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The Existence and Stability of Standing Waves for the Inhomogeneous Nonlinear Schrödinger Equation with Magnetic Potential

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Abstract

In this paper, we study the Cauchy problem for the three dimensional inhomogeneous nonlinear Schrödinger equation (INLS) in the presence of a constant magnetic field. We establish the existence and orbital stability of normalized standing waves. Our approach introduces a novel methodology that circumvents the widely utilized concentration-compactness principle.

Keywords

Schrödinger Equation, Magnetic Field, Existence, Orbital Stability

1. Introduction

This paper is concerned with the Cauchy problem for inhomogeneous nonlinear Schrödinger equations with a constant magnetic field in three dimensions

$$\begin{cases} i\partial_t\psi + (\nabla + i\boldsymbol{A})^2\psi = -|x|^{-b}|\psi|^p\psi, & (t,x)\in\mathbb{R}^+\times\mathbb{R}^3, \\ \psi|_{t=0} = \psi_0, \end{cases} \tag{1.1}$$

where $\boldsymbol{A}$ is a vector-valued potential of the form

$$\boldsymbol{A}(x) = \frac{a}{2}(-x_2, x_1, 0), \quad x = (x_1, x_2, x_3)\in\mathbb{R}^3 \tag{1.2}$$

modeling the effect of an external magnetic field

$$\operatorname{curl}(\boldsymbol{A}) = (0,0,a), \quad a \neq 0. \tag{1.3}$$

As a fundamental equation of quantum mechanics, the time-independent Schrödinger equation Schrödinger [1] successfully describes the quantum evolution of particles without external fields or interparticle interactions. Practical systems are often driven by external sources, and homogeneous equations fail to describe the evolutionary behavior under such external driving. Therefore, it is necessary to investigate the inhomogeneous nonlinear Schrödinger equation with a magnetic potential (see [2]-[5] and references therein). The main objective of this paper is to study the existence and stability of solutions to the inhomogeneous nonlinear Schrödinger equation with a constant magnetic potential (1.1). Moreover, the Schrödinger equation with a uniform magnetic field serves as an effective model for describing the behavior of a single non-relativistic quantum particle subjected to an electromagnetic field (see, for example, [6] [7]). Avron, Herbst, and Simon [8]-[10] conducted a thorough mathematical analysis of the linear Schrödinger operator in the presence of a constant magnetic field.

Furthermore, there are conservation laws of mass and energy, namely

$$M\left(\psi(t)\right)=\left\|\psi(t)\right\|_{L^2}^2=M\left(\psi_0\right), \tag{Mass}$$

and

$$E\left(\psi(t)\right)=\frac{1}{2}\left\|(\nabla+iA)\psi(t)\right\|_{L^2}^2-\frac{1}{p+2}\int_{\mathbb{R}^3}|x|^{-b}|\psi|^{p+2}\,\mathrm{d}x=E\left(\psi_0\right), \tag{Energy}$$

for all $t\in\left[0,T^*\right)$. Moreover, we have the following useful identity (see e.g., [11], Lemma 2.2)

$$\left\|(\nabla+iA)u\right\|_{L^2}^2=\left\|\nabla u\right\|_{L^2}^2+aR(u)+\frac{a^2}{4}\left\|\rho u\right\|_{L^2}^2, \tag{1.4}$$

where $\rho:=\sqrt{x_1^2+x_2^2}$ and

$$R(u):=i\int\left(x_2\partial_{x_1}u-x_1\partial_{x_2}u\right)\bar{u}=\int L_z u\bar{u}\mathrm{d}x, \tag{1.5}$$

where $L_z:=i\left(x_2\partial_{x_1}-x_1\partial_{x_2}\right)$. Next, we focus on the existence and stability of standing waves with a specified mass for Equation (1.1). By standing waves, we mean solutions to (1.1) of the form $\psi(t,x)=\mathrm{e}^{i\omega t}\phi(x)$, where $\omega\in\mathbb{R}$ and ϕ is a solution to

