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Formulas for Matrix Representations of su(3) and sl(3,C) Lie Algebras

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Abstract

Topological groups have wide application in the study of continuous symmetries in mathematics and the sciences. The Lie group SU(3) is one such group. The Lie algebra su(3) incorporates the generators that produce the elements of SU(3) by exponentiation. Many other Lie algebras, including su(2), have known formulas that may be utilized to build matrix bases that span the generators of those algebras. This article contributes formulas for matrices that form bases of irreducible representations of the su(3) and sl(3,C) Lie algebras. Having access to these matrix bases may provide benefits similar to those provided by the well-known formulas for the su(2) Lie algebra.

Keywords

Lie algebras, Matrix Representations, su(3), sl(3,C)

1. Introduction

The special unitary group SU(3) has extensive applications in physics, from particle physics [1]-[6] and nuclear physics [7] [8] to the isotropic 3D harmonic oscillator [9] and quark nuggets in astrophysics [10] [11]. Formulas for the spin 1/2 matrices of SU(2) are well known and their incorporation in research demonstrates the potential applications of the matrix formulas in this article.

The group SU(3) is exemplified by its prototype, the group of 3 × 3 unitary matrices with unit determinant with elements combined by matrix multiplication. A matrix version of an SU(3) irreducible representation (irrep) may have many more than 3 dimensions, with the larger matrices mimicking the behavior of the prototype 3 × 3 unitary matrices. There is an irrep for each pair of nonnegative integers (p,q) [3] [12]-[14].

Each element f of the SU(3) Lie group is generated by exponentiation of an element F of the $\mathfrak{su}(3)$ Lie algebra. The basis of the algebra consists of eight generators F_j, $j=1,\cdots,8$. Matrix representations have basis generators that are traceless matrices and hermitian. Being hermitian, the matrices have entries that may be complex numbers.

By a linear transformation with complex coefficients, the F_j basis can be transformed to the "spherical representation" of the F_j basis of $\mathfrak{su}(3)$, herein called the "$TYUV$ basis" [3] [4] [14]. Unlike the $\mathfrak{su}(3)$ Lie algebra, matrices for the $TYUV$ basis can have exclusively real-valued components. However, the basis $TYUV$ is the basis of the Lie algebra $\mathfrak{sl}(3,\mathbb{C})$, not $\mathfrak{su}(3)$. Many of the generators are not hermitian, and, by exponentiation, the elements of the algebra spanned by the basis generators $TYUV$ yield the generally nonunitary elements of the Lie group $\mathrm{SL}(3,\mathbb{C})$.

We present formulas that produce a real-valued $TYUV$ matrix basis for finite-dimensional irreducible highest-weight representations of the $\mathfrak{sl}(3,\mathbb{C})$ Lie algebra. The irreps are labeled with two nonnegative integers (p,q). A linear transformation yields the complex matrices for the F_j basis of the corresponding $\mathfrak{su}(3)$ irrep.

The $TYUV$ basis formulas were derived directly from the commutation relations (CR) of the $\mathfrak{sl}(3,\mathbb{C})$ Lie algebra. The CR's property of invariance under similarity transformations requires additional assumptions to counter the invariance and arrive at definite formulas. Introducing additional assumptions might introduce error, so an Appendix is included to verify that the basis satisfies the CRs.

There is a well-known alternative procedure. The groups SU(3) and $\mathrm{SL}(3,\mathbb{C})$ are subgroups of the general linear group $\mathrm{GL}(3,\mathbb{C})$ of 3 × 3 complex matrices. The group $\mathrm{GL}(3,\mathbb{C})$ itself has irreps, which we denote as $\mathrm{GL}_3(\bar{N},\mathbb{C})$ with dimensions $\bar{N}\geq 3$. The subscript 3 in GL_3 indicates that these are irreps of the group of 3 × 3 complex matrices. Gelfand and Tsetlin devised a method to determine matrix bases for the $\mathrm{GL}_3(\bar{N},\mathbb{C})$. [15] Unlike the TYUV approach directly from CRs, the GT basis approach needs to sort the subgroup's eigenvectors or states, from the larger GL_3 set. Nevertheless, restricting the matrix bases of $\mathrm{GL}_3(\bar{N},\mathbb{C})$ to the irreps of the subgroups SU(3) and $\mathrm{SL}(3,\mathbb{C})$ results in formulas for Gelfand-Tsetlin matrix bases for irreps with dimensions $\bar{N}$ of SU(3) and $\mathrm{SL}(3,\mathbb{C})$. [16]

A major consideration in the construction of the formulas are the subalgebras of $\mathfrak{sl}(3,\mathbb{C})$. The $\mathfrak{sl}(3,\mathbb{C})$ basis generators T^3, Y, $T^{\pm}$ form a basis for the $\mathfrak{u}(2)$ Lie subalgebra. The generators T^3 and $T^{\pm}$ make a basis for the Lie algebra $\mathfrak{su}(2)$ associated with massive particle spin.

The $\mathfrak{u}(2)$ subalgebra structure is important both for the approach here and for the GT formalism. In this article, the $\mathfrak{u}(2)$ Lie subalgebras are described by removing boxes from the $\mathfrak{sl}(3,\mathbb{C})$ Young diagram to give the $\mathfrak{u}(2)$ Young diagrams. With the box removal parameters, we devise a sequence function n that gives an integer between 1 and the dimension of the irrep. The sequence function n allows us to write row and column indices as functions of the parameters of the Young diagram. The same parameters are also incorporated into the function for the matrix entry. The matrix entries and their row and column indices are presented as functions of the Young diagram box removal parameters.

Each matrix $TYUV$ is a two dimensional array of entries arranged into rows and columns. The set of formulas includes a formula for each potentially nonzero entry and two formulas to locate the entry in a row and column. Given the integers (p,q) that determine an irrep, the formulas for each matrix generator produce numerical results for each entry and for the row and column indices of that entry.

Section 2 develops the 28 commutation relations of the $\mathfrak{sl}(3,\mathbb{C})$ Lie algebra for the basis $TYUV$. The CRs are quadratic equations that must be satisfied by the $TYUV$ matrices and, if satisfied, show that the $TYUV$ matrices form a basis of the Lie algebra $\mathfrak{sl}(3,\mathbb{C})$.

Section 3 obtains a list of the spins of the $\mathfrak{u}(2)$ Lie subalgebras in a general $\mathfrak{sl}(3,\mathbb{C})$ irrep. The list of spins can be related to the removal of boxes from a Young diagram, which offers a visual display of the subalgebra structure.

The list of spins determines a function n called the "sequence function." The sequence function n produces a sequence of integers that covers a range equal to the dimension d of the matrices, $n=1,\cdots,d$. The function n depends on two parameters a,b for removing Young diagram boxes and the eigenvalue β of T^3, which is a spin component.

Let M be a matrix that is a linear combination of $TYUV$ matrices. An entry M^{rc} in M has a row index $r=n(a_r,b_r,\beta_r)$ determined by the parameters a_r,b_r,β_r and a column index $c=n(a_c,b_c,\beta_c)$ determined by a_c,b_c,β_c.

Section 4 presents the formulas for 12 matrices that are used to define the eight $TYUV$ matrix generators for the basis of an irrep of the $\mathfrak{sl}(3,\mathbb{C})$ Lie algebra. The twelve matrices are "sparse monomial (SM) matrices." An SM matrix has at most one nonzero entry in each row and at most one nonzero entry in each column. The eight $TYUV$ matrices need one SM matrix for each of the four T,Y matrices and two SM matrices for each of the four U,V matrices.

Let M be one of the twelve SM matrices. The value of the single possibly nonzero entry M^{rc} in the row r of M is presented as a function of the six parameters a_r,b_r,β_r, a_c,b_c,β_c. However, the six parameters are constrained, so the entry M^{rc} is a function of three a,b,β parameters.

The eight $TYUV$ matrices of the basis for the $\mathfrak{sl}(3,\mathbb{C})$ Lie algebra are linear combinations of the 12 SM matrices. Also in Section 4, we write the eight F_j matrices for the basis of the $\mathfrak{su}(3)$ Lie algebra in terms of the twelve SM matrices.

The Appendix verifies that the eight $TYUV$ matrices obey the 28 CRs of the $\mathfrak{sl}(3,\mathbb{C})$ Lie algebra, and therefore form a basis of the $\mathfrak{sl}(3,\mathbb{C})$ Lie algebra.

2. Lie Algebras

In this section, the commutation relations (CRs) of the basis $TYUV$ of the Lie algebra $\mathfrak{sl}(3,\mathbb{C})$ are derived. We start with a matrix basis F_j of $\mathfrak{su}(3)$ taken from the literature. The transformation is then made from the base F_j of $\mathfrak{su}(3)$ to the base $TYUV$ of $\mathfrak{sl}(3,\mathbb{C})$. The CRs of the $TYUV$ matrices are calculated and displayed. These are the CRs that must be satisfied by the matrices in Section 4 to have a matrix representation of $\mathfrak{sl}(3,\mathbb{C})$.

