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# Fractional Operators with Homogeneous Kernels in Two Weighted Herz Spaces with Variable Exponents

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## Abstract

By using the function decomposition of the two weighted Herz spaces with variable exponents and hierarchical estimation methods, as well as the boundedness of the fractional operators with homogeneous kernels on the Lebesgue spaces with variable exponent. The boundedness of these operators on the two weighted Herz spaces with variable exponents is obtained.

## Keywords

Fractional Integral, Muckenhoupt Weight, Herz Space, Variable Exponent

## 1. Introduction

In order to better study the solutions of the Poisson equation $\Delta u = f$ in partial differential equations, Sobolev [1] showed that the fractional integral operator is bounded from the classical Lebesgue space $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$. After that, Harboure and Salinas [2] provided necessary and sufficient conditions for the boundedness of fractional integral operators and their commutators on $L^p(\omega)$. Capone, Cruz-Uribe and Fiorenza [3] have proved the boundedness on $L^{p(\cdot)}$, provided that the exponents satisfy the log-Hölder continuous conditions. Bernardis, Dalmasso and Pradolini [4] proved the boundedness of fractional integral operators and their commutators on $L^{p(\cdot)}(\omega)$. Cruz-Uribe and Wang [5] have also obtained the boundedness of fractional integrals on weighted Lebesgue spaces with variable exponent by applying the extrapolation theorem. We recall the so-called fractional integral operators.

Let $\mathbb{S}^{n-1}$, with $n \geq 2$, be the unit sphere in $\mathbb{R}^n$ with the normalized Lebesgue measure $\mathrm{d}\sigma(x')$. Assuming that $\Omega \in L^s(\mathbb{S}^{n-1})$, with $s > n/(n-\beta)$, the fractional integral operator with homogeneous kernel $T_{\Omega,\beta}$ is defined by

$$T_{\Omega,\beta} f(x) = \int_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^{n-\beta}} f(y)\mathrm{d}y$$

where $0 < \beta < n$.

If $\beta = 1$, the integral returns the well-known Cauchy principal value.

If $\Omega = 1$, $T_{\Omega,\beta}$ is the fractional integral operator (Riesz potential operator)

$$T_{\Omega,\beta} f(x) = \int_{\mathbb{R}^n} \frac{|f(y)|}{|x-y|^{n-\beta}} \mathrm{d}y.$$

If $\beta = 0$ and Ω satisfy the vanishing condition on $\mathbb{S}^{n-1}$:

$$\int_{\mathbb{S}^{n-1}} \Omega(x')\mathrm{d}\sigma(x') = 0\,\mathrm{s}$$

then $T_{\Omega,\beta}$ becomes a Calderon-Zygmund operator.

If $b \in L^1_{\text{loc}}(\mathbb{R}^n)$, set

$$\|b\|_{\text{BMO}} := \sup_B \frac{1}{|B|} \int_B |b(x) - b_B| \mathrm{d}x,$$

where $b_B = (1/|B|)\int_B b(y)\mathrm{d}y$ and the supremum is taken over all $B \subset \mathbb{R}^n$, and what follows $|B|$ is the Lebesgue measure of measurable set $B$ in $\mathbb{R}^n$. A function $b$ is called bounded mean oscillation if $\|b\|_* < \infty$. Denote by $\text{BMO}(\mathbb{R}^n)$ the set of all bounded mean oscillation functions on $\mathbb{R}^n$.

Let $b \in \text{BMO}(\mathbb{R}^n)$, the commutator of fractional integral operator $[b, T_{\Omega,\beta}]$ is defined by

$$[b, T_{\Omega,\beta}] = b(x) T_{\Omega,\beta} f(x) - T_{\Omega,\beta}(bf)(x).$$

Variable exponent function spaces play a vital role in fluid dynamics, elasticity dynamics, and differential equations with nonstandard growth. Since Kováčik and Rákosník developed the theory of variable exponent function spaces in [6], the variable exponent Lebesgue spaces $L^{p(\cdot)}(\mathbb{R}^n)$ have been extensively investigated, see [7]-[9]. A class of function spaces called Herz spaces has played an important role in real analysis because the interesting norm includes explicitly both local and global information of the function. Izuki introduced the variable exponent Herz spaces in [10] [11] and considered the boundedness of commutators of fractional integrals [12]. In [13], Izuki has proved the boundedness of commutators with fractional integrals on Herz spaces with variable exponent. In 2016, Izuki and Noi [14] defined weighted Herz spaces with variable exponent and proved the boundedness of fractional integrals on those spaces under proper assumptions on weights and exponents.

Recently, Izuki and Noi introduced two weight Herz spaces with variable exponents in [15]. Motivated by the mentioned works, we will consider the boundedness of the fractional operator with homogeneous kernels on two weight Herz spaces with variable exponents. Compared with reference [15], this paper introduces the two weight settings in the variable exponent Herz space for the first time. Although [15] established the theoretical framework of the two weight variable exponent Herz space, it did not involve the fractional integral operator. Compared with reference [16], this paper extends its single weight results to a more general and challenging two weight case, while retaining the homogeneity assumption of the kernel function. In terms of the proof method, it combines the fine characterization of variable exponent Muckenhoupt weight classes with the ring decomposition technique.

In this paper we use the following symbols and notations:

1) For any measurable set $E, |E|$ denotes the Lebesgue measure and $\chi_E$ means the characteristic function.

2) A locally integrable and positive function defined on $\mathbb{R}^n$ is said to be a weight. We write $w(E) := \int_E w(x)\,\mathrm{d}x$ for a weight $w$ and a measurable set $E$.

3) Given $k \in \mathbb{Z}$, we write $B_k := \overline{B(0,2^k)} = \{x \in \mathbb{R}^n : |x| \le 2^k\}$.

4) Give $k \in \mathbb{Z}$, we define $B_k := \{x \in \mathbb{R}^n : |x| \le 2^k\}$, $D_k := B_k \setminus B_{k-1} = \{x \in \mathbb{R}^n : 2^{k-1} < |x| \le 2^k\}$, $\chi_k := \chi_{D_k}$.

5) For any quantities $A$ and $B$, if there exists a constant $C > 0$ such that $A \le CB$, we write $A \lesssim B$. If $A \lesssim B$ and $B \lesssim A$, we write $A \approx B$.

## 2. Variable Lebesgue Spaces

We introduce Lebesgue spaces with variable exponent. Let $p(\cdot)$ be a measurable function on $\mathbb{R}^n$ taking values in $[1,\infty)$, the Lebesgue space with variable exponent $L^{p(\cdot)}(\mathbb{R}^n)$ is defined by

$$L^{p(\cdot)}(\mathbb{R}^n) := \left\{ f \text{ is measurable on } \mathbb{R}^n : \int_{\mathbb{R}^n} \left( \frac{|f(x)|}{\lambda} \right)^{p(x)} \mathrm{d}x < \infty \text{ for some } \lambda > 0 \right\}.$$

Then $L^{p(\cdot)}(\mathbb{R}^n)$ is a Banach function space equipped with the norm

$$\|f\|_{L^{p(\cdot)}} := \inf \left\{ \lambda > 0 : \int_{\mathbb{R}^n} \left( \frac{|f(x)|}{\lambda} \right)^{p(x)} \mathrm{d}x \le 1 \right\}.$$

Denote by $\mathscr{P}(\mathbb{R}^n)$ the set of all measurable functions $p(\cdot): \mathbb{R}^n \to (1,\infty)$ such that

$$1 < p^- := \operatorname*{ess\,inf}_{x \in \mathbb{R}^n} p(x), \quad p^+ := \operatorname*{ess\,sup}_{x \in \mathbb{R}^n} p(x) < \infty.$$

and $\mathscr{P}_0(\mathbb{R}^n)$ consists of all $p(\cdot)$ satisfying $p^- > 0$ and $p^+ < \infty$.

**Definition 2.1** ([9]) Let $\alpha(\cdot)$ be a real-valued function on $\mathbb{R}^n$.

(i) For any $x, y \in \mathbb{R}^n$, $|x - y| < 1/2$, if

$$|\alpha(x) - \alpha(y)| \lesssim \frac{1}{\log(e + 1/|x - y|)},$$

then $\alpha(\cdot)$ is said local log-Hölder continuous on $\mathbb{R}^n$.

(ii) For all $x \in \mathbb{R}^n$, if

$$\left|\alpha(x)-\alpha(0)\right| \lesssim \frac{1}{\log\left(e+1/|x|\right)},$$

then $\alpha(\cdot)$ is said log-Hölder continuous functions at origin and denote by $\mathscr{P}_0^{\log}\left(\mathbb{R}^n\right)$.