$$-(\nabla+iA)^2\phi+\omega\phi-|x|^{-b}|\phi|^p\phi=0. \tag{1.6}$$

This study recalls the research on Cauchy solutions for the nonlinear Schrödinger equation. When $b>0$, a compact embedding exists (see Lemma 2.3 [12]), which aids in establishing the compactness of Palais-Smale sequences and the existence of solutions. However, the inclusion of the $|x|^{-b}$ term for $b>0$ complicates the analysis of the symmetry and decay properties of solutions, necessitating different approaches and careful treatment. It is important to emphasize that the parameter $b>0$ plays a crucial and significant role throughout the discussion.

The presence of standing waves for Equation (1.1) can be established by minimizing the energy functional $E(u)$ subject to the mass constraint

$$J(m):=\left\{u\in H_A^1\left(\mathbb{R}^3\right):M(u)=m\right\},$$

where $m>0$. Specifically, we focus on the minimization problem

$$I(m):=\inf\left\{E(u):u\in J(m)\right\}.$$

Theorem 1.1 [13] *Let* $0 < p < \frac{4-2b}{3}$. *For any* $m > 0$, *there exists a minimizer for* $I(m)$.

Theorem 1.2 [14] *The set*

$$\mathcal{M}(m) := \{\phi \in J(m) : E(\phi) = I(m)\}$$

is orbitally stable with respect to the flow of Equation (1.1), meaning that for any $\varepsilon > 0$, there exists $\delta > 0$ such that for any initial condition $u_0 \in H_A^1(\mathbb{R}^3)$ satisfying

$$\inf_{\phi \in \mathcal{M}(c)} \|\psi_0 - \phi\|_{H_A^1} \le \delta,$$

the corresponding solution to (1.1) exists globally in time and satisfies

$$\inf_{\phi \in \mathcal{M}(c)} \inf_{y \in \mathbb{R}^3} \left\| e^{iA(y)\cdot x} \psi(t, \cdot + y) - \phi \right\|_{H_A^1} \le \varepsilon, \ \forall t \ge 0.$$

Remark **1.1** Lee and Seo [15] established the existence of solutions under critical conditions by constructing a complete metric space using the contraction mapping principle and applying the fixed point theorem to obtain local solutions. However, this approach, which circumvents compactness issues by selecting a sufficiently small time interval T, is unable to analyze the long-term energy behavior of solutions or address the lack of compactness. Consequently, it cannot directly resolve problems related to non-compactness.

In contrast, we employ the concentration compactness lemma along with suitable scaling parameters to prove compactness by ruling out vanishing—demonstrated by showing that every minimizing sequence of I(m) has an L^{p+2}-norm bounded away from zero (see Lemma 3.1)—and dichotomy.

2. Preliminaries

In this section, let us recall some fundamental properties of the magnetic Sobolev space $H_A^1(\mathbb{R}^3)$ and provide several lemmas to support the subsequent proofs.

Lemma 2.1 [14] *Suppose* $A \in L_{loc}^2(\mathbb{R}^3, \mathbb{R}^3)$. *Then* $H_A^1(\mathbb{R}^3)$ *equipped with the inner product*

$$\langle u, v \rangle_{H_A^1} := \int_{\mathbb{R}^3} u\bar{v}\,dx + \int_{\mathbb{R}^3} (\nabla + iA)u \cdot \overline{(\nabla + iA)v}\,dx$$

is a Hilbert space.

