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Practical Applied Mathematics for Scientific Research: Application of ACP Mathematical Methodology in Analyzing Algebraic Functions and Physical Experimental Data (Applications 11 and 12)

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Abstract

We introduce the ACP (asymptotic curve-based and proportionality-based) mathematical methodology to analyze the face, shape, and proportionality of several algebraic forms, including power and inverse functions as first-order nonlinear phenomena, and sigmoidal curves in physical experiments as second-order nonlinear phenomena. The goal is to express both types of nonlinear behavior using simple, straight-line-oriented proportionality graphs supported by appropriate nonlinear equations. In Part I, we examine first-order nonlinear phenomena using algebraic power and inverse functions and demonstrate the need for a combined linear and nonlinear logarithmic graph to obtain a complete representation. In Part II, we model fluidized-bed experimental data exhibiting various sigmoidal profiles and show that a second-order nonlinear equation can represent the full range of S- and C-shaped curves. The resulting formulation yields a concise straight-line graph and a unified nonlinear rate equation that describes the two-variable relationship.

Keywords

Asymptotic Concave and Convex Curve, Upper and Baseline Asymptote, Coefficient of Determination, Proportionality and Position Constant, Asymmetric-Bell and Sigmoid Curve

1. Introduction

We previously introduced the ACP (asymptotic curve-based and proportionality-oriented) nonlinear mathematical methodology for analyzing mathematical and physical data across scientific disciplines [1]-[4]. In this article, we extend that framework to explain two long-standing issues:

1) why inverse functions can be plotted on a rectilinear (Cartesian) graph in all four quadrants, yet only first-quadrant data appear on a nonlinear logarithmic graph; and

2) How to analyze nonlinear physical data, including sigmoidal and C-shaped curves, in a unified manner.

In Part I, we use algebraic power and inverse functions to illustrate the need for a combined linear and nonlinear logarithmic graph to fully characterize first-order nonlinear phenomena. This approach clarifies the origins of concave and convex curves and reveals the underlying proportionality relationships dictated by natural laws.

In Part II, we organize continuous sigmoidal data into a second-order nonlinear rate equation that captures the entire family of S- and C-shaped curves. The outcome is a concise straight-line representation of the two-variable relationship, accompanied by a single nonlinear rate equation.

2. Fundamentals of ACP Mathematics

ACP mathematics is built on two principles that classify numbers and define the notions of linear and nonlinear zero.

Principle I—Continuity. Continuous numbers are dynamic, non-terminating, and maintain continuity without interruption.

Principle II—Asymptotes. Asymptotes are never part of nonlinear numbers; they are approachable but unattainable.

Continuous numbers are classified as linear or nonlinear depending on whether they possess asymptotes. Two types of zero follow:

- Linear zero: the zero located between positive and negative numbers (…, −6, −4, −2, 0, 2, 4, 6, …).
- Nonlinear zero: the baseline asymptote for nonlinear numbers, which can be approached but never reached.

Ideal representation places linear numbers on linear scales and nonlinear numbers on nonlinear (logarithmic) scales. In practice, mismatches between number type and graph type often lead to incomplete or misleading interpretations.

Nonlinear numbers are characterized by the presence of a baseline or upper asymptote (or both), such as 0.001, 0.01, 0.1, 1, 10, 100, 1000, …—values that approach but never touch their asymptotes.

3. Lesson from the Nonlinear Numbers 0.99999…

The nonlinear number **0.99999**… provides an important lesson in ACP mathematics (Table 1). As shown in Lai [1], this number is a **one-sided nonlinear number**

Table 1. Interpreting the nonlinear number 0.99999…

Common Expression	Correct Expression
0.9 is not equal to 1	$0.9 \neq 1$
0.99 is not equal to 1	$0.99 \neq 1$
0.999 is not equal to 1	$0.999 \neq 1$
-	-
0.99999… is never equal to 1	$0.99999\ldots = 0.\dot{9} \neq 1$
Correct expression:	0.99999… →1 (asymptote) 0.99999… ~1 (asymptote)

with an **upper asymptote** at $Yu = 1$. It increases continuously but never touches or crosses its asymptote. It is incorrect to write 0.999999… or 0.9, 0.99, 0.9999… = $0.\dot{9}$ = 1, because these values are **dynamic, non-terminating, continuously moving nonlinear numbers**, whereas the number **1** is **static** and **not part of the nonlinear sequence**. Thus, the correct mathematical relationship is: 0.9, 0.99, 0.999, 0.9999, 0.99999…→ 1, or equivalently 0.9, 0.99, 0.999, 0.9999, 0.99999…~1. The error lies in the misuse of the **equal sign**. A dynamic number cannot be equated to a static number without violating the physical principle that continuous nonlinear numbers never reach their asymptotes. The symbols "→" or "~" are appropriate; the symbol "=" is not. A dynamic moving number cannot equate to a static number; Newton's law cannot be violated.

In Table 2, the variables are defined as follows:

- ***x***: elementary independent number
- ***X***: cumulative of x
- ***y***: elementary dependent number
- ***Y***: cumulative of y corresponding to X
- ***Yu***: active upper asymptote (entered in Cell I3)
- **COD**: coefficient of determination (Cell I4), computed as "=CORREL(B3:B7, F3:F7)^2".
- **Column E**: face value (Yu- Y), measurement of Y relative to Yu
- **Column F**: authentic value q(Yu- Y) = log(Yu- Y), logarithmic transformation of (Yu- Y), or put (Yu- Y) in logarithmic scale.

Excel Procedure for Determining the Unique/Optimal Asymptote *Yu*

To determine the unique or optimal value of Yu:

1) Enter an initial guess slightly larger than 0.99999…, e.g., **0.999992**, into Cell I3.

2) Enable Solver

- File → Options → Add-ins
- Choose *Excel Add-ins* → Go
- Check *Solver Add-in* → OK

Table 2. Structure of nonlinear numbers 0.99999…

I4 =CORREL(B3:B7,F3:F7)^2

	A	B	C	D	E	F	G	H	I
1	x	X	y	Y	(Yu - Y)	q(Yu - Y)			
2									
3	1	1	0.9	0.9	0.099992	-1.000034745	Active Yu =		0.999992
4	1	2	0.09	0.99	0.009992	-2.000347575	R^2 =		0.98678
5	1	3	0.009	0.999	0.000992	-3.003488328			
6	1	4	0.0009	0.9999	9.2E-05	-4.036212173			
7	1	5	0.00009	0.99999	2.0000000E-06	-5.698970004			

I4 =CORREL(B3:B7,F3:F7)^2

	A	B	C	D	E	F	G	H	I
1	x	X	y	Y	(Yu - Y)	q(Yu - Y)			
2									
3	1	1	0.9	0.9	0.1	-1	Active Yu =		1
4	1	2	0.09	0.99	0.01	-2	R^2 =		1.00000
5	1	3	0.009	0.999	0.001	-3			
6	1	4	0.0009	0.9999	1E-04	-4			
7	1	5	0.00009	0.99999	1E-05	-5			

3) Set up Solver

- Data → Solver
- **Set Objective:** I4 (COD)
- **To:** Max
- **By Changing Variable Cells:** I3 (Active *Yu*)
- **Constraints:**
  - I3 ≥ D7 (asymptote must exceed the largest observed *Y*)
  - Optional: I3 ≤ 6000
- **Solving Method:** GRG Nonlinear

4) Click Solve

5) Result: $Yu = 1$, $R^2 = 1$.

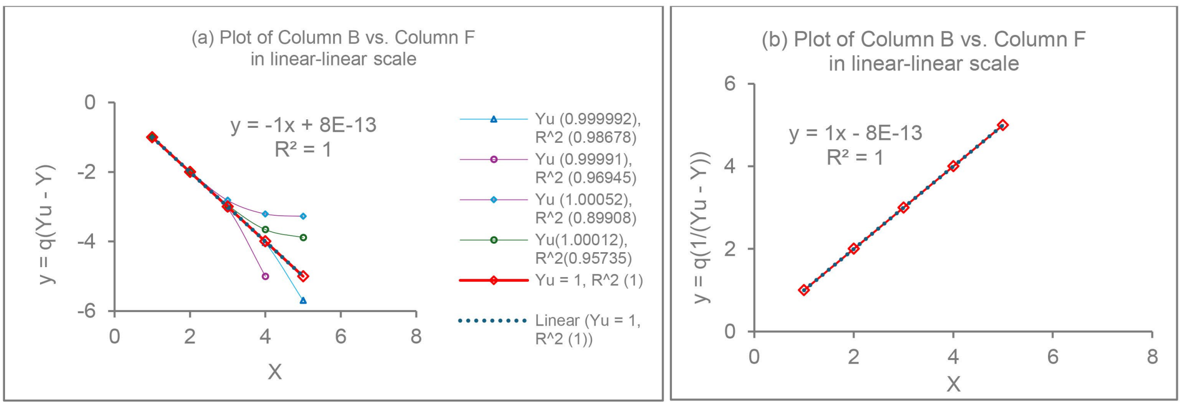


Figure 1. Proportionality plot, (a): $q(Yu - Y)$ versus X; (b): $q(1/(Yu - Y))$ versus X.