One basis F_j of the Lie algebra $\mathfrak{su}(3)$ consists of the following eight matrices, [3] [4] [17]

$$F_1 = \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad F_2 = \frac{1}{2}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad F_3 = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{1}$$

$$F_4 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad F_5 = \frac{1}{2}\begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \quad F_6 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

$$F_7 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \quad F_8 = \frac{1}{2\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

By inspection, the F_j s are hermitian and traceless.

The group element f of SU(3) can be expressed as the matrix exponent of an element of the $\mathfrak{su}(3)$ Lie algebra. We have

$$f = \mathrm{e}^{i\sum_{j=1}^{8}\theta_j F_j}, \tag{2}$$

where the matrix exponent is defined by its series expansion, $\exp A = \mathbf{1} + A + \cdots + A^N/N! + \cdots$, the unit matrix is denoted $\mathbf{1}$, and the coefficients θ_j are real. The matrix $\sum_j \theta_j F_j$ is the element of the Lie algebra $\mathfrak{su}(3)$ that generates the group element f. The eight matrices F_j form a basis for the generators of $\mathfrak{su}(3)$.

Many discussions of the SU(3) Lie group apply a transformation to its "spherical representation." The basis of the spherical representation can be chosen to be the $TYUV$ matrices determined by

$$T^3 = F_3;\; Y = \frac{2}{\sqrt{3}}F_8;\; T^{\pm} = F_1 \pm iF_2;\; U^{\pm} = F_6 \pm iF_7;\; V^{\pm} = F_4 \pm iF_5. \tag{3}$$

By (1) to (3), one finds

$$T^3 = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},\; Y = \frac{1}{3}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix},\; T^{+} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{4}$$

$$T^{-} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\; U^{+} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},\; U^{-} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

$$V^{+} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\; V^{-} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

The transformation is invertible, so one can determine the basis of F_j s from the $TYUV$ matrices.

The $TYUV$ matrices are not a basis for $\mathfrak{su}(3)$. They are traceless and the $TYUV$ matrices generate matrices with unit determinant. However, not all $TYUV$ generators are hermitian, so they do not generate unitary matrices, in general. The $TYUV$ matrices are a basis for the Lie algebra $\mathfrak{sl}(3,\mathbb{C})$, not $\mathfrak{su}(3)$ [12].

The notation $TYUV$ is retained for every irreducible representation (irrep) of the $\mathfrak{sl}(3,\mathbb{C})$ Lie algebra. The context should make it clear whether a matrix generator $TYUV$ is one of the matrices in the representation (4) or a matrix generator in the basis of some other representation of $\mathfrak{sl}(3,\mathbb{C})$.

The eight $TYUV$ matrices in (4) produce $8\times 7/2=28$ commutation relations (CR). The 28 CRs for the basis $TYUV$ of the Lie algebra $\mathfrak{sl}(3,\mathbb{C})$ can be grouped into three sets:

- CRs of the T,Y subalgebra $\mathfrak{u}(2)$:

$$\left[T^{+},T^{-}\right]=2T^{3};\ \left[T^{3},T^{\pm}\right]=\pm T^{\pm}; \tag{5}$$

$$\left[Y,T^{\pm}\right]=0;\ \left[Y,T^{3}\right]=0. \tag{6}$$

- CRs with commutators that mix generators T,Y and U,V:

$$\left[T^{3},U^{\pm}\right]=\mp\frac{1}{2}U^{\pm};\ \left[T^{3},V^{\pm}\right]=\pm\frac{1}{2}V^{\pm};\ \left[Y,U^{\pm}\right]=\pm U^{\pm};\ \left[Y,V^{\pm}\right]=\pm V^{\pm} \tag{7}$$

$$\left[T^{\pm},U^{\mp}\right]=\left[T^{\pm},V^{\pm}\right]=0;\ \left[T^{\pm},U^{\pm}\right]=\pm V^{\pm};\ \left[T^{\pm},V^{\mp}\right]=\mp U^{\mp}. \tag{8}$$

- CRs with commutators that involve U,V generators:

$$\left[U^{+},U^{-}\right]=\frac{3}{2}Y-T^{3};\ \left[V^{+},V^{-}\right]=\frac{3}{2}Y+T^{3}; \tag{9}$$

$$\left[U^{\pm},V^{\mp}\right]=\pm T^{\mp};\ \left[U^{\pm},V^{\pm}\right]=0, \tag{10}$$

where the commutator $[A,B]$ of two matrices is the difference of their dot products, $[A,B]\equiv AB-BA$.

Any set of $TYUV$ matrices that satisfy the 28 CRs in (5) to (10) form a basis of the $\mathfrak{sl}(3,\mathbb{C})$ Lie algebra. The formulas in Section 4 produce sets of $TYUV$ matrices that satisfy the 28 CRs in (5) to (10) and, therefore, form bases of representations of the $\mathfrak{sl}(3,\mathbb{C})$ Lie algebra.

As discussed in the next section, the reduction of the T,Y generators to $\mathfrak{u}(2)$ irreps provides parameters for the formulas in Section 4 that produce the matrices of an $\mathfrak{sl}(3,\mathbb{C})$ irrep.

3. The Sequence Function

In this section, a sequence function n is determined for an irreducible representation (irrep) of $\mathfrak{sl}(3,\mathbb{C})$. The sequence function n produces an integer in the range $1,\cdots,d$ when given a trio of parameters a,b,β. Here, d is the dimension of the matrices for the irrep.

The commutation relations (CR) are invariant under similarity transformations. By applying similarity transformations, one can rearrange the rows and columns of a matrix representation in many ways. The sequence function n sets the arrangement of the rows and columns of the matrices for the $TYUV$ basis of the irrep of $\mathfrak{sl}(3,\mathbb{C})$.

We start by considering subalgebras. Observe that the CRs (5) and (6) involve the generators T,Y exclusively. Thus, the four generators T^3, T^+, T^-, and Y form the basis of a subalgebra. That subalgebra can be shown to be the $\mathfrak{u}(2)$ Lie algebra. [12]

It follows that the four T,Y matrices can be reduced to a direct sum of $\mathfrak{u}(2)$ irreps. The T,Y matrices take a block-diagonal form, with each diagonal block being a T,Y matrix for one $\mathfrak{u}(2)$ irrep.

We follow convention and take T^3 and Y to be diagonal matrices, so their diagonal blocks are diagonal matrices. Their diagonal entries are the eigenvalues of eigenvectors, taking the d eigenvectors to be the columns of the unit matrix, δ^{rc}, one eigenvector for each column $c=1,\cdots,d$. We have

$$\sum_s {}^3T^{rs}\delta^{sc} = \beta_r\delta^{rc};\ \sum_s Y^{rs}\delta^{sc} = y_r\delta^{rc}, \tag{11}$$

where the diagonal entries are $\beta_r = {}^3T^{rr}$ and $y_r = Y^{rr}$ and the repeated indices on the right are not summed.

Note that, by (5), the subalgebra $\mathfrak{u}(2)$ has its own subalgebra, $\mathfrak{su}(2)$, whose basis is the set of three generators $\{T^3,T^+,T^-\}$. Familiarity with $\mathfrak{su}(2)$ is assumed. Be aware that what we have called the algebra "$\mathfrak{su}(2)$" is actually "$\mathfrak{sl}(2,\mathbb{C})$", via a spherical representation. The matrix T^+ generates $\exp i\theta T^+ = 1 + i\theta T^+$ which has a unit determinant, but is not hermitian. Here, we choose to follow the conventions that invoke "spherical representations" and do not distinguish $\mathfrak{su}(2)$ from $\mathfrak{sl}(2,\mathbb{C})$.

A $\mathfrak{u}(2)$ irrep has a spin t and its generators can be represented by square matrices of dimension $2t+1$. These square matrices form the diagonal blocks of the T,Y matrices. For one of the diagonal blocks of T^3, we know that its diagonal entries run from $-t$ to $+t$ in unit steps. The generator Y commutes with the T matrices, so we can assume that Y is proportional to the unit matrix with the dimension of the block's $\mathfrak{u}(2)$ irrep. We have

$$-t \le \beta_r \le +t;\ y_r = \text{constant}\ (\text{in one block}) \tag{12}$$

i.e. y_r is constant in any one diagonal block of Y.

The spins t of the u(2) irreps in the reduction of the T,Y matrices of an $\mathfrak{sl}(3,\mathbb{C})$ irrep are well-known in the literature [3] [13]. The result can be visualized with Young diagrams. The scheme produces a list of the $\mathfrak{u}(2)$ subalgebras for a given $\mathfrak{sl}(3,\mathbb{C})$ irrep. See **Figure 1**.

The Young diagram in **Figure 1(a)** represents the (p,q) $\mathfrak{sl}(3,\mathbb{C})$ irrep. The integer p gives the number of boxes in the first row that extend beyond the second row, while q is the number of boxes in the second row.

To determine the diagrams for the $\mathfrak{u}(2)$ subalgebras, we remove boxes from the diagram in **Figure 1(a)** [3]. As indicated, we take a boxes from the first row and we take b boxes from the second row of the Young diagram for $\mathfrak{sl}(3,\mathbb{C})$.

$$0 \le a \le p;\ 0 \le b \le q \tag{13}$$

The resulting diagram has $p-a+b$ excess boxes in the upper row, and the diagram has $q-b$ columns that are two boxes tall.

