

# Regularity Theory for Elliptic Equations with Anisotropic Diffusion

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## Abstract

For a class of strongly competitive elliptic systems with anisotropic diffusion posed on an open set  $\Omega \subset \mathbb{R}^N$  ( $N \geq 3$ ), we establish regularity estimates for weak solutions by a contradiction argument, via the construction of blow-up sequences and the use of monotonicity formulas together with Liouville-type theorems.

## Keywords

Anisotropic Diffusion, Blow-Up, Strongly Competitive Elliptic Systems, Regularity of Weak Solutions

## 1. Introduction

Systems of elliptic equations with strong interspecific competition arise naturally in several physical and biological models, such as multi-component Gross-Pitaevskii and nonlinear Schrödinger equations. The interaction between different components is typically governed by a real parameter  $\beta$ , whose sign and magnitude describe the nature and intensity of the competition. For instance, in [1], Noris, Tavares, Terracini, and Verzini (2010) established uniform Hölder estimates for the following strongly competitive nonlinear Schrödinger system:

$$\begin{cases} -\Delta u_\beta + \lambda_\beta u_\beta = \omega_1 u_\beta^3 - \beta u_\beta v_\beta^2, \\ -\Delta v_\beta + \mu_\beta v_\beta = \omega_2 v_\beta^3 - \beta u_\beta^2 v_\beta, \\ u_\beta, v_\beta \in H_0^1(\Omega). \end{cases}$$

In this setting, the strong competition regime corresponds to the limit  $\beta \rightarrow +\infty$  (note that the interaction term carries a negative sign), in which the interaction term strongly penalizes the coexistence of different components. In this limit, solutions are expected to exhibit phase separation phenomena, giving rise

to segregated limiting profiles whose supports are essentially disjoint. They showed that uniform  $L^\infty$  bounds for  $u_\beta$  and  $v_\beta$  imply uniform  $C^{0,\alpha}$  bounds for every  $\alpha \in (0,1)$ , and their proof is based on a blow-up analysis combined with the Alt-Caffarelli-Friedman monotonicity formula. Up to now, strongly competing models with isotropic diffusion have been extensively studied by many authors [2]-[4]. However, extending these results to the case of anisotropic operators entails significant additional difficulties, mainly due to the lack of rotational invariance and the more complex underlying geometry. Anisotropic diffusion refers to diffusion processes whose rates depend on direction, in [5], Terracini and Soave established an anisotropic monotonicity formula and further investigated anisotropic two-phase problems of the form

$$\operatorname{div}(A_1 \nabla u) \geq 0, \operatorname{div}(A_2 \nabla v) \geq 0,$$

where  $A_1$  and  $A_2$  are symmetric positive definite  $N \times N$  matrices with constant coefficients. Moreover, by means of a suitable change of variables, one may assume without loss of generality that  $A_1$  is diagonal and  $A_2$  is the identity matrix. This is also the system studied in the present paper.

In this paper, we study a genuinely anisotropic two-phase problem

$$\begin{cases} -\operatorname{div}(A \nabla u_{1,k}) = u_{1,k}(1-u_{1,k}) - k u_{1,k}^\gamma u_{2,k}^{\gamma+1} & \text{in } \Omega \subset \mathbb{R}^N, \\ -\Delta u_{2,k} = u_{2,k}(1-u_{2,k}) - k u_{2,k}^\gamma u_{1,k}^{\gamma+1} & \text{in } \Omega \subset \mathbb{R}^N, \\ u_{1,k} = u_{2,k} = 0 & \text{on } \partial\Omega, \end{cases} \quad (1.1)$$

where  $\Omega \subset \mathbb{R}^N$  is an open set with  $N \geq 3$ ,  $k > 1$  is the competition parameter. We assume that  $A$  is a positive definite  $N \times N$  diagonal matrix with constant coefficients, with the lowest eigenvalue 1:  $A := \operatorname{diag}(a_1, a_2, \dots, a_N)$  with  $1 = a_1 \leq \dots \leq a_N$ . The exponent  $\gamma > 1$  describes higher-order competitive interactions. The aim of this paper is to prove the uniform boundedness of positive solutions to the above family of equations in Hölder norms, via the construction of blow-up sequences and the use of monotonicity formulas together with Liouville-type theorems.

## 2. Preliminaries

To prove the regularity of solutions to the system we study, it is crucial to identify an appropriate monotonicity formula. Accordingly, we proceed to consider the positive solutions  $v_1, v_2$  of the following coupled system, which do not have disjoint supports and belong to  $H^1_{loc}(\mathbb{R}^N) \cap C(\mathbb{R}^N)$ , where the system is given by

$$\begin{cases} -\operatorname{div}(A \nabla v_1) = -v_1^\gamma v_2^{\gamma+1} & \text{in } \mathbb{R}^N, \\ -\Delta v_2 = -v_1^{\gamma+1} v_2^\gamma & \text{in } \mathbb{R}^N. \end{cases} \quad (2.1)$$

In order to obtain a Liouville-type result, we need to employ a suitable monotonicity formula. For  $N \geq 3$  and  $\delta > 0$ , we introduce a  $C^1$  auxiliary function

$$\Phi_\delta(x) = \begin{cases} \frac{N}{2} \delta^{2-N} + \frac{2-N}{2} \delta^{-N} |x|^2 & \text{if } |x| \leq \delta, \\ |x|^{2-N} & \text{if } |x| > \delta. \end{cases}$$

Moreover, we define

$$m_A(x) = -\operatorname{div}(A\nabla\Phi_{A,\delta}(x))/2, \quad m(x) = -\Delta\Phi_{Id,\delta}(x)/2,$$

where  $\Phi_{A,\delta}(x) = \Phi_\delta(A^{-1/2}x)$ ,  $\Phi_{Id,\delta}(x) = \Phi_\delta(x)$ . Then we can observe that  $m_A(x)$  and  $m(x)$  are both bounded on  $\mathbb{R}^N$ , vanish in  $\mathbb{R}^N \setminus B_\delta$  and are nonnegative for almost every  $x$ . And since  $\Phi_\delta$  coincides with the fundamental solution of the Laplacian away from the origin,  $\Phi_{A,\delta}$  agrees in the set  $\mathcal{E}_\delta^c = \{|A^{-1/2}x| \geq \delta\}$  with the fundamental solution associated with the operator  $\operatorname{div}(A\nabla)$ . We denote this fundamental solution by  $\Gamma_A$ , which admits the explicit representation

$$\Gamma_A(x) := \left( \sum_{i=1}^N \frac{x_i^2}{a_i} \right)^{\frac{2-N}{2}}.$$

In particular,  $\Phi_{A,\delta} = \Gamma_A$  in the set  $\mathcal{E}_\delta^c$ .

The Alt-Caffarelli-Friedman (ACF) monotonicity formula plays a fundamental role in the analysis of two-phase and multi-phase free boundary problems. Its original version was introduced in [6]. In the following, we develop an ACF-type monotonicity formula adapted to the system (2.1) under consideration.

**Theorem 2.1 (Monotonicity formula)** *Let  $v_1, v_2$  be positive solutions of (2.1) and let  $\varepsilon > 0$  be fixed. Then there exists an exponent  $\nu_{A,N} \in (0, 2)$  depending on  $A$  and  $N$ , and  $\bar{r} = \bar{r}(v_1, v_2, \varepsilon) > \delta$  such that the function*

$$J(r) := \frac{1}{r^{2(\nu_{A,N}-\varepsilon)}} \int_{B_r(x_0)} \left[ \Gamma_A(x-x_0) (\langle A\nabla v_1, \nabla v_1 \rangle + v_1^{\gamma+1} v_2^{\gamma+1}) + m_A(x) v_1^2 \right] \\ \cdot \int_{B_r(x_0)} \left[ |x-x_0|^{2-N} (|\nabla v_2|^2 + v_1^{\gamma+1} v_2^{\gamma+1}) + m(x) v_2^2 \right]$$

is increasing for  $r \in (\bar{r}, +\infty)$ .