(iii) For some real number $\alpha_\infty \in \mathbb{R}$, for $x \in \mathbb{R}^n$, if

$$\left|\alpha(x)-\alpha_\infty\right| \lesssim \frac{1}{\log\left(e+|x|\right)},$$

then $\alpha(\cdot)$ is said log-Hölder continuous at infinity and denote by $\mathscr{P}_\infty^{\log}\left(\mathbb{R}^n\right)$.

(iv) The function $\alpha(\cdot)$ satisfying (ii) and (iii) is denoted by $\mathscr{P}^{\log}\left(\mathbb{R}^n\right)$. It is also well known that the Hardy-Littlewood maximal operator $M$, defined by

$$Mf(x) := \sup_{x\in B} \frac{1}{|B|}\int_B \left|f(y)\right| \mathrm{d}y$$

is bounded on $L^{p(\cdot)}\left(\mathbb{R}^n\right)$ whenever $p(\cdot) \in \mathscr{P}^{\log}\left(\mathbb{R}^n\right) \cap \mathscr{P}\left(\mathbb{R}^n\right)$ [17].

Now we define the Muckenhoupt classes. We begin with the classical Muckenhoupt $A_1$ weight.

**Definition 2.2** ([18]) Let $\omega$ be a weighted function on $\mathbb{R}^n$, that is, $\omega$ is real-valued, non-negative and locally integrable. $\omega$ is said to be a Muckenhoupt $A_1$ weight if

$$M\omega(x) \lesssim \omega(x) \ \ a.e.\ x \in \mathbb{R}^n.$$

For $1 < p < \infty$, we say that $\omega$ is an $A_p$ weight if

$$\sup_B \left(\frac{1}{|B|}\int_B \omega(x)\,\mathrm{d}x\right)\left(\frac{1}{|B|}\int_B \omega(x)^{1-p'}\,\mathrm{d}x\right)^{p-1} < \infty.$$

**Definition 2.3** ([19]) Let $p(\cdot) \in \mathcal{P}\left(\mathbb{R}^n\right)$. For some constant $C$, a weight $\omega$ is said to be an $A_{p(\cdot)}$ weight, if for all balls $B$ in $\mathbb{R}^n$ such that

$$\frac{1}{|B|}\left\|\omega\chi_B\right\|_{L^{p(\cdot)}\left(\mathbb{R}^n\right)}\left\|\omega^{-1}\chi_B\right\|_{L^{p'(\cdot)}\left(\mathbb{R}^n\right)} \le C.$$

Diening and Hästä [20], have proved the next monotone property.

**Lemma 2.1** ([20]) If $p(\cdot), q(\cdot) \in \mathscr{P}^{\log}\left(\mathbb{R}^n\right) \cap \mathscr{P}\left(\mathbb{R}^n\right)$ and $p(\cdot) \le q(\cdot)$, then we have

$$A_1 \subset A_{p(\cdot)} \subset A_{q(\cdot)}.$$

In order to state the boundedness of fractional integrals on weighted function spaces we shall define the class $A\left(p_1(\cdot), p_2(\cdot)\right)$ as follows.

**Definition 2.4** Let $0 < \beta < n$ and $p_1(\cdot), p_2(\cdot) \in \mathscr{P}\left(\mathbb{R}^n\right)$ such that $1/p_2(x) \equiv 1/p_1(x) - \beta/n$. A weight $w$ is said to be an $A\left(p_1(\cdot), p_2(\cdot)\right)$ weight if

$$\left\|w\chi_B\right\|_{L^{p_2(\cdot)}}\left\|w^{-1}\chi_B\right\|_{L^{p_1'(\cdot)}} \lesssim |B|^{1-\frac{\beta}{n}},$$

holds for all balls $B \subset \mathbb{R}^n$.

**Lemma 2.2** ([5]) Let $0<\beta<n$ and $p_1(\cdot), p_2(\cdot) \in \mathscr{P}(\mathbb{R}^n)$ such that $1/p_2(x) \equiv 1/p_1(x) - \beta/n$. Then $w \in A(p_1(\cdot), p_2(\cdot))$ if and only if $w^{p_2(\cdot)} \in A_{1+p_2(\cdot)/p_1'(\cdot)}$.

**Definition 2.5** Let $p(\cdot) \in \mathscr{P}(\mathbb{R}^n)$ and $w \in A_{p(\cdot)}$. The weight variable exponent Lebesgue space $L^{p(\cdot)}(w)$ denotes the set of all complex-valued measurable functions f satisfying

$$L^{p(\cdot)}(w) = \left\{ f : f\omega^{\frac{1}{p(\cdot)}} \in L^{p(\cdot)}(\mathbb{R}^n) \right\}.$$

This is a Banach space equipped with the norm:

$$\|f\|_{L^{p(\cdot)}(w)} := \left\| f w^{1/p(\cdot)} \right\|_{L^{p(\cdot)}}.$$

**Lemma 2.3** ([16]) Let $p_1(\cdot) \in \mathscr{P}^{\log}(\mathbb{R}^n) \cap \mathscr{P}(\mathbb{R}^n)$ and $0<\beta<n/p_1^+$. Define $p_2(\cdot)$ by $1/p_2(\cdot) \equiv 1/p_1(\cdot) - \beta/n$. If $w \in A(p_1(\cdot), p_2(\cdot))$, then $T_{\Omega,\beta}$ is bounded from $L^{p_1(\cdot)}(w^{p_1(\cdot)})$ to $L^{p_2(\cdot)}(w^{p_2(\cdot)})$.

## 3. Two Weighted Herz Spaces with Variable Exponents

Let $w$ be a non-negative measurable function and let $\{f_j\}_{j\in\mathbb{Z}}$ be a sequence of functions. The modulus is defined by

$$\rho_{\ell^{q(\cdot)}\left(L^{p(\cdot)}(\omega)\right)}\left(\{f_j\}_j\right) := \sum_{j\in\mathbb{Z}} \inf\left\{ \lambda_j : \int_{\mathbb{R}^n} \left( \frac{|f_j(x)\omega(x)|}{\lambda_j^{\frac{1}{q(x)}}} \right)^{p(x)} dx \le 1 \right\},$$

where $\lambda^{\frac{1}{\infty}} = 1$. If $q^+ < \infty$ or $q(\cdot) \le p(\cdot)$, the above can be written as

$$\rho_{\ell^{q(\cdot)}\left(L^{p(\cdot)}(\omega)\right)}\left(\{f_j\}_j\right) = \sum_{j\in\mathbb{Z}} \left\| f_j \omega^{q(\cdot)} \right\|_{\frac{p(\cdot)}{L^{q(\cdot)}}}.$$

The norm is

$$\left\|\{f_j\}_j\right\|_{\ell^{q(\cdot)}\left(L^{p(\cdot)}(\omega)\right)} := \inf\left\{ \mu > 0 : \rho_{\ell^{q(\cdot)}\left(L^{p(\cdot)}(\omega)\right)}\left( \left\{ \frac{f_j}{\mu} \right\}_j \right) \le 1 \right\}.$$

**Definition 3.1** ([15]) Let $w_1 \in A_p$, $w_2 \in A_{p(\cdot)}$, $p(\cdot), q(\cdot) \in \mathscr{P}_0(\mathbb{R}^n)$ and $\alpha(\cdot) \in L^\infty(\mathbb{R}^n)$.

