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Riemann Integrals Containing Differentials within the Integrand

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Abstract

A certain class of infinite sums leading to Riemann integrals that involve additional differentials inside their integrands is investigated. It is shown that these integrals can easily be evaluated by applying a simple algorithm.

Keywords

Infinite, Sum, Riemann, Integral, Integrand, Differential

1. Introduction

There are situations where additional differential expressions appear inside the integrand of a definite integral, namely

$$\int_a^b f\left(x, \mathrm{d}x\right)\mathrm{d}x \tag{1}$$

where $f\left(x, \mathrm{d}x\right)$ is a function of both the variable x and its differential $\mathrm{d}x$. Although these situations are extremely rare, they do happen in certain areas of mathematics, probability, information theory, and science [1].

Consider an interval $\left[a, b\right]$ of real numbers x. The interval is divided into n small subintervals, each of width Δx_i ($i = 1, 2, 3, \cdots, n$), as shown in **Figure 1**. Let the value of x at the midpoint of slice i be x_i.

Suppose we have a function of the independent variables x and its increment Δx, denoted by $f\left(x, \Delta x\right)$. Then for the interval i, the value of this function is approximately $f\left(x_i, \Delta x_i\right)$. We construct the sum,

$$S_n = \sum_{i=1}^{n} f\left(x_i, \Delta x_i\right)\Delta x_i \tag{2}$$

We now take the limit $n \to \infty$ such that each $\Delta x_i \to 0$,

$$S = \lim_{n\to\infty} S_n = \lim_{n\to\infty}\sum_{i=1}^{n} f\left(x_i, \Delta x_i\right)\Delta x_i \tag{3}$$

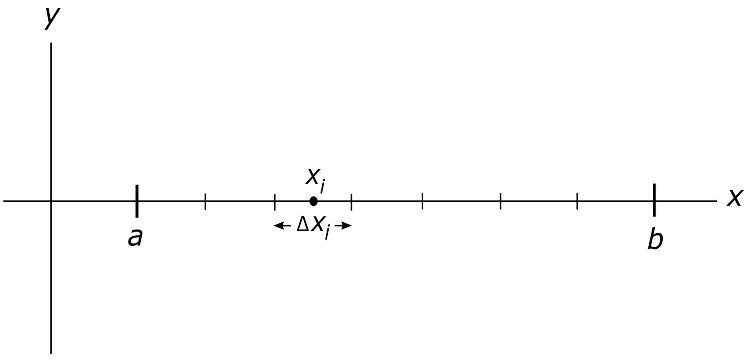

Figure 1. An interval $[a,b]$ is divided into n subintervals, each of width Δx_i. The coordinate x_i is the midpoint of subinterval i.

Sums and limits like these, although rare, appear in some areas such as information theory. For example, Shannon entropy S for discrete probabilities is defined by [2]-[5]

$$S = -\sum_{i=1}^{n} p_i \ln p_i \tag{4}$$

where p_i is the probability of the outcome E_i in a random event with outcomes $E_1, E_2, \cdots, E_n$. Extension of this definition to a continuous random variable with a probability density function $g(x)$ results in the equation

$$S_n = -\sum_{i=1}^{n} g(x_i)\Delta x_i \ln\left[g(x_i)\Delta x_i\right] = -\sum_{i=1}^{n} g(x_i)\ln\left[g(x_i)\Delta x_i\right]\Delta x_i \tag{5}$$

where here the function $g(x_i)\ln\left[g(x_i)\Delta x_i\right]$ plays the role of $f(x_i,\Delta x_i)$ of Equation (3).

In what follows, we show that the limit of a sum like Equation (3) can be converted into a Riemann integral just as the limit of ordinary sums [6],

$$\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x_i = \int_a^b f(x)\,\mathrm{d}x \tag{6}$$

except that for evaluation of the resulting integral, the $\mathrm{d}x$ in the *integrand* should be set equal to zero, namely,

$$\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i,\Delta x_i)\Delta x_i = \int_a^b f(x,\mathrm{d}x)_{\mathrm{d}x=0}\,\mathrm{d}x \tag{7}$$

where $[a,b]$ is the support set of the random variable x. But, if the integrand becomes undefined at $\mathrm{d}x = 0$, its limit should be considered for $\mathrm{d}x \to 0$. Note that this is not a new integration rule, it is simply Riemann integral, in which $\mathrm{d}x$ is set equal to zero in its integrand.

