Journal of Applied Mathematics and Physics, 2026, 14(4), 1457-1465
https://www.scirp.org/journal/jamp
ISSN Online: 2327-4379
ISSN Print: 2327-4352

Tropical (0, −1) Idempotent Matrix

Chonghua Cheng, Dandan Su*

Public Basic Teaching Department, Foshan Polytechnic, Foshan, China
Email: *17709626@qq.com

How to cite this paper: Cheng, C.H. and Su, D.D. (2026) Tropical (0, −1) Idempotent Matrix. *Journal of Applied Mathematics and Physics*, **14**, 1457-1465.
https://doi.org/10.4236/jamp.2026.144068

Received: March 25, 2026
Accepted: April 13, 2026
Published: April 16, 2026

Abstract

For an $n \times n$ tropical $(0,-1)$ matrix A, we first give the necessary and sufficient conditions for A to be an idempotent matrix when the diagonal elements of A are all 0 and all −1, respectively. Then we give some necessary conditions for some special tropical $(0,-1)$ matrices to be idempotent matrices. Finally, we discuss the relationship between the matrix set $M_n(T)$ under the tropical $(0,-1)$ semiring and the matrix set $M_n(\mathcal{B})$ under the binary Boolean semiring $\mathcal{B}$.

Keywords

Tropical Algebra, Idempotent Matrix, Binary Boolean Semirings

1. Introduction

The tropical semiring $\overline{\mathbb{R}}$ (see [1]-[3]) is the set of real numbers $\mathbb{R}$ together with $-\infty$, equipped with the operations of tropical addition and tropical multiplication defined respectively by

$$a \oplus b = \max\{a,b\}, \quad a \otimes b = a + b.$$

The theory of tropical algebra is an algebraic theory developed on tropical semirings. Its research began with the related work of Cuninghame-Green in the 1960s [4] [5]. At present, the theory of tropical algebra has developed into an important branch of algebra. Its research is widely used in optimization problems, control and geometric group theory. Recently, some scholars have been conducting research on tropical matrix theory. For example, in 2018, Izhakian, Johnson and Kambites [6] studied tropical matrix multiplicative semigroups without $-\infty$, and proved that any subgroup of tropical matrix multiplicative semigroups can be embedded in the multiplicative group of $n \times n$ tropical invertible matrices. In the same year, Yu, Zhao and Zeng [7] discussed the properties of idempotent normal tropical matrices, introduced a congruence related to Kleene stars, and then proved that the congruence is a double semilattice congruence. In 2020, Bakhadly, Guterman and De La Puente [8] studied mutually orthogonal tropical $(0,-1)$ normal matrix pairs for tropical multiplication, and characterized the minimum orthogonal pairs. In 2022, Wang [9] studied the idempotents of $n\times n$ tropical matrix multiplicative semigroups over the semiring $\overline{\mathbb{R}}$, and gave the idempotent classification of 3 × 3 and 4 × 4 tropical matrix multiplicative semigroups, respectively. In 2023, Bakhadly, Guterman and De La Puente [10] studied the orthogonality of normal matrices over the set $\{0,-1\}$, and then discussed the orthogonal set of a matrix, that is, the set of all matrices orthogonal to the matrix.

Wang [9] studied the idempotents of $n\times n$ tropical matrix multiplication semigroups over the semiring $\overline{\mathbb{R}}$, and gave the matrix form of low-order idempotents, but did not characterize the high-order idempotents. One of the reasons is that the 'types' of elements in the matrix are too rich. Since there are only two cases of elements in a matrix over a tropical $(0,-1)$ semiring, it is possible to characterize higher-order idempotents over a tropical $(0,-1)$ semiring. Therefore, this paper will study idempotent matrices over a tropical $(0,-1)$ semiring composed of sets $\{0,-1\}$.

In this paper, we first give the necessary and sufficient conditions for A to be an idempotent matrix when the diagonal elements of A are all 0 and all −1, respectively. Then we give some necessary conditions for some special tropical $(0,-1)$ matrices to be idempotent matrices. Finally, the relationship between the matrix set $M_n(T)$ under tropical $(0,-1)$ semiring and the matrix set $M_n(\mathcal{B})$ under binary Boolean semiring $\mathcal{B}$ is discussed. Since tropical $(0,-1)$ semirings are isomorphic to binary Boolean semirings, this study indirectly enriches the content of idempotent matrices over binary Boolean semirings.