The homogeneous two weight Herz space $\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(w_1, w_2)$ and non-homogeneous two weight Herz space $K_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(w_1, w_2)$ are defined respectively by

$$\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(w_1, w_2) := \left\{ f \in L_{\mathrm{loc}}^{p(\cdot)}\left(\mathbb{R}^n \setminus \{0\}, w_2\right) : \|f\|_{\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(w_1,w_2)} < \infty \right\},$$

and

$$K_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}\left(w_1,w_2\right):=\left\{f\in L_{\mathrm{loc}}^{p(\cdot)}\left(\mathbb{R}^n,w_2\right):\|f\|_{K_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(w_1,w_2)}<\infty\right\},$$

where

$$\|f\|_{\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(w_1,w_2)}:=\left\|\left\{w_1\left(B_k\right)^{\alpha(\cdot)/n}f\chi_k\right\}_{k\in\mathbb{Z}}\right\|_{\ell^{q(\cdot)}\left(L^{p(\cdot)}(w_2)\right)},$$

and

$$\|f\|_{K_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(w_1,w_2)}:=\left\|\left\{w_1\left(B_k\right)^{\alpha(\cdot)/n}f\tilde{\chi}_k\right\}_{k\ge 0}\right\|_{\ell^{q(\cdot)}\left(L^{p(\cdot)}(w_2)\right)}.$$

Regarding function spaces, we have the following Lemma:

**Lemma 3.1** ([15]) If $\alpha(\cdot)\in L^\infty\left(\mathbb{R}^n\right)$ and $\alpha(\cdot)\in\mathscr{P}^{\log}\left(\mathbb{R}^n\right)$, $\omega\in A_p$ for some $p\in[1,\infty)$ then for all $k\in\mathbb{Z}$ and $x\in D_k$,

$$w\left(B_k\right)^{\alpha(x)}\approx w\left(B_k\right)^{\alpha_\infty},\text{ if } k\ge 0,$$
$$w\left(B_k\right)^{\alpha(x)}\approx w\left(B_k\right)^{\alpha(0)},\text{ if } k\le -1.$$

**Lemma 3.2** ([15]) Let $\alpha(\cdot)\in L^\infty\left(\mathbb{R}^n\right)$, $p(\cdot),q(\cdot)\in\mathscr{P}_0\left(\mathbb{R}^n\right)$, $w_1\in A_p$ for some $p\in[1,\infty)$ and $w_2\in A_{p(\cdot)}$. If $q(\cdot),\alpha(\cdot)\in\mathscr{P}_\infty^{\log}\left(\mathbb{R}^n\right)$, then

$$K_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}\left(w_1,w_2\right)=K_{p(\cdot)}^{\alpha_\infty,q_\infty}\left(w_1,w_2\right).$$

Additionally, if $q(\cdot),\alpha(\cdot)\in\mathscr{P}^{\log}\left(\mathbb{R}^n\right)$, then

$$\|f\|_{\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(w_1,w_2)}\approx\left(\sum_{k\le 0}w_1\left(B_k\right)^{\alpha(0)q(0)/n}\|f\chi_k\|_{L^{p(\cdot)}(w_2)}^{q(0)}\right)^{\frac{1}{q(0)}}$$
$$+\left(\sum_{k>0}w_1\left(B_k\right)^{\alpha_\infty q_\infty/n}\|f\chi_k\|_{L^{p(\cdot)}(w_2)}^{q_\infty}\right)^{\frac{1}{q_\infty}}.$$

**Lemma 3.3** ([15]) Let $k,l\in\mathbb{Z}$, $w\in A_p$, where $p\in[1,\infty)$ and $\delta\in(0,1)$. $w^-:=\begin{cases}\delta & \alpha^-\ge 0\\ q & \alpha^-<0\end{cases}$, $w^+:=\begin{cases}q & \alpha^+\ge 0\\ \delta & \alpha^+<0\end{cases}$ If $\alpha(\cdot)\in L^\infty\left(\mathbb{R}^n\right)$ and $\alpha(\cdot)\in\mathscr{P}^{\log}\left(\mathbb{R}^n\right)$, then for any $x\in C_k$ and $y\in C_l$, we have

$$\left[w\left(B_k\right)\right]^{\alpha(x)}\lesssim\left[w\left(B_l\right)\right]^{\alpha(y)}\times\begin{cases}2^{(k-l)nw^+\alpha^+} & 0<2^l\le 2^{k-1}\\ 1 & 2^{k-1}<2^l\le 2^{k+1}\\ 2^{(k-l)mw^-\alpha^-} & 2^l>2^{k+1}\end{cases}$$

where the implicit constants are independent of $x,y,k$ and $l$.

**Lemma 3.4** ([21]) Let $X$ be a Banach function space. Suppose that the Hardy-Littlewood maximal operator $M$ is weakly bounded on $X$, that is,

$$\left\|\chi_{\{Mf>\lambda\}}\right\|_X\lesssim\lambda^{-1}\|f\|_X.$$

is true for all $f\in X$ and all $\lambda>0$. Then we have

$$\sup_{B:\text{ball}}\frac{1}{|B|}\|\chi_B\|_X\|\chi_B\|_{X'}<\infty.$$

The weighted Banach function space $X\left(\mathbb{R}^n, W\right)$ is a Banach function space equipped

$$\|f\|_{X\left(\mathbb{R}^n, W\right)} := \|fW\|_X .$$

The associate space of $X\left(\mathbb{R}^n, W\right)$ is a Banach function space and equals $X'\left(\mathbb{R}^n, W^{-1}\right)$. The properties above naturally arise from those of the usual Banach function spaces and the proof is found in [22].

If we take $X = L^{p(\cdot)}\left(\mathbb{R}^n\right)$ and $W = w$, then we have $L^{p(\cdot)}\left(\mathbb{R}^n, w\right) = L^{p(\cdot)}\left(w^{p(\cdot)}\right)$. If we take $X = L^{p'(\cdot)}\left(\mathbb{R}^n\right)$ and $W = w^{-1}$, then we have $L^{p'(\cdot)}\left(\mathbb{R}^n, w^{-1}\right) = L^{p'(\cdot)}\left(w^{-p'(\cdot)}\right)$.

$$\left(L^{p(\cdot)}\left(w^{p(\cdot)}\right)\right)' = \left(L^{p(\cdot)}\left(\mathbb{R}^n, w\right)\right)' = L^{p'(\cdot)}\left(\mathbb{R}^n, w^{-1}\right) = L^{p'(\cdot)}\left(w^{-p'(\cdot)}\right).$$

Thus we have:

**Lemma 3.5** ([15]) If $p(\cdot) \in \mathscr{P}^{\log}\left(\mathbb{R}^n\right) \cap \mathscr{P}\left(\mathbb{R}^n\right)$ and $\omega \in A_{p(\cdot)}$, then there exist constants $\delta_1, \delta_2 \in (0,1)$, such that for all balls $B$ in $\mathbb{R}^n$ and all measurable subsets $S \subset B$,

$$\frac{\|\chi_E\|_{L^{p(\cdot)}\left(w^{p(\cdot)}\right)}}{\|\chi_B\|_{L^{p(\cdot)}\left(w^{p(\cdot)}\right)}} = \frac{\|\chi_E\|_{\left(p^{p'(\cdot)}\left(w^{-p'(\cdot)}\right)\right)'}}{\|\chi_B\|_{\left(L^{p'(\cdot)}\left(w^{-p'(\cdot)}\right)\right)'}} \lesssim \left(\frac{|E|}{|B|}\right)^{\delta_1},$$

$$\frac{\|\chi_E\|_{\left(L^{p(\cdot)}\left(w^{p(\cdot)}\right)\right)'}}{\|\chi_B\|_{\left(L^{p(\cdot)}\left(w^{p(\cdot)}\right)\right)'}} \lesssim \left(\frac{|E|}{|B|}\right)^{\delta_2}.$$

**Lemma 3.6** ([23]) Let $0 < p < \infty, \delta > 0$. Then there is a positive constant $C$ such that

$$\left(\sum_{j=-\infty}^{\infty}\left(\sum_{k=-\infty}^{\infty} 2^{-|k-j|\delta} a_k\right)^p\right)^{1/p} \le C\left(\sum_{j=-\infty}^{\infty} a_j^p\right)^{1/p},$$

for non-negative sequences $\{a_j\}_{j=-\infty}^{\infty}$.

## 4. The Main Results

**Theorem 4.1** *Let* $0 < q_1(\cdot) < q_2(\cdot) < \infty$, $p(\cdot) \in \mathscr{P}^{\log}\left(\mathbb{R}^n\right) \cap \mathscr{P}\left(\mathbb{R}^n\right)$, $q(\cdot) \in \mathscr{P}^{\log}\left(\mathbb{R}^n\right) \cap \mathscr{P}_0\left(\mathbb{R}^n\right)$, $\alpha(\cdot) \in L^{\infty}\left(\mathbb{R}^n\right)$ *and* $\alpha(\cdot) \in \mathscr{P}^{\log}\left(\mathbb{R}^n\right)$, $v \in A_p$ *for some* $p \in [1, \infty)$, $w^{p_2(\cdot)} \in A_{(p_1(\cdot), p_2(\cdot))}$ *and* $\delta_1, \delta_2 \in (0,1)$. $0 < \beta < n(\delta_1 + \delta_2)$, $-n\delta_1 < w^- \alpha^- - n/r$ *and* $\beta + w^+ \alpha^+ + n/r < n\delta_2$. *Define* $p_1(\cdot)$ *by* $1/p_2(\cdot) \equiv 1/p_1(\cdot) - \beta/n$. *Then the fractional integral operator* $T_{\Omega,\beta}$ *is a bounded operator from* $\dot{K}_{p_2(\cdot)}^{\alpha(\cdot), q_2(\cdot)}\left(v, w^{p_2(\cdot)}\right)$ *to* $\dot{K}_{p_1(\cdot)}^{\alpha(\cdot), q_1(\cdot)}\left(v, w^{p_1(\cdot)}\right)$ *for all* $f \in \dot{K}_{p_2(\cdot)}^{\alpha(\cdot), q_2(\cdot)}\left(v, w^{p_2(\cdot)}\right)$.

