\alpha_{p-1}\right) \tag{42}$$

$$\varepsilon = \frac{T}{N} \tag{43}$$

and

$$\alpha_0 = \alpha_i, \quad \alpha_N^* = \alpha_f^* \tag{44}$$

If we let $N \to \infty$ we obtain the following path integral form

$$\left\langle \alpha_f^* \middle| U_0(T) \middle| \alpha_i \right\rangle = \int_{\alpha(0)=\alpha_i}^{\alpha^*(T)=\alpha_f^*} \frac{D\alpha^*(\tau) D\alpha(\tau)}{2\pi i} \exp\left[iS_0\left(\alpha^*(\tau), \alpha(\tau)\right)\right] \tag{45}$$

The action has the form

$$\begin{aligned} S_0\left(\alpha^*(\tau), \alpha(\tau)\right) = -i\alpha_f^*\alpha(T) + \int_0^T \mathrm{d}\tau \Big\{ &\alpha^*(\tau)\left[i\dot{\alpha}(\tau) - \omega\alpha(\tau)\right] \\ &- \alpha^*(\tau) j(\tau) - j^*(\tau)\alpha(\tau) \Big\} \end{aligned} \tag{46}$$

The path integral (45) with the action (46) is Gaussian and can be evaluated exactly. By varying $\alpha^*(\tau)$ the saddle point equation yields

$$i\dot{\alpha}(\tau)-\omega\alpha(\tau)-j(\tau)=0 \tag{47}$$

with solution

$$\alpha(t)=\mathrm{e}^{-i\omega t}\alpha_i-i\int_0^t \mathrm{d}\tau\,\mathrm{e}^{-i\omega(t-\tau)}j(\tau) \tag{48}$$

For completeness we give relations for $\alpha^*(t)$ as well. So, on varying $\alpha(t)$ we get

$$-i\dot{\alpha}^*(\tau)-\omega\alpha^*(\tau)-j^*(\tau)=0 \tag{49}$$

with solution

$$\alpha^*(t)=\mathrm{e}^{-i\omega(T-t)}\alpha_f^*-i\int_t^T \mathrm{d}\tau\,\mathrm{e}^{-i\omega(\tau-t)}j^*(\tau) \tag{50}$$

In order to derive the above equations, we have taken into account the boundary conditions given in Equation (44).

Now we can use the differential Equation (47) to write Equation (46) in the form

$$S_0\left(\alpha^*(\tau),\alpha(\tau)\right)=-i\alpha_f^*\alpha(T)-\int_0^T \mathrm{d}\tau\, j^*(\tau)\alpha(\tau) \tag{51}$$

Finally, since the integrals are Gaussian, we get

$$\left\langle\alpha_f^*\middle|U_0(T)\middle|\alpha_i\right\rangle=F(T)\exp\left[iS_0\left(\alpha^*(\tau),\alpha(\tau)\right)\right] \tag{52}$$

So, on using Equations (48) (51) (52) we get

$$\begin{aligned}\left\langle\alpha_f^*\middle|U_0(T)\middle|\alpha_i\right\rangle=\exp\Bigg\{&\alpha_f^*\mathrm{e}^{-i\omega T}\alpha_i-i\int_0^T \mathrm{d}t\left(\alpha_f^*\mathrm{e}^{-i\omega(T-t)}j(t)+j^*(t)\mathrm{e}^{-i\omega t}\alpha_i\right)\\&-\int_0^T\int_0^t j^*(t)\mathrm{e}^{-i\omega(t-t')}j(t')\mathrm{d}t'\mathrm{d}t\Bigg\}\end{aligned} \tag{53}$$

where the semigroup property of the path integral implies $F(T)=1$.

The Hamiltonian (30) is a superposition of Hamiltonians of the form (36). So, we can obtain the coherent states propagator of the Hamiltonian (30) in the form

$$\begin{aligned}&U_0\left(\alpha_f^{(\lambda)*}(\boldsymbol{k}),\alpha_i^{(\lambda)}(\boldsymbol{k}),\alpha_f^{(0)*}(\boldsymbol{k}),\alpha_i^{(0)}(\boldsymbol{k}),T;J\right)\\&=\exp\Bigg\{\int\frac{\mathrm{d}^{d-1}k}{(2\pi)^{d-1}2\omega_{\boldsymbol{k}}^m}\sum_{\lambda=1}^{d-1}\Bigg[\alpha_f^{(\lambda)*}(\boldsymbol{k})\mathrm{e}^{-i\omega_{\boldsymbol{k}}^m T}\alpha_i^{(\lambda)}(\boldsymbol{k})\\&\quad-i\int_0^T\mathrm{d}t\left(\alpha_f^{(\lambda)*}(\boldsymbol{k})\mathrm{e}^{-i\omega_{\boldsymbol{k}}^m(T-t)}j^{(\lambda)}(t,\boldsymbol{k})+j^{(\lambda)*}(t,\boldsymbol{k})\mathrm{e}^{-i\omega_{\boldsymbol{k}}^m t}\alpha_i^{(\lambda)}(\boldsymbol{k})\right)\\&\quad-\int_0^T\int_0^t j^{(\lambda)*}(t,\boldsymbol{k})\mathrm{e}^{-i\omega_{\boldsymbol{k}}^m(t-t')}j^{(\lambda)}(t',\boldsymbol{k})\mathrm{d}t'\mathrm{d}t\Bigg]\\&\quad+\int\frac{\mathrm{d}^{d-1}k}{(2\pi)^{d-1}2\omega_{\boldsymbol{k}}^M}\Bigg[\alpha_f^{(0)*}(\boldsymbol{k})\mathrm{e}^{i\omega_{\boldsymbol{k}}^M T}\alpha_i^{(0)}(\boldsymbol{k})\\&\quad-i\int_0^T\mathrm{d}t\left(\alpha_f^{(0)*}(\boldsymbol{k})\mathrm{e}^{i\omega_{\boldsymbol{k}}^M(T-t)}j^{(0)}(t,\boldsymbol{k})+j^{(0)*}(t,\boldsymbol{k})\mathrm{e}^{i\omega_{\boldsymbol{k}}^M t}\alpha_i^{(0)}(\boldsymbol{k})\right)\\&\quad-\int_0^T\int_0^t j^{(0)*}(t,\boldsymbol{k})\mathrm{e}^{i\omega_{\boldsymbol{k}}^M(t-t')}j^{(0)}(t',\boldsymbol{k})\mathrm{d}t'\mathrm{d}t\Bigg]\Bigg\}\end{aligned} \tag{54}$$

In Equation (30), Equation (54) notice the different sign in front of the symbol $\omega_{\boldsymbol{k}}^{M}$ compared with the sign in front of the symbol $\omega_{\boldsymbol{k}}^{m}$.