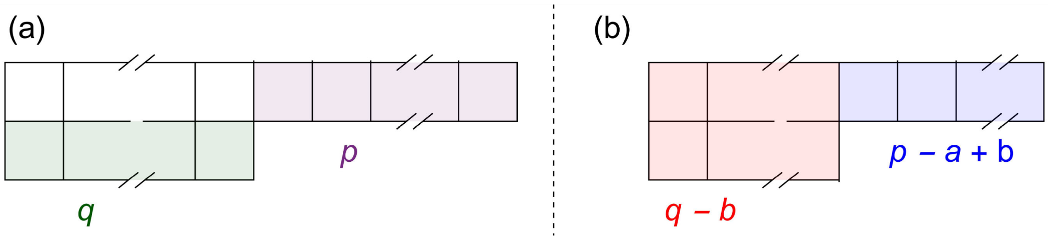

Figure 1. *Young Diagrams.* (a) This diagram represents the (p,q) irrep of the $\mathfrak{sl}(3,\mathbb{C})$ Lie algebra. The upper row has p more boxes than the lower row, which has q boxes. (b) The T,Y matrices can be reduced to direct sums of the $\mathfrak{u}(2)$ irreps that are represented by this diagram. The diagrams (b) result when boxes are removed from the diagram in (a). We take a boxes from the upper row overhang, $0 \le a \le p$, and we take b boxes from the second row, $0 \le b \le q$, as shown. Since the $q-b$ double box columns drop out for a diagram of $\mathfrak{u}(2)$, the diagrams for the $\mathfrak{u}(2)$ irreps are represented by the single row of $p-a+b$ boxes.

For a proper Young diagram of $\mathfrak{u}(2)$, the $q-b$ double box columns in (b) must be discarded. That leaves a single row consisting of $p'=p-a+b$ boxes. It is known that the dimension of the $\mathfrak{u}(2)$ irrep is $p'+1$, which means the dimension of the $\mathfrak{u}(2)$ block is $p-a+b+1$.

Knowing the list of subalgebras $\mathfrak{u}(2)$ and the dimension of each, allows us to calculate the dimension of the (p,q) $\mathfrak{sl}(3,\mathbb{C})$ irrep. The sum of the dimensions of the collection of diagonal blocks in the T,Y matrices is

$$d=\sum_{b=0}^{q}\sum_{a=0}^{p}(p-a+b+1)=\frac{1}{2}(p+1)(q+1)(p+q+2), \tag{14}$$

which coincides with the well-known expression for the dimension of the (p,q) $\mathfrak{sl}(3,\mathbb{C})$ irrep [3] [12]-[14]. This result supports the validity of the box removal process illustrated in **Figure 1**.

Each diagonal block matrix in the reduction of the T,Y matrices to $\mathfrak{u}(2)$ irreps has a spin t. Since the dimension of that block is $2t+1$ and we have just shown that the block has dimension $p'+1 = p-a+b+1$, we have

$$2t=p-a+b, \tag{15}$$

which determines the spin t as a function of the number of boxes removed from the first and second rows of the Young diagram in **Figure 1(a)**. The parameters a,b are not negative and a is at most p with b at most q. Thus, the spin t has a range t of

$$0 \le t \le (p+q)/2, \tag{16}$$

which agrees with the well-known value.

It is also well-known that the eigenvalue y has the value $y=(p-q)/3$ when the T^3 eigenvalue β_r reaches its maximum value $\beta_r = t = (p+q)/2$ from (16). Since t is related to the difference $a-b$ by (15), assume that y is a function of the sum $a+b$. By (15), we have $a=0$ and $b=q$ at max t. These considerations determine the dependence of y on a and b. We find

$$y=(p+2q)/3-a-b. \tag{17}$$

Equations (15) and (17) show how spin t and eigenvalue y depend on the box-reduction parameters a and b in Figure 1.

Now that we know which $\mathfrak{u}(2)$ irreps are in the reduction of the T,Y matrices, we find a place for each one in a sequence. There is one $\mathfrak{u}(2)$ irrep for each pair of box-removal parameters (a,b). So, a sequence of the parameter pairs (a,b) determines a sequence of $\mathfrak{u}(2)$ irreps.

The set of pairs of nonnegative integers (a,b) for a given (p,q) $\mathfrak{sl}(3,\mathbb{C})$ irrep form the rectangle from $(a,b)=(0,0)$ to $(a,b)=(p,q)$. Orient the rectangle so that $a=0,\cdots,p$ in the b^{th} row. We order $\mathfrak{u}(2)$ irreps row by row, starting with $k=1$ for $(a,b)=(0,0)$ and running to $k=(p+1)(q+1)$ for $(a,b)=(p,q)$. See Figure 2 for the case with $(p,q)=(5,3)$

For the (p,q) $\mathfrak{sl}(3,\mathbb{C})$ irrep, the (a,b) $\mathfrak{u}(2)$ subalgebra takes the k^{th} diagonal block of the T,Y matrices, with

$$k=1+a+b(p+1), \tag{18}$$

where $a=0,\cdots,p$, $b=0,\cdots,q$. The index k runs through successive whole numbers from 1 to $(q+1)(p+1)$. The point (a,b) in Figure 2 is marked by the index k of the (a,b) $\mathfrak{u}(2)$ irrep in the sequence.

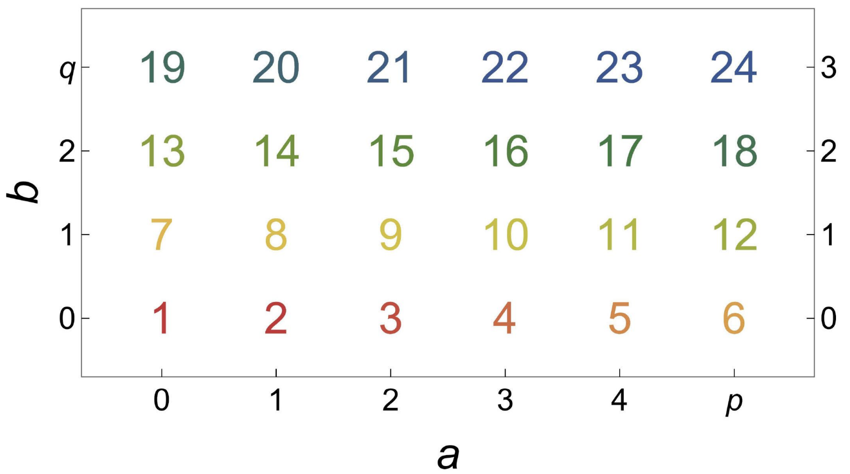

Figure 2. *The sequence of* $\mathfrak{u}(2)$ *irreps for the* $(p,q)=(5,3)$ $\mathfrak{sl}(3,\mathbb{C})$ *irrep*. Each u(2) subalgebra irrep can be identified by the numbers a,b of boxes removed from the first and second rows of the su(3) Young diagram in Figure 1(a). The allowed pairs (a,b) are plotted here. The irreps are ordered row-by-row from lower rows to upper rows and from left to right along each row. The first irrep has $(a,b)=(0,0)$ and the last, the 24th, has $(a,b)=(5,3)$. Each point (a,b) in the plot is marked by the place of its irrep in the sequence of $\mathfrak{u}(2)$ irreps.

Each pair of nonnegative integers (a,b) represents one $\mathfrak{u}(2)$ subalgebra irrep. Since $t = p - a + b$ is the spin and $y = (p + 2q)/3 - a - b$ is constant for the $\mathfrak{u}(2)$ irrep, the eigenvalues of the T^3, Y matrices of $\mathfrak{u}(2)$ irrep are

$$(\beta, y) = (-t, y), \cdots, (+t, y). \quad (19)$$

Recall that the eigenvalues are the diagonal components of the matrices T^3 and Y. We take (19) to be the order of the eigenvalues in the (a,b) $\mathfrak{u}(2)$ block of the T^3, Y matrices. In detail, let the eigenvector with eigenvalues (β, y) be the m^{th} of the $2t+1$ eigenvectors in the block. We find

$$m = t + \beta + 1 = (p - a + b)/2 + \beta + 1. \quad (20)$$

The place number m is a positive integer, $m = 1, \cdots, p - a + b + 1$. By (15), in terms of the spin t of the block, we have $m = 1, \cdots, 2t + 1$.

Combining the order of the (a,b) $\mathfrak{u}(2)$ blocks in the T, Y matrices from (18) and the order of the eigenvectors in each $\mathfrak{u}(2)$ block from (20), we get an expression for the place n of the eigenvector of T^3, Y that has the eigenvalue pair (β, y).