**Proof of Theorem 4.1** Let $f \in \dot{K}_{p_2(\cdot)}^{\alpha(\cdot), q_2(\cdot)}\left(v, w^{p_2(\cdot)}\right)$. Then, by the Jensen inequality, we have

$$\begin{aligned}\left\|T_{\Omega,\beta} f\right\|_{\dot{K}_{p_2(\cdot)}^{\alpha(\cdot),q_2(\cdot)}\left(v,w^{p_2(\cdot)}\right)} &\approx \left(\sum_{k=-\infty}^{-1} v\left(B_k\right)^{\alpha(0)q_2(0)/n} \left\|\left(T_{\Omega,\beta} f\right)\chi_k\right\|_{L^{p_2(\cdot)}\left(w^{p_2(\cdot)}\right)}^{q_2(0)}\right)^{\frac{1}{q_2(0)}} \\ &\quad + \left(\sum_{k=0}^{\infty} v\left(B_k\right)^{\alpha_\infty q_{2\infty}/n} \left\|\left(T_{\Omega,\beta} f\right)\chi_k\right\|_{L^{p_2(\cdot)}\left(w^{p_2(\cdot)}\right)}^{q_{2\infty}}\right)^{\frac{1}{q_{2\infty}}} \\ &\lesssim \left(\sum_{k=-\infty}^{-1} v\left(B_k\right)^{\alpha(0)q_1(0)/n} \left\|\left(T_{\Omega,\beta} f\right)\chi_k\right\|_{L^{p_2(\cdot)}\left(w^{p_2(\cdot)}\right)}^{q_1(0)}\right)^{\frac{1}{q_1(0)}} \\ &\quad + \left(\sum_{k=0}^{\infty} v\left(B_k\right)^{\alpha_\infty q_{1\infty}/n} \left\|\left(T_{\Omega,\beta} f\right)\chi_k\right\|_{L^{p_2(\cdot)}\left(w^{p_2(\cdot)}\right)}^{q_{1\infty}}\right)^{\frac{1}{q_{1\infty}}} \\ &:= E + F.\end{aligned}$$

Let $f_j := f\chi_j$ for any $j \in \mathbb{Z}$. Then $f = \sum_{j=-\infty}^{\infty} f_j$. So we have By Lemma 3.3, we decompose $f_i$ into the following three parts as

$$f = \sum_{j=-\infty}^{k-2} f_j + \sum_{j=k-1}^{k+1} f_j + \sum_{j=k+2}^{\infty} f_j.$$

It is easy to see that

$$E \le C\sum_{i=1}^{3} E_i,\ F \le C\sum_{i=1}^{3} F_i.$$

where

$$E_1 := \left(\sum_{k=-\infty}^{-1} v\left(B_k\right)^{\alpha(0)q_1(0)/n} \left\|\sum_{j=-\infty}^{k-2}\left(T_{\Omega,\beta} f_j\right)\chi_k\right\|_{L^{p_2(\cdot)}\left(w^{p_2(\cdot)}\right)}^{q_1(0)}\right)^{\frac{1}{q_1(0)}},$$

$$E_2 := \left(\sum_{k=-\infty}^{-1} v\left(B_k\right)^{\alpha(0)q_1(0)/n} \left\|\sum_{j=k-1}^{k+1}\left(T_{\Omega,\beta} f_j\right)\chi_k\right\|_{L^{p_2(\cdot)}\left(w^{p_2(\cdot)}\right)}^{q_1(0)}\right)^{\frac{1}{q_1(0)}},$$

$$E_3 := \left(\sum_{k=-\infty}^{-1} v\left(B_k\right)^{\alpha(0)q_1(0)/n} \left\|\sum_{j=k+2}^{\infty}\left(T_{\Omega,\beta} f_j\right)\chi_k\right\|_{L^{p_2(\cdot)}\left(w^{p_2(\cdot)}\right)}^{q_1(0)}\right)^{\frac{1}{q_1(0)}},$$

$$F_1 := \left(\sum_{k=0}^{\infty} v\left(B_k\right)^{\alpha_\infty q_{1\infty}/n} \left\|\sum_{j=-\infty}^{k-2}\left(T_{\Omega,\beta} f_j\right)\chi_k\right\|_{L^{p_2(\cdot)}\left(w^{p_2(\cdot)}\right)}^{q_{1\infty}}\right)^{\frac{1}{q_{1\infty}}},$$

$$F_2 := \left(\sum_{k=0}^{\infty} v\left(B_k\right)^{\alpha_\infty q_{1\infty}/n} \left\|\sum_{j=k-1}^{k+1}\left(T_{\Omega,\beta} f_j\right)\chi_k\right\|_{L^{p_2(\cdot)}\left(w^{p_2(\cdot)}\right)}^{q_{1\infty}}\right)^{\frac{1}{q_{1\infty}}},$$

$$F_3 := \left(\sum_{k=0}^{\infty} v\left(B_k\right)^{\alpha_\infty q_{1\infty}/n} \left\|\sum_{j=k+2}^{\infty}\left(T_{\Omega,\beta} f_j\right)\chi_k\right\|_{L^{p_2(\cdot)}\left(w^{p_2(\cdot)}\right)}^{q_{1\infty}}\right)^{\frac{1}{q_{1\infty}}}.$$

We estimate $E_1$. Note that if $x \in B_k, y \in B_j$, and $j \le k-2$, then $|x-y| \approx |x| \approx 2^k$. By generalized Hölder inequality, we obtain

$$\left|\left(T_{\Omega,\beta} f_j\right)(x)\right| \chi_k(x) \lesssim 2^{k(\beta-n)} \chi_k(x) \|\Omega(x-y)\|_{L^r\left(\mathbb{R}^n\right)} \left\|f_j(y)\right\|_{L^{r'}\left(\mathbb{R}^n\right)}.$$

By virtue of the generalized Hölder's inequality, we have

$$\left\|f_j(y)\right\|_{L^{r'}} \lesssim \left|B_j\right|^{-(1/r)} \left\|f_j w\right\|_{L^{p_1(\cdot)}} \left\|w^{-1}\chi_j\right\|_{L^{p_1'(\cdot)}}$$

Then, for any $\Omega \in L^s\left(\mathbb{S}^{n-1}\right)$, we have the following inequality

$$\left|\left(T_{\Omega,\beta} f_j\right)(x)\right| \chi_k(x) \lesssim 2^{k(\beta-n)} 2^{(k-j)(n/r)} \chi_k(x) \left\|f_j\right\|_{L^{p_1(\cdot)}\left(w^{p_1(\cdot)}\right)} \left\|\chi_j\right\|_{\left(L^{p_1(\cdot)}\left(w^{p_1(\cdot)}\right)\right)'}$$

By taking the $L^{p_2(\cdot)}\left(w^{p_2(\cdot)}\right)$-norm and using Lemma 3.4, we have

$$\begin{aligned}
&\left\|\left(T_{\Omega,\beta} f_j\right)\chi_k\right\|_{L^{p_2(\cdot)}\left(w^{p_2(\cdot)}\right)} \\
&\lesssim 2^{k(\beta-n)} 2^{(k-j)(n/r)} \left\|f_j\right\|_{L^{p_1(\cdot)}\left(w^{p_1(\cdot)}\right)} \left\|\chi_j\right\|_{\left(L^{p_1(\cdot)}\left(w^{p_1(\cdot)}\right)\right)'} \left\|\chi_{B_k}\right\|_{L^{p_2(\cdot)}\left(w^{p_2(\cdot)}\right)} \\
&= 2^{k\beta} 2^{(k-j)(n/r)} \left\|f_j\right\|_{L^{p_1(\cdot)}\left(w^{p_1(\cdot)}\right)} \left\|\chi_j\right\|_{\left(L^{p_1(\cdot)}\left(w^{p_1(\cdot)}\right)\right)'} 2^{-kn} \left\|\chi_{B_k}\right\|_{L^{p_2(\cdot)}\left(w^{p_2(\cdot)}\right)} \\
&\lesssim 2^{k\beta} 2^{(k-j)(n/r)} \left\|f_j\right\|_{L^{p_1(\cdot)}\left(w^{p_1(\cdot)}\right)} \left\|\chi_j\right\|_{\left(L^{p_1(\cdot)}\left(w^{p_1(\cdot)}\right)\right)'} \left\|\chi_{B_k}\right\|^{-1}_{\left(L^{p_2(\cdot)}\left(w^{p_2(\cdot)}\right)\right)'}.
\end{aligned}$$