To extract the generating functional of the correlations functions of the present system we have to integrate diagonally [14] [15] from vacuum to vacuum. Then

$$
\begin{aligned}
Z(J) &= U(0,0,0,0,T;J) \\
&= \prod_{\boldsymbol{k}}\prod_{\kappa=0}^{d-1}\left[\int \frac{\mathrm{d}^2\alpha^{(\kappa)}(\boldsymbol{k})}{\pi}\right]\prod_{\boldsymbol{k}}\prod_{\kappa=0}^{d-1}\left[\left\langle 0\middle|\alpha^{(\kappa)}(\boldsymbol{k})\right\rangle\right] \\
&\quad\times U_0\left(\alpha^{(\lambda)*}(\boldsymbol{k}),\alpha^{(\lambda)}(\boldsymbol{k}),\alpha^{(0)*}(\boldsymbol{k}),\alpha^{(0)}(\boldsymbol{k}),T;J\right)\prod_{\boldsymbol{k}}\prod_{\kappa=0}^{d-1}\left[\left\langle \alpha^{(\kappa)}(\boldsymbol{k})\middle|0\right\rangle\right]
\end{aligned}
\tag{55}
$$

So since

$$\left\langle 0\middle|\alpha^{(\kappa)}(\boldsymbol{k})\right\rangle = \exp\left[-\frac{1}{2}\left|\alpha^{(\kappa)}(\boldsymbol{k})\right|^2\right] \tag{56}$$

and

$$\left\langle \alpha^{(\kappa)}(\boldsymbol{k})\middle|0\right\rangle = \exp\left[-\frac{1}{2}\left|\alpha^{(\kappa)}(\boldsymbol{k})\right|^2\right] \tag{57}$$

we get

$$
\begin{aligned}
Z(J) = \exp\Bigg\{ &-\int \frac{\mathrm{d}^{d-1}k}{(2\pi)^{d-1}2\omega_{\boldsymbol{k}}^{m}}\sum_{\lambda=1}^{d-1}\ln\left(1-\mathrm{e}^{-i\omega_{\boldsymbol{k}}^{m}T}\right) - \int \frac{\mathrm{d}^{d-1}k}{(2\pi)^{d-1}2\omega_{\boldsymbol{k}}^{M}}\ln\left(1-\mathrm{e}^{i\omega_{\boldsymbol{k}}^{M}T}\right) \\
&- i\frac{1}{2}\int \frac{\mathrm{d}^{d-1}k}{(2\pi)^{d-1}2\omega_{\boldsymbol{k}}^{m}}\sum_{\lambda=1}^{d-1}\int_0^T\int_0^T j^{(\lambda)*}(t,\boldsymbol{k})\frac{\cos\left(\omega_{\boldsymbol{k}}^{m}|t-t'|-\frac{\omega_{\boldsymbol{k}}^{m}T}{2}\right)}{\sin\left(\frac{\omega_{\boldsymbol{k}}^{m}T}{2}\right)}j^{(\lambda)}(t',\boldsymbol{k})\mathrm{d}t'\mathrm{d}t \\
&+ i\frac{1}{2}\int \frac{\mathrm{d}^{d-1}k}{(2\pi)^{d-1}2\omega_{\boldsymbol{k}}^{M}}\int_0^T\int_0^T j^{(0)*}(t,\boldsymbol{k})\frac{\cos\left(\omega_{\boldsymbol{k}}^{M}|t-t'|-\frac{\omega_{\boldsymbol{k}}^{M}T}{2}\right)}{\sin\left(\frac{\omega_{\boldsymbol{k}}^{M}T}{2}\right)}j^{(0)}(t',\boldsymbol{k})\mathrm{d}t'\mathrm{d}t\Bigg\}
\end{aligned}
\tag{58}
$$

We can get the finite time interval Green function of vector fields from the present considerations. We do that in the next section.

3. Green Function

We procced to the extraction of the finite time domain Green function of the Stueckelberg lagrangian describing vector fields in a series of possible representations.

According to the discussion of the previous section the d dimensional Green function is given as

$$G^{(d)\mu\nu}(\boldsymbol{x}-\boldsymbol{x}',t-t';T) = -\frac{1}{Z(0)}\frac{\delta^2}{\delta J_\mu \delta J_\nu}Z(J)\bigg|_{J=0} \tag{59}$$

So, on performing the functional derivations according to Equations (58)-(59) we get

$$G^{(d)\mu\nu}\left(\boldsymbol{x}-\boldsymbol{x}',t-t';T\right)$$
$$=-i\int\frac{d^{d-1}k}{(2\pi)^{d-1}}e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}\left[\frac{\cos\left(\omega_{\vec{k}}^{m}\left|t-t'\right|-\frac{\omega_{\vec{k}}^{m}T}{2}\right)}{2\omega_{\vec{k}}^{m}\sin\left(\frac{\omega_{\vec{k}}^{m}T}{2}\right)}\left(g^{\mu\nu}-\frac{k^{\mu}k^{\nu}}{m^{2}}\right)\Bigg|_{k^{0}=\omega_{\vec{k}}^{m}}\right.$$
$$\left.+\frac{\cos\left(\omega_{\boldsymbol{k}}^{M}\left|t-t'\right|-\frac{\omega_{\boldsymbol{k}}^{M}T}{2}\right)}{2\omega_{\boldsymbol{k}}^{M}\sin\left(\frac{\omega_{\boldsymbol{k}}^{M}T}{2}\right)}\frac{k^{\mu}k^{\nu}}{m^{2}}\Bigg|_{k^{0}=\omega_{\boldsymbol{k}}^{M}}\right] \tag{60}$$

where $0\le t,t'\le T$. To derive Equation (60) we have applied Equation (35).

Further, we remove $\frac{k^{\mu}k^{\nu}}{m^{2}}$ by replacing the various k^{μ},k^{ν} with appropriate derivatives. So, we get the forms

$$G^{(d)0i}\left(\boldsymbol{x}-\boldsymbol{x}',t-t';T\right)=G^{(d)i0}\left(\boldsymbol{x}-\boldsymbol{x}',t-t';T\right)$$
$$=i\frac{1}{2m^{2}}\int\frac{d^{d-1}k}{(2\pi)^{d-1}}e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}k^{i}\left[\frac{\cos\left(\omega_{\vec{k}}^{m}\left|t-t'\right|-\frac{\omega_{\vec{k}}^{m}T}{2}\right)}{2\omega_{\vec{k}}^{m}\sin\left(\frac{\omega_{\vec{k}}^{m}T}{2}\right)}\right.$$
$$\left.-\frac{\cos\left(\omega_{\boldsymbol{k}}^{M}\left|t-t'\right|-\frac{\omega_{\boldsymbol{k}}^{M}T}{2}\right)}{\sin\left(\frac{\omega_{\boldsymbol{k}}^{M}T}{2}\right)}\right] \tag{61a}$$