The eigenvector in the (a,b) $\mathfrak{u}(2)$ irrep with eigenvalues (β, y) is the m^{th} eigenvector in the k^{th} $\mathfrak{u}(2)$ irrep. By (18) and (20), we find a formula for the place n of the eigenvector in the resulting sequence of eigenvectors. We have

$$\begin{aligned} n(a,b,\beta) &= \sum_{\bar{b}=0}^{b-1} \sum_{\bar{a}=0}^{p} \left(p - \bar{a} + \bar{b} + 1\right) + \sum_{\bar{a}=0}^{a-1} \left(p - \bar{a} + b + 1\right) + \left[(p - a + b)/2 + \beta + 1 \right] \\ &= 1 + a - a^2/2 + b + ab + b^2/2 + p/2 + ap + bp + b^2 p/2 + bp^2/2 + \beta, \end{aligned} \quad (21)$$

where $a = 0, \cdots, p$, $b = 0, \cdots, q$, and, by (15) and (19), $\beta = -(p - a + b)/2, \cdots, (p - a + b)/2$. For each allowed choice of parameters (a, b, β), the sequence function $n(a,b,\beta)$ yields a unique integer n, with $n = 1, \cdots, d$, where d is the dimension (14) of the irrep (p,q) $\mathfrak{sl}(3,\mathbb{C})$.

The sequence function $n(a,b,\beta)$ can be applied to the row and column indices of the matrices in the $\mathfrak{u}(2)$ Lie algebra. Consider an entry M^{rc} of one of the basis $TYUV$ matrices that we are constructing. Since the indices r and c are a pair of integers in the range $1, \cdots, d$, we must have

$$r = n(a_r, b_r, \beta_r); \; c = n(a_c, b_c, \beta_c), \quad (22)$$

for some allowed choices of the six parameters. In the following section, the formulas for the various entries M^{rc} are presented as functions of the two sets of parameters $\{a_r, b_r, \beta_r\}$ and $\{a_c, b_c, \beta_c\}$.

4. Matrix Generator Formulas

This section presents formulas for the construction of twelve matrices. The eight $TYUV$ generators for the basis of the $\mathfrak{sl}(3,\mathbb{C})$ Lie algebra and the eight F_j generators of the basis for the $\mathfrak{su}(3)$ Lie algebra are linear combinations of these twelve matrices.

The twelve matrices listed in **Table 1** are sparse monomial (SM) matrices.

"Sparse monomial matrix" is another name for a "sparse generalized permutation matrix." A permutation matrix results from the permutation of the columns of a unit matrix. This matrix has one entry equal to the number one in each row and one entry equal to one in each column. A "generalized" permutation matrix allows any nonzero number to take the place of the number one. The qualifier "sparse" reduces the constraint to "at most" one nonzero entry in a row or column. Thus, an SM matrix may have some rows or columns that are completely null.

There is an SM matrix for each of the four generators T,Y and two SM matrices each for the four generators U,V of the basis $TYUV$. The SM matrices for the generators U,V are distinguished by a subscript g or h. We have

$$U^{\pm} = U_g^{\pm} + U_h^{\pm};\ V^{\pm} = V_g^{\pm} + V_h^{\pm}. \tag{23}$$

The subscripts g and h indicate that the functions g and h appear in the formulas. The functions are defined by

$$g(a,b) \equiv (p-a)(p+q-a+1)(a+1)/[(p-a+b)(p-a+b+1)] \tag{24}$$

$$h(a,b) \equiv b(q-b+1)(p+b+1)/[(p-a+b)(p-a+b+1)].$$

See Table 1.

SM matrices like T^+ and T^- on lines 4 and 5 of the table differ by exchanging row index r with column index c are each other's transpose. Further inspection of Table 1 uncovers many transpose relations. We find that

$$(T^+)^{\mathrm{T}} = T^-;\ (U_g^+)^{\mathrm{T}} = U_g^-;\ (U_h^+)^{\mathrm{T}} = U_h^-;\ (V_g^+)^{\mathrm{T}} = V_g^-;\ (V_h^+)^{\mathrm{T}} = V_h^-. \tag{25}$$

These transpose relationships yield relationships among the CRs for the $TYUV$ matrices.

We know that the transpose of a commutator $[A,B]$ is the negative of the commutator of the transposes, $[A,B]^{\mathrm{T}} = -[A^{\mathrm{T}},B^{\mathrm{T}}]$. The CRs (5) to (10) are either invariant under transposition or yield another of the CRs. Thus, if the $TYUV$ matrices satisfy one of the CRs, then the matrices satisfy the CR's transpose. That reduces the number of CRs that must be considered when showing that the $TYUV$ matrices satisfy the $TYUV$ algebra.

Now, consider a different aspect of Table 1. The sequence function $n(a,b,\beta)$ runs from 1 to the dimension d of the matrices when the parameters have any combination of values in certain ranges $0 \le a \le p$, $0 \le b \le q$ and $-t \le \beta \le t$, where $t = (p-a+b)/2$. These ranges are called the "default" ranges in Table 1.

In Table 1, whenever there are nonzero changes in the a,b,β parameters from row to column, the allowed ranges of a,b,β change. For example, the parameter a can run from 0 to p. However, for U_g^+, the parameter a_c for the column c differs from a_r for its row r by one, *i.e.* $a_c = a_r + 1$. It follows that a_r cannot be equal to p because a_c cannot be $p+1$, so we must restrict a_r to $a_r = 0,\cdots,p-1$. Thus, for U_g^+, the rows $r = n(a_r,b_r,\beta_r)$ with $a_r = p$ are null. The other nonzero changes in parameters a,b,β in Table 1 have similar consequences. Changes in the range of the parameters a,b,β occupy the right column of Table 1.

Patterns appear in the formulas in **Table 1**. For the U,V SM matrices, just one of the parameters a_c or b_c differs by ±1 from its counterpart a_r or b_r. By (15), it follows that there is a half integer spin difference $t_c - t_r = \pm 1/2$. In the third column of **Table 1**, the row to column β parameters for the U,V SM formulas have a difference $\beta_c - \beta_r = \pm 1/2$, which is the smallest value allowed for the half integer spin difference $t_c - t_r$. Thus, nonzero entries for the U,V SM matrices are located where the differences in the parameters a,b,β from row to column are minimal.

Table 1. *The 12 matrices M that form the eight TYUV generators.* A sparse monomial (SM) matrix has at most one possibly nonzero entry in each row. For each row r of each SM matrix M, the table has a formula for its possibly nonzero entry. The row and column indices r,c of the entry are written in terms of the sequence function n in (21).

M	$\left(M^{rc}\right)$ [1]	(r,c) [2]	Ranges[3]
T^3	β_r	(n_r,n_r)	default
Y	$(p+2q)/3 - a_r - b_r$	(n_r,n_r)	default
T^+	$-\sqrt{t_c(1+t_c)-\beta_c(1+\beta_c)}$	$\left(n(a_c,b_c,\beta_c+1),n_c\right)$	$\beta_c \le t_c - 1$
T^-	$-\sqrt{t_r(1+t_r)-\beta_r(1+\beta_r)}$	$\left(n_r,n(a_r,b_r,\beta_r+1)\right)$	$\beta_r \le t_r - 1$
U_g^+	$-\left[g(a_r,b_r)(t_r-\beta_r)\right]^{1/2}$	$\left(n_r,n(a_r+1,b_r,\beta_r+1/2)\right)$	$a_r \le p-1$
			$\beta_r \le t_r - 1$
U_h^+	$-\left[h(a_c,b_c)(t_c+\beta_c)\right]^{1/2}$	$\left(n(a_c,b_c-1,\beta_c-1/2),n_c\right)$	$1 \le b_c$
			$-t_c+1 \le \beta_c$
U_g^-	$-\left[g(a_c,b_c)(t_c-\beta_c)\right]^{1/2}$	$\left(n(a_c+1,b_c,\beta_c+1/2),n_c\right)$	$a_c \le p-1$
			$\beta_c \le t_c - 1$
U_h^-	$-\left[h(a_r,b_r)(t_r+\beta_r)\right]^{1/2}$	$\left(n_r,n(a_r,b_r-1,\beta_r-1/2)\right)$	$1 \le b_r$
			$-t_r+1 \le \beta_r$
V_g^+	$+\left[g(a_r,b_r)(t_r+\beta_r)\right]^{1/2}$	$\left(n_r,n(a_r+1,b_r,\beta_r-1/2)\right)$	$a_r \le p-1$
			$-t_r+1 \le \beta_r$
V_h^+	$-\left[h(a_c,b_c)(t_c-\beta_c)\right]^{1/2}$	$\left(n(a_c,b_c-1,\beta_c+1/2),n_c\right)$	$1 \le b_c$
			$\beta_c \le t_c - 1$
V_g^-	$+\left[g(a_c,b_c)(t_c+\beta_c)\right]^{1/2}$	$\left(n(a_c+1,b_c,\beta_c-1/2),n_c\right)$	$a_c \le p-1$
			$-t_c+1 \le \beta_c$
V_h^-	$-\left[h(a_r,b_r)(t_r-\beta_r)\right]^{1/2}$	$\left(n_r,n(a_r,b_r-1,\beta_r+1/2)\right)$	$1 \le b_r$
			$\beta_r \le t_r - 1$

[1] $t=(p-a+b)/2$. [2] $n_r = n(a_r,b_r,\beta_r)$; $n_c = n(a_c,b_c,\beta_c)$. [3] By Equations. (13) and (19), the default ranges are $0 \le a \le p$; $0 \le b \le q$; $-t \le \beta \le t$.