Figure 1(a) demonstrates that only when $Yu = 1$ does the proportionality plot yield a perfect straight line with a COD of 1, confirming the law-of-nature proportionality. Figure 1(b) gives the plot of $q(1/(Yu - Y))$ versus X, indicating that the inverse of the face value is proportional to X.

Lessons from the Nonlinear Number 0.99999…

The analysis confirms:

(3a) Nonlinear numbers preserve continuity forever; they continuously add real positive increments.

(3b) Nonlinear numbers are cumulative, monotonically increasing sequences.

(3c) Nonlinear numbers are inherently associated with asymptotes.

(3d) A nonlinear number Y may approach Yu but can never reach or cross it.

(3e) Changes in nonlinear numbers must be measured relative to their asymptote, e.g., the face value ($Yu - Y$).

(3f) Plotting the face value or its logarithmic transformation versus the independent variable produces a straight line when proportionality exists.

4. Lesson from Eastern Philosopher Lie Tzu's Dichotomy Philosophy

Beyond the one-sided nonlinear number with an **upper asymptote**, we now examine Lie Tzu's classic example of a one-sided nonlinear number with a bottom asymptote, Yb (or 0) [5].

The Eastern philosopher **Lie Tzu (列子; Lie Zi, 450 - 375 BCE)** wrote:

"百尺之竿，日折其半，永世不休"

Given a 100-foot pole, if you halve it each day, continuing through infinite generations, the task can never be completed. This describes a nonlinear sequence that approaches a bottom asymptote but never reaches it.

Mathematical Interpretation

Let the initial value be $Y = 100$.

Let the proportionality constant be $K = 0.3$.

Let the bottom asymptote be $Yb = 0$.

The differential form of the proportionality equation: $d(q(Y - Yb)) = -KdX$. Integrating: $q(Y - Yb) = -KX + qC$, where $C = 100$ at $X = 0$.

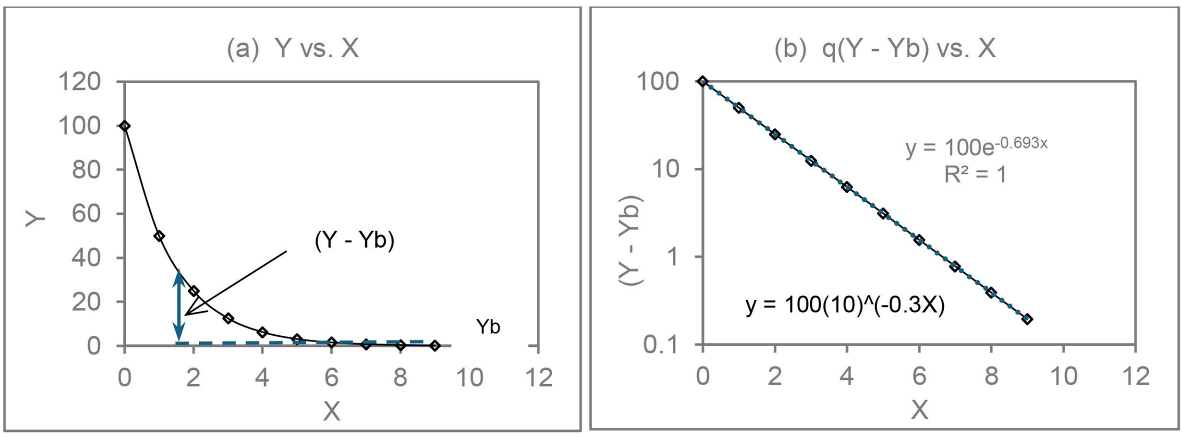


Figure 2. (a): Y vs. X; (b): $q(Yu - Y)$ vs. X.

- **Figure 2(a)**: Plotting nonlinear Y on a **linear scale** produces a **concave curve**, reflecting a mismatch between nonlinear numbers and linear scales.
- **Figure 2(b)**: Plotting ($Y - Yb$) on a **logarithmic scale** produces a **straight line**, revealing the underlying proportionality. The corresponding equations are $q(Y - Yb) = 100\ (10)^{-0.3X}$, or equivalently, $q(Y - Yb)=100\ e^{-0.693X}$ (Table 3).

Table 3. Data calculation for the regression equation in **Figure 2(b)** with Excel.

X	0	1	2	3	4	5	6	7	8	9	---
100 $(10)^{-0.3X}$	100	50.119	25.119	12.589	6.310	3.162	1.585	0.794	0.398	0.200	
100 exp $(-0.693X)$	100	50.007	25.007	12.506	6.254	3.127	1.564	0.782	0.391	0.196	

Lessons from Lie Tzu's Dichotomy

(4a) Nonlinear numbers preserve continuity forever; halving continues without termination.

(4b) Nonlinear numbers are associated with asymptotes.

(4c) A nonlinear number Y may approach Yb but can never reach or cross it.

(4d) Changes in nonlinear numbers must be measured relative to their asymptote, e.g., $(Y - Yb)$.

(4e) Plotting nonlinear numbers on a linear scale produces concave curves due to scale mismatch.

(4f) The logarithm of a nonlinear number (or its face value) plotted against the independent variable yields a straight line when proportionality exists.

5. Face, Shape, and Proportionality of a Few Mathematical Forms

Two types of numbers exist in nature: linear and nonlinear. Linear numbers exhibit linear change, whereas nonlinear numbers exhibit nonlinear change. Consequently, two types of graphs are required for proper representation: linear graphs for linear numbers and nonlinear logarithmic graphs for nonlinear numbers. When the number type and graph type are mismatched—for example, plotting nonlinear numbers on a linear scale—the resulting graph provides an incomplete or inconclusive picture.

Some mathematical functions, including power and inverse functions, are inherently nonlinear and require both linear and nonlinear graphs to fully express their shape, content, and proportionality. In this article, plain X and Y denote linear numbers, while boldface $\boldsymbol{X}$ and $\boldsymbol{Y}$ denote nonlinear numbers. The symbol K is used as both a proportionality constant and a rate constant. We begin with the linear-by-linear case as a reference, then extend the discussion to power and inverse functions.

5.1. Linear Graphs, Nonlinear Graphs, and Nonlinear Zero

To compare linear graphs and nonlinear graphs, it is helpful to establish a clear banner of principles that guide their correct use in ACP mathematics.

Linear-by-Linear Phenomena

In a general linear-by-linear relationship, when the change in linear Y is proportional to the change in linear X, the differential and integral equations are: $dY = KdX$, and $Y = KX + C$.

Figure 3(a) illustrates three straight lines corresponding to three proportionality constants, 1.5, −3, and 3, and three position constants, 64, 0, and 30.

Key characteristics:

- Each straight line extends continuously in both directions.
- All lines pass through the linear zero, which lies between positive and negative numbers.
- The slope K determines the direction and steepness; the constant C determines the vertical position.

This is the classical behavior of linear numbers plotted on a linear scale.

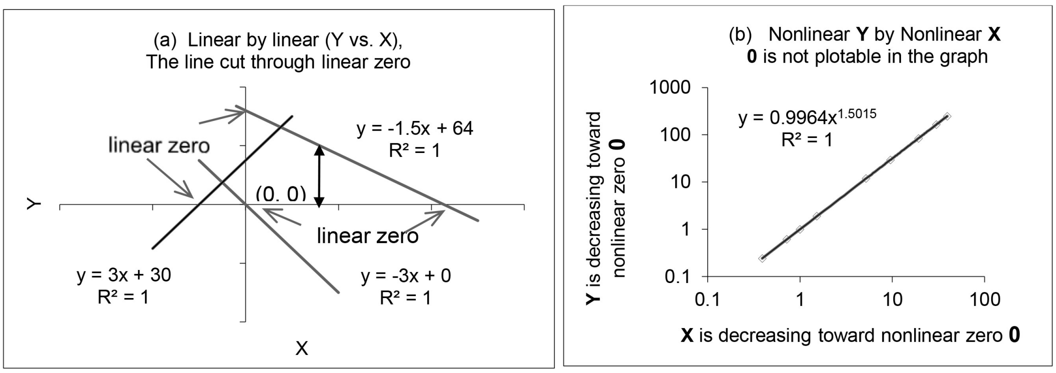


Figure 3. Comparison of two phenomena: (a) Linear by linear; (b) Nonlinear by nonlinear phenomena.

Nonlinear-by-Nonlinear Phenomena

In contrast, when both variables are nonlinear numbers, the proportionality relationship must be expressed on a log-log graph. The differential and integral equations become $d(\log Y) = Kd(\log X)$, and $\log Y = K\log X + \log C$, or equivalently, $Y = CX^K$. In Figure 3(b), the straight line corresponds to $C = 0.9964$, $K = 1.5015$.