**Proof** Our proof is based on Lemma 2.5 in [1]. Without loss of generality, we consider  $x_0 = 0$ . We introduce

$$\Lambda_1(0, r) = \frac{r^2 \int_{\partial B_r} (\langle A\nabla_\theta^A v_1, \nabla_\theta^A v_1 \rangle + v_1^{\gamma+1} v_2^{\gamma+1}) d\sigma}{\int_{\partial B_r} v_1^2 \mu d\sigma},$$

$$\Lambda_2(0, r) = \frac{r^2 \int_{\partial B_r} (|\nabla_\theta v_2|^2 + v_1^{\gamma+1} v_2^{\gamma+1}) d\sigma}{\int_{\partial B_r} v_2^2 d\sigma},$$

where  $\langle A\nabla_\theta^A v_1, \nabla_\theta^A v_1 \rangle = \langle A\nabla v_1, \nabla v_1 \rangle - \frac{\langle A\nabla v_1, \nu \rangle^2}{\mu(x)}$ ,  $|\nabla_\theta v_2|^2 = |\nabla v_2|^2 - |\partial_\nu v_2|^2$ .

Then, we test the equations for  $v_1$  and  $v_2$  in  $B_r$  with  $v_1\Phi_{A,\delta}$  and  $v_2\Phi_{Id,\delta}$ , respectively. By integration by parts, the definition of  $m_A(x)$  and Young's inequality, we obtain the estimate

$$J(r) \leq \frac{a_N^{N/2} r^2}{4r^{2(\nu_{A,N}-\varepsilon)} \gamma(\Lambda_1(0, r)) \gamma(\Lambda_2(0, r))} \int_{\partial B_r} (\langle A\nabla v_1, \nabla v_1 \rangle + v_1^{\gamma+1} v_2^{\gamma+1}) \Gamma_A \\ \cdot \int_{\partial B_r} (|\nabla v_2|^2 + v_1^{\gamma+1} v_2^{\gamma+1}) |x|^{2-N},$$

where  $\gamma(\Lambda_i(0, r)) = \sqrt{\left(\frac{N-2}{2}\right)^2 + \Lambda_i(0, r)} - \frac{N-2}{2}, i = 1, 2$ . Then, we can compute the derivative of  $J(r)$  for  $r > \delta$ . Using the above estimate, we obtain that

$$\frac{J'(r)}{J(r)} \geq -\frac{2(v_{A,N} - \varepsilon)}{r} + \frac{2\gamma(\Lambda_1(0, r))}{a_N^{N/2}r} + \frac{2\gamma(\Lambda_2(0, r))}{r}.$$

We proceed by contradiction, assuming that there exists a sequence  $r_n \rightarrow +\infty$  such that

$$a_N^{-N/2}\gamma(\Lambda_1(0, r_n)) + \gamma(\Lambda_2(0, r_n)) < v_{A,N} - \varepsilon.$$

To study the behavior as  $r_n \rightarrow +\infty$ , we perform a suitable normalized rescaling of the solution. By energy boundedness, the rescaled functions are bounded in  $H^1(\partial B_1(0))$ . Then by compactness, we can extract a subsequence that converges weakly to a limiting pair of functions. The crucial step lies in exploiting the competitive term in the system, which allows one to show that the limiting profiles have disjoint supports. This property reduces the problem to the classical Alt-Cafarelli-Friedman framework. Finally, we can obtain that

$$v_{A,N} \leq \liminf_{n \rightarrow \infty} (a_N^{-N/2}\gamma(\Lambda_1(0, r_n)) + \gamma(\Lambda_2(0, r_n))) \leq v_{A,N} - \varepsilon,$$

which contradicts the assumption and thus yields the monotonicity. □

Thus, having obtained a suitable monotonicity formula, we can now prove a Liouville-type result for the system under consideration.

**Theorem 2.2 (Liouville-type theorem)** *Let  $v_1, v_2$  be non negative solutions of (2.1). Suppose that the functions  $v_1, v_2$  grow at most like  $|x|^{\alpha_i}$ , namely  $|v_1(x)| \leq C(1 + |x|^{\alpha_1}), |v_2(x)| \leq C(1 + |x|^{\alpha_2})$  for every  $|x| \in \mathbb{R}^N$ , for some  $C > 0$ , with*

$$\alpha_1 > 0, \alpha_2 > 0, \alpha_1 + \alpha_2 < v_{A,N}.$$

*Then one of the functions is identically zero and the other is a constant.*

**Proof** Owing to the structure of system (2.1), if one of the functions is 0 or a positive constant, then the other must be a constant or 0 respectively. By contradiction we assume that neither  $v_1$  nor  $v_2$  is constant. For simplicity of notation, we write  $B_r = B_r(0)$ . Then by the maximum principle  $v_1$  and  $v_2$  are positive, and by Theorem 2.1 we know that there exists  $C > 0$  such that

$$\int_{B_r} \left[ \Gamma_A(x) (\langle A \nabla v_1, \nabla v_1 \rangle + v_1^{\gamma+1} v_2^{\gamma+1}) + m_A(x) v_1^2 \right] \cdot \int_{B_r} \left[ |x|^{2-N} (|\nabla v_2|^2 + v_1^{\gamma+1} v_2^{\gamma+1}) + m(x) v_2^2 \right] \geq Cr^{2(v_{A,N} - \varepsilon)} \tag{2.2}$$

for  $r$  sufficiently large. Consider a radial smooth cut-off function  $\eta$  such that  $0 \leq \eta \leq 1, \eta = 1$  in  $B_r(0), \eta = 0$  in  $\mathbb{R}^N \setminus B_{2r}(0)$ , and  $|\nabla \eta| \leq C/r$ . By testing the equation for  $v_1$  with  $\eta^2 \Phi_{A,\delta}(x) v_1$  on  $B_{2r}$ , we have

$$\int_{B_{2r}} -\operatorname{div}(A \nabla v_1) \eta^2 \Phi_{A,\delta}(x) v_1 = - \int_{B_{2r}} v_1^{\gamma+1} v_2^{\gamma+1} \eta^2 \Phi_{A,\delta}(x),$$

then integrating by parts further, we obtain

$$\begin{aligned} & \int_{B_{2r}} -\operatorname{div}(A\nabla v_1) \eta^2 \Phi_{A,\delta}(x) v_1 \\ &= \int_{B_{2r}} \left\langle A\nabla v_1, \nabla \left( \eta^2 \Phi_{A,\delta}(x) v_1 \right) \right\rangle = - \int_{B_{2r}} v_1^{\gamma+1} v_2^{\gamma+1} \eta^2 \Phi_{A,\delta}(x). \end{aligned} \quad (2.3)$$

Expanding the right-hand side further, we obtain

$$\begin{aligned} & \int_{B_{2r}} \left\langle A\nabla v_1, \nabla \left( \eta^2 \Phi_{A,\delta}(x) v_1 \right) \right\rangle \\ &= \int_{B_{2r}} \eta^2 \Phi_{A,\delta}(x) \left\langle A\nabla v_1, \nabla v_1 \right\rangle + 2\eta \Phi_{A,\delta}(x) v_1 \left\langle A\nabla v_1, \nabla \eta \right\rangle \\ & \quad + \left\langle A\nabla \left( \frac{v_1^2}{2} \right), \nabla \Phi_{A,\delta}(x) \right\rangle \eta^2. \end{aligned}$$

Substituting the above expression into (2.3),

$$\begin{aligned} & \int_{B_{2r}} \eta^2 \Phi_{A,\delta}(x) \left( \left\langle A\nabla v_1, \nabla v_1 \right\rangle + v_1^{\gamma+1} v_2^{\gamma+1} \right) \\ &= - \int_{B_{2r}} \left( 2\eta \Phi_{A,\delta}(x) v_1 \left\langle A\nabla v_1, \nabla \eta \right\rangle + \left\langle A\nabla \left( \frac{v_1^2}{2} \right), \nabla \Phi_{A,\delta}(x) \right\rangle \eta^2 \right). \end{aligned}$$

