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Global Existence and Finite Time Blow-Up for the Nonlinear Schrödinger Hartree Equation with a Constant Magnetic Field

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Abstract

The paper addresses the global existence and finite-time blow-up phenomena for the nonlinear Schrödinger Hartree equation in the presence of a constant magnetic field, which is given by: $i\partial_t\psi+(\nabla+iA)^2\psi-\left(|x|^{-1}*|\psi|^2\right)\psi+|\psi|^p\psi=0$, $(t,x)\in\mathbb{R}^+\times\mathbb{R}^3$. This equation is considered in three-dimensional space. Under mass-critical and supercritical conditions, we determine the precise thresholds for global existence and finite-time blow-up in the L^2-supercritical regime, specifically for the range $\frac{4}{3}<p<4$.

Keywords

Nonlinear Schrödinger Hartree Equation, Global Existence, Finite-Time Blow-Up

1. Introduction

Recently, we have studied a class of nonlinear Schrödinger Hartree equations featuring a constant magnetic field

$$\begin{cases} i\partial_t\psi+(\nabla+iA)^2\psi-\left(|x|^{-1}*|\psi|^2\right)\psi+|\psi|^p\psi=0, & (t,x)\in\mathbb{R}^+\times\mathbb{R}^3, \\ \psi|_{t=0}=\psi_0. \end{cases} \tag{1.1}$$

where

$$A(\boldsymbol{x})=\frac{b}{2}(-x_2,x_1,0),\quad \boldsymbol{x}=(x_1,x_2,x_3)\in\mathbb{R}^3. \tag{1.2}$$

A vector-valued potential is employed to model the influence of an external magnetic field

$$B = curl(A) = (0,0,b), \ b \neq 0.$$

The nonlinear Schrödinger equation (NLS) and its generalizations are widely used in quantum mechanics (see e.g., [1]), Bose-Einstein condensation (see e.g., [2] [3]), and nonlinear optics. These equations combine three key mechanisms: the magnetic Laplace operator for quantum dynamics in external magnetic fields, the Hartree-type nonlocal term for long-range interactions, and a local power nonlinearity for short-range interactions.

For blow-up theory, Glassey [4] introduced the virial identity to show finite-time blow-up for solutions with negative initial energy. Weinstein [5] provided sharp interpolation estimates that define blow-up thresholds. These results have been extended to nonlocal nonlinearity by Feng [6], and others.

In global existence theory, Cazenave [7], and Sulem [8] developed well-posedness frameworks and a priori bounds. Lions [9] concentration-compactness principle helped prove the existence of ground states and analyze their dynamics. For equations with mixed nonlinearities, Feng, Zhang, and Zhu [10]-[12] studied how competing effects influence blow-up. Additionally, magnetic Schrödinger operators have been studied by Arioli [13], Esteban [14], and Kurata [15]; spectral properties under magnetic fields were explored by Helffer [16], and Bouclet [17] established Strichartz estimates in this setting.

Before presenting our findings in this regard, it is essential to revisit the local theory pertaining to Equation (1.1). The local well-posedness of Equation (1.1) for initial data within the space $H_A^1(\mathbb{R}^3)$ was proven by Cazenave and Esteban in [7] [18]. Here the space $H_A^1(\mathbb{R}^3)$ is defined as

$$H_A^1(\mathbb{R}^3) := \left\{ f \in L^2(\mathbb{R}^3) : \left|(\nabla + iA) f\right| \in L^2(\mathbb{R}^3) \right\}$$

is a Hilbert space endowed with a specific norm

$$\|f\|_{H_A^1}^2 = \|(\nabla + iA) f\|_{L^2}^2 + \|f\|_{L^2}^2 .$$

More formally, when the parameter p is confined to the interval $0 < p < 4$ and the initial data ψ_0 is selected from the Hilbert space $H_A^1(\mathbb{R}^3)$, there exists a time $T^* \in (0,\infty)$, together with a uniquely defined maximal solution.