When the diagonal elements are 0 or −1, we obtain a necessary and sufficient condition for the matrix to be an idempotent matrix, which is a gap that the high-order tropical idempotent matrix does not fill. The main corollary of Boolean semirings is that the isomorphism of tropical $(0,-1)$ semirings and binary Boolean semirings enables us to transform the idempotency criterion of tropical $(0,-1)$ matrices into binary Boolean matrices, which indirectly enriches the theoretical content of idempotent matrices over binary Boolean semirings and provides a new perspective for related research.

2. Preliminaries

By a semiring we mean a nonempty set S with two binary operations $\oplus$ (addition) and $\otimes$ (multiplication) such that

- $(S,\oplus)$ is a commutative semigroup;
- $(S,\otimes)$ is a semigroup;
- $a\otimes(b\oplus c)=a\otimes b\oplus a\otimes c$, $(b\oplus c)\otimes a=b\otimes a\oplus c\otimes a$ for any $a,b,c\in S$.

Let $M_n(S)$ be the set of all square matrices of order n $(n\geq 2)$ over S. For $A=(a_{ij})$, $B=(b_{ij})\in M_n(S)$, we define:

$$A \oplus B = \left(a_{ij} \oplus b_{ij}\right), \quad A \otimes B = \left(\underset{k=1}{\overset{n}{\oplus}} a_{ik} \otimes b_{kj}\right).$$

Let $T = \{0, -1\}$, and define addition $\oplus$ and multiplication $\otimes$ on T as follows:

$$\text{For any } x, y \in T, \quad x \oplus y = \max\{x, y\}, \quad x \otimes y = x + y,$$

where $(-1) \otimes (-1) = (-1) + (-1) = -1$ (see [8]). According to the definition of semiring, we know that $(T, \oplus, \otimes)$ is a semiring, sometimes $(T, \oplus, \otimes)$ is also called Tropical $(0, -1)$ semiring.

Define the following binary relation $\leq$ on T:

$$a \leq b \Leftrightarrow a \oplus b = b.$$

Then $\leq$ is a partial order relation on T, and the partial order relation is consistent with respect to the addition $\oplus$ and multiplication $\otimes$ on T (see [7] [11]), that is, for any $a, b, c \in T$

$$a \leq b \Rightarrow a \oplus c = b \oplus c, \quad a \otimes c = b \otimes c \quad \text{and} \quad c \otimes a = c \otimes b.$$

Furthermore, if $a \oplus b = c$, then we have $a \leq c$ and $b \leq c$.

For the convenience of subsequent description, the symbol $[n]$ represents the set $\{1, 2, \cdots, n\}$, and $[n_1, n_2]$ represents the set of all integers that are not less than n_1 and not greater than n_2. For $a, b \in T$, $a \otimes b$ is denoted by ab. For $A = (a_{ij}), B = (b_{ij}) \in M_n(T)$, $A \otimes B$ is denoted by AB, and $A \otimes A$ is denoted by A^2. We use $a_{i,j}$ and $a_{i,j}^{(2)}$ to represent the elements at the (i, j) position of matrix A and matrix A^2, respectively.

For $A \in M_n(T)$, if matrix A satisfies $A^2 = A$, then A is called an idempotent matrix. If the diagonal elements of matrix A are all 0, then A is called a normal matrix. The set of all normal matrices over $M_n(T)$ is denoted by M_n^N. Let J_n denote the set of matrices with all elements -1 on the diagonal in $M_n(T)$. In particular, if every element in matrix A is -1, we denote $A = -1$. If the elements on the diagonal of an n-order matrix are all 0 and the other elements are all -1, then the matrix is called the identity matrix I_n. It can be seen that for any $A \in M_n(T)$, $AI_n = I_n A = A$. An n-order matrix is called a permutation matrix if it is obtained by permuting the rows or columns of the identity matrix. It can be seen from Reference [1] that the product of the inverse P^{-1} of the n-order permutation matrix P and itself is an identity matrix, that is, $P^{-1}P = PP^{-1} = I_n$.