By using the Cauchy-Schwarz and Young's inequalities, we have

$$\begin{aligned} & \int_{B_{2r}} \left| -2\eta \Phi_{A,\delta}(x) v_1 \left\langle A\nabla v_1, \nabla \eta \right\rangle \right| \\ & \leq \int_{B_{2r}} \frac{1}{2} \eta^2 \Phi_{A,\delta}(x) \left\langle A\nabla v_1, \nabla v_1 \right\rangle + 2\Phi_{A,\delta}(x) v_1^2 \left\langle A\nabla \eta, \nabla \eta \right\rangle. \end{aligned}$$

Therefore we can obtain that

$$\begin{aligned} & \int_{B_{2r}} \eta^2 \Phi_{A,\delta}(x) \left( \left\langle A\nabla v_1, \nabla v_1 \right\rangle + v_1^{\gamma+1} v_2^{\gamma+1} \right) \\ & \leq \int_{B_{2r}} \frac{1}{2} \eta^2 \Phi_{A,\delta}(x) \left\langle A\nabla v_1, \nabla v_1 \right\rangle + 2\Phi_{A,\delta} v_1^2 \left\langle A\nabla \eta, \nabla \eta \right\rangle \\ & \quad - \left\langle A\nabla \left( \frac{v_1^2 \eta^2}{2} \right), \nabla \Phi_{A,\delta}(x) \right\rangle + \left\langle A\nabla \left( \frac{\eta^2}{2} \right), \nabla \Phi_{A,\delta}(x) \right\rangle v_1^2. \end{aligned}$$

Recalling that  $\operatorname{div}(A\nabla \Phi_{A,\delta}(x)) = -2m_A(x)$  and testing it with  $\eta^2 v_1^2/2$  in  $B_{2r}$ , we can obtain

$$\begin{aligned} & \int_{B_{2r}} \operatorname{div}(A\nabla \Phi_{A,\delta}(x)) \frac{\eta^2 v_1^2}{2} \\ &= - \int_{B_{2r}} \left\langle A\nabla \Phi_{A,\delta}(x), \nabla \left( \frac{\eta^2 v_1^2}{2} \right) \right\rangle = - \int_{B_{2r}} m_A(x) \eta^2 v_1^2. \end{aligned}$$

Plugging this into the above gives

$$\begin{aligned} & \int_{B_{2r}} \eta^2 \left[ \Phi_{A,\delta}(x) \left( \left\langle A\nabla v_1, \nabla v_1 \right\rangle + v_1^{\gamma+1} v_2^{\gamma+1} \right) + m_A(x) v_1^2 \right] \\ & \leq \int_{B_{2r}} \frac{1}{2} \eta^2 \Phi_{A,\delta}(x) \left\langle A\nabla v_1, \nabla v_1 \right\rangle + 2\Phi_{A,\delta}(x) v_1^2 \left\langle A\nabla \eta, \nabla \eta \right\rangle \\ & \quad + \left\langle A\nabla \left( \frac{\eta^2}{2} \right), \nabla \Phi_{A,\delta}(x) \right\rangle v_1^2 \\ & \leq 2 \int_{B_{2r}} \left[ 2\Phi_{A,\delta}(x) v_1^2 \left\langle A\nabla \eta, \nabla \eta \right\rangle + \left\langle A\nabla \left( \frac{\eta^2}{2} \right), \nabla \Phi_{A,\delta}(x) \right\rangle v_1^2 \right]. \end{aligned}$$

Using the definitions of  $\eta$  and  $\Phi_{A,\delta}(x)$ , the left-hand side of the preceding

inequality can be restricted to  $B_r$ ,

$$\begin{aligned} & \int_{B_{2r}} \left[ \Phi_{A,\delta}(x) (\langle A \nabla v_1, \nabla v_1 \rangle + v_1^{\gamma+1} v_2^{\gamma+1}) + m_A(x) v_1^2 \right] \\ & \leq \int_{B_{2r} \setminus B_r} C v_1^2 \left[ \Gamma_A \langle A \nabla \eta, \nabla \eta \rangle + \langle A \nabla \eta, \nabla \Gamma_A \rangle \right] \leq C r^{2\alpha_1}, \end{aligned}$$

the final inequality here follows from  $|v_1(x)| < C(1+|x|)^{\alpha_1}$ . By taking the limits as  $\delta \rightarrow 0^+$ , we infer that

$$\int_{B_r} \left[ \Gamma_A(x) (\langle A \nabla v_1, \nabla v_1 \rangle + v_1^{\gamma+1} v_2^{\gamma+1}) + m_A(x) v_1^2 \right] \leq C r^{2\alpha_1}.$$

Similarly, we obtain

$$\int_{B_r(0)} \left[ |x|^{2-N} (|\nabla v_2|^2 + v_1^{\gamma+1} v_2^{\gamma+1}) + m(x) v_2^2 \right] \leq C r^{2\alpha_2},$$

which contradicts (2.2) for  $r$  large enough. □

In order to carry out the blow-up analysis, it is crucial to obtain a uniform control on the rescaled solutions. To this aim, we rely on the following technical lemma, which prevents the possible blow-up of subsolutions in the presence of strong absorption terms.

**Lemma 2.3** *Let  $x_0 \in \mathbb{R}^N$ ,  $M, \delta, \rho, B > 0$  and  $u \in H^1(B_{2\rho}(x_0)) \cap C(\bar{B}_{2\rho}(x_0))$  be a subsolution to*

$$\begin{cases} -\operatorname{div}(A \nabla u) \leq -M u^p + \delta & \text{in } B_{2\rho}(x_0), \\ u \leq B & \text{in } B_{2\rho}(x_0), \end{cases} \tag{2.4}$$

for some  $p \geq 1$ . Then there exists  $C > 0$ , depending only on the dimension  $N$ , such that

$$M u^p(x) \leq \frac{C B a_N}{\rho^2} + \delta \quad \text{for every } x \in B_\rho(x_0).$$

**Proof** Let  $T := A^{\frac{1}{2}}$ , and make the linear change of variables  $x = x_0 + T y$ , then we get  $y = T^{-1}(x - x_0)$ . Define a new function  $v(y) := u(x_0 + T y) = u(x)$ , we have that  $\nabla_y v(y) = T \nabla_x u(x)$ ,  $\nabla_x u(x) = T^{-1} \nabla_y v(y)$ , hence

$$\operatorname{div}_x(A \nabla_x u)(x) = \operatorname{div}_y(T^{-1}(A \nabla_y v(y))T^{-1}) = \operatorname{div}_y(\nabla_y v)(y) = \Delta_y v(y).$$

Thus, in the  $y$ -coordinates,  $v$  satisfies

$$-\Delta_y v(y) \leq -M v(y)^p + \delta \quad \text{in } \tilde{B} := \{y : |T y| < 2\rho\}.$$

Notice that  $\tilde{B}$  is an ellipsoid, which satisfies  $a_1 |y|^2 \leq |T y|^2 \leq a_N |y|^2$ , thus the inclusion holds

$$B_{\frac{2\rho}{\sqrt{a_N}}}(0) \subset \tilde{B} \subset B_{\frac{2\rho}{\sqrt{a_1}}}(0).$$

In order to directly apply Lemma 2.2 in [2], take  $\rho_y = \frac{\rho}{\sqrt{a_N}}$ , then

$B_{\frac{2\rho}{\sqrt{a_N}}}(0) \subset \tilde{B}$ . Therefore, we can deduce that

$$M v(y)^p \leq C \frac{B}{(\rho/\sqrt{a_N})^2} + \delta = \frac{C B a_N}{\rho^2} + \delta \quad \text{for every } y \in B_{\rho/\sqrt{a_N}}(0).$$

Since  $v(y) = u(x)$ , the estimate holds for all  $y$  such that  $|y| \leq \rho/\sqrt{a_N}$ . For such  $y$ , we have

$$|x - x_0| = |Ty| \leq \sqrt{a_N} |y| \leq \rho,$$

which implies that  $x \in B_\rho(x_0)$ . Therefore, we conclude that

$$Mu^p(x) \leq \frac{CBa_N}{\rho^2} + \delta \text{ for all } x \in B_\rho(x_0).$$

□

### 3. Regularity of Weak Solutions and Its Proof

The Liouville-type theorem for the limiting system, combined with the monotonicity formula, rules out the existence of nontrivial blow-up limits. This fact will be used in the next section to infer regularity properties of solutions to the original system.