Lemma 2.2 (Gagliardo-Nirenberg inequality [12]) *Let* $0 < p < 4-2b$, *and* $0 < b < 2$, *then the Gagliardo-Nirenberg inequality*

$$\int_{\mathbb{R}^3} |x|^{-b} |\psi(x)|^{p+2}\,dx \le K_{opt} \|\nabla \psi\|_{L^2(\mathbb{R}^3)}^{\frac{3p}{2}+b} \|\psi\|_{L^2(\mathbb{R}^3)}^{2-\frac{p}{2}-b} \tag{2.1}$$

holds, and the sharp constant $K_{opt} > 0$ *is explicitly given by*

$$K_{opt} = \left(\frac{3p+2b}{4-2b-p}\right)^{\frac{4-3p-2b}{4}} \frac{2p+4}{(3p+2b)\|Q\|_{L^2(\mathbb{R}^3)}^{p}},$$

where Q *is the unique positive radial solution to*

$$\Delta Q - Q + |x|^{-b} |Q|^p Q = 0. \tag{2.2}$$

Moreover, the solution Q *satisfies the following relations*

$$\|\nabla Q\|_{L^2(\mathbb{R}^3)} = \left(\frac{3p+2b}{4-p-2b}\right)^{\frac{1}{2}} \|Q\|_{L^2(\mathbb{R}^3)}, \tag{2.3}$$

and

$$\int_{\mathbb{R}^3} |x|^{-b} |Q(x)|^{p+2} \mathrm{d}x = \left(\frac{2p+4}{4-p-2b}\right) \|Q\|^2_{L^2(\mathbb{R}^3)}. \tag{2.4}$$

Lemma 2.3 (Diamagnetic inequality [16]) *Let* $A \in L^2_{loc}(\mathbb{R}^3, \mathbb{R}^3)$ *and* $u \in H^1_A(\mathbb{R}^3)$. *Then* $|u| \in H^1(\mathbb{R}^3)$. *In particular, we have*

$$|\nabla |u|(x)| \le |(\nabla + iA) u(x)| \quad \text{a.e.} \quad x \in \mathbb{R}^3. \tag{2.5}$$

Lemma 2.4 [14] Let $A \in L^2_{loc}(\mathbb{R}^3, \mathbb{R}^3)$. Then the following properties hold:

1) $C_0^\infty(\mathbb{R}^3)$ *is dense in* $H^1_A(\mathbb{R}^3)$.

2) $H^1_A(\mathbb{R}^3)$ *is continuously embedded in* $L^k(\mathbb{R}^3)$ *for all* $2 \le k \le 6-2b$.

3) *Assume that* A *is linear, i.e.,* $A(x+y) = A(x) + A(y)$ *for all* $x, y \in \mathbb{R}^3$. *Let* $y \in \mathbb{R}^3$, $u \in H^1_A(\mathbb{R}^3)$, *and set*

$$\tilde{u}(x) := \mathrm{e}^{iA(y)\cdot x} u(x+y), \quad x \in \mathbb{R}^3.$$

Then $(\nabla + iA)\tilde{u}(x) = \mathrm{e}^{iA(y)\cdot x} (\nabla + iA) u(x+y)$. *In particular,*

$$\|(\nabla + iA)\tilde{u}\|_{L^2} = \|(\nabla + iA) u\|_{L^2}.$$

4) *If* $A \in L^3_{loc}(\mathbb{R}^3, \mathbb{R}^3)$, *then* $H^1_A(\mathbb{R}^3)$ *is continuously embedded in* $H^1_{loc}(\mathbb{R}^3)$. *In particular,* $H^1_A(\mathbb{R}^3)$ *is compactly embedded in* $L^k_{loc}(\mathbb{R}^3)$ *for all* $2 \le k < 6-2b$.

Lemma 2.5 *Let* $A \in W^{1,\infty}_{loc}(\mathbb{R}^3, \mathbb{R}^3)$ *and* $j, k \in \{1,2,3\}$. *Then for any* $u \in C_0^\infty(\mathbb{R}^3)$, *we have*

$$\left|\int (\partial_j A_k - \partial_k A_j) u\bar{u} \mathrm{d}x\right| \le \|(\partial_j + iA_j) u\|^2_{L^2} + \|(\partial_k + iA_k) u\|^2_{L^2}.$$