A binary Boolean semiring $\mathcal{B} = (\{0,1\}, \boxplus, \boxdot)$ is a semiring with only two elements 0 and 1, and satisfies $1 \boxplus 1 = 1$. Let $A = (a_{i,j}) \in M_n(T)$, $A' = (a'_{i,j}) \in M_n(\mathcal{B})$, where

$$a'_{i,j} = \begin{cases} 0, & \text{if } a_{i,j} = -1 \\ 1, & \text{if } a_{i,j} = 0 \end{cases}.$$

We define a mapping f from $M_n(T)$ to $M_n(\mathcal{B})$ as follows:

$$f: M_n(T) \to M_n(\mathcal{B})$$
$$A \to A'$$

Obviously, f is a bijection. For $A, B \in M_n(T)$, it is easy to deduce

$$f(A \oplus B) = f(A) \boxplus f(B),\ f(A \otimes B) = f(A) \boxdot f(B).$$

Therefore,

$$M_n(T) \cong M_n(\mathcal{B}).$$

3. Characterizations of Idempotent Matrices over J_n and M_n^N

In this section, we will give the characterization of idempotent matrices over J_n and M_n^N. The following is the characterization of idempotent matrices over J_n.

Proposition 1. Let $A = (a_{i,j}) \in J_n$. Then $A^2 = A$ if and only if $A = -1$.

Proof. The adequacy is obviously established. The following proves the necessity. If $n = 2$, $A^2 = A$, then $A = -1$ can be obtained by simple calculation. Now consider $n \geq 3$. Since $A^2 = A$, for any $i \in [n]$, we have

$$a_{i,i}^{(2)} = \oplus_{k=1}^{k=n} a_{i,k} \otimes a_{k,i} = a_{i,i} = -1,$$

for any $a \in T$, there is $0 \oplus a = a \oplus 0 = 0$, so for any $k \in [n]$, there is $a_{i,k} \otimes a_{k,i} = -1$, that is, $a_{i,k} = -1$ or $a_{k,i} = -1$. For any $a \in T$, we have $-1 \otimes a = a \otimes (-1) = -1$, $-1 \oplus a = a \oplus -1 = a$. It can be seen that the i-th row element of matrix A^2 satisfies the following equation:

$$\begin{cases} a_{i,1}^{(2)} = \oplus_{k \neq i,1}^{k=n} a_{i,k} \otimes a_{k,1} = a_{i,1} \\ \quad \vdots \\ a_{i,j}^{(2)} = \oplus_{k \neq i,j}^{k=n} a_{i,k} \otimes a_{k,j} = a_{i,j} \\ \quad \vdots \\ a_{i,n}^{(2)} = \oplus_{k \neq i,n}^{k=n} a_{i,k} \otimes a_{k,n} = a_{i,n} \end{cases} \tag{1}$$

Since all diagonal elements of matrix A are −1, we will now analyze the value of the element $a_{i,j}$ when $i \neq j$. Let $\lambda_1 \in [n]$ with $\lambda_1 \neq i, j$, substituting $a_{i,\lambda_1} = \oplus_{k \neq i,\lambda_1}^{k=n} a_{i,k} \otimes a_{k,\lambda_1}$ into $a_{i,j} = \oplus_{k \neq i,j}^{k=n} a_{i,k} \otimes a_{k,j}$, if $n = 3$, then $a_{i,j} = a_{i,j} \otimes a_{j,\lambda_1} \otimes a_{\lambda_1,j} = -1$. If $n \geq 4$, then