Here:

- Both Y and X are plotted on nonlinear logarithmic scales. (Figure 3(b) is a plot of Kepler's third law) [2].
- The straight line decreases toward the nonlinear zero, which cannot be plotted.
- The nonlinear zero acts as a bottom asymptotic zero Yb, a baseline asymptote, or a pivot asymptotic nonlinear zero when it serves as both upper and lower asymptote.

A fundamental ACP principle is that nonlinear zeros are never part of nonlinear numbers. They are approachable but never reachable or crossable.

Representing Nonlinear Numbers Using Face Values

Because boldface notation (Y, X) is cumbersome and nonlinear numbers are always associated with asymptotes, ACP mathematics uses face values to represent nonlinear quantities:

- First-order nonlinear phenomena: $(Yu - Y)$, $(Y - Yb)$
- Second-order nonlinear phenomena: $(qYu - qY)$

5.2. Graphs of Nonlinear-by-Nonlinear Phenomena: Power Functions and Inverse Functions

Nonlinear curved phenomena—such as power functions and inverse functions—

must follow the nonlinear rules established in Section 5.1.

In these functions, both X and Y are nonlinear variables. When plotted on a rectilinear (Cartesian) graph using plain X and Y, the result is a curved line. However, when nonlinear face values are used, and the data are plotted on a nonlinear logarithmic graph, the relationship becomes a straight line.

Key outcomes:

- The slope of the straight line equals the power exponent.
- The straight line decreases toward the nonlinear bottom asymptotes Xp and Yp. When the bottom asymptote also serves as the upper asymptote, we call it a pivot asymptote.
- These asymptotes cannot be reached, crossed, or plotted; they are only implied.

Conventional and ACP Forms of the Power Equation

The conventional power equation is

$$Y = X^K \tag{1}$$

The ACP form incorporates the pivot asymptotes Yp and Xp:

$$(Y - Yp) = (X - Xp)^K \tag{2}$$

or equivalently,

$$(Yp - Y) = (Xp - X)^K \tag{2a}$$

The differential form is:

$$d\left(q\left(Y - Yp\right)\right) = Kd\left(q\left(X - Xp\right)\right) \tag{3}$$

or equivalently,

$$d\left(q\left(Yp - Y\right)\right) = Kd\left(q\left(Xp - X\right)\right) \tag{3a}$$

Here:

- Yp and Xp are pivot asymptotes (nonlinear zeros).
- They cannot be plotted in Excel; they appear as blank cells.
- The exponent K may take values such as 3, 2, or 1/3, depending on the phenomenon.

This ACP formulation ensures that nonlinear behavior is expressed relative to its asymptotic structure, preserving the correct mathematical and physical interpretation.

5.2.1. Power Equation I: $Y = X^3$, ACP Form: $(Y - Yp) = (X - Xp)^3$, or $(Yp - Y) = (Xp - X)^3$

Power functions compare one **nonlinear variable** with another. When nonlinear numbers are plotted on a **linear** (rectilinear) graph, the result is a **curve**, because the scale does not match the nonlinear nature of the numbers.

Why do we get curves? Because we are plotting **nonlinear numbers on a linear scale**, this distorts the true proportionality.

Rectilinear Plot (Figure 4(a))

Plotting the linear values of $Y = X^3$ on a rectilinear graph yields:

- A **concave curve** in the first quadrant
- A **convex curve** in the third quadrant
- No visible reference to the proportionality constant $K = 1$
- No visible reference to the slope (power) of the equation, which is **3**

The rectilinear graph hides the proportionality structure.

Nonlinear Logarithmic Plot (Figure 4(b))

When the same function is plotted using **nonlinear numbers** on a **logarithmic scale**, the curve becomes a **straight line**:

- The slope of the straight line is the **power exponent**, here **3**
- The proportionality constant is **1**, so the ACP form is $Y = 1 \times X^3$
- The straight line approaches the **nonlinear zero asymptote** Yp
- The asymptote can be approached but never reached or crossed

The correct distance measure is the **face value**: $(Y - Yp)$.

Handling Negative Values (Third Quadrant)

Logarithmic scales cannot be used to plot negative numbers. Thus:

- In Figure 4(a), the third-quadrant values of $Y = X^3$ are negative
- When switching to a log scale, these values **disappear**

However, ACP mathematics resolves this:

- Measure the distance from the pivot asymptote: $(Yp - Y) = 0 - (-2) = 2 > 0$
- This positive face value **can** be plotted on a logarithmic scale
- The same method applies to all power functions in this section

In the graphs, $\boldsymbol{X^-}$ and $\boldsymbol{X^+}$ denote the negative and positive sides of X in the rectilinear graph.

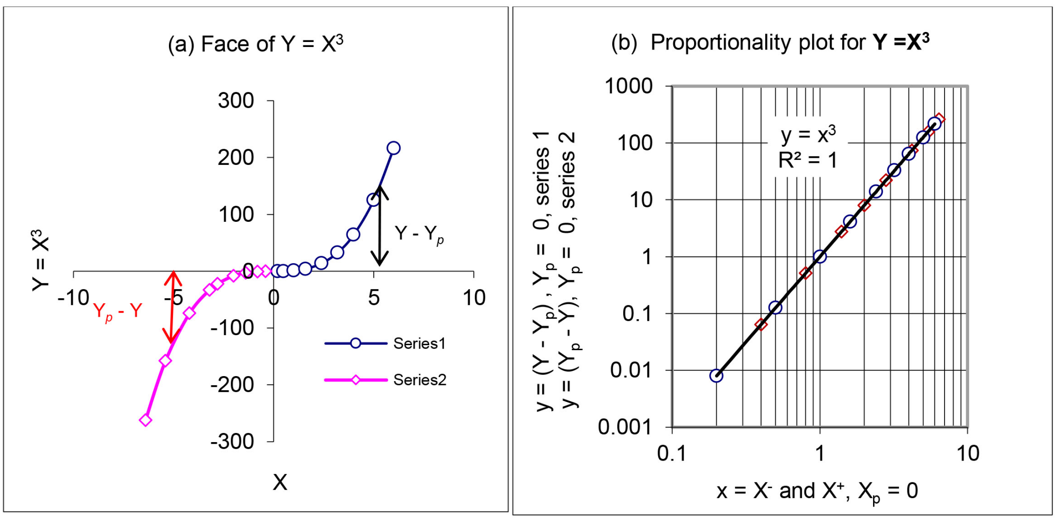


Figure 4. (a) Face of $Y = X^3$(rectilinear) (b) Proportionality plot of $\boldsymbol{Y = X^3}$ (logarithmic)

5.2.2. Power Equation II: $Y = -X^3$, or $Y = -X^3$

The function $Y = -X^3$ is simply the reflection of $Y = X^3$ across the X-axis.

Rectilinear Plot (Figure 5(a))

- The curve is inverted relative to Figure 4(a)
- The proportionality constant is still hidden

- The nonlinear behavior is distorted by the linear scale

Nonlinear Logarithmic Plot (Figure 5(b))

- Using face values relative to the pivot asymptote allows all data to be plotted
- The straight line again reveals the power exponent $K = 3$
- The line approaches the nonlinear zero asymptote but never reaches it

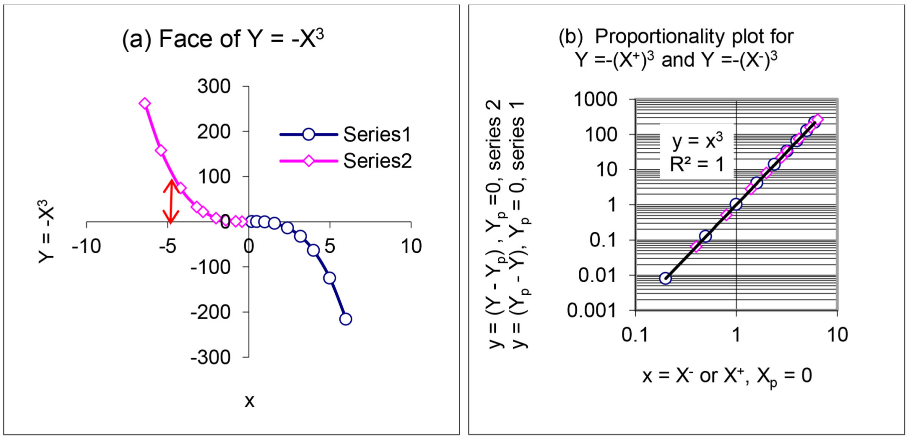


Figure 5. (a) Face of $Y = -X^3$; (b) Proportionality plot of $\boldsymbol{Y = -X^3}$,

5.2.3. Power Equation III: $Y = X^{1/3}$, or $Y = X^{1/3}$

The cube root function is another nonlinear function.