and

$$G^{(d)\mu\nu}\left(\boldsymbol{x}-\boldsymbol{x}',t-t';T\right)$$
$$=-i\left(g^{\mu\nu}+\frac{\partial_{x}^{\mu}\partial_{x}^{\nu}}{m^{2}}\right)\int\frac{\mathrm{d}^{d-1}k}{(2\pi)^{d-1}}\mathrm{e}^{i\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{x}')}\frac{\cos\left(\omega_{\boldsymbol{k}}^{m}\left|t-t'\right|-\frac{\omega_{\boldsymbol{k}}^{m}T}{2}\right)}{2\omega_{\boldsymbol{k}}^{m}\sin\left(\frac{\omega_{\boldsymbol{k}}^{m}T}{2}\right)}$$
$$+i\frac{\partial_{x}^{\mu}\partial_{x}^{\nu}}{m^{2}}\int\frac{\mathrm{d}^{d-1}k}{(2\pi)^{d-1}}\mathrm{e}^{i\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{x}')}\frac{\cos\left(\omega_{\boldsymbol{k}}^{M}\left|t-t'\right|-\frac{\omega_{\boldsymbol{k}}^{M}T}{2}\right)}{2\omega_{\boldsymbol{k}}^{M}\sin\left(\frac{\omega_{\boldsymbol{k}}^{M}T}{2}\right)} \tag{61b}$$

for the remaining matrix elements.

Equations (61a, 61b) give the Green function that describes the dynamics of vector fields within the Stueckelberg lagrangian. We can use it in the study of their generation and propagation in a spacetime of a finite time interval.

Proceeding to the study of the Green function (61a, 61b) we write the sine function in terms of exponentials. Then

$$G^{(d)0i}\left(\boldsymbol{x}-\boldsymbol{x}',t-t';T\right)=G^{(d)i0}\left(\boldsymbol{x}-\boldsymbol{x}',t-t';T\right)$$
$$=-\frac{1}{m^2}\int\frac{d^{d-1}k}{(2\pi)^{d-1}}e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}k^i\left[\frac{\cos\left(\omega_{\vec{k}}^m\left|t-t'\right|-\frac{\omega_{\vec{k}}^m T}{2}\right)}{1-e^{-i\omega_{\vec{k}}^m T}}e^{\frac{-i\omega_{\vec{k}}^m T}{2}}\right.$$
$$\left.-\frac{\cos\left(\omega_{\boldsymbol{k}}^M\left|t-t'\right|-\frac{\omega_{\boldsymbol{k}}^M T}{2}\right)}{1-\mathrm{e}^{-i\omega_{\boldsymbol{k}}^M T}}\mathrm{e}^{\frac{-i\omega_{\boldsymbol{k}}^M T}{2}}\right]\tag{62a}$$

and

$$G^{(d)\mu\nu}\left(\boldsymbol{x}-\boldsymbol{x}',t-t';T\right)$$
$$=\left(g^{\mu\nu}+\frac{\partial_x^\mu\partial_x^\nu}{m^2}\right)\int\frac{\mathrm{d}^{d-1}k}{(2\uppi)^{d-1}}\mathrm{e}^{i\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{x}')}\frac{\cos\left(\omega_{\boldsymbol{k}}^m\left|t-t'\right|-\frac{\omega_{\boldsymbol{k}}^m T}{2}\right)}{\omega_{\boldsymbol{k}}^m\left(1-\mathrm{e}^{-i\omega_{\boldsymbol{k}}^m T}\right)}\mathrm{e}^{\frac{-i\omega_{\boldsymbol{k}}^m T}{2}}\tag{62b}$$
$$-\frac{\partial_x^\mu\partial_x^\nu}{m^2}\int\frac{\mathrm{d}^{d-1}k}{(2\uppi)^{d-1}}\mathrm{e}^{i\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{x}')}\frac{\cos\left(\omega_{\boldsymbol{k}}^M\left|t-t'\right|-\frac{\omega_{\boldsymbol{k}}^M T}{2}\right)}{\omega_{\boldsymbol{k}}^M\left(1-\mathrm{e}^{-i\omega_{\boldsymbol{k}}^M T}\right)}\mathrm{e}^{\frac{-i\omega_{\boldsymbol{k}}^M T}{2}}$$

for the remaining matrix elements.

Now we can expand the denominators of Equations (62a, 62b) in geometric series to get

$$G^{(d)0i}\left(\boldsymbol{x},\tau;T\right)=G^{(d)i0}\left(\boldsymbol{x},\tau;T\right)$$
$$=-\frac{1}{2m^2}\int\frac{d^{d-1}k}{(2\pi)^{d-1}}e^{i\vec{k}\cdot\vec{x}}k^i\left[e^{i\omega_{\vec{k}}^m|\tau|}\sum_{n=0}^{\infty}e^{-i(n+1)\omega_{\vec{k}}^m T}+e^{-i\omega_{\vec{k}}^m|\tau|}\sum_{n=0}^{\infty}e^{-in\omega_{\vec{k}}^m T}\right.\tag{63a}$$
$$\left.-\mathrm{e}^{i\omega_{\boldsymbol{k}}^M|\tau|}\sum_{n=0}^{\infty}\mathrm{e}^{-i(n+1)\omega_{\boldsymbol{k}}^M T}-\mathrm{e}^{-i\omega_{\boldsymbol{k}}^M|\tau|}\sum_{n=0}^{\infty}\mathrm{e}^{-in\omega_{\boldsymbol{k}}^M T}\right]$$

and

$$G^{(d)\mu\nu}\left(\boldsymbol{x},\tau;T\right)$$
$$=\left(g^{\mu\nu}+\frac{\partial_x^\mu\partial_x^\nu}{m^2}\right)\int\frac{\mathrm{d}^{d-1}k}{(2\uppi)^{d-1}}\mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{x}}\frac{1}{2\omega_{i\boldsymbol{k}}^m}\left[\mathrm{e}^{i\omega_{i\boldsymbol{k}}^m|\tau|}\sum_{n=0}^{\infty}\mathrm{e}^{-i(n+1)\omega_{i\boldsymbol{k}}^m T}+\mathrm{e}^{-i\omega_{i\boldsymbol{k}}^m|\tau|}\sum_{n=0}^{\infty}\mathrm{e}^{-in\omega_{i\boldsymbol{k}}^m T}\right]\tag{63b}$$
$$-\frac{\partial_x^\mu\partial_x^\nu}{m^2}\int\frac{\mathrm{d}^{d-1}k}{(2\uppi)^{d-1}}\mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{x}}\frac{1}{2\omega_{\vec{k}}^M}\left[\mathrm{e}^{i\omega_{i\boldsymbol{k}}^M|\tau|}\sum_{n=0}^{\infty}\mathrm{e}^{-i(n+1)\omega_{i\boldsymbol{k}}^M T}+\mathrm{e}^{-i\omega_{i\boldsymbol{k}}^M|\tau|}\sum_{n=0}^{\infty}\mathrm{e}^{-in\omega_{i\boldsymbol{k}}^M T}\right]$$

for the remaining matrix elements.