In particular, if A *is as in* (1.2), *then*

$$|a| \|u\|^2_{L^2} \le \|(\nabla + iA) u\|^2_{L^2}. \tag{2.6}$$

Lemma 2.6 *Let* $A \in L^2_{loc}(\mathbb{R}^3, \mathbb{R}^3)$ *and* $(u_n)_{n\ge1}$ *be a bounded sequence in* $H^1_A(\mathbb{R}^3)$. *Assume that* $u_n \rightharpoonup u$ *weakly in* $H^1_A(\mathbb{R}^3)$. *Then we have*

$$\|(\nabla + iA) u_n\|^2_{L^2} = \|(\nabla + iA) u\|^2_{L^2} + \|(\nabla + iA)(u_n - u)\|^2_{L^2} + o_n(1),$$

$$\int_{\mathbb{R}^3} |x|^{-b} |u_n|^k \mathrm{d}x = \int_{\mathbb{R}^3} |x|^{-b} |u|^k \mathrm{d}x + \int_{\mathbb{R}^3} |x|^{-b} |u_n - u|^k \mathrm{d}x + o_n(1), \quad 2 \le k \le 6-2b.$$

Proof As $H^1_A(\mathbb{R}^3)$ is continuously embedded in $L^r(\mathbb{R}^3)$ for all $2 \le r \le 6-2b$. The second identity follows directly from the refined Fatou's lemma established by Brézis and Lieb [17]. Now, we will demonstrate the first identity. Set $v_n := u_n - u$. We see that $v_n \rightharpoonup 0$ weakly in $H^1_A(\mathbb{R}^3)$. We compute

$$\begin{aligned}\|(\nabla+iA)u_n\|_{L^2}^2 &= \|(\nabla+iA)(u+v_n)\|_{L^2}^2\\ &= \|(\nabla+iA)u\|_{L^2}^2+\|(\nabla+iA)v_n\|_{L^2}^2\\ &\quad+2\operatorname{Re}\int(\nabla+iA)u\cdot\overline{(\nabla+iA)v_n}\,\mathrm{d}x.\end{aligned}$$

Let $\varepsilon>0$. Since $C_0^\infty(\mathbb{R}^3)$ is dense in $H_A^1(\mathbb{R}^3)$, we take $\phi\in C_0^\infty(\mathbb{R}^3)$ so that $\|(\nabla+iA)(u-\phi)\|_{L^2}<\varepsilon/(2K)$, where $K:=\sup_{n\ge1}\|v_n\|_{H_A^1}<\infty$. Since $v_n\rightharpoonup 0$ weakly in $H_A^1(\mathbb{R}^3)$, we see that

$$\left|\int(\nabla+iA)\phi\cdot\overline{(\nabla+iA)v_n}\,\mathrm{d}x\right|\to0\ \text{ as } n\to\infty.$$

Thus there exists $n_0\in\mathbb{N}$ such that for $n\ge n_0$,

$$\begin{aligned}&\left|\int(\nabla+iA)u\cdot\overline{(\nabla+iA)v_n}\,\mathrm{d}x\right|\\ &\le\left|\int(\nabla+iA)(u-\phi)\cdot\overline{(\nabla+iA)v_n}\,\mathrm{d}x\right|+\left|\int(\nabla+iA)\phi\cdot\overline{(\nabla+iA)v_n}\,\mathrm{d}x\right|\\ &\le\|(\nabla+iA)(u-\phi)\|_{L^2}\|(\nabla+iA)v_n\|_{L^2}+\varepsilon/2\\ &<\varepsilon.\end{aligned}$$