$$\psi \in C\left(\left[0,T^*\right), H_A^1(\mathbb{R}^3)\right) \cap C^1\left(\left[0,T^*\right), H_A^{-1}(\mathbb{R}^3)\right),$$

where $H_A^{-1}(\mathbb{R}^3)$ is the dual space of $H_A^1(\mathbb{R}^3)$. For each $\psi_0 \in H_A^1(\mathbb{R}^3)$, the solution to (1.1) exists on a maximal time interval $(-T_*, T^*)$. The maximal time of existence satisfies the blow-up alternative: if $T^* < \infty$ (resp. $T_* < \infty$), then

$$\lim_{t \nearrow T^*} \|\nabla u(t)\|_{L^2} = \infty \ \left(\text{resp. } \lim_{t \searrow -T_*} \|\nabla u(t)\|_{L^2} = \infty\right).$$

Moreover there are conservation laws of mass and energy:

$$M(\psi(t)) = \|\psi(t)\|_{L^2}^2 = M(\psi_0),$$

$$E(\psi(t)) = \frac{1}{2}\|(\nabla + iA)\psi(t)\|_{L^2}^2 + \frac{1}{4}\int_{\mathbb{R}^3}\left(\frac{1}{|x|} * |\psi(t)|^2\right)|\psi(t)|^2 \,\mathrm{d}x - \frac{1}{p+2}\|\psi(t)\|_{L^{p+2}}^{p+2}$$
$$= E(\psi_0)$$

for all $t \in [0, T^*)$.

For stating the blow-up results associated with (1.1), let us define the subsequent Hilbert space

$$\Sigma_A(\mathbb{R}^3) := \left\{ f \in H_A^1(\mathbb{R}^3) : |x| f \in L^2(\mathbb{R}^3) \right\},$$

equipped with the norm

$$\|f\|_{\Sigma_A}^2 = \|(\nabla + iA) f\|_{L^2}^2 + \|xf\|_{L^2}^2 + \|f\|_{L^2}^2 .$$

It is well-established that the space $\Sigma_A(\mathbb{R}^3)$ coincides with the standard weighted Sobolev space

$$\Sigma(\mathbb{R}^3) := \left\{ f \in H^1(\mathbb{R}^3) : |x| f \in L^2(\mathbb{R}^3) \right\}.$$

Owing to this property, we obtain the following useful identity:

$$\|(\nabla + iA) f\|_{L^2}^2 = \|\nabla f\|_{L^2}^2 + bR(f) + \frac{b^2}{4}\|\rho f\|_{L^2}^2 , \tag{1.3}$$

where $\rho := \sqrt{x_1^2 + x_2^2}$ and

$$R(f) := i\int \left(x_2 \partial_{x_1} f - x_1 \partial_{x_2} f\right) \bar{f} \,\mathrm{d}x = \int L_z f \bar{f} \,\mathrm{d}x,\ L_z := i\left(x_2 \partial_{x_1} - x_1 \partial_{x_2}\right).$$

The structure of this paper is outlined below: Chapter 2 introduces the mathematical framework and lays out the relevant preliminary findings. Chapter 3 establishes precise thresholds for global existence and finite-time blow-up in the mass-critical and mass-supercritical regimes.

Theorem 1.1 [19] [20] The set

$\frac{4}{3} < p < 4$, $\psi_0 \in \Sigma_A(\mathbb{R}^3)$ and $\psi : [0, T^*) \times \mathbb{R}^3 \to \mathbb{C}$ be the corresponding solution to (1.1). Then the solution blows up in finite time, *i.e.* $T^* < \infty$ proved that one of the following conditions holds:

- [20] $E(\psi_0) < 0$;
- [19] $E_0(\psi_0) < 0$;
- [19] $E_0(\psi_0) = 0$ and $\operatorname{Im}\int x \cdot \nabla \psi_0(x) \bar{\psi}_0 \mathrm{d}x < 0$;
- [19] $E_0(\psi_0) > 0$ and $\operatorname{Im}\int x \cdot \nabla \psi_0(x) \bar{\psi}_0 \mathrm{d}x < -\sqrt{2E_0(\psi_0)}\|x\psi_0\|_{L^2}$.