The pattern extends to the functions g,h. The function g appears when the a parameter changes from row to column, and h is in the formula when b

changes. The numerator of g depends on a, not b, while the numerator of h is a function of b and not of a. And g and h share the same denominator, $(p-a+b)(p-a+b+1)=2t(2t+1)$, where $2t=p-a+b$ by (15).

The matrices F_j for a basis of the (p,q) irrep of $\mathfrak{su}(3)$ are found by inverting the transformation (3) and applying it to the basis $TYUV$ of the (p,q) $\mathfrak{sl}(3,\mathbb{C})$ irrep in (23) and **Table 1**. We have

$$F_1=\left(T^++T^-\right)/2; F_2=-i\left(T^+-T^-\right)/2; F_3=T^3;$$
$$F_4=\left(V_g^++V_h^++V_g^-+V_h^-\right)/2; F_5=-i\left(V_g^++V_h^+-V_g^--V_h^-\right)/2; \quad (26)$$
$$F_6=\left(U_g^++U_h^++U_g^-+U_h^-\right)/2; F_7=-i\left(U_g^++U_h^+-U_g^--U_h^-\right)/2; F_8=\sqrt{3}Y/2.$$

By (25) and **Table 1**, the matrices F_j are Hermitian and traceless. Thus, the F_j matrices generate unitary matrices by (2), and the matrices they generate have a determinant equal to one.

5. Discussion

The formulas in Section 4 provide the means to construct finite-dimensional irreducible highest-weight representations of the Lie algebras $\mathfrak{sl}(N,\mathbb{C})$ and $\mathfrak{su}(N)$. The representations are characterized by two nonnegative integers (p,q).

Possible topics for further investigations include extensions to reducible representations, infinite dimensional or continuous matrices. Non-integral cases of (p,q) may be explored.

Consider the observation that quadratic equations with real coefficients may not have real solutions. A standard example is the quadratic equation $x^2+1=0$, which has no real-valued solutions for x since the square of a real number is positive. The 28 commutation relations (CR) of the $\mathfrak{sl}(3,\mathbb{C})$ algebra are quadratic equations with real-valued coefficients. The solutions provided in Section 4 show that these quadratic equations have real-valued solutions.

Similarly, the CRs of the basis generators of $\mathfrak{sl}(N,\mathbb{C})$ and $\mathfrak{su}(N)$ for $N>3$ are quadratic equations and, therefore, likely solvable. Certainly, the Young diagrams in **Figure 1** can be generalized. As a consequence, one supposes that the sequence functions n for $N>3$ can be determined. It would be interesting to discover whether the patterns noted in Section 4 for the $N=3$ matrices persist with $N>3$. And formulas like those presented here, but for $N>3$, may provide useful versions of matrix bases for $\mathfrak{sl}(N,\mathbb{C})$ and $\mathfrak{su}(N)$.

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Conflicts of Interest

The author declares that he has no conflicts of interest.

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Appendix

Appendix A. Verification of the Formulas

To verify that the $TYUV$ matrices form a basis for the (p,q) $\mathfrak{sl}(3,\mathbb{C})$ irrep, the proposed $TYUV$ matrices in Section 4 are substituted in the 28 commutation relations (CR) (5) to (10). If the eight $TYUV$ matrices satisfy those 28 CRs, then they form a basis for the (p,q) irrep.

The eight $TYUV$ matrices are combinations of the twelve sparse monomial (SM) matrices in Table 1. It is convenient to expand the 28 CRs (5) to (10) into CRs for the twelve SM matrices. That increases the number of CRs to verify. However, as mentioned in Section 4, many CRs can be paired with their transposes. Since it suffices to verify just one CR of a transpose pair, that decreases the number of CRs to verify. In total, we must verify a total of 32 CRs each of which involves only SM matrices.

Verification calculations are separated into three tables, Tables A2-A4. Each calculation occupies a section in the table where we list the CR, followed by the relevant matrix dot products. The dot products of SM matrices are SM matrices. For each SM matrix, a formula is given for the possibly nonzero component in each row r and a second formula gives the column c where the nonzero entry is located in row r.

Consider the dot product $(M_1 \cdot M_2)^{rc} = M_1^{rs} M_2^{sc}$ for two SM matrices M_1^{rc}, M_2^{rc}, where the sum over the repeated index s is implied. For each row $r = n(a_r, b_r, \beta_r)$, the nonzero component is in column c, where $c = n(a_c, b_c, \beta_c)$. We have

$$c = n(a_r + \Delta a_1 + \Delta a_2, b_r + \Delta b_1 + \Delta b_2, \beta_r + \Delta\beta_1 + \Delta\beta_2). \tag{27}$$

In (27), $\Delta a_i, \Delta b_i, \Delta\beta_i$ are the parameter differences $(a_c - a_r, b_c - b_r, \beta_c - \beta_r)$ for the matrix M_i, $i = 1,2$. Since addition is commutative, we infer, by (27), that the dot products $M_1 \cdot M_2$ and $M_2 \cdot M_1$ and their commutator $M_1 \cdot M_2 - M_2 \cdot M_1$ make nonzero contributions to the same column c of row r.

The parameter differences can be retrieved from Table A1 for the twelve SM matrices in Table 1. Thus, the formulas for c for dot products in Tables A2-A4 result from adding the appropriate Δa, Δb, and $\Delta\beta$ in Table A1, as in (27). The formulas for c for multiples of individual SM matrices come directly from Table A1 or Table 1.

To illustrate the algebra that may be required to confirm the tabulated verifications, we detail a sample calculation for CR #23 in Table A4.

In Table A1, the SM matrices U_g^+ and U_h^- have row to column parameter differences $(a_c - a_r, b_c - b_r, \beta_c - \beta_r) = (+1, 0, +1/2)$ and $(0, -1, -1/2)$, respectively. It follows from (27) that the nonzero contributions of the dot products $U_g^+ \cdot U_h^-$ and $U_h^- \cdot U_g^+$ appear in column $c = n(a_r + 1, b_r - 1, \beta_r)$ of row $r = n(a_r, b_r, \beta_r)$. The dot products and the commutator $[U_g^+, U_h^-]$ contribute to the same column c in the row r.

Table A1. *The row/column* a,b,β *parameter differences for the matrices* M *in* **Table 1**. Two sparse monomial (SM) matrices M_1^{rc}, M_2^{rc} have a dot product with row/column a,b,β parameter differences that are the sum of the differences of the SM matrices M_1^{rc} and M_2^{rc}. For example, both the dot products $U_g^+ \cdot V_g^+$ and $V_g^+ \cdot U_g^+$ have $\Delta a, \Delta b, \Delta\beta = 2,0,0$. The two contribute to the same matrix entries.

M	Δa [1]	Δb	$\Delta\beta$	M	Δa	Δb	$\Delta\beta$
T^3	0	0	0	Y	0	0	0
T^+	0	0	−1	T^-	0	0	+1
U_g^+	+1	0	+1/2	U_g^-	−1	0	−1/2
U_h^+	0	+1	+1/2	U_h^-	0	−1	−1/2
V_g^+	+1	0	−1/2	V_g^-	−1	0	+1/2
V_h^+	0	+1	−1/2	V_h^-	0	−1	+1/2

[1] $\Delta a = a_c - a_r$; $\Delta b = b_c - b_r$; $\Delta\beta = \beta_c - \beta_r$.

Next, the formula for the dot product $U_g^+ \cdot U_h^-$ is just the product of the two formulas listed in **Table 1**. We have

$$\begin{aligned}\left(U_g^+ \cdot U_h^-\right)^{rc} &= {}^{+}U_g^{rs-}U_h^{sc} \\ &= -\left[g(a_r,b_r)(t_r-\beta_r)\right]^{1/2}(-1)\left[h(a_s,b_s)(t_s+\beta_s)\right]^{1/2} \\ &= \left[(t_r-\beta_r)(t_r-1/2+\beta_r+1/2)g(a_r,b_r)h(a_r+1,b_r)\right]^{1/2} \\ &= \left[(t_r-\beta_r)(t_r+\beta_r)g(a_r,b_r)h(a_r+1,b_r)\right]^{1/2},\end{aligned} \quad (28)$$

which agrees with the expression on line 8 of the calculations for CR 23, $\left[U^+,U^-\right]=3Y/2-T^3$, in **Table A4**.

For each row $r=n(a_r,b_r,\beta_r)$ of the matrix $U_g^+ \cdot U_h^-$, the entry (28) appears in column $c=n(a_r+1,b_r-1,\beta_r)$, as previously noted. Since we must have $a_c=a_r+1\le p$ and $b_c=b_r-1\ge 0$, the entry (28) does not appear in the rows with $a_r=p$ or $b_r=0$. Thus, the rows r with the entries (28) have restricted parameter ranges, $0\le a_r\le p-1$ and $1\le b_r\le q$. The rows r for $a_r=p$ or $b_r=0$ are filled with zeros in each column.