The proof is complete. □

Lemma 2.7 *Let $A\in L^3_{loc}(\mathbb{R}^3,\mathbb{R}^3)$ be linear. Let $(u_n)_{n\ge1}$ be a bounded sequence in $H_A^1(\mathbb{R}^3)$, i.e., $\sup_{n\ge1}\|u_n\|_{H_A^1}<\infty$. Assume that there exists $\varepsilon_0>0$ such that*

$$\inf_{n\ge1}\|u_n\|_{L^k}\ge\varepsilon_0 \tag{2.7}$$

for some $2<k<6-2b$. Then up to a subsequence, there exist $u\in H_A^1(\mathbb{R}^3)\setminus\{0\}$ and $\{y_n\}_{n\ge1}\subset\mathbb{R}^3$ such that

$$\mathrm{e}^{iA(y_n)\cdot x}u_n(x+y_n)\rightharpoonup u\quad\text{weakly in } H^1(\mathbb{R}^3).$$

3. Existence and Stability of Normalized Standing Waves

In this section, we prove the existence and orbital stability of normalized standing waves related to (1.1). To prove it, we present the relevant arguments in this section. A prerequisite for this proof is the following result, which is essential to rule out the possibility of vanishing.

Lemma 3.1 Let A be as in (1.2) and $0<p<\frac{4-2b}{3}$. Let $m>0$ and $(u_n)_{n\ge1}$ be a minimizing sequence for $I(m)$. Then there exists $M>0$ such that

$$\liminf_{n\to\infty}\|u_n\|_{L^{p+2}}\ge K>0.$$

Proof Assume by contradiction that there exists a subsequence still denoted by $(u_n)_{n\ge1}$ satisfying $\lim_{n\to\infty}\|u_n\|_{L^{p+2}}=0$. As a result of Hölder's inequality, we then know that

$$\begin{aligned}\int_{\mathbb{R}^3}|x|^{-b}|u_n|^{p+2}\,\mathrm{d}x&=\int_{B_R(0)}|x|^{-b}|u_n|^{p+2}\,\mathrm{d}x+\int_{\mathbb{R}^3\setminus B_R(0)}|x|^{-b}|u_n|^{p+2}\,\mathrm{d}x\\ &\le\left(\int_{B_R(0)}|x|^{-\frac{br}{r-1}}\,\mathrm{d}x\right)^{\frac{r-1}{r}}\left(\int_{B_R(0)}|u_n|^{(p+2)r}\,\mathrm{d}x\right)^{\frac1r}\\ &\quad+R^{-b}\int_{\mathbb{R}^3\setminus B_R(0)}|u_n|^{p+2}\,\mathrm{d}x,\end{aligned}$$

where $R>0$ is a constant. Note that

$$\left(\int_{B_R(0)}|x|^{-\frac{br}{r-1}}\mathrm{d}x\right)^{\frac{r-1}{r}}<\infty,$$

then we have $\lim_{n\to\infty}\int_{\mathbb{R}^3}|x|^{-b}|u_n|^{p+2}\mathrm{d}x=0$. Thanks to (2.6), we see that

$$I(m)=\lim_{n\to\infty}E(u_n)=\lim_{n\to\infty}\frac{1}{2}\|(\nabla+iA)u_n\|_{L^2}^2\geq\lim_{n\to\infty}\frac{|a|}{2}\|u_n\|_{L^2}^2=\frac{|a|m}{2}. \tag{3.1}$$

Denote $x=(x_\perp,x_3)$ with $x_\perp=(x_1,x_2)\in\mathbb{R}^2$ and $x_3\in\mathbb{R}$, and set $v(x_\perp):=\sqrt{\frac{|a|}{2\pi}}\mathrm{e}^{-\frac{|a|}{4}|x_\perp|^2}$. One can readily check that

$$\|v\|_{L^2(\mathbb{R}^2)}=1,\ \|\nabla_\perp v\|_{L^2(\mathbb{R}^2)}^2+\frac{a^2}{4}\|\rho v\|_{L^2(\mathbb{R}^2)}^2=|a|.$$