2. Proof of Equation 7

Let us expand each function $f(x_i,\Delta x_i)$ of the sum in Equation (3) with respect to Δx_i in a Taylor series about $\Delta x_i = 0$,

$$\begin{aligned} S &= \lim_{n\to\infty}\sum_{i=1}^{n}\left[f\left(x_i,0\right)+\left(\frac{\partial f}{\partial \Delta x_i}\right)_0 \Delta x_i+\frac{1}{2}\left(\frac{\partial^2 f}{\partial \Delta x_i^2}\right)_0 \Delta x_i^2+\cdots\right]\Delta x_i \\ &= \lim_{n\to\infty}\sum_{i=1}^{n}\left[f\left(x_i,0\right)\Delta x_i+\left(\frac{\partial f}{\partial \Delta x_i}\right)_0 \Delta x_i^2+\frac{1}{2}\left(\frac{\partial^2 f}{\partial \Delta x_i^2}\right)_0 \Delta x_i^3+\cdots\right] \end{aligned} \tag{8}$$

However, as each $\Delta x_i \to 0$, the second and higher powers of Δx_i become negligible compared to its first power, and we obtain

$$S = \lim_{n\to\infty}\sum_{i=1}^{n} f\left(x_i,0\right)\Delta x_i \tag{9}$$

But this, by definition, is the Riemann integral,

$$S = \int_a^b f\left(x,0\right)\mathrm{d}x \tag{10}$$

which, for more clarity, can be written as

$$S = \int_a^b f\left(x,\mathrm{d}x\right)_{\mathrm{d}x=0}\mathrm{d}x \tag{11}$$

which proves Equation (7).

3. Additional Examples

A simple example is the following sum over the interval $a=0$ and $b=1$,

$$\begin{aligned} \lim_{n\to\infty}\sum_{i=1}^{n}\sqrt{\mathrm{e}^{x_i}\cos\left(\Delta x_i\right)}\,\Delta x_i &= \int_0^1\sqrt{\mathrm{e}^{x}\cos\left(\mathrm{d}x\right)}\,\mathrm{d}x = \int_0^1\sqrt{\mathrm{e}^{x}\cos\left(0\right)}\,\mathrm{d}x \\ &= \int_0^1 \mathrm{e}^{x/2}\mathrm{d}x = 1.2974 \end{aligned}$$

Numerical evaluation of the left hand side of this equation verifies the result.

Evaluation of integrals involving $\mathrm{d}x$ in the integrand should adhere to the rule explained above. Otherwise, incorrect answers may result. More specifically, the differential $\mathrm{d}x$ within the integrand and that of the integral should be treated differently; the former should be set equal to zero while the latter should be considered as infinitesimal. For example, the integral

$$\int_1^2 \ln\left(\mathrm{d}x\right)\mathrm{d}x \tag{12}$$

may seem to be equal to zero because $\lim_{\mathrm{d}x\to 0}\mathrm{d}x\ln\left(\mathrm{d}x\right)=0$. However, this result is incorrect. The correct answer is

$$\int_1^2 \ln\left(\mathrm{d}x\right)\mathrm{d}x = \int_1^2 \ln\left(0\right)\mathrm{d}x = \ln\left(0\right)\int_1^2 \mathrm{d}x = -\infty \tag{13}$$

which can be verified by numerically evaluating the following sum over the indicated interval,

$$\lim_{n\to\infty}\sum_{i=1}^{n}\ln\left(\Delta x_i\right)\Delta x_i \tag{14}$$

4. Discussion and Conclusion

There are situations where the differential of a Riemann integral also appears in the integrand. This happens when the infinite sum whose limit yields the Riemann integral contains the term Δx in its function. However, these situations are highly uncommon. To the best of our knowledge, they are not mentioned in any mathematics textbooks, and a literature survey did not produce any results. In conclusion, because these cases have shown up occasionally in science and mathematics [1], they warrant an examination and explanation, which has been the objective of this article.

Returning to the example of Shannon entropy for a continuous random variable, taking the limit of Equation (5), we obtain

$$S = -\int_a^b g(x)\ln\left[g(x)\mathrm{d}x\right]_{\mathrm{d}x=0}\mathrm{d}x = -\ln(0)\int_a^b g(x)\mathrm{d}x = \infty \tag{15}$$

Consequently, the continuous Shannon entropy, as a direct limit of the discrete entropy, diverges without renormalization. However, this renormalization alters the physical meaning of the resulting equation, and hence should no longer be called Shannon entropy [7]. Nevertheless, a different type of continuous entropy, or differential entropy, has been defined by [8]-[10]

$$S = -\int_a^b g(x)\ln g(x)\mathrm{d}x \tag{16}$$

where the limits a and b define the support set, or the interval of the random variable x, and $g(x)$ is the probability density function for x. Despite its shortcomings, this entropy is applied in areas such as thermodynamics, statistical mechanics, and information theory [7].

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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