The dot product $U_g^+ \cdot U_h^-$ in (28) appears when we expand the CR #23 $\left[U^+,U^-\right]=\left[\left(U_g^+ + U_h^+\right)\cdot\left(U_g^- + U_h^-\right)\right]=3Y/2-T^3$. Unlike Y and T^3, the dot products $U_g^+ \cdot U_h^-$ and $U_h^- \cdot U_g^+$ are nonzero off-diagonal. They contribute, instead, to the column $c=n(a_r+1,b_r-1,\beta)$ of the row $r=n(a_r,b_r,\beta)$ and we have $r\ne c$. Both Y and T^3 have nonzero entries only on the diagonal $r=c$. Therefore, if the matrices obey the CR, then the commutator $\left[U_g^+ \cdot U_h^-\right]$ should vanish, $\left[U_g^+,U_h^-\right]=0$.

For the dot product $U_h^- \cdot U_g^+$, the steps that gave (28) produce the result

$$\left(U_h^- \cdot U_g^+\right)^{rc}=\left[(t_r-\beta_r)(t_r+\beta_r)g(a_r,b_r-1)h(a_r,b_r)\right]^{1/2}. \quad (29)$$

Thus, the expression for $U_h^- \cdot U_g^+$ differs from $U_g^+ \cdot U_h^-$ by the parameters in the functions g and h. There is a b_r-1 in one and an a_r+1 in the other.

Taking the g and h in (28), we have

$$
\begin{aligned}
& g(a_r,b_r)h(a_r+1,b_r) \\
&= \left[\frac{(p-a_r)(p+q-a_r+1)(a_r+1)}{(p-a_r+b_r)(p-a_r+b_r+1)}\right]\left[\frac{b_r(q-b_r+1)(p+b_r+1)}{(p-a_r-1+b_r)(p-a_r-1+b_r+1)}\right] \\
&= \left[\frac{(p-a_r)(p+q-a_r+1)(a_r+1)}{(p-a_r+b_r-1)(p-a_r+b_r-1+1)}\right]\left[\frac{b_r(q-b_r+1)(p+b_r+1)}{(p-a_r+b_r)(p-a_r+b_r+1)}\right] \\
&= g(a_r,b_r-1)h(a_r,b_r).
\end{aligned}
\tag{30}
$$

The g and h denominators trade places in the intermediate steps. It follows from (28), (29) and (30) that $\left[U_g^+,U_h^-\right]=U_g^+\cdot U_h^- - U_h^-\cdot U_g^+=0$. This successfully verifies one of the three CRs with SM matrices for CR #23.

The process applied in the example is followed throughout **Tables A2-A4**. Each of the 28 CRs is numbered and appears in their own sections of the tables. The CRs that are each other's transposes appear in the same section since verification of a CR also verifies its transpose. The CR is broken down into CRs with sparse monomial (SM) matrices, and each SM dot product is tabulated with its formula and column coordinate c. The dot products are sorted and collected together by column c. The sums of generators that the commutators are expected to equal are also listed.

The example of $U_g^+\cdot U_h^-$ occupies considerable space in this Appendix. Rather than repeat the process for all of the dot products and SM CRs, we leave the algebra to the reader. The complete calculations implied by the intermediate steps in **Tables A2-A4** confirm that the 28 CRs are satisfied by the $TYUV$ matrices. Therefore, the matrices obtained with the formulas in Section 4 constitute a basis for the generators of the Lie algebra $\mathfrak{sl}(3,\mathbb{C})$.

Table A2. *The commutation relations (CR) for the* T,Y *matrices.* This table verifies the CRs in (5) and (6). CRs that are each other's transpose are put in the same section. All dot products are sparse monomial matrices. The formula for a dot product's matrix entry results from multiplying two quantities in **Table 1**. The entry is to be placed in row $r=n(a_r,b_r,\beta_r)$ and column c which can be found on the far right. By (27), the formula for column c adds the appropriate quantities in **Table A1**. For a sample calculation, see (28), (29), (30).

Item	CR, formula for item[1]	c
1.	$\left[T^+,T^-\right]=2T^3$	
$T^+\cdot T^-$	$t_r(1+t_r)-(-1+\beta_r)\beta_r$	$n(a_r,b_r,\beta_r)$
$T^-\cdot T^+$	$t_r(1+t_r)-\beta_r(1+\beta_r)$	$n(a_r,b_r,\beta_r)$
$2T^3$	$2\beta_r$	$n(a_r,b_r,\beta_r)$
2, 3.	$\left[T^3,T^-\right]=-T^-$; $\left[T^3,T^+\right]=T^+$	
$T^3\cdot T^-$	$-\beta_r\sqrt{t_r(1+t_r)-\beta_r(1+\beta_r)}$	$n(a_r,b_r,\beta_r+1)$
$T^-\cdot T^3$	$-\left((1+\beta_r)\sqrt{t_r(1+t_r)-\beta_r(1+\beta_r)}\right)$	$n(a_r,b_r,\beta_r+1)$
$-T^-$	$\sqrt{t_r(1+t_r)-\beta_r(1+\beta_r)}$	$n(a_r,b_r,\beta_r+1)$

Continued

4, 5.	CRs: $\left[Y,T^{+}\right]=0$; $\left[Y,T^{-}\right]=0$	
$Y\cdot T^{+}$	$-\sqrt{t_r(1+t_r)-(-1+\beta_r)\beta_r}\,y_r$	$n(a_r,b_r,\beta_r-1)$
$T^{+}\cdot Y$	$-\sqrt{t_r(1+t_r)-(-1+\beta_r)\beta_r}\,y_r$	$n(a_r,b_r,\beta_r-1)$
0	0	$0\le c\le d$ (all c)
6.	CR: $\left[Y,T^{3}\right]=0$	
$Y\cdot T^{3}$	$\beta_r y_r$	$n(a_r,b_r,\beta_r)$
$T^{3}\cdot Y$	$\beta_r y_r$	$n(a_r,b_r,\beta_r)$
0	0	$0\le c\le d$ (all c)
	[1] $t_r=(p-a+b)/2$; $y_r=(p+2q)/3-a_r-b_r$	

Table A3. *The commutation relations (CR) for commutators with one* T,Y *generator and one* U,V *generator.* This table is set up like **Table A2**. However, by (23), each U,V matrix combines two sparse monomial (SM) matrices, *e.g.* $U^{+}=U_g^{+}+U_h^{+}$. Thus each CR splits into two SM CRs. For example, consider CR #21: $\left[T^{+},V^{-}\right]=\left[T^{+},V_g^{-}\right]+\left[T^{+},V_h^{-}\right]$. In row r, the entry's destination column c is different for $\left[T^{+},V_g^{-}\right]$ compared with $\left[T^{+},V_h^{-}\right]$, with the first c shaded blue and the second shaded orange. Since entries must be in the same row and column in order to combine, the commutator $\left[T^{+},V_g^{-}\right]$ with c shaded blue equals the SM matrix $-U_g^{-}$ portion of $-U^{-}$, while $+\left[T^{+},V_h^{-}\right]$ with its orange c makes the SM matrix $-U_h^{-}$ portion of $-U^{-}$.

Item	CR, formula for item[1]	c
7, 8.	CR: $\left[T^{3},U^{+}\right]=-U^{+}/2$; $\left[T^{3},U^{-}\right]=U^{-}/2$	
$T^{3}\cdot U_g^{+}$	$-\beta_r\sqrt{(t_r-\beta_r)g(a_r,b_r)}$	$n(a_r+1,b_r,\beta_r+1/2)$
$U_g^{+}\cdot T^{3}$	$-\left(\left(\frac{1}{2}+\beta_r\right)\sqrt{(t_r-\beta_r)g(a_r,b_r)}\right)$	$n(a_r+1,b_r,\beta_r+1/2)$
$-U_g^{+}/2$	$+(1/2)\left[g(a_r,b_r)(t_r-\beta_r)\right]^{1/2}$	$n(a_r+1,b_r,\beta_r+1/2)$
$T^{3}\cdot U_h^{+}$	$-\beta_r\sqrt{(1+t_r+\beta_r)h(a_r,1+b_r)}$	$n(a_r,b_r+1,\beta_r+1/2)$
$U_h^{+}\cdot T^{3}$	$-\frac{1}{2}(1+2\beta_r)\sqrt{(1+t_r+\beta_r)h(a_r,1+b_r)}$	$n(a_r,b_r+1,\beta_r+1/2)$
$-U_h^{+}/2$	$+(1/2)\left[h(a_c,b_c)(t_c+\beta_c)\right]^{1/2}$	$n(a_r,b_r+1,\beta_r+1/2)$
9, 10.	CR: $\left[T^{3},V^{+}\right]=V^{+}/2$; $\left[T^{3},V^{-}\right]=-V^{-}/2$	
$T^{3}\cdot V_g^{+}$	$\beta_r\sqrt{(t_r+\beta_r)g(a_r,b_r)}$	$n(a_r+1,b_r,\beta_r-1/2)$
$V_g^{+}\cdot T^{3}$	$\left(-\frac{1}{2}+\beta_r\right)\sqrt{(t_r+\beta_r)g(a_r,b_r)}$	$n(a_r+1,b_r,\beta_r-1/2)$
$+V_g^{+}/2$	$+(1/2)\left[g(a_r,b_r)(t_r+\beta_r)\right]^{1/2}$	$n(a_r+1,b_r,\beta_r-1/2)$
$T^{3}\cdot V_h^{+}$	$-\beta_r\sqrt{(1+t_r-\beta_r)h(a_r,1+b_r)}$	$n(a_r,b_r+1,\beta_r-1/2)$
$V_h^{+}\cdot T^{3}$	$\frac{1}{2}(1-2\beta_r)\sqrt{(1+t_r-\beta_r)h(a_r,1+b_r)}$	$n(a_r,b_r+1,\beta_r-1/2)$
$+V_h^{+}/2$	$-(1/2)\left[h(a_c,b_c)(t_c-\beta_c)\right]^{1/2}$	$n(a_r,b_r+1,\beta_r-1/2)$