By Lemma 3.5, we see that

$$\begin{aligned}
&2^{k\beta} 2^{(k-j)(n/r)} \left\|\chi_j\right\|_{\left(L^{p_1(\cdot)}\left(w^{p_1(\cdot)}\right)\right)'} \left\|\chi_{B_k}\right\|^{-1}_{\left(L^{p_2(\cdot)}\left(w^{p_2(\cdot)}\right)\right)'} \\
&= 2^{k\beta} 2^{(k-j)(n/r)} \left\|\chi_j\right\|_{\left(L^{p_1(\cdot)}\left(w^{p_1(\cdot)}\right)\right)'} \left\|\chi_{B_k}\right\|^{-1}_{\left(L^{p_2(\cdot)}\left(w^{p_2(\cdot)}\right)\right)'} \frac{\left\|\chi_{B_j}\right\|_{\left(L^{p_2(\cdot)}\left(w^{p_2(\cdot)}\right)\right)'}}{\left\|\chi_{B_k}\right\|_{\left(L^{p_2(\cdot)}\left(w^{p_2(\cdot)}\right)\right)'}} \\
&\lesssim 2^{k\beta} 2^{n\delta_2(j-k)} 2^{(k-j)(n/r)} \left\|\chi_j\right\|_{\left(L^{p_1(\cdot)}\left(w^{p_1(\cdot)}\right)\right)'} \left\|\chi_{B_j}\right\|_{\left(L^{p_2(\cdot)}\left(w^{p_2(\cdot)}\right)\right)'}. \\
&\lesssim 2^{k\beta} 2^{(k-j)\left(\frac{n}{r}-n\delta_2\right)} \left\|\chi_j\right\|_{\left(L^{p_1(\cdot)}\left(w^{p_1(\cdot)}\right)\right)'} \left\|\chi_{B_j}\right\|_{\left(L^{p_2(\cdot)}\left(w^{p_2(\cdot)}\right)\right)'}.
\end{aligned}$$

By the obvious inequality $2^{j\beta} \chi_{B_j}(x) \lesssim \left(T_{\Omega,\beta} f_{B_j}\right)(x)$ and the boundedness of $T_{\Omega,\beta} : L^{p_1(\cdot)}\left(w^{p_1(\cdot)}\right) \to L^{p_2(\cdot)}\left(w^{p_2(\cdot)}\right)$, we have

$$\left\|\chi_{B_j}\right\|_{L^{p_2(\cdot)}\left(w^{p_2(\cdot)}\right)} \lesssim 2^{-j\beta} \left\|T_{\Omega,\beta}\chi_{B_j}\right\|_{L^{p_2(\cdot)}\left(w^{p_2(\cdot)}\right)} \lesssim 2^{-j\beta} \left\|\chi_{B_j}\right\|_{L^{p_1(\cdot)}\left(w^{p_1(\cdot)}\right)}.$$

By using Lemma 3.5 again, we obtain

$$\begin{aligned}
\left\|\chi_{B_j}\right\|_{L^{p_2(\cdot)}\left(w^{p_2(\cdot)}\right)} &\lesssim 2^{-j\beta} \left\|\chi_{B_j}\right\|_{L^{p_1(\cdot)}\left(w^{p_1(\cdot)}\right)} \lesssim 2^{j(n-\beta)} \left\|\chi_{B_j}\right\|^{-1}_{\left(L^{p_1(\cdot)}\left(w^{p_1(\cdot)}\right)\right)'} \\
&\lesssim 2^{j(n-\beta)} \left\|\chi_j\right\|^{-1}_{\left(L^{p_1(\cdot)}\left(w^{p_1(\cdot)}\right)\right)'}.
\end{aligned}$$

Thus we obtain

$$
\begin{aligned}
&2^{k\beta}2^{(k-j)\left(\frac{n}{r}-n\delta_2\right)}\left\|\chi_j\right\|_{\left(L^{p_1(\cdot)}\left(w^{p_1(\cdot)}\right)\right)'}\left\|\chi_{B_j}\right\|_{\left(L^{p_2(\cdot)}\left(w^{p_2(\cdot)}\right)\right)'}\\
&\lesssim 2^{k\beta}2^{(k-j)\left(\frac{n}{r}-n\delta_2\right)}2^{j(n-\beta)}\left\|\chi_{B_j}\right\|^{-1}_{L^{p_2(\cdot)}\left(w^{p_2(\cdot)}\right)}\left\|\chi_{B_j}\right\|^{-1}_{\left(L^{p_2(\cdot)}\left(w^{p_2(\cdot)}\right)\right)'}\\
&=2^{\left(\beta+\frac{n}{r}-n\delta_2\right)(k-j)}\left(2^{-jn}\left\|\chi_{B_j}\right\|_{L^{p_2(\cdot)}\left(w^{p_2(\cdot)}\right)}\left\|\chi_{B_j}\right\|_{\left(L^{p_2(\cdot)}\left(w^{p_2(\cdot)}\right)\right)'}\right)^{-1}\\
&\lesssim 2^{\left(\beta+\frac{n}{r}-n\delta_2\right)(k-j)}\left(\left\|\chi_{B_j}\right\|^{-1}_{\left(L^{p_2(\cdot)}\left(w^{p_2(\cdot)}\right)\right)'}\left\|\chi_{B_j}\right\|_{\left(L^{p_2(\cdot)}\left(w^{p_2(\cdot)}\right)\right)'}\right)^{-1}\\
&\lesssim 2^{\left(\beta+\frac{n}{r}-n\delta_2\right)(k-j)}.
\end{aligned}
$$

Thus we get

$$
\left\|\left(T_{\Omega,\beta}f_j\right)\chi_k\right\|_{L^{p_2(\cdot)}\left(w^{p_2(\cdot)}\right)}\lesssim 2^{\left(\beta+\frac{n}{r}-n\delta_2\right)(k-j)}\left\|f_j\right\|_{L^{p_1(\cdot)}\left(w^{p_1(\cdot)}\right)}
$$

By using Lemma 3.3 we obtain

$$
\begin{aligned}
E_1&\lesssim\left\{\sum_{k=-\infty}^{-1}v\left(B_k\right)^{\alpha(0)q_1(0)/n}\left(\sum_{j=-\infty}^{k-2}2^{\left(\beta+\frac{n}{r}-n\delta_2\right)(k-j)}\left\|f_j\right\|_{L^{p_1(\cdot)}\left(w^{p_1(\cdot)}\right)}\right)^{q_1(0)}\right\}^{\frac{1}{q_1(0)}}\\
&\lesssim\left\{\sum_{k=-\infty}^{-1}\left(\sum_{j=-\infty}^{k-2}2^{\left(\beta+\frac{n}{r}+w^+\alpha^+-n\delta_2\right)(k-j)}\left\|v\left(B_j\right)^{\alpha(\cdot)/n}f_j\right\|_{L^{p_1(\cdot)}\left(w^{p_1(\cdot)}\right)}\right)^{q_1(0)}\right\}^{\frac{1}{q_1(0)}}.
\end{aligned}
$$

where in the last inequality we use the fact that the sets

$$
v\left(B_k\right)^{\alpha(0)/n}\left\|f_j\right\|_{L^{p_1(\cdot)}\left(w^{p_1(\cdot)}\right)}\lesssim 2^{w^+\alpha^+(k-j)}\left\|v\left(B_j\right)^{\alpha(\cdot)/n}f_j\right\|_{L^{p_1(\cdot)}\left(w^{p_1(\cdot)}\right)}.
$$