**Theorem 3.1** *Let  $\Omega \subset \mathbb{R}^N$  be an open set. Under the standing assumptions, assume that  $N \geq 3$  and  $\gamma > 1$ . Let  $\mathbf{u}_k = (u_{1,k}, u_{2,k})$  be a positive solution to the system (1.1) at fixed  $k > 1$ , satisfy that there exists  $M > 0$  with*

$$\sup_{k>1} \|\mathbf{u}_k\|_{L^\infty(\Omega)} \leq M.$$

*Then, there exists  $v_{A,N} \in (0, 2)$  depending only on  $A$  and  $N$  such that the following holds: for every  $\alpha \in \left(0, \frac{v_{A,N}}{2}\right)$  there exists  $C > 0$  independent of  $k$  such that*

$$\|\mathbf{u}_k\|_{C^{0,\alpha}(\bar{\Omega})} \leq C.$$

**Proof** Let  $\alpha \in \left(0, \frac{v_{A,N}}{2}\right)$ , and suppose for contradiction that  $\{\mathbf{u}_k\}$  is not bounded in  $C^{0,\alpha}(\bar{\Omega})$ , so there exists a sequence  $k \rightarrow +\infty$  such that

$$L_k := \max_{i=1,2} \max_{\substack{x,y \in \Omega \\ x \neq y}} \frac{|u_{i,k}(x) - u_{i,k}(y)|}{|x - y|^\alpha} \rightarrow +\infty.$$

We can assume that  $L_k$  is attained by  $u_{1,k}$  at the pair  $(x_k, y_k)$ , then

$$|x_k - y_k|^\alpha = \frac{|u_{1,k}(x_k) - u_{1,k}(y_k)|}{L_k} \leq \frac{2\|u_{1,k}\|_{L^\infty}}{L_k} \leq \frac{2M}{L_k} \rightarrow 0, \text{ as } k \rightarrow +\infty.$$

Then we introduce the following blow-up of  $\mathbf{u}_k$  with center  $x_k$ , with  $r_k \rightarrow 0^+$  to be chosen later:

$$\mathbf{v}_k(x) := \frac{1}{L_k r_k^\alpha} \mathbf{u}_k(x_k + r_k x), \quad x \in \Omega_k := \frac{\Omega - x_k}{r_k}.$$

Depending on the asymptotic behavior of the distance  $d(x_k, \partial\Omega)$ , and on  $r_k$ , we have  $\Omega_k \rightarrow \Omega_\infty$ , where  $\Omega_\infty$  is either  $\mathbb{R}^N$  or an half-space. The function  $\mathbf{v}_k = (v_{1,k}, v_{2,k})$  is a positive solution to

$$\begin{cases} -\operatorname{div}(A\nabla v_{1,k}) = \frac{r_k^2}{L_k r_k^\alpha} L_k r_k^\alpha v_{1,k}(x)(1 - L_k r_k^\alpha v_{1,k}(x)) - M_k v_{1,k}^\gamma v_{2,k}^{\gamma+1} & \text{in } \Omega_k, \\ -\Delta v_{2,k} = \frac{r_k^2}{L_k r_k^\alpha} L_k r_k^\alpha v_{2,k}(x)(1 - L_k r_k^\alpha v_{2,k}(x)) - M_k v_{2,k}^\gamma v_{1,k}^{\gamma+1} & \text{in } \Omega_k, \\ v_{1,k} = v_{2,k} = 0 & \text{on } \partial\Omega_k, \end{cases} \quad (3.1)$$

where  $M_k = kL_k^{2\gamma} r_k^{2\alpha\gamma+2}$ . And we denote

$$\begin{aligned} f_{i,k}(x, v_{i,k}) &= \frac{r_k^2}{L_k r_k^\alpha} L_k r_k^\alpha v_{i,k}(x)(1 - L_k r_k^\alpha v_{i,k}(x)), \\ f_{i,k}(x, u_{i,k}) &= \frac{r_k^{2-\alpha}}{L_k} u_{i,k}(x_k + r_k x)(1 - u_{i,k}(x_k + r_k x)). \end{aligned}$$

Since  $\|u_{i,k}\|_{L^\infty(\Omega_k)} \leq M$ ,  $L_k \rightarrow +\infty$ ,  $r_k \rightarrow 0^+$ , then we can deduce

$$\begin{aligned} \|f_{i,k}(\cdot, v_{i,k})\|_{L^\infty(\Omega_k)} &\rightarrow 0 \text{ as } k \rightarrow +\infty. \text{ Moreover, for every } k, \\ \max_{i=1,2} \max_{\substack{x,y \in \Omega \\ x \neq y}} \frac{|v_{i,k}(x) - v_{i,k}(y)|}{|x - y|^\alpha} &= \frac{1}{L_k r_k^\alpha} \frac{|u_{1,k}(x_k) - u_{1,k}(y_k)|}{\left|\frac{x_k - y_k}{r_k}\right|^\alpha} = \frac{1}{L_k} \cdot L_k = 1. \end{aligned} \quad (3.2)$$

In this way, by suitable translations and rescalings, we complete the blow-up procedure and normalize the Hölder seminorm of the solutions. At this point, two potential pathological behaviors still need to be excluded: the divergence of the rescaled values at the origin,  $\mathbf{v}_k(0)$ , and the loss of the competition effects in the limit. Lemma 3.2 and Lemma 3.3 are devoted precisely to ruling out these possibilities, thereby completing the blow-up analysis.

**Lemma 3.2** *Let  $r_k \rightarrow 0^+$  as  $k \rightarrow +\infty$  be such that*

- 1) *there exists  $R' > 0$  such that  $|x_k - y_k| \leq R' r_k$ ;*
- 2)  *$M_k \rightarrow 0$ .*

Then  $\{\mathbf{v}_k(0)\}$  is bounded in  $k$ .

**Proof** Assume by contradiction that  $\{\mathbf{v}_k(0)\}$  is unbounded, and Let  $R \geq R'$ , then we will show that both  $v_{1,k}$  and  $v_{2,k}$  must in fact be bounded.

To begin with, suppose that  $v_{2,k}(0)$  is unbounded. Since the  $v_{2,k}$  is uniformly Hölder continuous and vanish on  $\partial\Omega_k$ , we can consider  $k$  sufficiently large such that  $B_{2R} = B_{2R}(0) \subset \Omega_k$ , and let  $R > R'$ .

Let  $\varphi \in C_c^\infty(B_{2R})$  be a nonnegative function such that  $\varphi = 1$  in  $B_R$ . By testing the equation for  $v_{1,k}$  against  $v_{1,k}\varphi^2$ , we obtain that

$$\int_{B_{2R}} -\operatorname{div}(A\nabla v_{1,k}) v_{1,k} \varphi^2 = \int_{B_{2R}} r_k^2 v_{1,k}^2 \varphi^2 (1 - L_k r_k^\alpha v_{1,k}) - M_k v_{1,k}^{\gamma+1} v_{2,k}^{\gamma+1} \varphi^2. \quad (3.3)$$

By further integration by parts and expansion, we obtain

$$\int_{B_{2R}} \langle A\nabla v_{1,k}, \nabla(v_{1,k} \varphi^2) \rangle = \int_{B_{2R}} \langle A\nabla v_{1,k}, \nabla v_{1,k} \rangle \varphi^2 + \int_{B_{2R}} \langle A\nabla v_{1,k}, \nabla \varphi \rangle 2\varphi v_{1,k}.$$

Plugging the above into (3.3) and collecting terms yields

$$\begin{aligned} &\int_{B_{2R}} \langle A\nabla v_{1,k}, \nabla v_{1,k} \rangle \varphi^2 + M_k v_{1,k}^{\gamma+1} v_{2,k}^{\gamma+1} \varphi^2 \\ &= -\int_{B_{2R}} 2\varphi v_{1,k} \langle A\nabla v_{1,k}, \nabla \varphi \rangle + \int_{B_{2R}} r_k^2 v_{1,k}^2 \varphi^2 (1 - L_k r_k^\alpha v_{1,k}). \end{aligned} \quad (3.4)$$