So according to standard results

$$G^{(d)0i}\left(\boldsymbol{x},\tau;T\right)=G^{(d)i0}\left(\boldsymbol{x},\tau;T\right)=A^{(d)}\left(\boldsymbol{x}^2,\tau;T\right)x^i\tag{64}$$

where

$$A^{(d)}(\vec{x}^2,\tau;T)=-\frac{1}{2m^2}\frac{1}{\vec{x}^2}\int\frac{d^{d-1}k}{(2\pi)^{d-1}}\left(\vec{k}\cdot\vec{x}\right)e^{i\vec{k}\cdot\vec{x}}\left[e^{i\omega_{\vec{k}}^m|\tau|}\sum_{n=0}^{\infty}e^{-i(n+1)\omega_{\vec{k}}^m T}+e^{-i\omega_{\boldsymbol{k}}^m|\tau|}\sum_{n=0}^{\infty}e^{-in\omega_{\boldsymbol{k}}^m T}-e^{i\omega_{\boldsymbol{k}}^M|\tau|}\sum_{n=0}^{\infty}e^{-i(n+1)\omega_{\boldsymbol{k}}^M T}-e^{-i\omega_{\boldsymbol{k}}^M|\tau|}\sum_{n=0}^{\infty}e^{-in\omega_{\boldsymbol{k}}^M T}\right] \tag{65}$$

If $\mu=\nu=0$ we get

$$G^{(d)00}(\boldsymbol{x},\tau;T)=\left(1+\frac{\partial_x^0\partial_x^0}{m^2}\right)\int\frac{\mathrm{d}^{d-1}k}{(2\pi)^{d-1}}\mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{x}}\frac{1}{2\omega_{\boldsymbol{k}}^m}\left[\mathrm{e}^{i\omega_{\boldsymbol{k}}^m|\tau|}\sum_{n=0}^{\infty}\mathrm{e}^{-i(n+1)\omega_{\boldsymbol{k}}^m T}+\mathrm{e}^{-i\omega_{\boldsymbol{k}}^m|\tau|}\sum_{n=0}^{\infty}\mathrm{e}^{-in\omega_{\boldsymbol{k}}^m T}\right]-\frac{\partial_x^0\partial_x^0}{m^2}\int\frac{\mathrm{d}^{d-1}k}{(2\pi)^{d-1}}\mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{x}}\frac{1}{2\omega_{\boldsymbol{k}}^M}\left[\mathrm{e}^{i\omega_{\boldsymbol{k}}^M|\tau|}\sum_{n=0}^{\infty}\mathrm{e}^{-i(n+1)\omega_{\boldsymbol{k}}^M T}+\mathrm{e}^{-i\omega_{\boldsymbol{k}}^M|\tau|}\sum_{n=0}^{\infty}\mathrm{e}^{-in\omega_{\boldsymbol{k}}^M T}\right] \tag{66}$$

Moreover

$$G^{(d)ij}(\boldsymbol{x},\tau;T)=\left[B^{(d)}(\boldsymbol{x}^2,\tau;T)+C^{(d)}(\boldsymbol{x}^2,\tau;T)\right]\delta^{ij}+D^{(d)}(\boldsymbol{x}^2,\tau;T)x^i x^j \tag{67}$$

where

$$B^{(d)}(\boldsymbol{x}^2,\tau;T)=-\frac{1}{2m^2}\frac{1}{d-2}\frac{1}{\boldsymbol{x}^2}\int\frac{\mathrm{d}^{d-1}k}{(2\pi)^{d-1}}\left[-(\boldsymbol{k}\cdot\boldsymbol{x})^2+\boldsymbol{k}^2\boldsymbol{x}^2\right]\mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{x}}\times\left\{\frac{1}{\omega_{\boldsymbol{k}}^m}\left[\mathrm{e}^{i\omega_{\boldsymbol{k}}^m|\tau|}\sum_{n=0}^{\infty}\mathrm{e}^{-i(n+1)\omega_{\boldsymbol{k}}^m T}+\mathrm{e}^{-i\omega_{\boldsymbol{k}}^m|\tau|}\sum_{n=0}^{\infty}\mathrm{e}^{-in\omega_{\boldsymbol{k}}^m T}\right]-\frac{1}{\omega_{\boldsymbol{k}}^M}\left[\mathrm{e}^{i\omega_{\boldsymbol{k}}^M|\tau|}\sum_{n=0}^{\infty}\mathrm{e}^{-i(n+1)\omega_{\boldsymbol{k}}^M T}+\mathrm{e}^{-i\omega_{\boldsymbol{k}}^M|\tau|}\sum_{n=0}^{\infty}\mathrm{e}^{-in\omega_{\boldsymbol{k}}^M T}\right]\right\} \tag{68}$$

$$D^{(d)}(\boldsymbol{x}^2,\tau;T)=-\frac{1}{2m^2}\frac{1}{d-2}\frac{1}{(\boldsymbol{x}^2)^2}\int\frac{\mathrm{d}^{d-1}k}{(2\pi)^{d-1}}\left[(d-1)(\boldsymbol{k}\cdot\boldsymbol{x})^2-\boldsymbol{k}^2\boldsymbol{x}^2\right]\mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{x}}\times\left\{\frac{1}{\omega_{\boldsymbol{k}}^m}\left[\mathrm{e}^{i\omega_{\boldsymbol{k}}^m|\tau|}\sum_{n=0}^{\infty}\mathrm{e}^{-i(n+1)\omega_{\boldsymbol{k}}^m T}+\mathrm{e}^{-i\omega_{\boldsymbol{k}}^m|\tau|}\sum_{n=0}^{\infty}\mathrm{e}^{-in\omega_{\boldsymbol{k}}^m T}\right]-\frac{1}{\omega_{\boldsymbol{k}}^M}\left[\mathrm{e}^{i\omega_{\boldsymbol{k}}^M|\tau|}\sum_{n=0}^{\infty}\mathrm{e}^{-i(n+1)\omega_{\boldsymbol{k}}^M T}+\mathrm{e}^{-i\omega_{\boldsymbol{k}}^M|\tau|}\sum_{n=0}^{\infty}\mathrm{e}^{-in\omega_{\boldsymbol{k}}^M T}\right]\right\} \tag{69}$$

and

$$C^{(d)}(\boldsymbol{x}^2,\tau;T)=-\int\frac{\mathrm{d}^{d-1}k}{(2\pi)^{d-1}}\mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{x}}\frac{1}{2\omega_{\boldsymbol{k}}^m}\left[\mathrm{e}^{i\omega_{\boldsymbol{k}}^m|\tau|}\sum_{n=0}^{\infty}\mathrm{e}^{-i(n+1)\omega_{\boldsymbol{k}}^m T}+\mathrm{e}^{-i\omega_{\boldsymbol{k}}^m|\tau|}\sum_{n=0}^{\infty}\mathrm{e}^{-in\omega_{\boldsymbol{k}}^m T}\right] \tag{70}$$

In **Appendix A**, we give expressions for the above integrals for spacetime dimensions three and four.