Note that $\beta+\frac{n}{r}+w^+\alpha^+-n\delta_2<0$. We consider the two cases $1<q_1(0)<\infty$ and $0<q_1(0)\leq 1$. If $1<q_1(0)<\infty$, then by using the Hölder's inequality and Lemma 3.5, we obtain

$$
\begin{aligned}
E_1&\lesssim\left\{\sum_{k=-\infty}^{-1}\left(\sum_{j=-\infty}^{k-2}2^{\left(\beta+\frac{n}{r}+w^+\alpha^+-n\delta_2\right)(k-j)}\left\|v\left(B_j\right)^{\alpha(\cdot)/n}f_j\right\|_{L^{p_1(\cdot)}\left(w^{p_1(\cdot)}\right)}\right)^{q_1(0)}\right\}^{\frac{1}{q_1(0)}}\\
&\lesssim\left\{\sum_{k=-\infty}^{-1}\left(\sum_{j=-\infty}^{k-2}2^{\left(\beta+\frac{n}{r}+w^+\alpha^+-n\delta_2\right)(k-j)q_1(0)/2}\left\|v\left(B_j\right)^{\alpha(\cdot)/n}f_j\right\|^{q_1(0)}_{L^{p_1(\cdot)}\left(w^{p_1(\cdot)}\right)}\right)\right.\\
&\quad\left.\times\left(\sum_{j=-\infty}^{k-2}2^{\left(\beta+\frac{n}{r}+w^+\alpha^+-n\delta_2\right)(k-j)q_1(0)'/2}\right)^{q_1(0)/\left(q_1(0)\right)'}\right\}^{\frac{1}{q_1(0)}}\\
&\lesssim\left\{\sum_{k=-\infty}^{-1}\sum_{j=-\infty}^{k-2}2^{\left(\beta+\frac{n}{r}+w^+\alpha^+-n\delta_2\right)(k-j)q_1(0)/2}\left\|v\left(B_j\right)^{\alpha(\cdot)/n}f_j\right\|^{q_1(0)}_{L^{p_1(\cdot)}\left(w^{p_1(\cdot)}\right)}\right\}^{\frac{1}{q_1(0)}}
\end{aligned}
$$

$$
\begin{aligned}
&=\left\{\sum_{j=-\infty}^{-3}\left\|v\left(B_j\right)^{\alpha(\cdot)/n} f_j\right\|_{L^{p_1(\cdot)}\left(w^{p_1(\cdot)}\right)}^{q_1(0)}\sum_{k=-\infty}^{j+2}2^{\left(\beta+\frac{n}{r}+w^{+}\alpha^{+}-n\delta_2\right)(k-j)q_1(0)/2}\right\}^{\frac{1}{q_1(0)}}\\
&\lesssim\left\{\sum_{j=-\infty}^{-3}\left\|v\left(B_j\right)^{\alpha(\cdot)/n} f_j\right\|_{L^{p_1(\cdot)}\left(w^{p_1(\cdot)}\right)}^{q_1(0)}\right\}^{\frac{1}{q_1(0)}}\\
&\lesssim\|f\|_{\dot{K}_{p_1(\cdot)}^{\alpha(\cdot),q_1(\cdot)}\left(v,w^{p_1(\cdot)}\right)}.
\end{aligned}
$$

If $0<q_1(0)\le 1$, then by using the Jensen's inequality and Lemma 3.6, we obtain

$$
\begin{aligned}
E_1&\lesssim\left\{\sum_{k=-\infty}^{-1}\left(\sum_{j=-\infty}^{k-2}2^{\left(\beta+\frac{n}{r}+w^{+}\alpha^{+}-n\delta_2\right)(k-j)}\left\|v\left(B_j\right)^{\alpha(\cdot)/n} f_j\right\|_{L^{p_1(\cdot)}\left(w^{p_1(\cdot)}\right)}\right)^{q_1(0)}\right\}^{\frac{1}{q_1(0)}}\\
&\lesssim\left\{\sum_{k=-\infty}^{-1}\sum_{j=-\infty}^{k-2}2^{\left(\beta+\frac{n}{r}+w^{+}\alpha^{+}-n\delta_2\right)(k-j)q_1(0)}\left\|v\left(B_j\right)^{\alpha(\cdot)/n} f_j\right\|_{L^{p_1(\cdot)}\left(w^{p_1(\cdot)}\right)}^{q_1(0)}\right\}^{\frac{1}{q_1(0)}}\\
&\lesssim\left\{\sum_{j=-\infty}^{-3}\left\|v\left(B_j\right)^{\alpha(\cdot)/n} f_j\right\|_{L^{p_1(\cdot)}\left(w^{p_1(\cdot)}\right)}^{q_1(0)}\sum_{k=-\infty}^{j+2}2^{\left(\beta+\frac{n}{r}+w^{+}\alpha^{+}-n\delta_2\right)(k-j)q_1(0)}\right\}^{\frac{1}{q_1(0)}}\\
&\lesssim\left\{\sum_{j=-\infty}^{-3}\left\|v\left(B_j\right)^{\alpha(\cdot)/n} f_j\right\|_{L^{p_1(\cdot)}\left(w^{p_1(\cdot)}\right)}^{q_1(0)}\right\}^{\frac{1}{q_1(0)}}\\
&\lesssim\|f\|_{\dot{K}_{p_1(\cdot)}^{\alpha(\cdot),q_1(\cdot)}\left(v,w^{p_1(\cdot)}\right)}.
\end{aligned}
$$

Next we estimate $F_1$. By using the same argument as above, we see that

$$
\begin{aligned}
F_1&=\left(\sum_{k=0}^{\infty}v\left(B_k\right)^{\alpha_\infty q_{1\infty}/n}\left\|\sum_{j=-\infty}^{k-2}\left(T_{\Omega,\beta} f_j\right)\chi_k\right\|_{L^{p_2(\cdot)}\left(w^{p_2(\cdot)}\right)}^{q_{1\infty}}\right)^{\frac{1}{q_{1\infty}}}\\
&\lesssim\left\{\sum_{k=-\infty}^{-1}\left(\sum_{j=-\infty}^{k-2}2^{\left(\beta+\frac{n}{r}+w^{+}\alpha^{+}-n\delta_2\right)(k-j)}\left\|v\left(B_j\right)^{\alpha(\cdot)/n} f_j\right\|_{L^{p_1(\cdot)}\left(w^{p_1(\cdot)}\right)}\right)^{q_{1\infty}}\right\}^{\frac{1}{q_{1\infty}}}\\
&\lesssim\left\{\sum_{k=0}^{\infty}\left(\sum_{j=-\infty}^{-1}2^{\left(\beta+\frac{n}{r}+w^{+}\alpha^{+}-n\delta_2\right)(k-j)}\left\|v\left(B_j\right)^{\alpha(\cdot)/n} f_j\right\|_{L^{p_1(\cdot)}\left(w^{p_1(\cdot)}\right)}\right)^{q_{1\infty}}\right\}^{\frac{1}{q_{1\infty}}}\\
&\quad+\left\{\sum_{k=2}^{\infty}\left(\sum_{j=0}^{k-2}2^{\left(\beta+\frac{n}{r}+w^{+}\alpha^{+}-n\delta_2\right)(k-j)}\left\|v\left(B_j\right)^{\alpha(\cdot)/n} f_j\right\|_{L^{p_1(\cdot)}\left(w^{p_1(\cdot)}\right)}\right)^{q_{1\infty}}\right\}^{\frac{1}{q_{1\infty}}}\\
&:=I_1+I_2.
\end{aligned}
$$

By using Lemma 3.6 and note $\beta+\frac{n}{r}+w^{+}\alpha^{+}-n\delta_2<0$, we obtain

$$I_1=\left\{\sum_{k=0}^{\infty}\left(\sum_{j=-\infty}^{-1}2^{\left(\beta+\frac{n}{r}+w^{+}\alpha^{+}-n\delta_2\right)(k-j)}\left\|v\left(B_j\right)^{\alpha(\cdot)/n}\ f_j\right\|_{L^{p_1(\cdot)}\left(w^{p_1(\cdot)}\right)}\right)^{q_{1\infty}}\right\}^{\frac{1}{q_{1\infty}}}$$

$$\lesssim\|f\|_{\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(v,w)}\left\{\sum_{k=0}^{\infty}\left(\sum_{j=-\infty}^{-1}2^{\left(\beta+\frac{n}{r}+w^{+}\alpha^{+}-n\delta_2\right)(k-j)}\right)^{q_{1\infty}}\right\}^{\frac{1}{q_{1\infty}}}$$

$$\lesssim\|f\|_{\dot{K}_{p_1(\cdot)}^{\alpha(\cdot),q_1(\cdot)}\left(v,w^{p_1(\cdot)}\right)}.$$