Let $w\in C_0^\infty(\mathbb{R})$ be such that $\|w\|_{L^2(\mathbb{R})}^2=m$ and set

$$u_\lambda(x)=v(x_\perp)w_\lambda(x_3),\ w_\lambda(x_3)=\lambda^{\frac{1}{2}}w(\lambda x_3) \tag{3.2}$$

with $\lambda>0$ to be chosen later. We have $\|u_\lambda\|_{L^2}^2=c$ for all $\lambda>0$. Using (1.4), we see that

$$\begin{aligned}\|(\nabla+iA)u_\lambda\|_{L^2}^2&=\|\nabla u_\lambda\|_{L^2}^2+b\mathbf{R}(u_\lambda)+\frac{a^2}{4}\|\rho u_\lambda\|_{L^2}^2\\&=\|\nabla_\perp v\|_{L^2(\mathbb{R}^2)}^2\|w_\lambda\|_{L^2(\mathbb{R})}^2+\|v\|_{L^2(\mathbb{R}^2)}^2\|\partial_3 w_\lambda\|_{L^2(\mathbb{R})}^2\\&\quad+a\left(\int_{\mathbb{R}^2}L_z v\bar{v}\mathrm{d}x_\perp\right)\|w_\lambda\|_{L^2(\mathbb{R})}^2+\frac{a^2}{4}\|\rho v\|_{L^2(\mathbb{R}^2)}^2\|w_\lambda\|_{L^2(\mathbb{R})}^2\\&=m\left(\|\nabla_\perp v\|_{L^2(\mathbb{R}^2)}^2+\frac{a^2}{4}\|\rho v\|_{L^2(\mathbb{R}^2)}^2\right)+\lambda^2\|\partial_3 w\|_{L^2(\mathbb{R})}^2\\&=m|a|+\lambda^2\|\partial_3 w\|_{L^2(\mathbb{R})}^2,\end{aligned}$$

where

$$\int_{\mathbb{R}^2}L_z v\bar{v}\mathrm{d}x_\perp=-i\int_{\mathbb{R}^2}x_2 v\partial_1\bar{v}\mathrm{d}x_1\mathrm{d}x_2+i\int_{\mathbb{R}^2}x_2 v\partial_2\bar{v}\mathrm{d}x_1\mathrm{d}x_2,$$

as v is radially symmetric, we have

$$\partial_1\bar{v}=v'(x_\perp)\cdot\frac{x_1}{x_\perp}$$

$$\partial_2\bar{v}=v'(x_\perp)\cdot\frac{x_2}{x_\perp},$$

then we get $L_z v\bar{v}\mathrm{d}x_\perp=0$. It follows that

$$E(u_\lambda)=\frac{|a|m}{2}+\frac{\lambda^2}{2}\|\partial_3 w\|_{L^2(\mathbb{R})}^2-C\lambda^{\frac{p}{2}+b}\Phi(x).$$

where C is a constant, Φ is a function of x. As $p<4-2b$, by taking $\lambda>0$ sufficiently small, we have $E(f_\lambda)<\frac{|a|m}{2}$. In particular, $I(m)<\frac{|a|m}{2}$ which contradicts (3.1). Then the proof is complete. □

***Proof of Theorem* 1.1** We first show that $I(m)$ is well-defined, *i.e.*, $I(m) > -\infty$. Let $u \in J(m)$ and $0 < \varepsilon < \frac{1}{2}$, by the Young's inequality and Gagliardo-Nirenberg inequality (2.1), we have

$$E(u) \geq \frac{1}{2}\|(\nabla + iA)u\|_{L^2}^2 - M(\varepsilon)\|(\nabla + iA)u\|_{L^2}^2 - C(\varepsilon)\|u\|_{L^2}^{\frac{16-4p-2b}{4-3p-2b}}$$
$$= \left(\frac{1}{2} - M(\varepsilon)\right)\|(\nabla + iA)u\|_{L^2}^2 - C > -C$$

for all $u \in J(m)$, where $0 < C < \infty$. This shows that $I(m) > -\infty$. Now let $(u_n)_{n\geq 1}$ be a minimizing sequence for $I(m)$. From the above estimate, we have

$$\left(\frac{1}{2} - M(\varepsilon)\right)\|(\nabla + iA)u_n\|_{L^2}^2 \leq E(u_n) \to I(m) \text{ as } n \to \infty.$$

This shows that $(u_n)_{n\geq 1}$ is a bounded sequence in $H_A^1(\mathbb{R}^3)$.