We observe that if we let $T\to\infty$ the expressions in the above parentheses become $\mathrm{e}^{-i\omega_{\boldsymbol{k}}^m|\tau|}$ and $\mathrm{e}^{-i\omega_{\boldsymbol{k}}^M|\tau|}$. So, we obtain the standard infinite time domain results for the Stueckelberg lagrangian [1] [2]. In fact, if $F^{(d)\mu\nu}(\boldsymbol{x},t)$ is its corresponding Green function, then as we can check [1] [2]

$$F^{(d)\mu\nu}(\vec{x},t)=-i\int\frac{d^d k}{(2\pi)^d}e^{-ik\cdot x}\left[\frac{g^{\mu\nu}-\frac{k^\mu k^\nu}{m^2}}{k^2-m^2+i0}+\frac{\frac{k^\mu k^\nu}{m^2}}{k^2-M^2+i0}\right] \tag{71}$$

Then on comparing the various time dependences in the series in the above equations, it is easy to conclude that

$$G^{(d)0i}(\boldsymbol{x},t;T)=G^{(d)i0}(\boldsymbol{x},t;T)=\sum_{n=0}^{\infty}F^{(d)0i}(\boldsymbol{x},t+nT)+\sum_{n=0}^{\infty}F^{(d)0i}(\boldsymbol{x},(n+1)T-t) \tag{72a}$$

and

$$G^{(d)\mu\nu}(\boldsymbol{x},t;T)=\sum_{n=-\infty}^{\infty}F^{(d)\mu\nu}(\boldsymbol{x},t+nT) \tag{72b}$$

for the remaining matrix elements.

Moreover

$$G^{(d)\mu\nu}(\boldsymbol{x},t;T)=G^{(d)\mu\nu}(\boldsymbol{x},T-t;T)\quad 0\le t\le T \tag{73}$$

Therefore $G^{(d)\mu\nu}(\boldsymbol{x},t;T)$ obeys periodic boundary conditions with respect the time.

Further if $\lambda\to 0$ then $M\to\infty$ and $\omega_{\boldsymbol{k}}^{M}\to\infty$. So, the expressions containing $\omega_{\boldsymbol{k}}^{M}$ in the above parentheses become zero and we obtain the results of ref. [10].

If we let $m\to 0$ we obtain the photon Green function $G_{ph}^{(d)\mu\nu}(\boldsymbol{x},\tau;T)$ in the form

$$\begin{aligned}G_{ph}^{(d)i0}(\boldsymbol{x},\tau;T)&=G_{ph}^{(d)0i}(\boldsymbol{x},\tau;T)\\&=i\int\frac{d^{d-1}k}{(2\pi)^{d-1}}\mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{x}}k^{i}\frac{1-\lambda}{\lambda}\frac{1}{4|\boldsymbol{k}|\sin^{2}\left(\frac{|\boldsymbol{k}|T}{2}\right)}\\&\times\left[\left(|\tau|-\frac{T}{2}\right)\sin\left(|\boldsymbol{k}||\tau|-\frac{|\boldsymbol{k}|T}{2}\right)\sin\left(\frac{|\boldsymbol{k}|T}{2}\right)\right.\\&\left.+\frac{T}{2}\cos\left(|\boldsymbol{k}||\tau|-\frac{|\boldsymbol{k}|T}{2}\right)\cos\left(\frac{|\boldsymbol{k}|T}{2}\right)\right]\end{aligned} \tag{74}$$

$$\begin{aligned}G_{ph}^{(d)ij}(\boldsymbol{x},\tau;T)&=i\int\frac{\mathrm{d}^{d-1}k}{(2\pi)^{d-1}}\mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{x}}\left\{\frac{\cos\left(|\boldsymbol{k}||\tau|-\frac{|\boldsymbol{k}|T}{2}\right)}{2|\boldsymbol{k}|\sin\left(\frac{|\boldsymbol{k}|T}{2}\right)}\delta^{ij}+\frac{1-\lambda}{\lambda}\frac{k^{i}k^{j}}{|\boldsymbol{k}|^{2}}\frac{1}{4|\boldsymbol{k}|\sin^{2}\left(\frac{|\boldsymbol{k}|T}{2}\right)}\right.\\&\times\left[\left(|\tau|-\frac{T}{2}\right)|\boldsymbol{k}|\sin\left(|\boldsymbol{k}||\tau|-\frac{|\boldsymbol{k}|T}{2}\right)\sin\left(\frac{|\boldsymbol{k}|T}{2}\right)\right.\\&+\frac{T}{2}|\boldsymbol{k}|\cos\left(|\boldsymbol{k}||\tau|-\frac{|\boldsymbol{k}|T}{2}\right)\cos\left(\frac{|\boldsymbol{k}|T}{2}\right)\\&\left.\left.+\cos\left(|\boldsymbol{k}||\tau|-\frac{|\boldsymbol{k}|T}{2}\right)\sin\left(\frac{|\boldsymbol{k}|T}{2}\right)\right]\right\}\end{aligned} \tag{75}$$

and

$$G_{ph}^{(d)00}(\boldsymbol{x},\tau;T) = -i\int \frac{\mathrm{d}^{d-1}k}{(2\pi)^{d-1}} \mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{x}} \left\{ \frac{\cos\left(|\boldsymbol{k}||\tau| - \frac{|\boldsymbol{k}|T}{2}\right)}{2|\boldsymbol{k}|\sin\left(\frac{|\boldsymbol{k}|T}{2}\right)} - \frac{1-\lambda}{\lambda}\frac{1}{4|\boldsymbol{k}|\sin^2\left(\frac{|\boldsymbol{k}|T}{2}\right)} \right.$$
$$\times\left[\left(|\tau| - \frac{T}{2}\right)|\boldsymbol{k}|\sin\left(|\boldsymbol{k}||\tau| - \frac{|\boldsymbol{k}|T}{2}\right)\sin\left(\frac{|\boldsymbol{k}|T}{2}\right)\right.$$
$$+\frac{T}{2}|\boldsymbol{k}|\cos\left(|\boldsymbol{k}||\tau| - \frac{|\boldsymbol{k}|T}{2}\right)\cos\left(\frac{|\boldsymbol{k}|T}{2}\right)$$
$$\left.\left. -\cos\left(|\boldsymbol{k}||\tau| - \frac{|\boldsymbol{k}|T}{2}\right)\sin\left(\frac{|\boldsymbol{k}|T}{2}\right)\right]\right\} \tag{76}$$

4. Application

We proceed to the study of the generation of vector fields and more particularly of photons, from currents. Let the current be $J_\nu(\boldsymbol{x},t)$ where the vector $\boldsymbol{x}$ has dimension $d-1$. Then

$$A^\mu(t,\boldsymbol{x}) = -i\int_0^T \mathrm{d}t' \int \mathrm{d}^{d-1}\boldsymbol{x}' G^{(d)\mu\nu}(\boldsymbol{x}-\boldsymbol{x}', t-t'; T) J_\nu(\boldsymbol{x}', t') \tag{77}$$