By using the same argument as $E_1$, we have

$$I_2=\left\{\sum_{k=2}^{\infty}\left(\sum_{j=0}^{k-2}2^{\left(\beta+\frac{n}{r}+w^{+}\alpha^{+}-n\delta_2\right)(k-j)}\left\|v\left(B_j\right)^{\alpha(\cdot)/n}\ f_j\right\|_{L^{p_1(\cdot)}\left(w^{p_1(\cdot)}\right)}\right)^{q_{1\infty}}\right\}^{\frac{1}{q_{1\infty}}}$$

$$\lesssim\left\{\sum_{j=0}^{\infty}\left\|v\left(B_j\right)^{\alpha(\cdot)/n}\ f_j\right\|_{L^{p_1(\cdot)}\left(w^{p_1(\cdot)}\right)}^{q_{1\infty}}\right\}^{\frac{1}{q_{1\infty}}}$$

$$\lesssim\|f\|_{\dot{K}_{p_1(\cdot)}^{\alpha(\cdot),q_1(\cdot)}\left(v,w^{p_1(\cdot)}\right)}.$$

Therefore, we have $F_1\lesssim I_1+I_2\lesssim\|f\|_{\dot{K}_{p_1(\cdot)}^{\alpha(\cdot),q_1(\cdot)}\left(v,w^{p_1(\cdot)}\right)}$.

We estimate $E_3$. Note that if $x\in B_k, y\in B_j$, and $j\geq k+2$, then $|x-y|\approx|x|\approx 2^j$. Using the generalized Hölder inequality, we have,

$$\left|T_{\Omega,\beta}\left(f_j\right)\chi_k(x)\right|\lesssim 2^{j(\beta-n)}\chi_k(x)\left\|\Omega(x-y)\right\|_{L^r\left(\mathbb{R}^n\right)}\left\|f_j(y)\right\|_{L^{r'}\left(\mathbb{R}^n\right)}.$$

By virtue of the generalized Hölder's inequality, we have

$$\left\|f_j(y)\right\|_{L^{r'}}\lesssim\left|B_j\right|^{-(1/r)}\left\|f_j w\right\|_{L^{p_1(\cdot)}}\left\|w^{-1}\chi_j\right\|_{L^{p_1'(\cdot)}}.$$

Then, for any $\Omega\in L^s\left(\mathbb{S}^{n-1}\right)$, we have the following inequality

$$\left|\left(T_{\Omega,\beta}f_j\right)(x)\right|\chi_k(x)\lesssim 2^{j(\beta-n)}2^{(j-k)(n/r)}\chi_k(x)\left\|f_j\right\|_{L^{p_1(\cdot)}\left(w^{p_1(\cdot)}\right)}\left\|\chi_j\right\|_{\left(L^{p_1(\cdot)}\left(w^{p_1(\cdot)}\right)\right)'}$$

By taking the $L^{p_2(\cdot)}\left(w^{p_2(\cdot)}\right)$-norm, we have

$$\left\|\left(T_{\Omega,\beta}f_j\right)\chi_k\right\|_{L^{p_2(\cdot)}\left(w^{p_2(\cdot)}\right)}$$

$$\lesssim 2^{j(\beta-n)}2^{(j-k)(n/r)}\|f\|_{L^{p_1(\cdot)}\left(\omega^{p_1(\cdot)}\right)}\left\|\chi_j\right\|_{\left(L^{p_1(\cdot)}\left(w^{p_1(\cdot)}\right)\right)'}\left\|\chi_k\right\|_{L^{p_2(\cdot)}\left(w^{p_2(\cdot)}\right)}.$$

By Lemma 3.5, we see that

$$2^{j(\beta-n)}\left\|\chi_j\right\|_{L^{p_1'(\cdot)}\left(w^{-p_1'(\cdot)}\right)}\left\|\chi_k\right\|_{L^{p_2(\cdot)}\left(w^{p_2(\cdot)}\right)}$$

$$\lesssim 2^{j(\beta-n)}\left\|\chi_j\right\|_{L^{p_1'(\cdot)}\left(w^{-p_1'(\cdot)}\right)}\left\|\chi_j\right\|_{L^{p_2(\cdot)}\left(w^{p_2(\cdot)}\right)}\frac{\left\|\chi_k\right\|_{L^{p_2(\cdot)}\left(w^{p_2(\cdot)}\right)}}{\left\|\chi_j\right\|_{L^{p_2(\cdot)}\left(w^{p_2(\cdot)}\right)}}$$

$$\lesssim 2^{j(\beta-n)}2^{n\delta_1(k-j)}\left\|\chi_j\right\|_{L^{p_1'(\cdot)}\left(w^{-p_1'(\cdot)}\right)}\left\|\chi_j\right\|_{L^{p_2(\cdot)}\left(w^{p_2(\cdot)}\right)}.$$

By the definition 2.4 and using Lemma 3.4, we obtain

$$\begin{aligned}\left\|\chi_j\right\|_{L^{p_1'(\cdot)}\left(w^{-p_1'(\cdot)}\right)}\left\|\chi_j\right\|_{L^{p_2(\cdot)}\left(w^{p_2(\cdot)}\right)} &\lesssim \left\|\chi_{B_j}\right\|_{L^{p_1'(\cdot)}\left(w^{-p_1'(\cdot)}\right)}\left\|\chi_{B_j}\right\|_{L^{p_2(\cdot)}\left(w^{p_2(\cdot)}\right)} \\ &\lesssim \left\|w^{-1}\chi_{B_j}\right\|_{L^{p'(\cdot)}}\left\|w\chi_{B_j}\right\|_{L^{p_2(\cdot)}} \\ &\lesssim 2^{jn(1-\beta/n)}.\end{aligned}$$

Hence we have

$$2^{j(\beta-n)}\left\|\chi_j\right\|_{L^{p_1'(\cdot)}\left(w^{-p_1'(\cdot)}\right)}\left\|\chi_k\right\|_{L^{p_2(\cdot)}\left(w^{p_2(\cdot)}\right)} \lesssim 2^{n\delta_1(k-j)}.$$

Thus we get

$$\left\|\left(T_{\Omega,\beta}f_j\right)\chi_k\right\|_{L^{p_2(\cdot)}(w)} \lesssim 2^{(k-j)(n\delta_1-n/r)}\left\|f_jw\right\|_{L^{p_1(\cdot)}\left(\omega^{p_1(\cdot)}\right)}.$$

Therefore we see that

$$\begin{aligned}E_3 &\lesssim \left\{\sum_{k=-\infty}^{-1} v\left(B_k\right)^{\alpha(0)q(0)/n}\left(\sum_{j=k+2}^{\infty} 2^{(k-j)(n\delta_1-n/r)}\left\|f_j\right\|_{L^{p_1(\cdot)}\left(\omega^{p_1(\cdot)}\right)}\right)^{q_1(0)}\right\}^{\frac{1}{q_1(0)}} \\ &\lesssim \left\{\sum_{k=-\infty}^{-1}\left(\sum_{j=k+2}^{\infty} 2^{(k-j)\left(w^-\alpha^-+n\delta_1-n/r\right)}\left\|v\left(B_j\right)^{\alpha(\cdot)/n} f_j\right\|_{L^{p_1(\cdot)}\left(w^{p_1(\cdot)}\right)}\right)^{q_1(0)}\right\}^{\frac{1}{q_1(0)}}.\end{aligned}$$

Note that $w^-\alpha^-+n\delta_1-n/r>0$. We consider the two cases: $1<q_1(0)<\infty$ and $0<q_1(0)\le 1$.