Moreover, by Lemma 3.1, we see that up to a subsequence,

$$\inf_{n\geq 1}\|u_n\|_{L^{\frac{10-2b}{3}}} \geq K > 0.$$

By Lemma 2.7, up to a subsequence, there exist $u \in H_A^1(\mathbb{R}^3)\setminus\{0\}$ and $(y_n)_{n\geq 1} \subset \mathbb{R}^3$ such that

$$\tilde{u}_n(x) := \mathrm{e}^{iA(y_n)\cdot x}u_n(x + y_n) \rightharpoonup u \text{ weakly in } H_A^1(\mathbb{R}^3).$$

By the weak convergence in $H_A^1(\mathbb{R}^3)$, we have

$$0 < \|u\|_{L^2}^2 \leq \liminf_{n\to\infty}\|\tilde{u}_n\|_{L^2}^2 = \liminf_{n\to\infty}\|u_n\|_{L^2}^2 = m$$

and

$$\|(\nabla + iA)u\|_{L^2}^2 \leq \liminf_{n\to\infty}\|(\nabla + iA)\tilde{u}_n\|_{L^2}^2 = \liminf_{n\to\infty}\|(\nabla + iA)u_n\|_{L^2}^2.$$

Next we claim that

$$\|u\|_{L^2}^2 = m. \tag{3.3}$$

Let's delay verifying (3.2) for now and complete the proof of Theorem 1.1. By the weak convergence in $H_A^1(\mathbb{R}^3)$ and (3.2), we infer that $\tilde{u}_n \to f$ strongly in $L^2(\mathbb{R}^3)$. Using this strong convergence and the magnetic Gagliardo-Nirenberg inequality

$$\int_{\mathbb{R}^3}|x|^{-b}|u|^{p+2}\,\mathrm{d}x \leq K_{opt}\|(\nabla + iA)u\|_{L^2(\mathbb{R}^3)}^{\frac{3p}{2}+b}\|u\|_{L^2(\mathbb{R}^3)}^{2-\frac{p}{2}-b},$$

we see that $\lim_{n\to\infty}\int_{\mathbb{R}^3}|x|^{-b}|\tilde{u}_n|^{p+2}\,\mathrm{d}x = \int_{\mathbb{R}^3}|x|^{-b}|u|^{p+2}\,\mathrm{d}x$. Thus we get

$$I(m) \leq E(u) \leq \liminf_{n\to\infty}E(\tilde{u}_n) = \liminf_{n\to\infty}E(u_n) = I(m),$$

hence $E(u) = I(m)$ or u is a minimizer for $I(m)$. This also implies that $\tilde{u}_n \to f$ strongly in $H_A^1(\mathbb{R}^3)$.

It remains to prove (3.2). Assume by contradiction that it is not true, *i.e.*, $0<\|u\|_{L^2}^2<m$. We have for any $\lambda>0$,

$$E(\lambda u)=\lambda^2 E(u)+\frac{\lambda^2(1-\lambda^p)}{p+2}\int_{\mathbb{R}^3}|x|^{-b}|u|^{p+2}\,\mathrm{d}x$$

or

$$E(f)=\frac{1}{\lambda^2}E(\lambda f)+\frac{\lambda^p-1}{p+2}\int_{\mathbb{R}^3}|x|^{-b}|u|^{p+2}\,\mathrm{d}x.$$