Applying the Cauchy-Schwarz and Young's inequalities to the above equation,

$$\begin{aligned} & \int_{B_{2R}} \left| -2\varphi v_{1,k} \langle A\nabla v_{1,k}, \nabla \varphi \rangle \right| \\ & \leq 2 \int_{B_{2R}} |\varphi| |v_{1,k}| \left( \langle A\nabla v_{1,k}, \nabla v_{1,k} \rangle \right)^{1/2} \left( \langle A\nabla \varphi, \nabla \varphi \rangle \right)^{1/2} \\ & \leq \int_{B_{2R}} \frac{1}{2} \varphi^2 \langle A\nabla v_{1,k}, \nabla v_{1,k} \rangle + 2v_{1,k}^2 \langle A\nabla \varphi, \nabla \varphi \rangle. \end{aligned}$$

Substituting the above equation into Equation (3.4), we obtain

$$\begin{aligned} & \int_{B_{2R}} \langle A\nabla v_{1,k}, \nabla v_{1,k} \rangle \varphi^2 + M_k v_{1,k}^{\gamma+1} v_{2,k}^{\gamma+1} \varphi^2 \\ & \leq \int_{B_{2R}} \frac{1}{2} \varphi^2 \langle A\nabla v_{1,k}, \nabla v_{1,k} \rangle + 2v_{1,k}^2 \langle A\nabla \varphi, \nabla \varphi \rangle + \int_{B_{2R}} r_k^2 v_{1,k}^2 \varphi^2 (1 - L_k r_k^\alpha v_{1,k}). \end{aligned}$$

Therefore, we have

$$\frac{1}{2} \int_{B_{2R}} \varphi^2 \langle A\nabla v_{1,k}, \nabla v_{1,k} \rangle + M_k v_{1,k}^{\gamma+1} v_{2,k}^{\gamma+1} \varphi^2 \leq C \int_{B_{2R}} (v_{1,k}^2 + 1),$$

since  $\varphi = 1$  in  $B_R$ ,

$$\int_{B_R} M_k |v_{1,k}|^{\gamma+1} |v_{2,k}|^{\gamma+1} \leq C \int_{B_{2R}} (v_{1,k}^2 + 1).$$

Hence, by using assumption (ii) and afterwards the boundedness of the oscillation of  $v_{1,k}$ , we deduce that for every  $x$  in  $B_R$

$$v_{1,k}^{\gamma+1}(x) v_{2,k}^{\gamma+1}(x) \leq C (v_{1,k}^2(x) + 1), \tag{3.5}$$

where  $C > 0$  depends only on  $R$ . Evaluating this inequality at  $x = 0$ , we get

$$v_{1,k}^{\gamma+1}(0) v_{2,k}^{\gamma+1}(0) \leq C (v_{1,k}^2(0) + 1).$$

Since  $|v_{2,k}(0)| \rightarrow +\infty$  and  $\gamma > 0$ , we can see that  $v_{1,k}(0)$  is bounded. In other words, this implies that  $\inf_{B_{2R}} |v_{2,k}| \rightarrow +\infty$ ,  $\sup_{B_{2R}} |v_{1,k}|$  is bounded, and from (3.5) we have that actually  $\sup_{B_{2R}} v_{1,k} \rightarrow 0$ . Let now consider the quantity

$I_k := M_k \inf_{B_{2R}} v_{2,k}^{\gamma+1} \rightarrow +\infty$ , we have

$$-\operatorname{div}(A\nabla v_{1,k}) \leq \left\| r_k^2 v_{1,k} (1 - L_k r_k^\alpha v_{1,k}) \right\|_{L^\infty(B_R)} - |I_k v_{1,k}|^\gamma.$$

By applying the modified Lemma 2.3,

$$I_k \sup_{B_R} v_{1,k}^\gamma(x) \leq \frac{C a_N}{R^2} \sup_{B_R} |v_{2,k}| + \sup_{B_R} \left| r_k^2 v_{1,k} (1 - L_k r_k^\alpha v_{1,k}) \right| = o_n(1)$$

and hence  $M_k v_{1,k}^\gamma v_{2,k}^{\gamma+1} \rightarrow 0$ , as  $k \rightarrow +\infty$ . Moreover, since

$$\left\| r_k^2 v_{1,k} (1 - L_k r_k^\alpha v_{1,k}) \right\|_{L^\infty(B_R)} \rightarrow 0, \text{ we have}$$

$$\left\| \operatorname{div}(A\nabla v_{1,k}) \right\|_{L^\infty(B_R)} \rightarrow 0 \tag{3.6}$$

for every sufficiently large  $R > 0$ .

Consider now  $\tilde{v}_{1,k} := v_{1,k} - v_{1,k}(0)$ . The above discussion shows that  $\tilde{v}_{1,k} \rightarrow \tilde{v}_{1,\infty}$  locally uniformly in  $\mathbb{R}^N$ , where  $\tilde{v}_{1,\infty}$  is globally  $\alpha$ -Hölder continuous in  $\Omega_\infty$ , and  $\tilde{v}_{2,\infty} = 0$ . The uniform convergence of the  $A$ -Laplacians implies that  $\tilde{v}_{1,k} \rightarrow \tilde{v}_{1,\infty}$  in  $C^1_{loc}(\Omega_\infty)$ . By the assumption (i), we can assume

$\frac{|x_k - y_k|}{r_k} \rightarrow a$ . If  $a = 0$ , then

$$\frac{\left| v_{1,k}(0) - v_{1,k}\left(\frac{y_k - x_k}{r_k}\right) \right|}{\left| \frac{y_k - x_k}{r_k} \right|^\alpha} = \frac{\left| \tilde{v}_{1,k}(0) - \tilde{v}_{1,k}\left(\frac{y_k - x_k}{r_k}\right) \right|}{\left| \frac{y_k - x_k}{r_k} \right|^\alpha} \leq C \left| \frac{y_k - x_k}{r_k} \right|^{1-\alpha} \rightarrow 0,$$

which contradicts (3.2). Then  $a \neq 0$ , we have  $|\tilde{v}_{1,\infty}(0) - \tilde{v}_{1,\infty}(a)| = |a|^\alpha$ , so that  $\tilde{v}_{1,\infty}$  is a nonconstant  $A$ -harmonic function in  $\Omega_\infty$ , globally  $\alpha$ -Hölder continuous. Therefore Lemma 4.2 in [5] Provides a contradiction both for  $\Omega_\infty = \mathbb{R}^N$ , and for  $\Omega_\infty$  a half-space. We have shown that  $v_{2,k}(0)$  is bounded. Let now check that the same happens with  $v_{1,k}(0)$ . Assume that  $v_{1,k}(0)$  is unbounded, and text the equation for  $v_{1,k}$  in the same way. Let  $I_k := M_k \inf_{B_{2R}} v_{1,k}^{\gamma+1} \rightarrow +\infty$ . Thus we have

$$I_k \sup_{B_R} v_{2,k}^\gamma(x) \leq \frac{C}{R^2} \sup_{B_R} |v_{2,k}| + \sup_{B_R} \left| r_k^2 v_{2,k} (1 - L_k r_k^\alpha v_{2,k}) \right| = o_n(1).$$

In particular, for  $x \in B_R$ ,

$$M_k v_{1,k}^\gamma v_{2,k}^{\gamma+1} \leq o_n(1) \frac{\sup_{B_R} v_{1,k}^\gamma}{M_k^{1/\gamma} \inf_{B_{2R}} |v_{1,k}|^{\frac{(\gamma+1)^2}{\gamma}}} \rightarrow 0,$$

as  $(\gamma + 1)^2 / \gamma > \gamma$ . Once again we obtain that  $\|\Delta v_{2,k}\|_{L^\infty(B_R)} \rightarrow 0$  and the proof follows as before. □