Rectilinear Plot (Figure 6(a))

- Produces a curved line
- Does not reveal the proportionality constant
- Does not show the asymptotic behavior

Nonlinear Logarithmic Plot (Figure 6(b))

- Produces a straight line with slope $K = 1/3$
- The line approaches the pivot asymptotes Xp and Yp
- Negative values are handled via face values

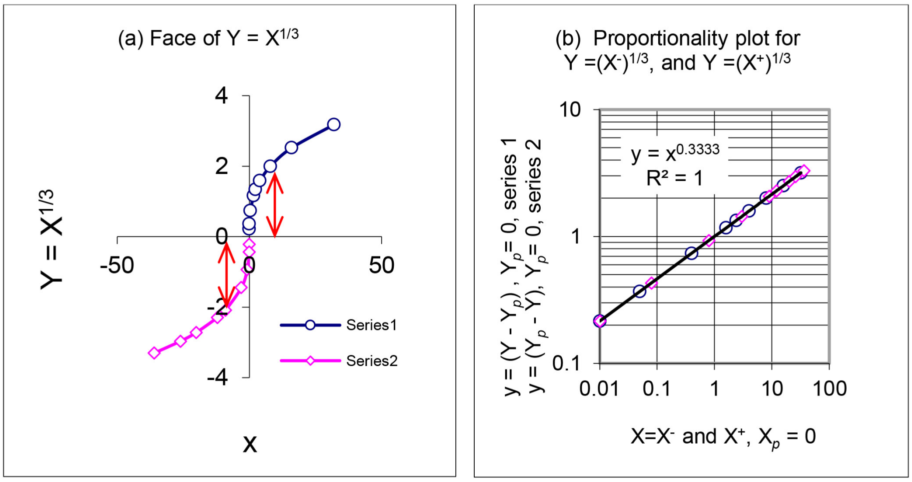


Figure 6. (a) Face of $Y = X^{1/3}$; (b) Proportionality plot of $\boldsymbol{Y = X^{1/3}}$.

5.2.4. Power Equation IV: $Y = X^2$, or $Y = X^2$ (*and Shifted Form* $Y = X^2 + 10$)

Rectilinear Plot (Figure 7(a))

- The parabola is curved
- The slope and proportionality constant are not visible
- The nonlinear nature is obscured

Nonlinear Logarithmic Plot (Figure 7(b))

- Produces a straight line with slope $K = 2$
- Approaches the nonlinear zero asymptote
- Reveals the true proportionality

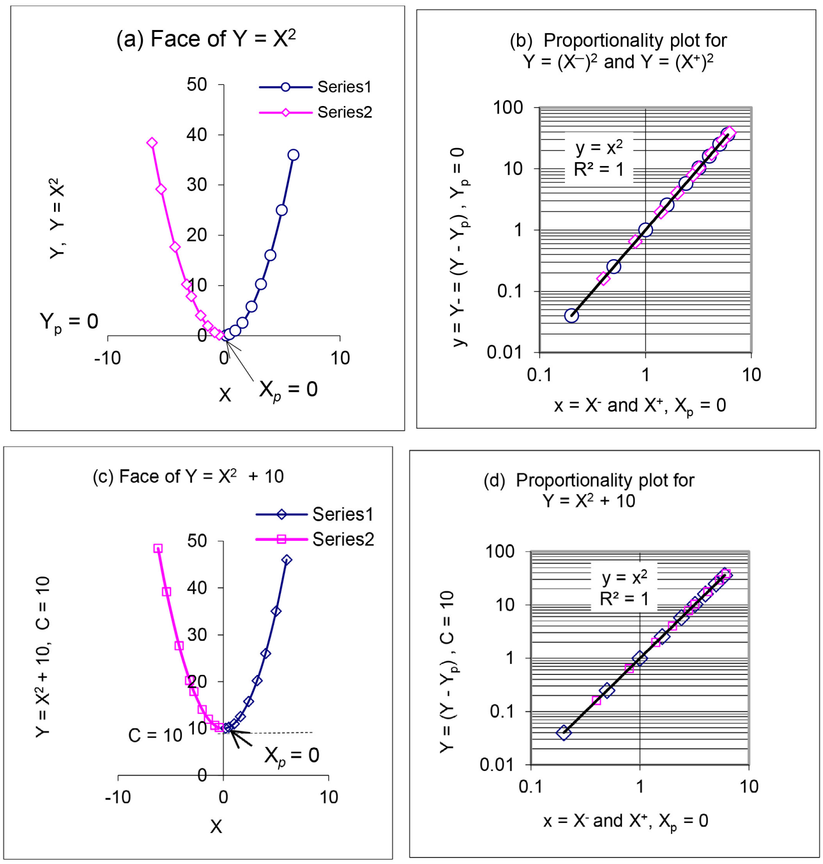


Figure 7. (a) Face of $Y = X^2$, and (b) Proportionality plot of $\boldsymbol{Y = X^2}$, (c) Face of $Y = X^2 + 10$; and (d) Proportionality plot of $\boldsymbol{Y = X^2 + 10}$.

Shifted Function $\boldsymbol{Y = X^2 + 10}$

The vertical shift does not change the power exponent; it only shifts the **curve's position**.

- **Figure 7(c):** Rectilinear plot shows a shifted parabola
- **Figure 7(d):** Logarithmic plot shows a straight line with the same slope, $K = 2$, but a different intercept

This demonstrates that ACP mathematics cleanly separates:

- **Power (slope)**
- **Position constant (intercept)**

5.2.5. Paradigm for Power Function $Y = X^3$, $Y = X^2$, and $Y = X^{1/3}$

Figure 8 presents the **paradigm** for power functions:

- All straight lines intersect at $C = 1$
- The slope of each line equals the **power exponent**
- The lines extend continuously in both directions
- All lines approach the **pivot asymptotic nonlinear zero**
- The paradigm unifies all power functions under ACP mathematics

This paradigm demonstrates the elegance of ACP methodology:

- Rectilinear graphs show curves
- Logarithmic graphs reveal straight-line proportionality
- Asymptotes define the nonlinear structure
- Face values provide the correct measurement system

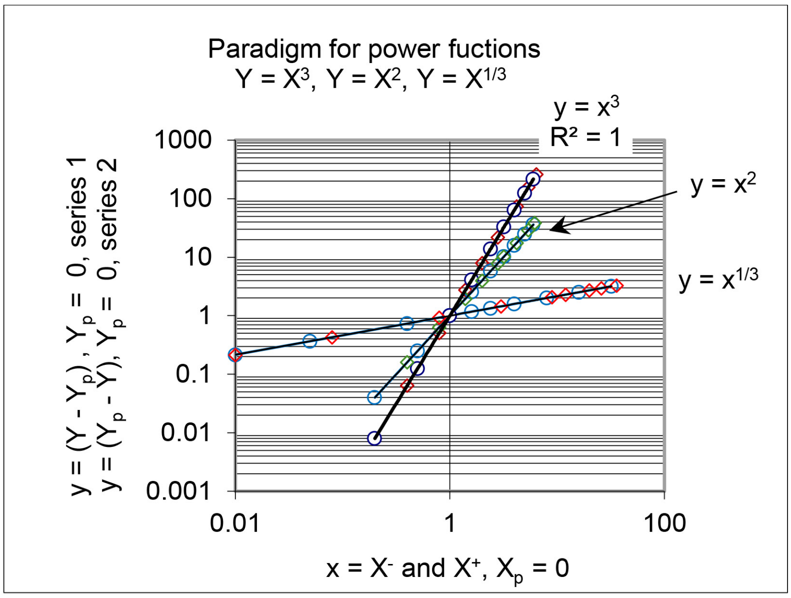
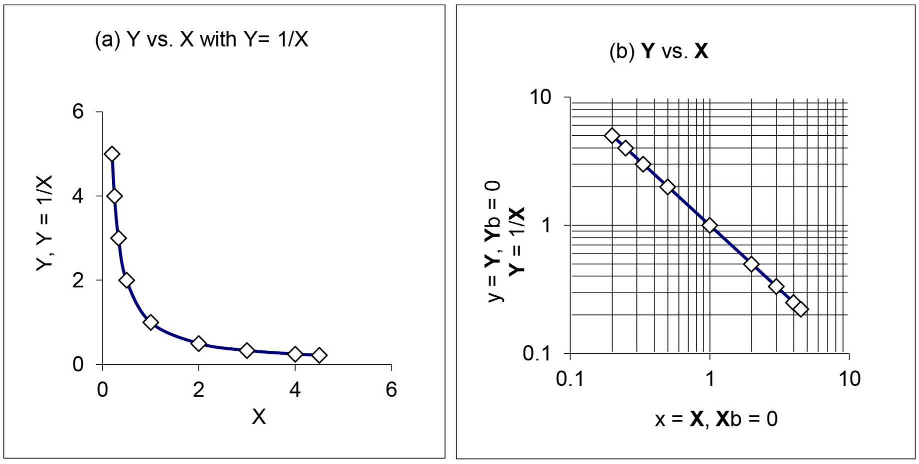


Figure 8. Paradigm for power function $Y = X^3$, $Y = X^2$, and $Y = X^{1/3}$.