Set $\lambda_0=\frac{\sqrt{m}}{\|u\|_{L^2}}>1$. We have $\|\lambda_0 u\|_{L^2}^2=m$ and

$$E(u)=\frac{\|u\|_{L^2}^2}{m}E(\lambda_0 u)+\frac{\lambda_0^p-1}{p+2}\int_{\mathbb{R}^3}|x|^{-b}|u|^{p+2}\,\mathrm{d}x>\frac{\|u\|_{L^2}^2}{m}I(m)$$

as $u\neq 0$ and $\lambda_0>1$. Similarly, set $\lambda_n:=\frac{\sqrt{m}}{\|\tilde{u}_n-u\|_{L^2}}$. By Lemma 2.6, we have $\|\tilde{u}_n-u\|_{L^2}^2\to m-\|u\|_{L^2}^2$ as $n\to\infty$, hence $\lambda_n\to\frac{\sqrt{m}}{\sqrt{m-\|u\|_{L^2}^2}}>1$ as $n\to\infty$. In particular, we have

$$\lim_{n\to\infty}E(\tilde{u}_n-u)=\lim_{n\to\infty}\left(\frac{1}{\lambda_n^2}E(\lambda_n(\tilde{u}_n-u))+\frac{\lambda_n^\alpha-1}{\alpha+2}\|\tilde{u}_n-u\|_{L^{\alpha+2}}^{\alpha+2}\right)\geq\frac{m-\|u\|_{L^2}^2}{m}I(m).$$

Using the refined Fatou's lemma (see Lemma 2.6), we get

$$\begin{aligned}I(m)&=\lim_{n\to\infty}E(u_n)=\lim_{n\to\infty}E(\tilde{u}_n)=E(u)+\lim_{n\to\infty}E(\tilde{u}_n-u)\\&>\frac{\|u\|_{L^2}^2}{m}I(m)+\frac{m-\|u\|_{L^2}^2}{m}I(m)=I(m),\end{aligned}$$

which is a contradiction. The proof is completed. □

***Proof of Theorem* 1.2** Let us now demonstrate that the set of minimizers $\mathcal{M}(m)$ is orbitally stable as described in Theorem 1.1. We will use an argument from [13]. Assume by contradiction that it is not true. Then there exist $\varepsilon_0>0$, $\phi_0\in\mathcal{M}(m)$, and a sequence of initial data $(\psi_{0,n})_{n\geq1}\subset H_A^1(\mathbb{R}^3)$ such that

$$\lim_{n\to\infty}\|\psi_{0,n}-\phi_0\|_{H_A^1}=0 \tag{3.4}$$

and a sequence of time $(t_n)_{n\geq1}\subset[0,\infty)$ such that

$$\inf_{\phi\in\mathcal{M}(m)}\inf_{y\in\mathbb{R}^3}\left\|\mathrm{e}^{iA(y)\cdot}\psi_n(t_n,\cdot+y)-\phi\right\|_{H_A^1}\geq\varepsilon_0, \tag{3.5}$$

where ψ_n is the solution to (1.1) with initial data $\psi_n|_{t=0}=\psi_{0,n}$. Since $\phi_0\in\mathcal{M}(m)$, we have $E(\phi_0)=I(m)$. From (3.3) and the Sobolev embedding, we infer that

$$\|u_{0,n}\|_{L^2}^2\to\|\phi_0\|_{L^2}^2=m,\ E(u_{0,n})\to E(\phi_0)=I(m)\ \text{as } n\to\infty.$$

By the conservation laws of mass and energy, we have

$$\left\|\psi_n(t_n)\right\|_{L^2}^2 \to m,\ E\left(\psi_n(t_n)\right) \to I(m) \text{ as } n \to \infty.$$

In particular, $\left(\psi_n(t_n)\right)_{n\geq 1}$ is a minimizing sequence for $I(m)$.As argued in Step 1, we see that up to a subsequence, there exist $\phi \in \mathcal{M}(m)$ and $(y_n)_{n\geq 1} \subset \mathbb{R}^3$ such that

$$\left\|e^{iA(y_n)\cdot}\psi_n(t_n,\cdot+y_n)-\phi\right\|_{H_A^1} \to 0 \text{ as } n \to \infty.$$

However, this contradicts (3.4). Thus, the proof is finished. □

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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