Where

$$E_0(f) := \frac{1}{2}\|\nabla f\|_{L^2}^2 + \frac{b^2}{8}\|\rho f\|_{L^2}^2 + \frac{1}{4}\int_{\mathbb{R}^3}\left(\frac{1}{|x|} * |f|^2\right)|f|^2 \,\mathrm{d}x - \frac{1}{p+2}\|f\|_{L^{p+2}}^{p+2} . \tag{1.4}$$

Remark 1.1 Thanks to the conservation of angular momentum, *i.e.*, $R(\psi(t)) = R(\psi_0)$, we observe that

$$E_0(\psi(t)) = E_0(\psi_0),\ \forall t \in [0, T^*). \tag{1.5}$$

Theorem 1.2 Let $p=\frac{4}{3}$.

1) If $\psi_0 \in H_A^1(\mathbb{R}^3)$ satisfies $\|\psi_0\|_{L^2} < \|Q\|_{L^2}$, where Q is the unique positive radial solution to

$$-\Delta Q + Q + \frac{1}{|x|} * |Q|^2 Q - |Q|^p Q = 0. \tag{1.6}$$

then the corresponding solution to (1.1) exists globally in time, *i.e.* $T^* = \infty$.

2) For $c > \|Q\|_{L^2}$, there exists $\psi_0 \in \Sigma_A(\mathbb{R}^3)$ such that the corresponding solution to (1.1) with initial data $\psi|_{t=0} = \psi_0$ blow up in finite time, *i.e.* $T^* = \infty$.

Lemma 1.2 The equation

$$-\Delta Q + Q + \frac{1}{|x|} * |Q|^2 Q - |Q|^p Q = 0, \ x \in \mathbb{R}^3$$

admits a unique positive radial solution $Q > 0$ in $H^1(\mathbb{R}^3)$ satisfying $Q(x) \to 0$ as $|x| \to \infty$, and $Q \in \Sigma_A(\mathbb{R}^3)$.

proof. The right-hand side is a spherically symmetric nonlinear term that is locally Lipschitz continuous with respect to Q, satisfying the core conditions of the integral moving plane method. By the radial symmetry conclusion and the rotational invariance of the equation, all positive solutions are radially symmetric about the origin, *i.e.*, $Q = Q(r)$ $(r = |x|)$, and the radial solution is strictly monotonic with $Q'(r) < 0$ for all $r > 0$. Thus, it suffices to prove the uniqueness in the radial Sobolev space $H_A^1(\mathbb{R}^3)$. For the radial function $Q = Q(r)$, the radial form of the three-dimensional Laplace operator is $\Delta Q = Q'' + \frac{2}{r} Q'$. Denote the radial nonlocal term by

$$V(r) = \left(\frac{1}{|x|} * |Q|^2 \right)(r) = \int_0^\infty \frac{s^2 Q(s)^2}{\max(r,s)} \mathrm{ds}.$$

Then the original equation is reduced to a second-order nonlinear ordinary differential equation (ODE):

$$Q'' + \frac{2}{r} Q' - Q + V(r) Q - Q^{\frac{7}{3}} = 0, \ r > 0,$$

endowed with the regularity and decay conditions: $Q'(0) = 0$, $Q(r) > 0$ for all $r > 0$, and $Q(r) \to 0$ as $r \to \infty$.

By the local existence and uniqueness theorem for ODEs, there exists a unique local solution $Q(\cdot;a)$ for any initial value $Q(0) = a > 0$. Combining with the Pohozaev identity in [21], this local solution can be extended to all $r > 0$, and $Q(r)$ decays exponentially as $Q(r) \sim C \frac{\mathrm{e}^{-r}}{r}$ for $r \to \infty$, hence $Q \in \Sigma_A(\mathbb{R}^3)$. Define the radial energy density matching the form of the energy functional in this paper:

$$E(r) = \frac{1}{2}(Q')^2 + \frac{1}{2}Q^2 + \frac{1}{4}V(r)Q^2 - \frac{3}{10}Q^{\frac{10}{3}}.$$

Taking the derivative and substituting the transformed radial ODE: $Q''+Q+V(r)Q-Q^{\frac{7}{3}}=-\frac{2}{r}Q'$, we simplify to obtain

$$E'(r)=Q'\left(-\frac{2}{r}Q'\right)+\frac{1}{2}V'(r)Q^2=-\frac{2}{r}(Q')^2+\frac{1}{2}V'(r)Q^2.$$