5.2.6. Inverse Function $Y = 1/X$ and $Y = 1/X$



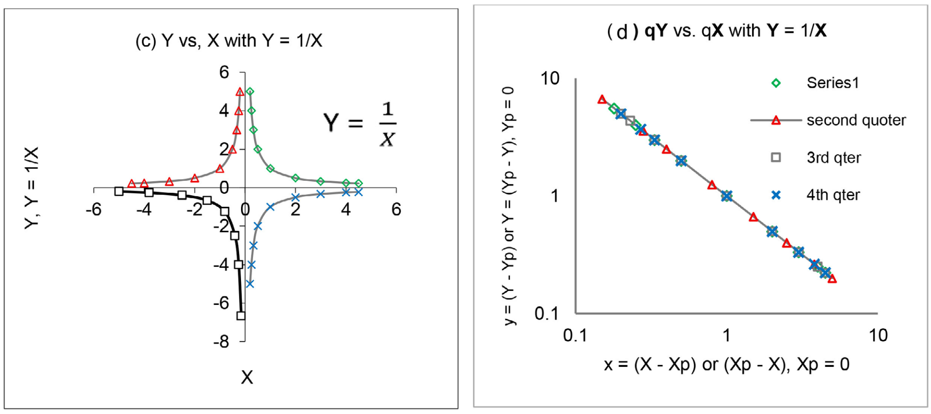


Figure 9. Inverse Function: (a) in the first quadrant, $Y = 1/X$ (rectilinear); (b): $Y = 1/X$ (logarithmic); (c): $Y = 1/X$ in four quadrants; (d): $qY = 1/qX$ in log-log scale.

In conventional mathematics, the inverse function $Y = 1/X$ is plotted on a rectilinear (Cartesian) graph across all four quadrants. This produces the familiar hyperbolic curves shown in Figure 9(c). Positive and negative values of X and Y appear naturally in this representation.

However, when the axes are converted to logarithmic scales, only the first-quadrant portion of the graph remains visible. The curves in the second, third, and fourth quadrants disappear because:

- Negative values cannot be plotted on a logarithmic scale
- Zero cannot be plotted on a logarithmic scale

Thus, the log-log plot (Figure 9(d)) shows only a single straight line corresponding to the first-quadrant data.

Understanding the Confusion: Mismatch of Numbers and Scales

The confusion arises from a mismatch:

- The rectilinear graph is appropriate for linear numbers, but
- The inverse function involves nonlinear numbers, which must be plotted on a nonlinear logarithmic scale

ACP mathematics resolves this by emphasizing two principles:

1) Nonlinear numbers are always associated with asymptotes

2) Nonlinear change must be measured relative to the asymptote, using face values such as $(Yu - Y)$, $(Xu - X)$ for first-order phenomena, and $(qYu - qY)$, $(qXu - qX)$ for second-order phenomena.

Asymptotes in the Four Quadrants

For the inverse function, the asymptotic structure changes depending on the quadrant:

Y-asymptotes

- First and Second Quadrants: have a bottom asymptote (the *x*-axis)
- Third and Fourth Quadrants: The same *x*-axis becomes the upper asymptote of

X-asymptotes

- First and Fourth Quadrants: have a bottom asymptote (the *y*-axis)
- Second and Third Quadrants: The same *y*-axis becomes the upper asymptote

of X

Because both variables share the same asymptotes, we call them pivot asymptotes: $Yp = 0$, $Xp = 0$

ACP Interpretation of the Inverse Function

In the first quadrant, the nonlinear change in Y is negatively proportional to the nonlinear change in X. This relationship is expressed as: $Y = 1/X$

These are the ACP integral equations for the inverse function:

$$q(Y - Yp) = Kq(X - Xp) + qC \tag{3c}$$

$$q(Yp - Y) = Kq(Xp - X) + qC \tag{3d}$$

Conventional Mathematical Forms are:

$$(Y - Yp) = C(X - Xp)^K \tag{4a}$$

$$(Yp - Y) = C(Xp - X)^K \tag{4b}$$

The above power-law form equations:

- Apply to inverse functions,
- Apply to root functions,
- Apply to power functions,
- And apply to many physical nonlinear phenomena.

6. Construction Sequence for Horizontal Asymmetric-Bell, Standard Sigmoidal Curve, and Straight-Line Proportionality Graphs

To prepare for the discussion of vertical sigmoidal curves in Section 7, we first examine the standard horizontal sigmoidal curve and the sequence of transformations that lead to a straight-line proportionality graph. This section uses simulated data to illustrate how second-order nonlinear phenomena evolve through four stages:

1) Asymmetric bell-shaped elementary data
2) Standard horizontal sigmoidal curve
3) Asymptotically convex curve
4) Straight-line proportionality graph

Each stage reveals a different aspect of the nonlinear structure.

6.1. From Elementary Data to Cumulative Connectivity

Stage 1—Asymmetric Bell-Shaped Curve (Figure 10(a) and Figure 10(b))

The elementary variables y and x are not mathematically connected. When plotted as y vs. x, the result is an asymmetric bell-shaped curve. These elementary values represent second-order nonlinear numbers, but their structure is not yet visible.

- Figure 10(a): X treated as categorical
- Figure 10(b): X treated as linear

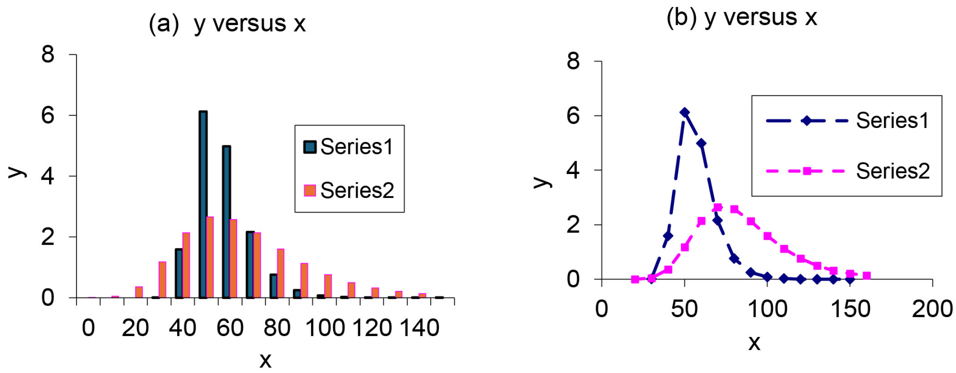


Figure 10. Second-order nonlinear numbers *y*. (10a) *X* treated as categorical; (10b) *X* treated as linear.

Stage 2—Standard Horizontal Sigmoidal Curve (Figure 11(a))

When the elementary values *y* are cumulatively summed to form *Y*, and the corresponding *x* values are cumulatively summed to form *X*, the resulting plot of cumulative *Y* vs. cumulative *X* produces a standard horizontal sigmoidal curve.

This transformation introduces mathematical connectivity:

- *y* → elementary, disconnected
- *Y* → cumulative, connected
- *x* → elementary
- *X* → cumulative

The sigmoidal curve in Figure 11(a) is equivalent to the area under the curve (AUC) of the elementary data (see Appendix A [6] [7]). The plot of *Y* on a log scale in Figure 11(b) accounts for one order of nonlinearity, and the measurement of face value ($qYu - qY$) accounts for additional orders of nonlinearity, resulting in a total of second-order nonlinearity phenomena.

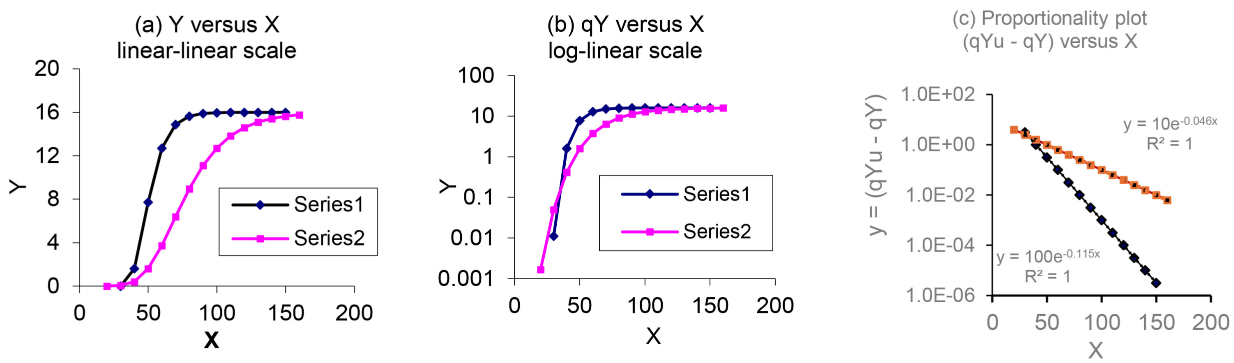


Figure 11. Transformation Sequence. (a) Sigmoidal curve (linear-linear scale); (b) Asymptotically convex curve (log-linear scale); (c) Straight-line proportionality graph (according to Equation (5) in section 6.4).