If $1<q_1(0)<\infty$, then by using the Hölder inequality and Lemma 3.6, we obtain

$$\begin{aligned}E_3 &\lesssim \left\{\sum_{k=-\infty}^{-1}\left(\sum_{j=k+2}^{\infty} 2^{(k-j)\left(w^-\alpha^-+n\delta_1-n/r\right)}\left\|v\left(B_j\right)^{\alpha(\cdot)/n} f_j\right\|_{L^{p_1(\cdot)}\left(w^{p_1(\cdot)}\right)}\right)^{q_1(0)}\right\}^{\frac{1}{q_1(0)}} \\ &\lesssim \left\{\sum_{k=-\infty}^{-1}\left(\sum_{j=k+2}^{\infty} 2^{(k-j)\left(w^-\alpha^-+n\delta_1-n/r\right)q_1(0)/2}\left\|v\left(B_j\right)^{\alpha(\cdot)/n} f_j\right\|_{L^{p_1(\cdot)}\left(w^{p_1(\cdot)}\right)}^{q_1(0)}\right)\right. \\ &\qquad \left.\times\left(\sum_{j=k+2}^{\infty} 2^{(k-j)\left(w^-\alpha^-+n\delta_1-n/r\right)q_1(0)/2}\right)^{q_1(0)/(q_1(0))'}\right\}^{\frac{1}{q_1(0)}} \\ &\lesssim \left\{\sum_{j=-\infty}^{1}\left\|v\left(B_j\right)^{\alpha(\cdot)/n} f_j\right\|_{L^{p_1(\cdot)}\left(w^{p_1(\cdot)}\right)}^{q_1(0)}\sum_{k=-\infty}^{j-2} 2^{(k-j)\left(w^-\alpha^-+n\delta_1-n/r\right)q_1(0)}\right\}^{\frac{1}{q_1(0)}} \\ &\lesssim \|f\|_{\dot{K}_{p_1(\cdot)}^{\alpha(\cdot),q_1(\cdot)}\left(v,w^{p_1(\cdot)}\right)}.\end{aligned}$$

If $0<q_1(0)\le 1$, then by using the Jensen's inequality and Lemma 3.6, we obtain

$$E_3 \lesssim \left\{ \sum_{k=-\infty}^{-1} \left( \sum_{j=k+2}^{\infty} 2^{(k-j)\left(w^-\alpha^- + n\delta_1 - n/r\right)} \left\| v\left(B_j\right)^{\alpha(\cdot)/n} f_j \right\|_{L^{p_1(\cdot)}\left(w^{p_1(\cdot)}\right)} \right)^{q_1(0)} \right\}^{\frac{1}{q_1(0)}}$$

$$\lesssim \left\{ \sum_{j=-\infty}^{-3} \left\| v\left(B_j\right)^{\alpha(\cdot)/n} f_j \right\|_{L^{p_1(\cdot)}\left(w^{p_1(\cdot)}\right)}^{q_1(0)} \sum_{k=j+2}^{\infty} 2^{(k-j)\left(w^-\alpha^- + n\delta_1 - n/r\right)q_1(0)} \right\}^{\frac{1}{q_1(0)}}$$

$$\lesssim \|f\|_{\dot{K}_{p_1(\cdot)}^{\alpha(\cdot),q_1(\cdot)}\left(v,w^{p_1(\cdot)}\right)}.$$

Next we estimate $F_3$. By using the same argument as above, we see that

$$F_3 = \left( \sum_{k=0}^{\infty} v\left(B_k\right)^{\alpha_\infty q_{1\infty}/n} \left\| \sum_{j=k+2}^{\infty} \left(T_{\Omega,\beta} f_j\right) \chi_k \right\|_{L^{p_2(\cdot)}\left(w^{p_2(\cdot)}\right)}^{q_{1\infty}} \right)^{\frac{1}{q_{1\infty}}}$$

$$\lesssim \left\{ \sum_{k=0}^{\infty} \left( \sum_{j=k+2}^{\infty} 2^{(k-j)\left(w^-\alpha^- + n\delta_1 - n/r\right)} \left\| v\left(B_j\right)^{\alpha(\cdot)/n} f_j \right\|_{L^{p_1(\cdot)}\left(w^{p_1(\cdot)}\right)} \right)^{q_{1\infty}} \right\}^{\frac{1}{q_{1\infty}}}$$

$$\lesssim \left( \sum_{j=0}^{\infty} \left\| v\left(B_j\right)^{\alpha_2(\cdot)/n} f_j \right\|_{L^{p_1(\cdot)}\left(w^{p_1(\cdot)}\right)}^{q_{1\infty}} \right)^{\frac{1}{q_{1\infty}}}$$

$$\lesssim \|f\|_{\dot{K}_{p_1(\cdot)}^{\alpha(\cdot),q_1(\cdot)}\left(v,w^{p_1(\cdot)}\right)}.$$

Last we estimate $E_2$. It is easy to see that

$$v\left(B_k\right)^{\alpha(0)/n} \left\| \sum_{j=k-1}^{k+1} \left(T_{\Omega,\beta} f_j\right) \chi_k \right\|_{L^{p_2(\cdot)}\left(w^{p_2(\cdot)}\right)}$$

$$\lesssim 2^{\left(\beta + w^+\alpha^+ + n/r - n\delta_2\right)} \left\| v\left(B_{k-1}\right)^{\alpha(\cdot)/n} f\chi_{k-1} \right\|_{L^{p_1(\cdot)}\left(w^{p_1(\cdot)}\right)} + \left\| v\left(B_k\right)^{\alpha(\cdot)/n} f\chi_k \right\|_{L^{p_1(\cdot)}\left(w^{p_1(\cdot)}\right)}$$

$$+ 2^{-\left(w^-\alpha^- + n\delta_1 - n/r\right)} \left\| v\left(B_{k+1}\right)^{\alpha(\cdot)/n} f\chi_{k+1} \right\|_{L^{p_1(\cdot)}\left(w^{p_1(\cdot)}\right)}$$

$$\lesssim \sum_{j=k-1}^{k+1} 2^{(j-k)n} \left\| v\left(B_j\right)^{\alpha(\cdot)/n} f_j \right\|_{L^{p_1(\cdot)}\left(w^{p_1(\cdot)}\right)}.$$

By above equation.

$$E_2 \lesssim \left\{ \sum_{k=-\infty}^{-1} \left( \sum_{j=k-1}^{k+1} 2^{(j-k)n} \left\| v\left(B_j\right)^{\alpha(\cdot)/n} f_j \right\|_{L^{p_1(\cdot)}\left(w^{p_1(\cdot)}\right)} \right)^{q_1(0)} \right\}^{\frac{1}{q_1(0)}}$$

$$\lesssim \left\{ \sum_{k=-\infty}^{-1} \left\| v\left(B_k\right)^{\alpha(\cdot)/n} f\chi_k \right\|_{L^{p_1(\cdot)}\left(w^{p_1(\cdot)}\right)}^{q_1(0)} \right\}^{\frac{1}{q_1(0)}}$$

$$\lesssim \|f\|_{\dot{K}_{p_1(\cdot)}^{\alpha(\cdot),q_1(\cdot)}\left(v,w^{p_1(\cdot)}\right)}.$$

Now we estimate $F_2$. By using the same argument as above, we see that

$$
\begin{aligned}
F_2 &= \left( \sum_{k=0}^{\infty} v\left(B_k\right)^{\alpha_\infty q_{1\infty}/n} \left\| \sum_{j=k-1}^{k+1} \left(T_{\Omega,\beta} f_j\right) \chi_k \right\|_{L^{p_2(\cdot)}\left(w^{p_2(\cdot)}\right)}^{q_{1\infty}} \right)^{\frac{1}{q_{1\infty}}} \\
&\lesssim \left\{ \sum_{k=0}^{\infty} \left( \sum_{j=k-1}^{k+1} 2^{(j-k)n} \left\| v\left(B_j\right)^{\alpha(\cdot)/n} f_j \right\|_{L^{p_1(\cdot)}\left(w^{p_1(\cdot)}\right)} \right)^{q_{1\infty}} \right\}^{\frac{1}{q_{1\infty}}} \\
&\lesssim \left\{ \sum_{k=0}^{\infty} \left\| v\left(B_k\right)^{\alpha(\cdot)/n} f \chi_k \right\|_{L^{p_1(\cdot)}\left(w^{p_1(\cdot)}\right)}^{q_{1\infty}} \right\}^{\frac{1}{q_{1\infty}}} \\
&\lesssim \|f\|_{\dot{K}_{p_1(\cdot)}^{\alpha(\cdot),q_1(\cdot)}\left(v,w^{p_1(\cdot)}\right)}.
\end{aligned}
$$

Now, by putting the estimated values of $E_i$ and $F_i$ together, we can obtain

$$
\left\|T_{\Omega,\beta} f\right\|_{\dot{K}_{p_2(\cdot)}^{\alpha(\cdot),q_2(\cdot)}\left(v,w^{p_2(\cdot)}\right)} := E + F \lesssim \|f\|_{\dot{K}_{p_1(\cdot)}^{\alpha(\cdot),q_1(\cdot)}\left(v,w^{p_1(\cdot)}\right)}.
$$

This completes the proof of the boundedness of the fractional integral operator on the two weighted Herz spaces with variable exponents.

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## Conflicts of Interest

The authors have no relevant financial or non-financial interests to disclose.

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