**Lemma 3.3** *Under the previous notation, we have*

$$\limsup_{k \rightarrow +\infty} k L_k^{2\gamma} |x_k - y_k|^{2\alpha\gamma+2} = +\infty.$$

**Proof** By contradiction, let us assume that  $k L_k^{2\gamma} |x_k - y_k|^{2\alpha\gamma+2}$  is bounded. Then, we choose

$$r_k = \left( k L_k^{2\gamma} \right)^{-\frac{1}{2\alpha\gamma+2}}.$$

We can observe that  $M_k = L_k^{2\gamma} r_k^{2\alpha\gamma+2} k = L_k^{2\gamma} \left( k L_k^{2\gamma} \right)^{-1} k = 1$ . Moreover, from  $|x_k - y_k|^{2\alpha\gamma+2} \leq \frac{C}{k L_k^{2\gamma}} = C r_k^{2\alpha\gamma+2}$ , it follows that  $|x_k - y_k| \leq C r_k$ . These two conditions exactly satisfy Lemma 3.2, hence the sequence  $\{v_k(0)\}$  is bounded. Then by Lemma 5.1 in [5], we know  $v_k \rightarrow v_\infty$  locally uniformly in  $\Omega_\infty$ . Moreover, since  $M_k = 1$ , we find that  $L_i v_{i,k}$  converges locally uniformly, and hence  $v_k \rightarrow v_\infty$  in  $C_{loc}^1(\Omega_\infty)$ , with  $v$  globally  $\alpha$ -Hölder continuous in  $\Omega_\infty$ . By uniform convergence and (3.1) we have that

$$\begin{cases} -\operatorname{div}(A \nabla v_{1,\infty}) = -v_{1,\infty}^\gamma v_{2,\infty}^{\gamma+1} & \text{in } \Omega_\infty, \\ -\Delta v_{2,\infty} = -v_{2,\infty}^\gamma v_{1,\infty}^{\gamma+1} & \text{in } \Omega_\infty. \end{cases} \tag{3.7}$$

Thus,  $v_{i,\infty}$  is a subharmonic. By the strong maximum principle, either  $v_{i,\infty} \equiv 0$  or  $v_{i,\infty} > 0$  in  $\Omega_\infty$ . Finally, as in the intermediate part of the proof of Lemma 3.2, we can also deduce that  $v_{1,\infty}$  is non-constant in  $\Omega_\infty$ .

If  $\Omega_\infty = \mathbb{R}^N$ , by the above proof (Theorem 2.2), we observe that  $v_{i,\infty}$  is constant, a contradiction. In case  $\Omega_\infty$  is a half-space, we know that at most one component  $v_{i,\infty}$  does not vanish identically. Since  $v_{1,\infty}$  is nonconstant, we infer that  $v_{2,\infty} = 0$ , and  $v_{1,\infty}$  is a nonconstant  $A$ -harmonic function in a half-space, globally  $\alpha$ -Hölder continuous, which attains a constant boundary datum on  $\partial\Omega_\infty$ . This contradicts the conclusion that, in a half-space, any solution of  $\operatorname{div}(A\nabla v) = 0$  which is constant on the boundary and globally Hölder continuous must be constant.  $\square$

Lemma 3.3 ensures that the distance  $|x_k - y_k|$  provides a genuine blow-up scale, in the sense that the rescaled competition effects do not vanish trivially. Fixing  $r_k = |x_k - y_k|$ , we can now perform the blow-up analysis and describe the asymptotic behavior of the rescaled solutions. This is the content of the following lemma.

**Lemma 3.4** *Let*

$$r_k = |x_k - y_k|.$$

*There exists  $v_{1,\infty}, v_{2,\infty}$ , which are global  $\alpha$ -Hölder continuous, such that, as  $k \rightarrow \infty$ , the following hold up to a subsequence:*

- 1)  $v_{1,k} \rightarrow v_{1,\infty}, v_{2,k} \rightarrow v_{2,\infty}$  uniformly in compact subsets of  $\Omega_\infty = \mathbb{R}^N$ ;
- 2) for any fixed  $r > 0$  and  $x_0 \in \mathbb{R}^N$ ,

$$\int_{B_r(x_0)} M_k v_{i,k}^{\gamma+1} v_{j,k}^{\gamma+1} \rightarrow 0;$$

- 3)  $\|v_{1,k} - v_{1,\infty}\|_{H^1(B_r(x_0))} \rightarrow 0, \|v_{2,k} - v_{2,\infty}\|_{H^1(B_r(x_0))} \rightarrow 0$ .

For the proof of this lemma, we refer the reader to Lemmas 3.6 and 3.7 in [1]. In the next lemma, we summarize the main properties satisfied by the limit functions  $v_{1,\infty}, v_{2,\infty}$ .

**Lemma 3.5** *We have that*

- 1)  $v_{2,\infty} \equiv 0$  in  $\mathbb{R}^N$ ,  $v_{1,\infty}$  is nonconstant;
- 2) it results  $\operatorname{div}(A\nabla v_{1,\infty}) = 0$  in  $\{v_{1,\infty} > 0\}$ ;
- 3)  $\{v_{1,\infty} = 0\} \neq \emptyset$ , and  $\{v_{1,\infty} > 0\}$  is connected.

The proof is similar to that of Lemma 3.7 and Remark 3.8 in [1]. The lack of symmetry in the exponents of the competition terms prevents the construction of an Almgren-type frequency function for the whole system. We therefore exploit the geometric invariance of the energy functional and, after passing to the limit, derive a structural identity satisfied by the unique nontrivial component  $v_{1,\infty}$  in the whole space.

**Lemma 3.6** *Let  $Q \in C_c^\infty(\mathbb{R}^N, \mathbb{R}^N)$ , then*

$$\int_{\mathbb{R}^N} \left( 2 \langle dQ \nabla v_{1,\infty}, \nabla v_{1,\infty} \rangle - \operatorname{div} Q |\nabla v_{1,\infty}|^2 \right) = 0. \tag{3.8}$$

**Proof** Multiply the equation in (3.1) for  $v_{1,k}$  by the test function  $\langle \nabla v_{1,k}, Q \rangle$  and integrate to obtain

$$\int_{\Omega_k} \left\langle A \nabla v_{1,k}, \nabla \left( \langle \nabla v_{1,k}, Q \rangle \right) \right\rangle + M_k v_{1,k}^\gamma v_{2,k}^{\gamma+1} \langle \nabla v_{1,k}, Q \rangle - f_{1,k}(x, v_{1,k}) \langle \nabla v_{1,k}, Q \rangle = 0. \tag{3.9}$$

By integration by parts, we have

$$\int_{\Omega_k} \left\langle A \nabla v_{1,k}, \nabla \left( \langle \nabla v_{1,k}, Q \rangle \right) \right\rangle = - \int_{\Omega_k} \frac{1}{2} \langle A \nabla v_{1,k}, \nabla v_{1,k} \rangle \operatorname{div} Q + \langle dQ A \nabla v_{1,k}, \nabla v_{1,k} \rangle.$$

Substituting the above expression into (3.9), we obtain

$$\begin{aligned} & \int_{\Omega_k} \langle dQ A \nabla v_{1,k}, \nabla v_{1,k} \rangle - \frac{1}{2} \langle A \nabla v_{1,k}, \nabla v_{1,k} \rangle \operatorname{div} Q \\ & + M_k v_{1,k}^\gamma v_{2,k}^{\gamma+1} \langle \nabla v_{1,k}, Q \rangle - f_{1,k}(x, v_{1,k}) \langle \nabla v_{1,k}, Q \rangle = 0. \end{aligned} \tag{3.10}$$

Similarly, performing the same procedure on the equation involving  $v_{2,k}$ , then adding it to (3.10), and simplifying the resulting expression before passing to the limit as  $k \rightarrow +\infty$ , we obtain

$$\int_{\mathbb{R}^N} \left( 2 \langle dQ \nabla v_{1,\infty}, \nabla v_{1,\infty} \rangle - \operatorname{div} Q |\nabla v_{1,\infty}|^2 \right) = 0,$$

where we have used Lemma 3.4 and Lemma 3.5, and the fact that

$$\|f_{i,k}(\cdot, v_{i,k})\|_{L^\infty(\Omega_k)} \rightarrow 0 \text{ as } k \rightarrow +\infty. \quad \square$$