Let us consider a charged particle on a trajectory $\boldsymbol{y}(t)$. Then, the current d-vector has the form

$$J_\nu(\boldsymbol{x},t) = Q\mathrm{v}_\nu(t)\delta^{(d-1)}(\boldsymbol{x}-\boldsymbol{y}(t)) \tag{78}$$

$\mathrm{v}_\nu(t)$ $\nu = 0,1,\cdots,d-1$ is its velocity and Q its charge. Further, we set

$$\mathrm{v} = (1, \mathrm{v}_1, \cdots, \mathrm{v}_{d-1}) = (1, \mathbf{v}) \tag{79}$$

The present current is conserved as

$$\begin{aligned} \partial_\nu J^\nu(\boldsymbol{x},t) &= \partial_0 J^0(\boldsymbol{x},t) - \partial_i J^i(\boldsymbol{x},t) \\ &= Q\partial_0 y^i(t)\partial^{x_i}\left[\delta^{(d-1)}(\boldsymbol{x}-\boldsymbol{y}(t))\right] - Q\mathrm{v}^i(t)\partial^{x_i}\left[\delta^{(d-1)}(\boldsymbol{x}-\boldsymbol{y}(t))\right] \\ &= 0 \end{aligned} \tag{80}$$

We have used the fact that $\partial_0\boldsymbol{y}(t) = \mathbf{v}$.

Therefore, the above theory is applicable and we get

$$\begin{aligned} A^0(t,\boldsymbol{x}) = -iQ\int_0^T \Big\{ & G^{(d)00}(\boldsymbol{x}-\boldsymbol{y}(t'), t-t'; T) \\ & - A^{(d)}\left((\boldsymbol{x}-\boldsymbol{y}(t'))^2, t-t'; T\right)(\boldsymbol{x}-\boldsymbol{y}(t'))^j \mathrm{v}_j(t')\Big\}\mathrm{d}t' \end{aligned} \tag{81}$$

and

$$\begin{aligned} A^i(t,\boldsymbol{x}) = -iQ\int_0^T \Big\{ & A^{(d)}\left((\boldsymbol{x}-\boldsymbol{y}(t'))^2, t-t'; T\right)(\boldsymbol{x}-\boldsymbol{y}(t'))^i \\ & -\left[B^{(d)}\left((\boldsymbol{x}-\boldsymbol{y}(t'))^2, t-t'; T\right) + C^{(d)}\left((\boldsymbol{x}-\boldsymbol{y}(t'))^2, t-t'; T\right)\right]\mathrm{v}^i(t') \\ & -D^{(d)}\left((\boldsymbol{x}-\boldsymbol{y}(t'))^2, t-t'; T\right)(\boldsymbol{x}-\boldsymbol{y}(t'))^i(\boldsymbol{x}-\boldsymbol{y}(t'))^j \mathrm{v}_j(t')\Big\}\mathrm{d}t' \end{aligned} \tag{82}$$

Source-generated potentials are expected to be gauge dependent objects but physical results should not be affected by the gauge choice. Therefore, they must be independent of λ.

In the case of photons when $\lambda=1$ we obtain the Feynman gauge. The limiting $\lambda\to\infty$ is the Landau gauge.

Now we study photons and we consider the Feynman gauge. Then

$$\begin{aligned} G_{ph}^{(d)\mu\nu}(\boldsymbol{x},\tau;T) &= -ig^{\mu\nu}\int\frac{\mathrm{d}^{d-1}k}{(2\pi)^{d-1}}\mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{x}}\frac{\cos\left(|\boldsymbol{k}||\tau|-\frac{|\boldsymbol{k}|T}{2}\right)}{2|\boldsymbol{k}|\sin\left(\frac{|\boldsymbol{k}|T}{2}\right)} \\ &= g^{\mu\nu}\int\frac{\mathrm{d}^{d-1}k}{(2\pi)^{d-1}}\mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{x}}\frac{\cos\left(|\boldsymbol{k}||\tau|-\frac{|\boldsymbol{k}|T}{2}\right)}{|\boldsymbol{k}|\left(1-\mathrm{e}^{-i|\vec{k}|T}\right)}\mathrm{e}^{-i\frac{|\boldsymbol{k}|T}{2}} \\ &= g^{\mu\nu}\int\frac{\mathrm{d}^{d-1}k}{(2\pi)^{d-1}}\mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{x}}\frac{1}{2|\boldsymbol{k}|}\left[\mathrm{e}^{i|\boldsymbol{k}||\tau|}\sum_{n=0}^{\infty}\mathrm{e}^{-i(n+1)|\boldsymbol{k}|T}+\mathrm{e}^{-i|\boldsymbol{k}||\tau|}\sum_{n=0}^{\infty}\mathrm{e}^{-in|\boldsymbol{k}|T}\right] \end{aligned} \tag{83}$$

So, in the case of a uniformly moving charge with velocity v_0 along the x-direction, Equation (79) becomes

$$\mathrm{v}=(1,\mathrm{v}_0,0,\cdots,0) \tag{84}$$

and therefore

$$\boldsymbol{y}(t)=(\mathrm{v}_0 t,0,\cdots,0) \tag{85}$$

If $\mathrm{v}_0=0$ then the particle is stationary. We proceed to applications concerning that case.

If $d=4$ we get

$$\begin{aligned} A^0(T,\boldsymbol{x}) &= -iQ\int_0^T\mathrm{d}\tau\int\frac{\mathrm{d}^3k}{(2\pi)^3}\mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{x}}\frac{1}{2|\boldsymbol{k}|}\left[\mathrm{e}^{i|\boldsymbol{k}|(T-\tau)}\sum_{n=0}^{\infty}\mathrm{e}^{-i(n+1)|\boldsymbol{k}|T}+\mathrm{e}^{-i|\boldsymbol{k}|(T-\tau)}\sum_{n=0}^{\infty}\mathrm{e}^{-in|\boldsymbol{k}|T}\right] \\ &= -iQ\frac{1}{4\pi^2}\int_0^T\mathrm{d}\tau\int_0^{\infty}\mathrm{d}k\,kj_0(k|\boldsymbol{x}|)\left[\mathrm{e}^{-ik\tau}\sum_{n=0}^{\infty}\mathrm{e}^{-inkT}+\mathrm{e}^{ik\tau}\sum_{n=0}^{\infty}\mathrm{e}^{-i(n+1)kT}\right] \\ &= -iQ\frac{1}{4\pi^2}\int_0^T\mathrm{d}\tau\int_0^{\infty}\mathrm{d}k\frac{\sin(k|\boldsymbol{x}|)}{|\boldsymbol{x}|}\left[\mathrm{e}^{-ik\tau}\sum_{n=0}^{\infty}\mathrm{e}^{-inkT}+\mathrm{e}^{ik\tau}\sum_{n=0}^{\infty}\mathrm{e}^{-i(n+1)kT}\right] \\ &= -iQ\frac{1}{4\pi^2}\int_0^T\mathrm{d}\tau\sum_{n=-\infty}^{\infty}\frac{1}{|\boldsymbol{x}|^2-(\tau+nT)^2} \\ &= iQ\frac{1}{4\pi^2}\int_{-\infty}^{\infty}\mathrm{d}\tau\frac{1}{\tau^2-|\boldsymbol{x}|^2+i0}=Q\frac{1}{4\pi}\frac{1}{|\boldsymbol{x}|} \end{aligned} \tag{86}$$

That result corresponds to the electric potential of a single point charge in a three-dimensional space.