6.2. Introducing the Nonlinear Logarithmic Scale

Stage 3—Asymptotically Convex Curve (Figure 11(b))

When the vertical axis of Figure 11(a) is converted from a linear scale to a nonlinear logarithmic scale, the sigmoidal curve becomes an asymptotically convex curve.

This transformation reveals:

- The presence of an upper asymptote qYu
- The nonlinear nature of the cumulative variable qY
- The approach toward the asymptote without ever reaching it

The convexity reflects the second-order nonlinear behavior of qY.

6.3. Straight-Line Proportionality Graph

Stage 4—Proportionality Plot (**Figure 11(c)**)

To obtain a straight-line representation, ACP mathematics uses the face value of the second-order nonlinear number: ($qYu - qY$), where:

- $qY = \log(Y)$
- $qYu = \log(Yu)$
- Yu is the upper asymptote of Y

Plotting ($qYu - qY$) on a logarithmic scale against X produces a straight-line -oriented proportionality graph, as shown in **Figure 11(c)**.

This is the hallmark of ACP methodology: nonlinear phenomena become straight lines when expressed relative to their asymptotes.

6.4. Governing Equations for Second-Order Nonlinearity

ACP mathematics provides two proportionality equations for second-order nonlinear phenomena:

Equation (5): Nonlinear Y vs. Linear X

$$d\left(q\left(qYu - qY\right)\right) = -KdX \tag{5}$$

This states that the nonlinear change of the phase value ($qYu - qY$) is proportional to the linear change in X.

Equation (6): Nonlinear Y vs. Nonlinear X

$$d\left(q\left(qYu - qY\right)\right) = -Kd\left(q\left(X - Xb\right)\right) \tag{6}$$

This states that the nonlinear change of the phase value is proportional to the nonlinear change of ($X - Xb$), where Xb is the bottom asymptote of X [1] [2].

These equations describe the second-order nonlinear relationship between the cumulative dependent variable Y and the independent variable X.

7. Simulation of Fluidized Bed Experiments Having Vertical Sigmoidal Curves

In the fluidization literature, researchers have often used different mathematical equations to describe the S-shaped and C-shaped voidage profiles observed under the same physical setup and similar operating conditions in circulating fluidized-bed (CFB) risers [8] [9]. This practice is inconsistent with the **law-of-nature principle that a single physical phenomenon—under the same geometry and testing conditions—should be described by a single governing equation**, with differences appearing only in the parameters (e.g., C and K), not in the mathematical form.

Appendix B reproduces the voidage-versus-mass-flux data from Monazam and Shadle [8] [9]. Their own summary highlights the inconsistency in traditional curve-fitting:

"The decrease in solids fraction at the bottom of the riser C-shaped profile was fit to an exponential equation as developed by Kunii and Levenspiel (1990) [10], while the S-shaped profile was fit to the equation presented by Li and Kwauk (1980)" [11].

In reality, these data represent **one nonlinear phenomenon** and can be described by a **single ACP second-order nonlinear equation**, given as Equation (7), with its integral form in Equation (7a). These equations are the **transpose** of Equations (5) and (5a), exchanging the roles of X and Y, with one additional requirement: the diameter of the CFB column, Xd, must be subtracted from X to obtain the correct nonlinear distance ($X - Xd$).

7.1. Governing ACP Equation for Vertical Sigmoidal Curves

The ACP second-order nonlinear equation for vertical sigmoidal curves is:

$$d\left(q\left(Y-Yb\right)\right)=-Kd\left(q\left(qXu-q\left(X-Xd\right)\right)\right) \tag{7}$$

with the integral form:

$$q\left(Y-Yb\right)=-K\left(q\left(qXu\right)-q\left(X-Xd\right)\right)+qC \tag{7a}$$

Here:

- Yb is the **bottom asymptote** of Y
- Xu is the **upper asymptote** of X
- Xd is the **lower asymptote** of X, corresponding to the **left wall** of the CFB column
- K is the proportionality constant
- C is the position constant
- $q(\cdot)$ denotes the logarithmic transformation

These equations describe **all** S-shaped and C-shaped voidage profiles using a single **unified nonlinear model**.

7.2. Asymptotic Structure of the CFB Test Column

For a vertical CFB riser:

- The **right-side column wall** is the **upper asymptote** of X, assigned as $Xu = 1$.
- The **left-side column wall** is the **bottom asymptote**, represented as $Xd = 0.8$.

Thus, the nonlinear distance in the horizontal direction is: $X - Xd$.

This ensures that the nonlinear variable X is always measured **relative to its asymptote**, consistent with ACP methodology.

7.3. Simulation of S- and C-Shaped Curves

Using:

- $Xu = 1$

- $Xd = 0.8$
- $K = 0.2$
- $C = 0.2$, 0.1, 0.2, 0.5; 1, 2, 5, 10, 17

We generate a family of **vertical sigmoidal curves** (Figure 12(a)) and their corresponding **straight-line proportionality plots** (Figure 12(b)).

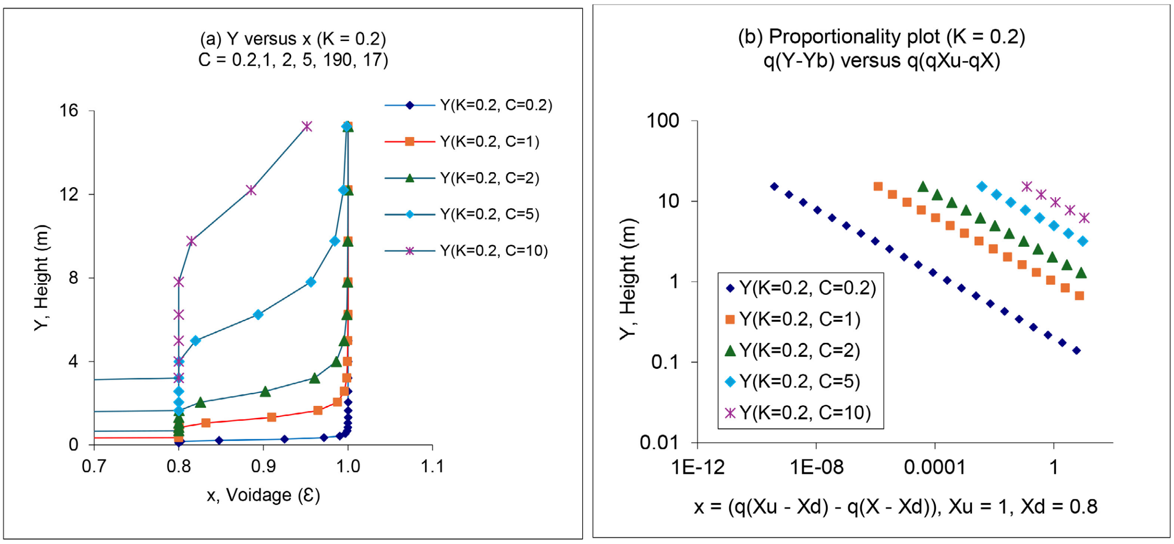


Figure 12. Simulated Fluidized Bed Graphs. (a) S- and C-curves. (b) Proportionality plot for $q(Y - Yb)$ versus $q(q(Xu - Xd) - q(X - Xd))$.

Figure 12(a)—Cartesian Graph

- Shows S-shaped and C-shaped voidage profiles
- All curves arise from the **same equation**, differing only in C

Figure 12(b)—Proportionality Graph

- Shows **parallel straight lines**
- Parallelism indicates the same K
- Vertical shifting corresponds to changes in C

This demonstrates that **all observed curve shapes**—S-curves, C-curves, and transitional forms—are simply **different parameterizations** of the same ACP equation.

7.4. Excel Implementation (Table 4)

Table 4 provides the Excel worksheet for the case:

- $Xu = 1$
- $K = 0.2$
- $C = 1$
- $Xd = 0.8$

Key formulas:

- **Column E** (recursive calculation of X):
- Cell E3: = E4 × 1.25
  - Drag E3 → E25

- Cell E26 gives initial $Y = 0.09$
- **Column D** (phase value $q(Yu - Y)$):
  - Cell D3: = (LOG($G$3-$G$6)-LOG(A3-$G$6))
  - Drag D3 → D26
- **Column B** (theoretical cumulative Y): =($G$3-$G$6)/(10^(($G$5/E3)^(1/$G$4)))
- **Column C** (theoretical X):
- Cell C3: = B3 − B4
- Drag C3 → C26

Plots:

- **Column D vs. Column E** → Figure 12(b) (straight lines)
- **Column A vs. Column E** → Figure 12(a) (sigmoidal curves)

7.5. Interpretation

By varying the parameter C:

- The straight lines in Figure 12(b) **shift up and down**
- The corresponding sigmoidal curves in Figure 12(a) **shift accordingly**
- The slope K remains constant, preserving parallelism
- All curves remain solutions of the **same ACP equation**

This confirms the ACP principle:

One physical phenomenon → One nonlinear equation → Many curves via parameter variation

This resolves the inconsistency in the traditional literature and provides a unified mathematical framework for CFB voidage profiles.