*Proof of Theorem 3.1.* To complete the proof of Theorem 3.1, we exploit the domain variation formula to recover an Almgren-type frequency monotonicity for the limiting component  $v_{1,\infty}$ , which yields a precise description of its nodal set. This allows us to rule out nontrivial blow-up limits, leading to a contradiction and hence to the desired regularity result. From (3.8) we can derive

$$\int_{\partial B_r(x_0)} |\nabla v_{1,\infty}|^2 = \frac{N-2}{r} \int_{B_r} |\nabla v_{1,\infty}|^2 + 2 \int_{\partial B_r(x_0)} \langle v_{1,\infty}, \nu \rangle^2$$

for every  $x_0 \in \mathbb{R}^N$  and almost every  $r > 0$ . And by Lemma 2.8 in [6], we obtain

$$\frac{d}{dr} \left( \int_{\partial B_r(x_0)} v_{1,\infty}^2 \right) = \frac{N-1}{r} \int_{\partial B_r(x_0)} v_{1,\infty}^2 + 2 \int_{\partial B_r(x_0)} v_{1,\infty} \langle v_{1,\infty}, \nu \rangle.$$

To this end, we need to introduce the Almgren frequency function. We define

$$N(x_0, r) = \frac{rD(x_0, r)}{H(x_0, r)} = \frac{r \int_{B_r(x_0)} |\nabla v_{1,\infty}|^2}{\int_{\partial B_r(x_0)} v_{1,\infty}^2}, \text{ where } H(x_0, r) = \int_{\partial B_r(x_0)} v_{1,\infty}^2.$$

Combining the two identities above with the Cauchy-Schwarz inequality, we can show that  $N(x_0, r)$  is nondecreasing, and that  $N(x_0, r)$  is constant and equal to  $c$  if and only if  $v_{1,\infty}$  is  $c$ -homogeneous. Then we can prove that  $\{v_{1,\infty} = 0\}$  is a linear subspace of dimension at most  $N - 2$ , and in particular has local capacity 0.

**Step 1.** The interior of the set  $\{v_{1,\infty} = 0\}$  is empty. That is, if  $v_{1,\infty}(x_0) = 0$ , then in every small neighborhood  $B_\rho(x_0)$  around that point, we have  $v_{1,\infty} \neq 0$ . By contradiction, assume that  $x_0$  is an interior point of the set  $\{v_{1,\infty} = 0\}$ . Then

there exists a sufficiently small ball  $B_{r_1}(x_0)$  such that  $v_{1,\infty} = 0$  in  $B_{r_1}(x_0)$ . This implies that  $H(x_0, r_1) = 0$  on the boundary of the ball. Therefore, there exist  $r_1, r_2 > 0$  such that for all  $r \in (r_1, r_2]$ , we have  $H(x_0, r) > 0$ . Since  $v_{1,\infty} \in H^1_{loc}(\mathbb{R}^N)$ , by the monotonicity formula, we obtain that

$$\frac{d}{dr} \log \frac{H(x_0, r)}{r^{N-1}} = \frac{H'(x_0, r)r^{N-1} - H(x_0, r)(N-1)r^{N-2}}{H(x_0, r)r^{N-1}}.$$

Then we have

$$\frac{d}{dr} \log \frac{H(x_0, r)}{r^{N-1}} = \frac{2D(x_0, r)}{H(x_0, r)} = \frac{2N(x_0, r)}{r} \leq \frac{2N(x_0, r_2)}{r} = \frac{C}{r}.$$

Thus  $\frac{d}{dr} \log \frac{H(x_0, r)}{r^{N-1}} \leq \frac{C}{r}$ . Integrating both sides of the above equation from

$r$  to  $R$ , and assuming  $r_1 < r < R < r_2$ , we have  $\frac{H(x_0, R)}{R^{N-1+C}} \leq \frac{H(x_0, r)}{r^{N-1+C}}$ . As

$r \rightarrow r_1^+$ , we know that  $\lim_{r \rightarrow r_1^+} H(x_0, r) = 0$ . Therefore, we deduce that

$H(x_0, R) = 0$ . This contradicts the assumption that  $H(x_0, R) > 0$ .

**Step 2.** Prove that for any  $x_0 \in \mathbb{R}^N$  and any radius  $r > 0$ , we have  $H(x_0, r) \neq 0$ . By contradiction, suppose there exists a radius  $r_1 > 0$  such that  $H(x_0, r_1) = 0$ . That implies  $v_{1,\infty} = 0$  on  $\partial B_{r_1}(x_0)$ . Since  $H(x_0, r) = \int_{\partial B_r(x_0)} v_{1,\infty}^2$ ,  $H = 0$  means  $v_{1,\infty}$  vanishes on the entire sphere. Because  $v_{1,\infty} \in H^1_{loc}$ , continuity yields that  $v_{1,\infty}$  is zero on an open region inside the ball—in other words  $v_{1,\infty} \equiv 0$  in  $B_{r_1}(x_0)$ . This contradicts the result in step 1. Moreover, we can find that  $\frac{H(x_0, r)}{r^{N-1}}$  is a non-decreasing function,

$$\frac{d}{dr} \left( \frac{H(x_0, r)}{r^{N-1}} \right) = \frac{H'(x_0, r) - \frac{N-1}{r} H(x_0, r)}{r^{N-1}} = \frac{2}{r^{N-1}} \int_{B_r(x_0)} |\nabla v_{1,\infty}|^2.$$

Thus, if  $H(x_0, r_1) = 0$  for some  $r_1$ , then  $H(x_0, r) = 0, \forall 0 < r < r_1$ . That is,  $v_{1,\infty}$  identically in the entire ball of radius  $r_1$  centered at  $x_0$ , which creates an interior region of the zero set, this contradicts the result in step 1 again. Therefore, we can rigorously conclude that  $H(x_0, r) > 0, \forall x_0 \in \mathbb{R}^N, r > 0$ .

**Step 3.** The function  $v_{1,\infty}$  attains the value 0 at least at some point in  $\mathbb{R}^N$ . By contradiction, suppose that  $v_{1,\infty} > 0$  holds throughout  $\mathbb{R}^N$ . Then, by Lemma 3.5, we know that

$$-\operatorname{div}(A \nabla v_{1,\infty}) = 0 \text{ in } \{v_{1,\infty} > 0\}.$$

Since the coefficient matrix  $A$  is constant, symmetric, and positive definite, it can be transformed into  $-\Delta v_{1,\infty} = 0$  in  $\{v_{1,\infty} > 0\}$ , through a suitable linear change of variables. At this point, we only know that  $v_{1,\infty}$  is harmonic in its positivity set. Now, by the assumption  $\{v_{1,\infty} > 0\} = \mathbb{R}^N$ , the equation becomes

$-\Delta v_{1,\infty} = 0$  in  $\mathbb{R}^N$ , so  $v_{1,\infty}$  is a globally bounded harmonic function in the whole space  $\mathbb{R}^N$ . By the classical Liouville theorem, we have  $v_{1,\infty} \equiv C$ , which contradicts the fact that  $v_{1,\infty}$  is non-constant. Therefore, there must exist some point  $x_0$  such that  $v_{1,\infty}(x_0) = 0$ .