$$\begin{aligned} a_{i,j} &= \oplus_{k \neq i,j,\lambda_1}^{k=n} a_{i,k} \otimes a_{k,j} \oplus \left[a_{i,\lambda_1} \otimes a_{\lambda_1,j} \right] \\ &= \oplus_{k \neq i,j,\lambda_1}^{k=n} a_{i,k} \otimes a_{k,j} \oplus \left(\oplus_{k \neq i,\lambda_1}^{k=n} a_{i,k} \otimes a_{k,\lambda_1} \right) \otimes a_{\lambda_1,j} \\ &= \oplus_{k \neq i,j,\lambda_1}^{k=n} a_{i,k} \otimes a_{k,j} \oplus \left(\oplus_{k \neq i,\lambda_1,j}^{k=n} a_{i,k} \otimes a_{k,\lambda_1} \right) \otimes a_{\lambda_1,j} \oplus a_{i,j} a_{j,\lambda_1} a_{\lambda_1,j}, \end{aligned}$$

since for any $k \neq i, j, \lambda_1$, we have $a_{k,\lambda_1} \otimes a_{\lambda_1,j} \leq a_{k,j}$, that is, $a_{i,k} a_{k,\lambda_1} a_{\lambda_1,j} \leq a_{i,k} a_{k,j}$. From $a_{j,\lambda_1} a_{\lambda_1,j} = -1$, we can get that

$$a_{i,j} = \oplus_{k \neq i,j,\lambda_1}^{k=n} a_{i,k} \otimes a_{k,j}.$$

Through the above analysis, it can be seen that if $a_{i,\lambda_1} = \oplus_{k\neq i,\lambda_1}^{k=n} a_{i,k} \otimes a_{k,\lambda_1}$ is substituted into each equation except itself in Equation (1), then each equation except itself in Equation (1) will eliminate the term containing a_{i,λ_1}. In Equation (1), $a_{i,\lambda_1} = \oplus_{k\neq i,\lambda_1}^{k=n} a_{i,k} \otimes a_{k,\lambda_1}$ and $a_{i,i} = \oplus_{k\neq i}^{k=n} a_{i,k} \otimes a_{k,i}$ are deleted, then Equation (1) becomes

$$\begin{cases} \vdots \\ a_{i,h} = \oplus_{k\neq i,h,\lambda_1}^{k=n} a_{i,k} \otimes a_{k,h} \\ \vdots \\ a_{i,j} = \oplus_{k\neq i,j,\lambda_1}^{k=n} a_{i,k} \otimes a_{k,j} \\ \vdots \end{cases} \tag{2}$$

where $h \neq i, j, \lambda_1$. If the number of restrictions on the corner marker k in the above (2) is not $n-1$, the operation similar to the above is performed again, that is, for any $\lambda_2 \in [n]$, where $\lambda_2 \neq i, j, \lambda_1$, $a_{i,\lambda_2} = \oplus_{k\neq i,\lambda_2,\lambda_1}^{k=n} a_{i,k} \otimes a_{k,\lambda_2}$ is substituted into $a_{i,j} = \oplus_{k\neq i,j,\lambda_1}^{k=n} a_{i,k} \otimes a_{k,j}$, then $a_{i,j} = \oplus_{k\neq i,j,\lambda_1,\lambda_2}^{k=n} a_{i,k} \otimes a_{k,j}$ can be obtained. Repeat the above steps until the limited number of corner k is $n-1$. Finally, we can get

$$\begin{cases} a_{i,\mu} = a_{i,j} \otimes a_{j,\mu} \\ a_{i,j} = a_{i,\mu} \otimes a_{\mu,j} \end{cases},$$

where $\mu \neq i, j, \lambda_1, \cdots, \lambda_l, l \in [n-3]$. Substituting $a_{i,\mu} = a_{i,j} \otimes a_{j,\mu}$ into $a_{i,j} = a_{i,\mu} \otimes a_{\mu,j}$, then $a_{i,j} = a_{i,j} \otimes a_{j,\mu} \otimes a_{\mu,j} = -1$ is obtained.

In summary, $A = -1$. □

Proposition 2. Let $A = (a_{i,j}) \in M_n^N$. Then $A^2 = A$ if and only if when $a_{i,j} = -1$, there is $a_{i,k} = -1$ or $a_{k,j} = -1$, where $k \neq i, j$.

Proof. When $n = 2$, the proposition is obvious. Now consider $n \geq 3$.