Table 4. Excel worksheet for $Xu = 1$, $K = 0.2$, $C = 1$, and $Xd = 0.8$.

D3 =(LOG($G$3-$G$6)-LOG(A3-$G$6))

	A	B	C	D	E	F	G
1	Theor Cum X + $H$7	Theor Cum X	Theor X	(qXu-qX)	Y(K=0.2, C=0.2)		
2							
3	1.0000	0.199999	0.000001	0.0000	15.2466	Xu =	1
4	1.0000	0.199998	0.000003	0.0000	12.1973	K=	0.2
5	1.0000	0.199995	0.000011	0.0000	9.7578	C=	1
6	1.0000	0.199984	0.000033	0.0000	7.8063	Xd=	0.8
7	1.0000	0.199952	0.000099	0.0001	6.2450	Y(i)=Y(j)*1.25	
8	0.9999	0.199852	0.000303	0.0003	4.9960		
9	0.9995	0.199549	0.000922	0.0010	3.9968		
10	0.9986	0.198627	0.002788	0.0030	3.1974		
11	0.9958	0.195839	0.008269	0.0091	2.5580		
12	0.9876	0.187570	0.023138	0.0279	2.0464		
13	0.9644	0.164432	0.054405	0.0850	1.6371		
14	0.9100	0.110027	0.077742	0.2595	1.3097		
15	0.8323	0.032286	0.031520	0.7920	1.0477		
16	0.8008	0.000766	0.000766	2.4171	0.8382		
17	0.8000	0.000000	0.000000	7.3763	0.6706		
18	0.8000	0.000000	0.000000	#NUM!	0.5364		
19	0.8000	0.000000	0.000000	#NUM!	0.4292		
20	0.8000	0.000000	#NUM!	#NUM!	0.3433		
21	#NUM!	#NUM!	#NUM!	#NUM!	0.2747		
22	#NUM!	#NUM!	#NUM!	#NUM!	0.2197		
23	#NUM!	#NUM!	#NUM!	#NUM!	0.1758		
24	#NUM!	#NUM!	#NUM!	#NUM!	0.1406		
25	#NUM!	#NUM!	#NUM!	#NUM!	0.1125		
26	#NUM!	#NUM!	#NUM!	#NUM!	0.0900		

8. Discussions

8.1. Recognize the Existence of Nonlinear Numbers

For centuries, students have been taught that 0.9999… = $0.\dot{9}$ = 1. This traditional view arises from a lack of recognition of nonlinear numbers. Linear and nonlinear numbers coexist, but nonlinear numbers must be measured relative to their asymptotes.

For first-order nonlinear phenomena, the relevant measure is with ($Yu - Y$); For second-order nonlinear phenomena, the measure becomes with ($qYu - qY$).

Recognizing nonlinear numbers provides an opportunity to modernize mathematical education and adopt the ACP methodology as a systematic approach for analyzing nonlinear behavior in science and engineering. The ACP concepts should be introduced to students as early as high school.

8.2. Nonlinear Equations are Derived from Concave and Convex Asymptotic Curves

Linear changes in linear phenomena are straightforward to measure. However, measuring change along a curved line in a nonlinear phenomenon requires a different mathematical framework. **Figure 13** illustrates this contrast by comparing:

- a linear line in a linear-by-linear graph (**Figure 13(a)**),
- an asymptotically convex curve in a linear-by-linear graph (**Figure 13(b)**), and
- an asymptotically convex curve in a log-by-linear graph (**Figure 13(c)**).

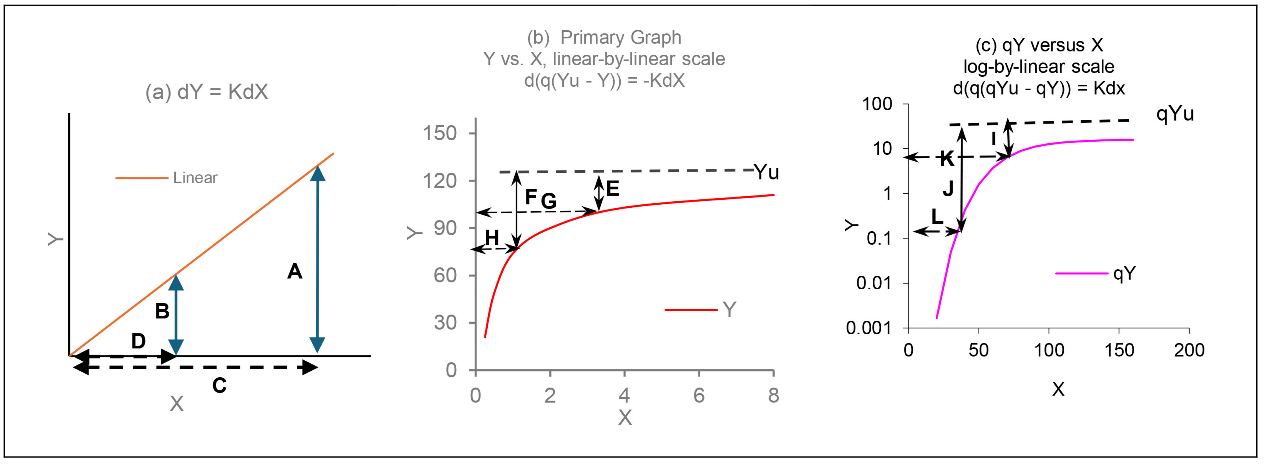


Figure 13. Linear versus Nonlinear Change: (a) Linear line in linear graph; (b) Convex curve in linear graph; (c) Convex curve in log-linear graph.

8.2.1. Linear Change vs. Nonlinear Change

Linear Case (**Figure 13(a)**)

For a straight line, the change in Y is proportional to the change in X: $dY = KdX$. In ratio form, the proportionality is: $\frac{A}{B} = \frac{C}{D} = k$. where the double-arrow distances A, B, C, D represent linear increments. This is the classical linear-by-linear proportionality.

8.2.2. Nonlinear Case: Asymptotically Convex Curves

Convex Curve in Linear Scale (**Figure 13(b)**)

In **Figure 13(b)**, the curve is asymptotically convex. The variable Y approaches its upper asymptote, Yu, but never reaches or touches it. The nonlinear distance is measured by the face value: ($Yu - Y$)

Convex Curve in Log-Linear Scale (Figure 13(c))

In Figure 13(c), the logarithmic transformation produces a curve approaching the asymptote qYu. Again, the curve approaches but never reaches the asymptote.

8.2.3. Law-of-Nature Proportionality in Nonlinear Phenomena

In nonlinear phenomena, the vertical nonlinear distance and the horizontal nonlinear distance are inversely proportional:

- As the vertical double-arrow increases, the horizontal double-arrow decreases
- As the vertical double-arrow decreases, the horizontal double-arrow increases

This inverse relationship is the foundation for deriving nonlinear differential equations.

Figure 13(b) Interpretation

When the vertical solid arrow changes from $E \to F$, the horizontal dashed arrow changes from $G \to H$. This yields the first-order nonlinear differential equation:

$$d\left(q\left(Yu - Y\right)\right) = -KdX \tag{8}$$

Integrating:

$$q\left(Yu - Y\right) = -KX + C \tag{8a}$$

Here: $q(Yu - Y)$ is the first-order nonlinear authentic number. C is the position constant, determining the vertical placement of the straight line in the proportionality graph

8.2.4. Nonlinear Measurement of *X*

In many physical systems, the independent variable X is nonlinear as well. In such cases, the differential term dX must be replaced by a nonlinear distance: ($X - Xb$), yielding:

$$d\left(q\left(Yu - Y\right)\right) = -Kd\left(q\left(X\right)\right) \tag{9}$$

$$d\left(q\left(Yu - Y\right)\right) = -Kd\left(q\left(X - Xb\right)\right) \tag{9a}$$

Note 1: Meaning of qX

- qX is shorthand for the authentic nonlinear number of X
- It represents $q(X - Xb)$, where Xb is the bottom asymptote (nonlinear zero)
- This nonlinear zero behaves like a baseline asymptote or black hole asymptote
- It can be approached but never reached or plotted
- Excel enforces this rule: negative and zero values cannot be plotted on log charts

8.2.5. Second-Order Nonlinear Equations

So far, we have discussed first-order nonlinear equations, where the dependent variable contains one "q". We now extend this to second-order nonlinear equations, where the dependent variable contains two "q", such as:

$$d\left(q\left(qYu - qY\right)\right) = -KdX \tag{10}$$

$$q\left(qYu - qY\right) = -KX + C \tag{10a}$$

These equations are parallel to Equations (8) and (9).