We observe that $A^0(T,\boldsymbol{x})$ is independent of T. Moreover, since $g^{i0}=0$ we get

$$A^i(T,\boldsymbol{x})=0 \tag{87}$$

If $d = 3$ $A^0(T, \boldsymbol{x})$ obeys the relations

$$\begin{aligned} A^0(T,\boldsymbol{x}) &= -iQ\int_0^T \mathrm{d}\tau \int \frac{\mathrm{d}^2 k}{(2\pi)^2} \mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{x}} \frac{1}{2|\boldsymbol{k}|}\left[\mathrm{e}^{i|\boldsymbol{k}|(T-\tau)}\sum_{n=0}^{\infty}\mathrm{e}^{-i(n+1)|\boldsymbol{k}|T} + \mathrm{e}^{-i|\boldsymbol{k}|(T-\tau)}\sum_{n=0}^{\infty}\mathrm{e}^{-in|\boldsymbol{k}|T}\right] \\ &= -iQ\frac{1}{4\pi}\int_0^T \mathrm{d}\tau \int_0^\infty \mathrm{d}k\, J_0(k|\boldsymbol{x}|)\left[\mathrm{e}^{-ik\tau}\sum_{n=0}^{\infty}\mathrm{e}^{-inkT} + \mathrm{e}^{ik\tau}\sum_{n=0}^{\infty}\mathrm{e}^{-i(n+1)kT}\right] \\ &= -iQ\frac{1}{4\pi}\int_0^T \mathrm{d}\tau \sum_{n=-\infty}^{\infty}\frac{1}{\sqrt{|\boldsymbol{x}|^2-(\tau+nT)^2}} \\ &= -iQ\frac{1}{4\pi}\int_{-\infty}^{\infty}\mathrm{d}\tau \frac{1}{\sqrt{|\boldsymbol{x}|^2-\tau^2}} = Q\frac{1}{4\pi}\int_{-\infty}^{\infty}\mathrm{d}\rho \frac{1}{\sqrt{|\boldsymbol{x}|^2+\rho^2}} \end{aligned} \tag{88}$$

where in the last equality we have performed an appropriate Wick rotation. The final result corresponds to the electric potential of a single point charge in a 2 + 1-dimensional spacetime. $\boldsymbol{x}$ is a vector in a 2-dimensional space. Moreover

$$A^i(T, \boldsymbol{x}) = 0 \tag{89}$$

5. Conclusions

In the present paper, we studied the finite-time dynamics of vector fields using a Stueckelberg-type lagrangian, Hamiltonian quantization, and holomorphic path-integral methods. Within that structure, we integrated and derived finite time-interval Green functions in general dimension and gave explicit forms for selected cases, including the massless photon limit. Additionally, we gave integral as well as series representations. We then applied these kernels to source-generated potentials in three- and four-dimensional spacetime. These results can be used in the extraction of finite time predictions.

As far as gauge invariance is concerned, we can observe that the present Stueckelberg formulation with an auxiliary scalar field is invariant under local gauge transformations.

Finally, we notice that the present approach appears as an alternative compared to the numerical integration of the corresponding equations or to the employment of lattice field techniques.

In subsequent work, we intend to consider the dynamics of other fields, interacting or free, and study them.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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Appendix

Appendix A: Numerical Forms

Here we present the integrals (65), (66), (68), (69), (70) for spacetime dimensions equal to three and four. So, we obtain the expressions below concerning $G^{(d)00}(\boldsymbol{x},\tau;T)$, $A^{(d)}(\boldsymbol{x}^2,\tau;T)$, $B^{(d)}(\boldsymbol{x}^2,\tau;T)$, $C^{(d)}(\boldsymbol{x}^2,\tau;T)$ and $D^{(d)}(\boldsymbol{x}^2,\tau;T)$.

So, if $d=3$ $G^{(3)00}(\boldsymbol{x},\tau;T)$ takes the form

$$\begin{aligned} &G^{(3)00}(\boldsymbol{x},\tau;T) \\ &= -i\frac{1}{4\pi m^2}\sum_{n=-\infty}^{\infty}\frac{1}{\left[(nT+|\tau|)^2-|\boldsymbol{x}|^2\right]^{\frac{5}{2}}}\Big[2(nT+|\tau|)^2+|\boldsymbol{x}|^2 \\ &+im\left[2(nT+|\tau|)^2+|\boldsymbol{x}|^2\right]\sqrt{(nT+|\tau|)^2-|\boldsymbol{x}|^2} \\ &-m^2(nT+|\tau|)^4+m^2(nT+|\tau|)^2|\boldsymbol{x}|^2\Big]\exp\left[-im\sqrt{(nT+|\tau|)^2-|\boldsymbol{x}|^2}\right] \\ &-\{m\to M\}-i\frac{1}{4\pi}\sum_{n=-\infty}^{\infty}\frac{1}{\left[(nT+|\tau|)^2-|\boldsymbol{x}|^2\right]^{1/2}}\exp\left[-im\sqrt{(nT+|\tau|)^2-|\boldsymbol{x}|^2}\right] \end{aligned} \tag{A1}$$

In the equations the symbol $\{m\to M\}$ represents the previous expression where in the sum and only the sum, we replace the symbol m with the symbol M.

Proceeding if $d=4$ we get for the $G^{(4)00}(\boldsymbol{x},\tau;T)$, the expression

$$\begin{aligned} &G^{(4)00}(\boldsymbol{x},\tau;T) \\ &= -i\frac{1}{4\pi^2 m^2}\sum_{n=-\infty}^{\infty}\frac{m}{\left[(nT+|\tau|)^2-|\boldsymbol{x}|^2\right]^{\frac{5}{2}}}\Big\{im\sqrt{(nT+|\tau|)^2-|\boldsymbol{x}|^2} \\ &\times\left(3(nT+|\tau|)^2+|\boldsymbol{x}|^2\right)K_0\left(im\sqrt{(nT+|\tau|)^2-|\boldsymbol{x}|^2}\right)+\Big[2\left(3(nT+|\tau|)^2+|\boldsymbol{x}|^2\right) \\ &-m^2(nT+|\tau|)^2\left((nT+|\tau|)^2-|\boldsymbol{x}|^2\right)\Big]K_1\left(im\sqrt{(nT+|\tau|)^2-|\boldsymbol{x}|^2}\right)\Big\} \\ &-\{m\to M\}-i\frac{m}{4\pi^2}\sum_{n=-\infty}^{\infty}\frac{1}{\left[(nT+|\tau|)^2-|\boldsymbol{x}|^2\right]^{\frac{1}{2}}}K_1\left(im\sqrt{(nT+|\tau|)^2-|\boldsymbol{x}|^2}\right) \end{aligned} \tag{A2}$$