Continued

11, 12.	CR: $\left[Y,U^{+}\right]=U^{+}$; $\left[Y,U^{-}\right]=-U^{-}$	
$Y\cdot U_{g}^{+}$	$-\sqrt{(t_r-\beta_r)g(a_r,b_r)}y_r$	$n(a_r+1,b_r,\beta_r+1/2)$
$U_{g}^{+}\cdot Y$	$-\sqrt{(t_r-\beta_r)g(a_r,b_r)}(-1+y_r)$	$n(a_r+1,b_r,\beta_r+1/2)$
U_{g}^{+}	$-\left[g(a_r,b_r)(t_r-\beta_r)\right]^{1/2}$	$n(a_r+1,b_r,\beta_r+1/2)$
$Y\cdot U_{h}^{+}$	$-\sqrt{(1+t_r+\beta_r)h(a_r,1+b_r)}y_r$	$n(a_r,b_r+1,\beta_r+1/2)$
$U_{h}^{+}\cdot Y$	$-\sqrt{(1+t_r+\beta_r)h(a_r,1+b_r)}(-1+y_r)$	$n(a_r,b_r+1,\beta_r+1/2)$
U_{h}^{+}	$-\left[h(a_c,b_c)(t_c+\beta_c)\right]^{1/2}$	$n(a_r,b_r+1,\beta_r+1/2)$
13, 14.	CR: $\left[Y,V^{+}\right]=V^{+}$; $\left[Y,V^{-}\right]=-V^{-}$	
$Y\cdot V_{g}^{+}$	$\sqrt{(t_r+\beta_r)g(a_r,b_r)}y_r$	$n(a_r+1,b_r,\beta_r-1/2)$
$V_{g}^{+}\cdot Y$	$\sqrt{(t_r+\beta_r)g(a_r,b_r)}(-1+y_r)$	$n(a_r+1,b_r,\beta_r-1/2)$
$+V_{g}^{+}$	$+\left[g(a_r,b_r)(t_r+\beta_r)\right]^{1/2}$	$n(a_r+1,b_r,\beta_r-1/2)$
$Y.V_{h}^{+}$	$-\sqrt{(1+t_r-\beta_r)h(a_r,1+b_r)}y_r$	$n(a_r,b_r+1,\beta_r-1/2)$
$V_{h}^{+}\cdot Y$	$-\sqrt{(1+t_r-\beta_r)h(a_r,1+b_r)}(-1+y_r)$	$n(a_r,b_r+1,\beta_r-1/2)$
$+V_{h}^{+}$	$-\left[h(a_c,b_c)(t_c-\beta_c)\right]^{1/2}$	$n(a_r,b_r+1,\beta_r-1/2)$
15, 16.	CR: $\left[T^{+},U^{-}\right]=0$; $\left[T^{-},U^{+}\right]=0$	
$T^{+}\cdot U_{g}^{-}$	$\sqrt{(2+t_r-\beta_r)w(t_r,\beta_r-1)g(-1+a_r,b_r)}$	$n(a_r-1,b_r,\beta_r-3/2)$
$U_{g}^{-}\cdot T^{+}$	$\sqrt{(1+t_r-\beta_r)w(t_c,\beta_c)g(-1+a_r,b_r)}$,	$n(a_r-1,b_r,\beta_r-3/2)$
	where $t_c=t_r+1/2$; $\beta_c=\beta_r-3/2$	
0	0	$0\le c\le d$ (all c)
$T^{+}\cdot U_{h}^{-}$	$\sqrt{(-1+t_r+\beta_r)w(t_r,\beta_r-1)h(a_r,b_r)}$	$n(a_r,b_r-1,\beta_r-3/2)$
$U_{h}^{-}\cdot T^{+}$	$\sqrt{(t_r+\beta_r)w(t_c,\beta_c)h(a_r,b_r)}$	$n(a_r,b_r-1,\beta_r-3/2)$
0	0	$0\le c\le d$ (all c)
17, 18.	CR: $\left[T^{+},V^{+}\right]=0$; $\left[T^{-},V^{-}\right]=0$	
$T^{+}\cdot V_{g}^{+}$	$-\sqrt{(-1+t_r+\beta_r)w(t_r,\beta_r-1)g(a_r,b_r)}$	$n(a_r+1,b_r,\beta_r-3/2)$
$V_{g}^{+}\cdot T^{+}$	$-\sqrt{(t_r+\beta_r)w(t_c,\beta_c)g(a_r,b_r)}$,	$n(a_r+1,b_r,\beta_r-3/2)$
	where $t_c=t_r-1/2$; $\beta_c=\beta_r-3/2$	
0	0	$0\le c\le d$ (all c)
$T^{+}\cdot V_{h}^{+}$	$\sqrt{(2+t_r-\beta_r)w(t_r,\beta_r-1)h(a_r,1+b_r)}$	$n(a_r,b_r+1,\beta_r-3/2)$
$V_{h}^{+}\cdot T^{+}$	$\sqrt{(1+t_r-\beta_r)w(t_c,\beta_c)h(a_r,1+b_r)}$,	$n(a_r,b_r+1,\beta_r-3/2)$
	where $t_c=t_r+1/2$; $\beta_c=\beta_r-3/2$	
0	0	$0\le c\le d$ (all c)
19, 20.	CR: $\left[T^{+},U^{+}\right]=V^{+}$; $\left[T^{-},U^{-}\right]=-V^{-}$	
$T^{+}\cdot U_{g}^{+}$	$\sqrt{(1+t_r-\beta_r)w(t_r,\beta_r-1)g(a_r,b_r)}$	$n(a_r+1,b_r,\beta_r-1/2)$
$U_{g}^{+}\cdot T^{+}$	$\sqrt{(t_r-\beta_r)w(t_c,\beta_c)g(a_r,b_r)}$,	$n(a_r+1,b_r,\beta_r-1/2)$
	where $t_c=t_r-1/2$; $\beta_c=\beta_r-1/2$	

Continued

$+V_g^+$	$+\left[g(a_r,b_r)(t_r+\beta_r)\right]^{1/2}$	$n(a_r+1,b_r,\beta_r-1/2)$
$T^+\cdot U_h^+$	$\sqrt{(t_r+\beta_r)w(t_r,\beta_r-1)h(a_r,b_r+1)}$	$n(a_r,b_r+1,\beta_r-1/2)$
$U_h^+\cdot T^+$	$\sqrt{(1+t_r+\beta_r)w(t_c,\beta_c)h(a_r,b_r+1)}$,	$n(a_r,b_r+1,\beta_r-1/2)$
	where $t_c=t_r+1/2$; $\beta_c=\beta_r-1/2$	
$+V_h^+$	$-\left[h(a_c,b_c)(t_c-\beta_c)\right]^{1/2}$	$n(a_r,b_r+1,\beta_r-1/2)$
21, 22.	CR: $\left[T^+,V^-\right]=-U^-$; $\left[T^-,V^+\right]=+U^+$	
$T^+\cdot V_g^-$	$-\sqrt{(t_r+\beta_r)w(t_r,\beta_r-1)g(-1+a_r,b_r)}$	$n(a_r-1,b_r,\beta_r-1/2)$
$V_g^-\cdot T^+$	$-\sqrt{(1+t_r+\beta_r)w(t_c,\beta_c)g(-1+a_r,b_r)}$,	$n(a_r-1,b_r,\beta_r-1/2)$
	where $t_c=t_r+1/2$; $\beta_c=\beta_r-1/2$	
$-U_g^-$	$+\left[g(-1+a_r,b_r)(1+t_r-\beta_r)\right]^{1/2}$	$n(a_r-1,b_r,\beta_r-1/2)$
$T^+\cdot V_h^-$	$\sqrt{(1+t_r-\beta_r)w(t_r,\beta_r-1)h(a_r,b_r)}$	$n(a_r,b_r-1,\beta_r-1/2)$
$V_h^-\cdot T^+$	$\sqrt{(t_r-\beta_r)w(t_c,\beta_c)h(a_r,b_r)}$	$n(a_r,b_r-1,\beta_r-1/2)$
	where $t_c=t_r-1/2$; $\beta_c=\beta_r-1/2$	
$-U_h^-$	$+\left[h(a_r,b_r)(t_r+\beta_r)\right]^{1/2}$	$n(a_r,b_r-1,\beta_r-1/2)$
	[1] $t=(p-a+b)/2$	
	$y_r=(p+2q)/3-a_r-b_r$;	
	$w(t,\beta)=t(t+1)-\beta(\beta+1)$	

Table A4. *The commutation relations (CR) for commutators with two* U,V *generators.* This table is set up in much the same way as Table A2 and Table A3. Since a U,V matrix is the sum of two sparse monomial (SM) matrices, the dot products here could contribute to as many as four matrix columns of a given row. The most is three. To verify that the matrices satisfy a CR, calculate the commutators by subtracting the dot products and compare the result with the expected linear combination of generators. All of the calculations succeed, thereby showing that the $TYUV$ matrices in Section 4 form a basis for the Lie algebra $\mathfrak{sl}(3,\mathbb{C})$.