Adequacy. If $i \neq j$, then

$$\begin{aligned} a_{i,j}^{(2)} &= \oplus_{k=1}^{k=n} a_{i,k} \otimes a_{k,j} \\ &= (a_{i,i} \oplus a_{j,j}) \otimes a_{i,j} \oplus \left[\oplus_{k\neq i,j}^{k=n} a_{i,k} \otimes a_{k,j}\right] \\ &= a_{i,j} \oplus \left[\oplus_{k\neq i,j}^{k=n} a_{i,k} \otimes a_{k,j}\right]. \end{aligned}$$

If $a_{i,j} = -1$, then either $a_{i,k} = -1$ or $a_{k,j} = -1$ holds, where $k \neq i, j$, then

$$a_{i,j}^{(2)} = a_{i,j} = -1.$$

If $a_{i,j} = 0$, from $a \oplus 0 = 0 \oplus a = 0 \, (a \in T)$, we have $a_{i,j}^{(2)} = a_{i,j} = 0$. Furthermore, since $a_{i,i}^{(2)} = a_{i,i} = 0$ holds for any $i \in [n]$, we can conclude that $A^2 = A$.

Necessity. Obviously established. □

Theorem 3. Let $B \in M_n(T)$ and $B^2 = B$. If there exists a permutation matrix P such that $PBP^{-1} = \begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix}$, where $B_1 \in J_{n-p}$ and B_4 is a p-order matrix with all elements 0. Let $A = (a_{i,j}) = \begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix}$, then we have

1) $A=-1$ when $p=0$;

2) When $p=1$, for any $i, j\in[1,n-1]$, we have $a_{i,j}=a_{i,n}a_{n,j}$;

3) When $n\geq 3$ and $p\in[2,n-1]$, then for any $i,\mu\in[n-p+1,n]$ and $\lambda, j\in[1,n-p]$, we have $a_{i,j}=a_{i,\lambda}$, $a_{\lambda,\mu}=a_{\lambda,i}$.

Proof. 1) It can be obtained from Proposition 1.

2) When $n=2$ or $n=3$, the result clearly holds. Now consider $n\geq 4$. Since B is an idempotent matrix, $A=PBP^{-1}$ is also an idempotent matrix. In this case, for any $i, j\in[1,n-1]$, we have

$$a_{i,j}^{(2)}=\oplus_{k=1}^{k=n}a_{i,k}\otimes a_{k,j}=a_{i,j}.$$

When $i=j$. Since $a_{i,i}^{(2)}=a_{i,i}=-1$, and $0\oplus a=a\oplus 0=0$ for any $a\in T$, then $a_{i,k}\otimes a_{k,i}=-1$ for any $k\in[n]$. Therefore, $a_{i,i}=a_{i,n}a_{n,i}$.

When $i\neq j$. For any $k\in[1,n-1]$, we have $a_{k,k}=-1$, then for the first $n-1$ column elements of the i-th row of the matrix A^2, we can get that

$$\begin{cases} a_{i,1}^{(2)}=\oplus_{k\neq i,1}^{k=n}a_{i,k}\otimes a_{k,1}=a_{i,1} \\ \quad\vdots \\ a_{i,j}^{(2)}=\oplus_{k\neq i,j}^{k=n}a_{i,k}\otimes a_{k,j}=a_{i,j} \\ \quad\vdots \\ a_{i,n-1}^{(2)}=\oplus_{k\neq i,n-1}^{k=n}a_{i,k}\otimes a_{k,n-1}=a_{i,n-1} \end{cases} \tag{3}$$

The following procedure is similar to Proposition 1. Let $\lambda_1\in[1,n-1]$ be chosen arbitrarily, where $\lambda_1\neq i, j$. Substitute $a_{i,\lambda_1}=\oplus_{k\neq i,\lambda_1}^{k=n}a_{i,k}\otimes a_{k,\lambda_1}$ into $a_{i,j}=\oplus_{k\neq i,j}^{k=n}a_{i,k}\otimes a_{k,j}$, we obtain $a_{i,j}=\oplus_{k\neq i,j,\lambda_1}^{k=n}a_{i,k}\otimes a_{k,j}$. If we substitute $a_{i,\lambda_1}=\oplus_{k\neq i,\lambda_1}^{k=n}a_{i,k}\otimes a_{k,\lambda_1}$ into every term of Equation (3) except itself, then every term of Equation (3) except itself will eliminate the terms containing a_{i,λ_1}. If we remove the terms $a_{i,\lambda_1}=\oplus_{k\neq i,\lambda_1}^{k=n}a_{i,k}\otimes a_{k,\lambda_1}$ and $a_{i,i}=\oplus_{k\neq i}^{k=n}a_{i,k}\otimes a_{k,i}$ from Equation (4), Equation (4) becomes