Moreover, if we let $d=3$ we get for $A^{(3)}(\boldsymbol{x}^2,\tau;T)$

$$\begin{aligned} &A^{(3)}(\boldsymbol{x}^2,\tau;T) \\ &= \sqrt{\frac{2}{\pi}}\frac{1}{4\pi m^2}e^{\frac{i\pi}{4}}\left[\sum_{n=0}^{\infty}\frac{((n+1)T-|\tau|)m^{\frac{3}{2}}K_{\frac{5}{2}}\left(im\sqrt{((n+1)T-|\tau|)^2-|\boldsymbol{x}|^2}\right)}{\left(((n+1)T-|\tau|)^2-|\boldsymbol{x}|^2\right)^{\frac{7}{4}}}-\{m\to M\}\right. \\ &\left.+\sum_{n=0}^{\infty}\frac{(|\tau|+nT)m^{\frac{3}{2}}K_{\frac{5}{2}}\left(im\sqrt{(|\tau|+nT)^2-|\boldsymbol{x}|^2}\right)}{\left((|\tau|+nT)^2-|\boldsymbol{x}|^2\right)^{\frac{7}{4}}}-\{m\to M\}\right] \end{aligned} \tag{A3}$$

$A^{(4)}\left(\boldsymbol{x}^2,\tau;T\right)$ satisfies the following relations if $d=4$

$$A^{(4)}\left(\boldsymbol{x}^2,\tau;T\right)$$
$$=\frac{1}{4\pi^2m^2}e^{-\frac{i\pi}{4}}\left[\sum_{n=0}^{\infty}\frac{\left((n+1)T-|\tau|\right)m^{\frac{3}{2}}K_3\left(im\sqrt{\left((n+1)T-|\tau|\right)^2-|\boldsymbol{x}|^2}\right)}{\left(\left((n+1)T-|\tau|\right)^2-|\boldsymbol{x}|^2\right)^{\frac{9}{4}}}-\{m\to M\}\right.$$
$$\left.+\sum_{n=0}^{\infty}\frac{\left(|\tau|+nT\right)m^{\frac{3}{2}}K_3\left(im\sqrt{\left(|\tau|+nT\right)^2-|\boldsymbol{x}|^2}\right)}{\left(\left(|\tau|+nT\right)^2-|\boldsymbol{x}|^2\right)^{\frac{9}{4}}}-\{m\to M\}\right] \quad \text{(A4)}$$

Further $B^{(3)}\left(\boldsymbol{x}^2,\tau;T\right)$ where $d=3$ has the form

$$B^{(3)}\left(\boldsymbol{x}^2,\tau;T\right)=-i\frac{1}{4\pi m^2}\sum_{n=-\infty}^{\infty}\frac{1}{\left(\left(|\tau|+nT\right)^2-|\boldsymbol{x}|^2\right)^{\frac{3}{2}}}\left(1+im\sqrt{\left(|\tau|+nT\right)^2-|\boldsymbol{x}|^2}\right)$$
$$\times\exp\left(-im\sqrt{\left(|\tau|+nT\right)^2-|\boldsymbol{x}|^2}\right)-\{m\to M\} \quad \text{(A5)}$$

Proceeding further if $d=4$, $B^{(4)}\left(\boldsymbol{x}^2,\tau;T\right)$ becomes

$$B^{(4)}\left(\boldsymbol{x}^2,\tau;T\right)=\frac{1}{4\pi^2m^2}\sum_{n=-\infty}^{\infty}\frac{m^2}{\left(|\tau|+nT\right)^2-|\boldsymbol{x}|^2}K_2\left(im\sqrt{\left(|\tau|+nT\right)^2-|\boldsymbol{x}|^2}\right)-\{m\to M\} \quad \text{(A6)}$$

If $d=3$ $C^{(3)}\left(\boldsymbol{x}^2,\tau;T\right)$ has the form

$$C^{(3)}\left(\boldsymbol{x}^2,\tau;T\right)=i\frac{1}{4\pi}\sum_{n=-\infty}^{\infty}\frac{1}{\left(\left(|\tau|+nT\right)^2-|\boldsymbol{x}|^2\right)^{\frac{1}{2}}}\exp\left[-im\sqrt{\left(|\tau|+nT\right)^2-|\boldsymbol{x}|^2}\right] \quad \text{(A7)}$$

while if $d=4$ we get for $C^{(4)}\left(\boldsymbol{x}^2,\tau;T\right)$

$$C^{(4)}\left(\boldsymbol{x}^2,\tau;T\right)=i\frac{1}{2\pi^2}\sum_{n=-\infty}^{\infty}\frac{m}{\left(\left(|\tau|+nT\right)^2-|\boldsymbol{x}|^2\right)^{\frac{1}{2}}}K_1\left(im\sqrt{\left(|\tau|+nT\right)^2-|\boldsymbol{x}|^2}\right) \quad \text{(A8)}$$

For $d=3$ $D^{(3)}\left(\boldsymbol{x}^2,\tau;T\right)$ is

$$D^{(3)}\left(\boldsymbol{x}^2,\tau;T\right)$$
$$=-\sqrt{\frac{2}{\pi}}\frac{1}{4\pi m^2}e^{-\frac{i\pi}{4}}\sum_{n=-\infty}^{\infty}\frac{m^{\frac{5}{2}}}{\left(\left(|\tau|+nT\right)^2-|\boldsymbol{x}|^2\right)^{\frac{5}{4}}}K_{\frac{5}{2}}\left(im\sqrt{\left(|\tau|+nT\right)^2-|\boldsymbol{x}|^2}\right)-\{m\to M\} \quad \text{(A9)}$$

Finally, $D^{(4)}\left(\boldsymbol{x}^2,\tau;T\right)$ where to $d=4$ takes the form

$$D^{(4)}\left(\boldsymbol{x}^2,\tau;T\right)$$
$$=i\frac{1}{4\pi^2m^2}\sum_{n=-\infty}^{\infty}\frac{m^3}{\left(\left(|\tau|+nT\right)^2-|\boldsymbol{x}|^2\right)^{\frac{3}{2}}}K_3\left(im\sqrt{\left(|\tau|+nT\right)^2-|\boldsymbol{x}|^2}\right)-\{m\to M\} \quad \text{(A10)}$$

In the above equations we can apply the identity [16]

$$K_{\nu}\left(iz\right)=\frac{-\pi i}{2}\mathrm{e}^{-\frac{\pi}{2}\nu i}H_{-\nu}^{(2)}\left(z\right) \tag{A11}$$

In the equations (A1) to (A11) K_l are modified Bessel functions of the third kind and $H_l^{(2)}$ are Hankel functions of the second kind.