From $Q'(r)\le 0$ and the monotonicity of the nonlocal term $V'(r)=-r^2Q(r)^2\le 0$, we directly get $E'(r)\le 0$ for all $r>0$, which means the energy density is non-increasing along the radial direction. Suppose there exist two distinct positive radial decaying solutions $Q_1=Q(\cdot;a_1)$ and $Q_2=Q(\cdot;a_2)$ satisfying $0<a_1<a_2$ and the above regularity and decay conditions. At $r=0$, $Q_1'(0)=Q_2'(0)=0$, and the energy density is given by $E_i(0)=\frac{1}{2}a_i^2+\frac{1}{4}V_i(0)a_i^2-\frac{3}{10}a_i^{\frac{10}{3}}$. Define $\varphi(t)=\frac{1}{2}t^2+\frac{1}{4}V(0)t^2-\frac{3}{10}t^{\frac{10}{3}}$, which is strictly increasing for $t>0$, thus $E_1(0)<E_2(0)$. Since both Q_1 and Q_2 decay exponentially to 0, we have $\lim_{r\to\infty}E_1(r)=\lim_{r\to\infty}E_2(r)=0$. Combining the monotonicity of the energy density, we obtain $E_1(r)\le E_1(0)<E_2(0)\le E_2(r)$ for all $r>0$. Let $w(r)=Q_2(r)-Q_1(r)$, then $w(0)=a_2-a_1>0$ and $w'(0)=0$. Substitute Q_1 and Q_2 into the radial ODE and subtract the two equations; by the mean value theorem, we have

$$w''+\frac{2}{r}w'+\left(V_2(r)-1+\xi^{\frac{4}{3}}\right)w+\left(V_2(r)-V_1(r)\right)Q_1=0,$$

where ξ lies between Q_1 and Q_2. Combining the Sturm comparison theorem in [7] with the global energy inequality, if $a_2>a_1$ and the energy density is strictly larger for all r, then Q_2 either changes sign at some finite r or fails to decay at infinity, both of which contradict that Q_2 is a positive radial decaying solution. Therefore, the assumption is invalid, and there do not exist two distinct positive radial decaying solutions. In conclusion, the equation has a unique positive radial solution Q in $H^1(\mathbb{R}^3)$, and $Q\in\Sigma_A(\mathbb{R}^3)$. The proof is complete.

Remark 1.2 At present, it remains unclear whether a blow-up solution exists for the mass-critical Equation (1.1) when the initial mass equals the minimal mass, *i.e.*, $\|\psi_0\|_2=\|Q\|_2$.

Next we derive a sharp threshold for the mass-supercritical case that separates global existence from finite-time blow-up.

Theorem 1.3 Let $\frac{4}{3}<p<4$. Let $\psi_0\in\Sigma_A(\mathbb{R}^3)$ be such that $E_0(\psi_0)\ge 0$ and

$$E_0(\psi_0)\left[M(\psi_0)\right]^{\oe_{cr}}<E^0(Q)\left[M(Q)\right]^{\oe_{cr}}, \tag{1.7}$$

where E_0 is as in (1.4) and

$$E^0(f):=\frac{1}{2}\|\nabla f\|_{L^2}^2+\frac{1}{4}\int_{\mathbb{R}^3}\left(\frac{1}{|x|}*|f|^2\right)|f|^2\,\mathrm{d}x-\frac{1}{p+2}\|f\|_{L^{p+2}}^{p+2},\quad \oe_{cr}:=\frac{4-p}{3p-4}. \tag{1.8}$$

1) If

$$\left\|\nabla\psi_0\right\|_{L^2}\left\|\psi_0\right\|_{L^2}^{\text{œ}_{cr}}<\left\|\nabla Q\right\|_{L^2}\left\|Q\right\|_{L^2}^{\text{œ}_{cr}}, \tag{1.9}$$

then the corresponding solution to (1.1) exists globally in time, *i.e.*, $T^*=\infty$ and satisfies

$$\left\|\nabla\psi(t)\right\|_{L^2}\left\|\psi(t)\right\|_{L^2}^{\text{œ}_{cr}}<\left\|\nabla Q\right\|_{L^2}\left\|Q\right\|_{L^2}^{\text{œ}_{cr}},$$

for all $t\in[0,\infty)$.