Figure 13(c) Interpretation

When the vertical solid arrow changes from $I \rightarrow J$, the horizontal dashed arrow changes from $K \rightarrow L$, giving $d(q(qYu - qY)) = -KdX$; Integrating: $(q(qYu - qY)) = -KX + C$.

Here:

- $q(qYu - qY)$ is the second-order nonlinear authentic number
- C again determines the vertical position of the straight line

Nonlinear X in Second-Order Phenomena

When X is nonlinear:

$$d\left(q\left(qYu - qY\right)\right) = -Kd\left(q\left(X - Xb\right)\right) \tag{11}$$

$$q\left(qYu - qY\right) = -Kq\left(X - Xb\right) + qC \tag{11a}$$

These equations govern second-order nonlinear phenomena, such as sigmoidal curves and C-curves, as well as many physical processes.

8.2.6. Concave Asymptotic Curves

For concave asymptotic curves, cumulative numbers are replaced with demulative numbers (the opposite of cumulative), and a bottom asymptote Xb is introduced:

$$d\left(q\left(Y - Yb\right)\right) = -KdX \tag{12}$$

$$d\left(q\left(Y - Yb\right)\right) = -Kd\left(X - Xb\right) \tag{13}$$

These equations mirror the convex-curve equations but apply to concave asymptotic behavior.

8.3. First-Order Nonlinear Phenomena

Power functions represent first-order nonlinear-by-nonlinear phenomena. When plotted on a linear X-Y axis, both positive and negative values appear, producing concave and convex curves on rectilinear graphs (Figures 4(a)-7(a)). When plotted on nonlinear Y and X axes, only positive values appear, producing straight lines on nonlinear logarithmic graphs (Figure 4(b), Figure 6(b), Figure 7(b)).

Rectilinear plots of concave and convex curves have two limitations:

1) The slope of the equation is not visible.

2) The curves do not show their approach toward nonlinear asymptotic zeros.

Nonlinear logarithmic plots overcome both limitations: the straight lines reveal the slope and clearly show the approach toward the asymptotic nonlinear zero, which cannot be reached or plotted.

Figure 8 provides a concise paradigm for power functions: the slope corresponds to the power, all lines intersect at $C = 1$, and the lines extend in both directions to preserve continuity. ACP mathematics emphasizes continuity, in contrast to traditional treatments that often struggle with it, as noted by David Berlinski in A Tour of the Calculus (Chapter 26, "A Farewell to Continuity") [12].

8.4. Second-Order Nonlinear Phenomena

Many physical experiments are governed by second-order nonlinear equations, including the relationship between X-ray energy and X-ray transmission through diamond and Kapton films [1], and the electrostatic separation of fine particles [2]. Second-order nonlinear equations are distinguished by their ability to connect four characteristic representations of nonlinear behavior:

1) Asymmetric bell-shaped curves
2) Sigmoidal curves
3) Asymptotically convex curves
4) ACP straight-line proportionality graphs

These four forms are not separate phenomena; they are different manifestations of the same second-order nonlinear structure.

Figure 12(a) and Figure 12(b) illustrate the general application of Equation (7):

- Figure 12(a) shows a family of S- and C-shaped curves generated by varying the position constant
- Figure 12(b) shows the corresponding parallel straight lines in the proportionality graph

The transformation from Equation (6) to Equation (7) is achieved by:

1) Transporting the nonlinear structure from Y to X
2) Introducing the bottom asymptote Xd
3) Replacing the term $qXu - qX$ with $q(Xu - Xd) - q(X - Xd)$

This modification ensures that the nonlinear distance is measured correctly for a vertical sigmoidal curve.

8.5. Concise Explanation of Collecting Cumulative Numbers

In Excel tables, we always reserve a row above the data set as a blank cell to make it easy to calculate cumulative numbers and for back calculations. For example, in Table 2 column A, x is an elementary number. We calculate cumulative numbers X in Column B by inputting Cell B3 as "=B2 + A3", then drag B3 → B7 to complete the Column. That is, the relationship between the elementary number y and the cumulative number Y is: Target Cell = Top Cell + Left Cell. The same is for calculating Y in Column D (Table 5).

Table 5. Procedure for collecting cumulative numbers.

	Top Cell		B2
Left Cell	**Target Cell**	A3	**B3**

9. Conclusions

The ACP mathematical methodology provides a unified framework for developing graphs and equations across scientific disciplines. We have shown that combining rectilinear and logarithmic graphs enhances conceptual understanding of power and inverse functions, particularly regarding the role of the baseline asymptote. This approach clarifies the origins of concave and convex curves and yields the equations and parameters governing first-order nonlinear phenomena.

For second-order nonlinear phenomena, a single nonlinear equation effectively describes circular fluidized-bed (CFB) operation, capturing the full range of C- and S-shaped experimental curves. The resulting paradigm graph presents a family of straight lines and a unified nonlinear rate equation.

Together, these results demonstrate that the ACP methodology is versatile, intuitive, and broadly applicable in scientific research and engineering analysis.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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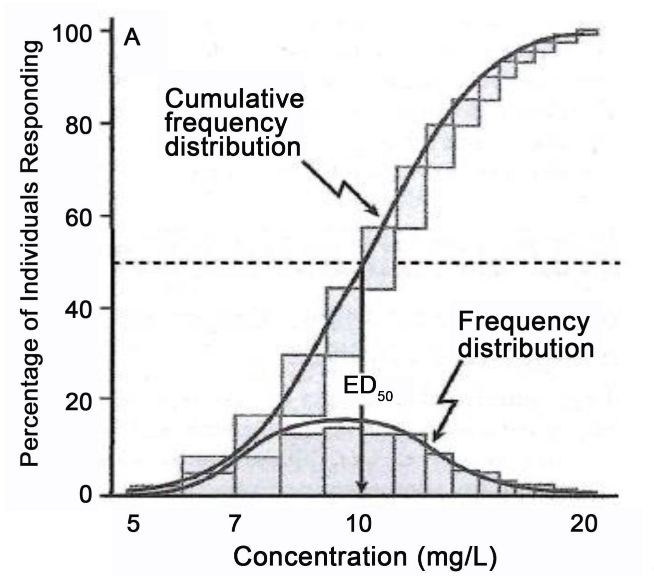
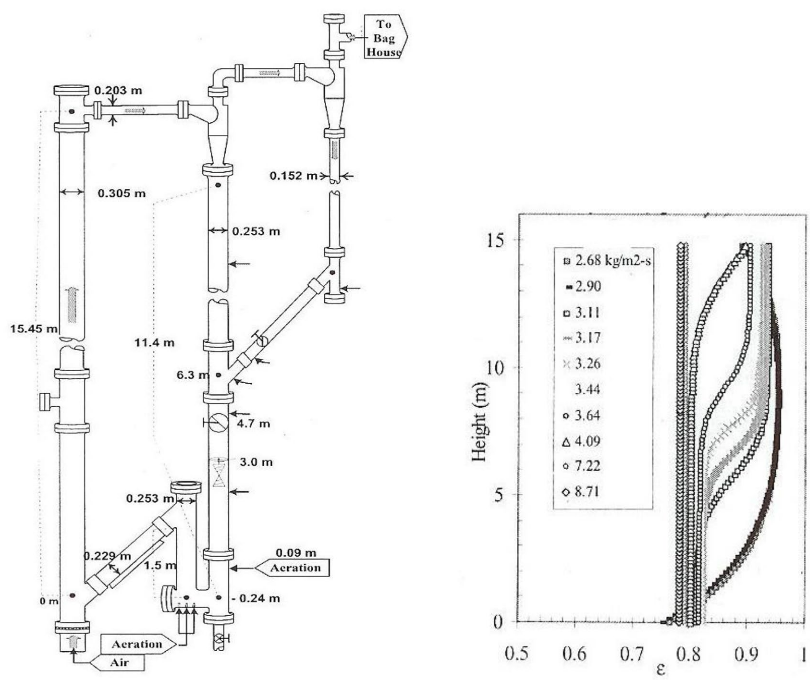
Appendix

Appendix A: Exhibition of Cumulative Curve, Sigmoidal Curve Is Area under the Curve (AUC), after Brunton [6]



Appendix B: Set up and Voidage Profile of Circular Fluidized Bed Test [9] [10]



Symbols: θ = 10; q = Log (nonlinear logarithmic); αβ (extension of XY); ϕ = (0) (nonlinear zero); x = elementary independent variable, y or (y) = elementary dependent variable or y = equation y (inside the graph); X = cumulative of x, Y = cumulative of y or (y).