**Step 4.** For the function  $v_{1,\infty}$ , if  $x_0 \in \{v_{1,\infty} = 0\}$ , then after rescaling  $v_{1,\infty}$  around  $x_0$ , it exhibits local  $\alpha$ -homogeneity. Without loss of generality, assume that  $0 \in \{v_{1,\infty} = 0\}$ . We aim to prove that  $N(x_0, r) = \alpha$  for all  $r$ , so that  $v_{1,\infty}$  is  $\alpha$ -homogeneous with respect to the origin. Since  $N(x_0, r)$  is non-decreasing, if it attains the value  $\alpha - \varepsilon$  at some point  $\tilde{r}$ , we can get  $N(x_0, r) \leq N(x_0, \tilde{r}) = \alpha - \varepsilon$ , for all  $r \in (0, \tilde{r}]$ . Substituting this upper bound into Almgren's monotonicity formula, we obtain

$$\frac{d}{dr} \log \frac{H(x_0, r)}{r^{N-1}} = \frac{2N(x_0, r)}{r} \leq \frac{2(\alpha - \varepsilon)}{r}, \quad \forall r \in (0, \tilde{r}].$$

Integrating both sides of the above equation from  $r$  to  $\tilde{r}$ , and further calculation and simplification, we obtain

$$\frac{H(x_0, r)}{r^{N-1}} \geq \frac{H(x_0, \tilde{r})}{\tilde{r}^{N-1}} \cdot \left(\frac{r}{\tilde{r}}\right)^{2(\alpha - \varepsilon)}.$$

Therefore, there exists a constant  $C_1 = \frac{H(x_0, \tilde{r})}{\tilde{r}^{N-1+2(\alpha - \varepsilon)}} > 0$ , such that for all  $0 < r < \tilde{r}$ , we have that

$$H(x_0, r) \geq C_1 r^{N-1+2(\alpha - \varepsilon)}.$$

In other words, since  $v_{1,\infty}$  is globally  $\alpha$ -Hölder continuous and  $v_{1,\infty} = 0$ , there exists a constant  $C > 0$  such that for any  $x$ , we have  $|v_{1,\infty}(x)| \leq C|x|^\alpha$ . Thus, on the sphere  $\partial B_r$ , we have  $v_{1,\infty}^2 \leq C^2 r^{2\alpha}$ . By integrating over  $\partial B_r$ , we get

$$H(x_0, r) = \int_{\partial B_r} v_{1,\infty}^2 \leq C^2 r^{2\alpha} |\partial B_r| = C' r^{N-1+2\alpha}.$$

The upper and lower bounds for  $H(r)$  lead to a contradiction as  $r \rightarrow 0$ , unless  $\varepsilon = 0$ . Hence, we conclude that there cannot exist any  $\tilde{r} > 0$  such that  $N(x_0, \tilde{r}) = \alpha - \varepsilon$  ( $\varepsilon > 0$ ). Therefore,  $N(x_0, r) \geq \alpha$ ,  $\forall r > 0$ . In the same way, assume that there exists some radius  $\tilde{r} > 0$  such that  $N(0, \tilde{r}) = \alpha + \varepsilon$ . Since  $N(x_0, r)$  is non-decreasing, for  $\forall r \geq \tilde{r}$ , we have  $N(x_0, r) \geq N(x_0, \tilde{r}) = \alpha + \varepsilon$ . From the Almgren monotonicity formula, we know that

$$\frac{d}{dr} \log \frac{H(x_0, r)}{r^{N-1}} = \frac{2N(x_0, r)}{r} \geq \frac{2(\alpha + \varepsilon)}{r}.$$

Integrating the above equation, we obtain  $H(x_0, r) \geq C_2 r^{N-1+2(\alpha + \varepsilon)}$ ,  $\forall r > \tilde{r}$ . Then we have

$$C_2 r^{N-1+2(\alpha + \varepsilon)} \leq H(x_0, r) \leq C' r^{N-1+2\alpha}.$$

Therefore, we arrive at a contradiction, which implies that  $N(x_0, r) \leq \alpha$ ,  $\forall r > 0$ . In conclusion, we have  $N(x_0, r) = \alpha$ ,  $\forall r > 0$ .

**Step 5.** We decompose  $v_{1,\infty}$  into its “positive part” and “negative part”, denoted by

$$v_{1,\infty}^+(x) = \max\{v_{1,\infty}(x), 0\}, \quad v_{1,\infty}^-(x) = \max\{-v_{1,\infty}(x), 0\}.$$

Then  $v_{1,\infty}^+$  and  $v_{1,\infty}^-$  still satisfy the properties of being weakly subharmonic, having disjoint supports, and being  $\alpha$ -Hölder continuous. We know that at least one of  $v_{1,\infty}^+$  and  $v_{1,\infty}^-$  must be identically zero. Without loss of generality, we assume that  $v_{1,\infty}^- = 0$ , then  $v_{1,\infty} = v_{1,\infty}^+ \geq 0$ . Moreover, step 4 shows that  $v_{1,\infty}$  is  $\alpha$ -homogeneous with respect to each of its zero points. Assuming  $x_0 \in \{v_{1,\infty} = 0\}$  and taking any  $y \in \{v_{1,\infty} = 0\}$  with  $y = x_0 + \zeta$ . Then, for  $\forall t > 0$ , we have

$$v_{1,\infty}(x_0 + t\zeta) = t^\alpha v_{1,\infty}(x_0 + \zeta) = t^\alpha v_{1,\infty}(y) = 0.$$

Therefore, every point  $x_0 + t\zeta$  belongs to the zero set, the set  $\{v_{1,\infty} = 0\}$  is a cone with any of its points as a vertex. We have already shown that 0 belongs to the zero set, and that for any point in the zero set, the zero set is a cone with respect to that point. Therefore, we conclude that the zero set must be a linear subspace. Suppose that the zero set is a hyperplane, which divides the space into two regions  $\Omega_1$  and  $\Omega_2$ . Then, in each region,  $v_{1,\infty}$  is either nonnegative or nonpositive everywhere. By restricting  $v_{1,\infty}$  to these two half-spaces, we see that  $v_{1,\infty}|_{\Omega_1}$  and  $v_{1,\infty}|_{\Omega_2}$  remain subharmonic, have disjoint supports, and satisfy the required regularity. Applying Theorem 3.1 in [5] again, we deduce that one of these two functions must be identically zero. However, from Step 1 we know that the zero set has no interior. If  $v_{1,\infty}$  were identically zero in one of the half-spaces, then the zero set would have a nonempty interior, which contradicts step 1. Therefore, the zero set cannot be a hyperplane. Then  $\{v_{1,\infty} = 0\}$  is a linear subspace of dimension  $N - 2$ . We denote this set by  $E := \{v_{1,\infty} = 0\}$ , and introduce the capacity associated with the operator  $\operatorname{div}(A\nabla \cdot)$ :

$$\operatorname{cap}_A(E) := \inf \left\{ \int_{\mathbb{R}^N} \langle A\nabla u, \nabla u \rangle : u \in C_c^\infty, u \geq 1 \text{ on } E \right\}.$$

Since the matrix  $A$  is uniformly elliptic, this capacity is equivalent to the classical Newtonian capacity. In particular, any linear subspace of dimension  $N - 2$  has zero  $A$ -capacity.

It follows from classical potential theory that sets of zero capacity are removable for  $A$ -harmonic functions. Therefore,  $v_{1,\infty}$  can be extended as an  $A$ -harmonic function across its zero set. Since we have already shown that  $v_{1,\infty}$  is harmonic in the region outside the zero set and that  $v_{1,\infty} \in H_{loc}^1(\mathbb{R}^N) \cap C(\mathbb{R}^N)$ , it follows that  $v_{1,\infty}$  can be extended to a harmonic function on the whole space  $\mathbb{R}^N$ . By Liouville's theorem, any bounded harmonic function on  $\mathbb{R}^N$  must be constant. This contradicts the fact that  $v_{1,\infty}$  is a nontrivial limit function. We have completed the proof of the boundedness of  $\{\mathbf{u}_k\}$  in  $C^{0,\alpha}(\bar{\Omega})$ .  $\square$

In this paper, we investigate a class of strongly competing elliptic systems with anisotropic diffusion, and prove the Hölder regularity of their solutions via a contradiction argument. The proof relies on the construction of blow-up sequences,

combined with monotonicity formulas and Liouville-type theorems. Our results extend the existing theory from the isotropic setting to a more general anisotropic framework, while also allowing for more flexible nonlinear interaction terms. Several interesting questions remain open for further study. In particular, the present work focuses on diffusion operators with constant coefficient matrices; extending the analysis to the case of variable coefficient matrices  $A(x)$  would be a meaningful direction for future research.

### Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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