Item	CR, formula for item[1]	c
23.	CR: $\left[U^+,U^-\right]=3Y/2-T^3$	
$U_g^+\cdot U_g^-$	$(t_r-\beta_r)g(a_r,b_r)$	$n(a_r,b_r,\beta_r)$
$U_g^-\cdot U_g^+$	$(1+t_r-\beta_r)g(-1+a_r,b_r)$	$n(a_r,b_r,\beta_r)$
$U_h^+\cdot U_h^-$	$(1+t_r+\beta_r)h(a_r,1+b_r)$	$n(a_r,b_r,\beta_r)$
$U_h^-\cdot U_h^+$	$(t_r+\beta_r)h(a_r,b_r)$	$n(a_r,b_r,\beta_r)$
$3Y/2$	y_r	$n(a_r,b_r,\beta_r)$
$-T^3$	$-\beta_r$	$n(a_r,b_r,\beta_r)$
$U_g^+\cdot U_h^-$	$\sqrt{(t_r-\beta_r)(t_r+\beta_r)g(a_r,b_r)h(a_r+1,b_r)}$	$n(a_r+1,b_r-1,\beta_r)$
$U_h^-\cdot U_g^+$	$\sqrt{(t_r-\beta_r)(t_r+\beta_r)g(a_r,b_r-1)h(a_r,b_r)}$	$n(a_r+1,b_r-1,\beta_r)$
0	0	$0\le c\le d$ (all c)

Continued

$U_h^+ \cdot U_g^-$	$\sqrt{\left[(1+t_r)^2-\beta_r^2\right]g(-1+a_r,1+b_r)h(a_r,1+b_r)}$	$n(a_r-1,b_r+1,\beta_r)$
$U_g^- \cdot U_h^+$	$\sqrt{\left[(1+t_r)^2-\beta_r^2\right]g(-1+a_r,b_r)h(-1+a_r,1+b_r)}$	$n(a_r-1,b_r+1,\beta_r)$
0	0	$0\le c\le d$ (all c)
24.	CR: $\left[V^+,V^-\right]=3Y/2+T^3$	
$V_g^+ \cdot V_g^-$	$(t_r+\beta_r)g(a_r,b_r)$	$n(a_r,b_r,\beta_r)$
$V_g^- \cdot V_g^+$	$(1+t_r+\beta_r)g(-1+a_r,b_r)$	$n(a_r,b_r,\beta_r)$
$V_h^+ \cdot V_h^-$	$(1+t_r-\beta_r)h(a_r,1+b_r)$	$n(a_r,b_r,\beta_r)$
$V_h^- \cdot V_h^+$	$(t_r-\beta_r)h(a_r,b_r)$	$n(a_r,b_r,\beta_r)$
$3Y/2$	y_r	$n(a_r,b_r,\beta_r)$
T^3	β_r	$n(a_r,b_r,\beta_r)$
$V_g^+ \cdot V_h^-$	$-\sqrt{(t_r+\beta_r)(t_r-\beta_r)g(a_r,b_r)h(1+a_r,b_r)}$	$n(a_r+1,b_r-1,\beta_r)$
$V_h^- \cdot V_g^+$	$-\sqrt{(t_r+\beta_r)(t_r-\beta_r)g(a_r,-1+b_r)h(a_r,b_r)}$	$n(a_r+1,b_r-1,\beta_r)$
0	0	$0\le c\le d$ (all c)
$V_h^+ \cdot V_g^-$	$-\sqrt{\left[(1+t_r)^2-\beta_r^2\right]g(-1+a_r,1+b_r)h(a_r,1+b_r)}$	$n(a_r-1,b_r+1,\beta_r)$
$V_g^- \cdot V_h^+$	$-\sqrt{\left[(1+t_r)^2-\beta_r^2\right]g(-1+a_r,b_r)h(-1+a_r,1+b_r)}$	$n(a_r-1,b_r+1,\beta_r)$
0	0	$0\le c\le d$ (all c)
25, 26.	CR: $\left[U^+,V^-\right]=T^-$; $\left[U^-,V^+\right]=-T^+$	
$U_g^+ \cdot V_g^-$	$-\sqrt{(t_r-\beta_r)(1+t_r+\beta_r)}g(a_r,b_r)$	$n(a_r,b_r,\beta_r+1)$
$V_g^- \cdot U_g^+$	$-\sqrt{(t_r-\beta_r)(1+t_r+\beta_r)}g(-1+a_r,b_r)$	$n(a_r,b_r,\beta_r+1)$
$U_h^+ \cdot V_h^-$	$\sqrt{(t_r-\beta_r)(1+t_r+\beta_r)}h(a_r,1+b_r)$	$n(a_r,b_r,\beta_r+1)$
$V_h^- \cdot U_h^+$	$\sqrt{(t_r-\beta_r)(1+t_r+\beta_r)}h(a_r,b_r)$	$n(a_r,b_r,\beta_r+1)$
T^-	$-\sqrt{t_r(1+t_r)-\beta_r(1+\beta_r)}$	$n(a_r,b_r,\beta_r+1)$
$U_g^+ \cdot V_h^-$	$\sqrt{(t_r-\beta_r)(-1+t_r-\beta_r)g(a_r,b_r)h(1+a_r,b_r)}$	$n(a_r+1,b_r-1,\beta_r+1)$
$V_h^- \cdot U_g^+$	$\sqrt{(-1+t_r-\beta_r)(t_r-\beta_r)g(a_r,-1+b_r)h(a_r,b_r)}$	$n(a_r+1,b_r-1,\beta_r+1)$
0	0	$0\le c\le d$ (all c)
$U_h^+ \cdot V_g^-$	$-\sqrt{s(t_r,\beta_r)g(-1+a_r,1+b_r)h(a_r,1+b_r)}$	$n(a_r-1,b_r+1,\beta_r+1)$
$V_g^- \cdot U_h^+$	$-\sqrt{s(t_r,\beta_r)g(-1+a_r,b_r)h(-1+a_r,1+b_r)}$	$n(a_r-1,b_r+1,\beta_r+1)$
0	0	$0\le c\le d$ (all c)
27, 28.	CR: $\left[U^+,V^+\right]=0$; $\left[U^-,V^-\right]=0$	
$U_g^+ \cdot V_g^+$	$-\sqrt{(t_r^2-\beta_r^2)g(a_r,b_r)g(1+a_r,b_r)}$	$n(a_r+2,b_r,\beta_r)$
$V_g^+ \cdot U_g^+$	$-\sqrt{(t_r^2-\beta_r^2)g(a_r,b_r)g(1+a_r,b_r)}$	$n(a_r+2,b_r,\beta_r)$
0	0	$0\le c\le d$ (all c)
$U_g^+ \cdot V_h^+$	$(t_r-\beta_r)\sqrt{g(a_r,b_r)h(1+a_r,1+b_r)}$	$n(a_r+1,b_r+1,\beta_r)$

Continued

$V_h^+ \cdot U_g^+$	$(1+t_r-\beta_r)\sqrt{g(a_r,1+b_r)h(a_r,1+b_r)}$	$n(a_r+1,b_r+1,\beta_r)$
$U_h^+ \cdot V_g^+$	$-(1+t_r+\beta_r)\sqrt{g(a_r,1+b_r)h(a_r,1+b_r)}$	$n(a_r+1,b_r+1,\beta_r)$
$V_g^+ \cdot U_h^+$	$-(t_r+\beta_r)\sqrt{g(a_r,b_r)h(1+a_r,1+b_r)}$	$n(a_r+1,b_r+1,\beta_r)$
0	0	$0 \le c \le d$ (all c)
$U_h^+ \cdot V_h^+$	$\sqrt{\left[(1+t_r)^2-\beta_r^2\right]h(a_r,1+b_r)h(a_r,2+b_r)}$	$n(a_r,b_r+2,\beta_r)$
$V_h^+ \cdot U_h^+$	$\sqrt{\left[(1+t_r)^2-\beta_r^2\right]h(a_r,1+b_r)h(a_r,2+b_r)}$	$n(a_r,b_r+2,\beta_r)$
0	0	$0 \le c \le d$ (all c)

[1] $t_r = (p-a_r+b_r)/2$

$y_r = (p+2q)/3 - a_r - b_r$

$s(t,\beta) = (2+t+\beta)(1+t+\beta)$