$$\begin{cases} \quad\vdots \\ a_{i,j}=\oplus_{k\neq i,j,\lambda_1}^{k=n}a_{i,k}\otimes a_{k,j} \\ \quad\vdots \end{cases} \tag{4}$$

If the limit number of the angle k in Equation (4) is not $n-1$, the above operation is performed again, and finally $a_{i,j}=a_{i,n}\otimes a_{n,j}$ can be obtained.

In summary, when $p=1$, for any $i, j\in[1,n-1]$, we have $a_{i,j}=a_{i,n}\otimes a_{n,j}$.

3) If $i=\mu$ and $\lambda=j$, the conclusion clearly holds. Now consider the case where $i\neq\mu$ and $\lambda\neq j$. Since for any $i,\mu\in[n-p+1,n]$ and $\lambda, j\in[1,n-p]$, we have

$$a_{i,\mu}=0, a_{\mu,i}=0, a_{i,j}=\oplus_{k=1}^{k=n}a_{i,k}\otimes a_{k,j} \text{ and } a_{\mu,j}=\oplus_{k=1}^{k=n}a_{\mu,k}\otimes a_{k,j}.$$

Thus

$$\oplus_{k=n-p+1}^{k=n}a_{k,j}\leq a_{i,j},\ \oplus_{k=n-p+1}^{k=n}a_{k,j}\leq a_{\mu,j},$$

that is, $a_{\mu,j} \le a_{i,j}$, $a_{i,j} \le a_{\mu,j}$. Therefore, $a_{i,j} = a_{\mu,j}$. Similarly, we have $a_{\lambda,\mu} = a_{\lambda,i}$. □

Theorem 4. Let $B \in M_n(T)$ and $B^2 = B$. Suppose there exists a permutation matrix P such that $PBP^{-1} = \begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix}$, where $B_1 \in M_p^N$, the diagonal elements of B_4 and all elements below the diagonal are −1, and the remaining elements are 0. Let $A = (a_{i,j}) = \begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix}$, then we have

1) When $p = n$, $a_{i,j} = -1$ implies that either $a_{i,k} = -1$ or $a_{k,j} = -1$, where $k \in [n]$;

2) When $p = n-1$, then for any $k \in [n]$, either $a_{n,k} = -1$ or $a_{k,n} = -1$, where $k \ne i, j$;

3) When $n \ge 3$ and $p \in [1, n-2]$, then for any $i, \lambda \in [p+1, n]$, and $j \in [p]$, if $i \ge \lambda$, then $a_{i,j} \le a_{\lambda,j}$, $a_{j,\lambda} \le a_{j,i}$.

Proof. 1) It can be obtained from Proposition 2.

2) Since B is an idempotent matrix, $A = PBP^{-1}$ is also an idempotent matrix. If $p = n-1$, it follows from $A^2 = A$ that

$$a_{n,n}^{(2)} = \oplus_{k=1}^{k=n} a_{n,k} \otimes a_{k,n} = a_{n,n} = -1$$

Then, for any $k \in [n]$, either $a_{n,k} = -1$ or $a_{k,n} = -1$.

3) Let $i \in [p+1, n-1]$ and $j \in [p]$, then

$$\begin{cases} a_{p+1,j}^{(2)} = \oplus_{k=1}^{k=n} a_{p+1,k} \otimes a_{k,j} = a_{p+1,j} \\ \quad \vdots \\ a_{i,j}^{(2)} = \oplus_{k=1}^{k=n} a_{i,k} \otimes a_{k,j} = a_{i,j} \\ \quad \vdots \\ a_{n-1,j}^{(2)} = \oplus_{k=1}^{k=n} a_{n-1,k} \otimes a_{k,j} = a_{n-1,j} \end{cases} \tag{5}$$

Since $B_1 \in M_p^N$, $a_{j,j} = 0$, therefore, $a_{i,j} \oplus a_{j,j} = a_{i,j}$, further, we have

$$a_{i,j} = \oplus_{k=j}^{k=n} a_{i,k} \otimes a_{k,j} = a_{i,j} \oplus \left(\oplus_{k=i+1}^{k=n} a_{i,k} \otimes a_{k,j} \right).$$

From the diagonal elements of B_4 and the elements below the diagonal are all −1, and the remaining elements are 0, it follows that $a_{i,k} = 0$ for any $k \in [i+1, n]$, so

$$a_{i,j} = \oplus_{k=i}^{k=n} a_{k,j}$$

Therefore, Equation (5) can be written as

$$\begin{cases} a_{p+1,j} = \oplus_{k=p+1}^{k=n} a_{k,j} \\ \quad \vdots \\ a_{i,j} = \oplus_{k=i}^{k=n} a_{k,j} \\ \quad \vdots \\ a_{n-1,j} = \oplus_{k=n-1}^{k=n} a_{k,j} \end{cases} \tag{6}$$

Analyzing Equation (6) from bottom to top, we can see that $a_{n,j} \le \cdots \le a_{i,j} \le \cdots \le a_{p+1,j}$. Similarly, we have $a_{j,p+1} \le \cdots \le a_{j,\lambda} \le \cdots \le a_{j,n}$. Therefore, for any $i,\lambda \in [p+1,n]$ and $j \in [p]$, we have $a_{i,j} \le a_{\lambda,j}$, and $a_{j,\lambda} \le a_{j,i}$ when $i \ge \lambda$. □

Let $Q_n = f(I_n)$ and $D_n = f(J_n)$. If P_1 is obtained by permuting the rows or columns of Q_n in $M_n(\mathcal{B})$, then Theorem 3 can be rewritten as follows:

Let $B \in M_n(\mathcal{B})$ and $B^2 = B$. If there exists a permutation matrix P_1 such that $P_1BP_1^{-1} = \begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix}$, where $B_1 \in D_{n-p}$ and B_4 is a p-order matrix with all elements 1. Let $A = (a_{i,j}) = \begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix}$, then we have

1) When $p = 0$, the elements in matrix A are all 0;

2) When $p = 1$, for any $i, j \in [1, n-1]$, we have $a_{i,j} = a_{i,n} \boxdot a_{n,j}$;

3) When $n \ge 3$ and $p \in [2, n-1]$, then for any $i,\mu \in [n-p+1, n]$ and $\lambda, j \in [1, n-p]$, we have $a_{i,j} = a_{i,\lambda}$, $a_{\lambda,\mu} = a_{\lambda,i}$.

Theorem 4 can be rewritten as follows:

Let $B \in M_n(\mathcal{B})$ and $B^2 = B$. Suppose there exists a permutation matrix P_1 such that $P_1BP_1^{-1} = \begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix}$, where B_1 is a square matrix of order p with all diagonal elements of 1, the diagonal and below diagonal elements of B_4 are all 0, and the remaining elements of B_4 are 1. Let $A = (a_{i,j}) = \begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix}$, then we have

1) When $p = n$, $a_{i,j} = 0$ implies that either $a_{i,k} = 0$ or $a_{k,j} = 0$, where $k \in [n]$;

2) When $p = n-1$, then for any $k \in [n]$, either $a_{n,k} = 0$ or $a_{k,n} = 0$, where $k \ne i, j$;

3) When $n \ge 3$ and $p \in [1, n-2]$, then for any $i,\lambda \in [p+1, n]$ and $j \in [p]$, if $i \ge \lambda$, then $a_{i,j} \le a_{\lambda,j}$, and $a_{j,\lambda} \le a_{j,i}$.

Note: "$\le$" is a general size relation at this time.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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