2) If

$$\left\|\nabla\psi_0\right\|_{L^2}\left\|\psi_0\right\|_{L^2}^{\text{œ}_{cr}}>\left\|\nabla Q\right\|_{L^2}\left\|Q\right\|_{L^2}^{\text{œ}_{cr}}, \tag{1.10}$$

then the corresponding solution to (1.1) satisfies

$$\left\|\nabla\psi(t)\right\|_{L^2}\left\|\psi(t)\right\|_{L^2}^{\text{œ}_{cr}}<\left\|\nabla Q\right\|_{L^2}\left\|Q\right\|_{L^2}^{\text{œ}_{cr}}$$

for all $t\in\left[0,T^*\right)$. Moreover, the solution blows up in finite time, *i.e.*, $T^*<\infty$.

Remark 1.3 We only consider the case $E_0(\psi_0)\geq 0$, since according to Theorem 1.1, any solution with $E_0(\psi_0)<0$ will blow up in finite time. Furthermore, it follows from (3.5) that there is no function $\psi_0\in\Sigma_A\left(\mathbb{R}^3\right)$ that satisfies both (1.9) and

$$\left\|\nabla\psi_0\right\|_{L^2}\left\|\psi_0\right\|_{L^2}^{\text{œ}_{cr}}=\left\|\nabla Q\right\|_{L^2}\left\|Q\right\|_{L^2}^{\text{œ}_{cr}}.$$

Here Theorem 1.3 indeed gives a sharp threshold for global existence versus finite time blow-up for (1.1).

Remark 1.4 In the case without a magnetic potential, Holmer and Roudenko have already proven this type of result. They further showed that as time t tends to infinity, the global solution scatters and asymptotically approaches the solution of the linear equation.

However, in the presence of a constant magnetic field, this scattering result is not expected to hold, because the Strichartz estimates associated with the magnetic Schrödinger operator are only available for finite times.

2. Preliminaries

In this section, we recall some basic properties of the magnetic Sobolev space $H_A^1\left(\mathbb{R}^3\right)$ and preliminary results that will be used later. Firstly, let us recall the local theory for the Cauchy problem (1.1).

Lemma 2.1 1) [5] Let $N=3$ and $\frac{4}{3}<p<4$, then the following sharp Gagliardo-Nirenberg inequality

$$\left\|\psi\right\|_{L^{p+2}}^{p+2}\leq C_{opt}\left\|\psi\right\|_{L^2}^{\frac{4-p}{2}}\left\|\nabla\psi\right\|_{L^2}^{3p/2} \tag{2.1}$$

holds for any $\psi\in H_A^1\left(\mathbb{R}^3\right)$. The sharp constant C_{opt} is

$$C_{opt}=\frac{(p+2)}{2}\frac{1}{\left\|R\right\|_{L^2}^{p}},$$

where R is a radically ground state solution of the elliptic equation

$$-\Delta R + R - |R|^{P} R = 0. \tag{2.2}$$

2) [22] Let $N = 3, \frac{4}{3} < p < 4$, then

$$\int_{\mathbb{R}^3}\left(\frac{1}{|x|} * |\psi|^2\right)|\psi|^2 \mathrm{d}x \le \mathcal{C}_{GN} \|\nabla\psi\|_{L^2} \|\psi\|_{L^2}^3 .$$

The best constant $\mathcal{C}_{GN}$ is defined by

$$\mathcal{C}_{GN} = \frac{2}{\|W\|_{L^2}^2},$$

where W is a radically ground state solution of the elliptic equation

$$-\Delta W + W = \left(\frac{1}{|x|} * W^2\right) W.$$

Lemma 2.2 [14] Let $A \in L^2_{loc}\left(\mathbb{R}^3, \mathbb{R}^3\right)$. Then $H^1_A\left(\mathbb{R}^3\right)$ equipped with the inner product

$$\langle f, g\rangle_{H^1_A} := \int_{\mathbb{R}^3} f \bar{g} \mathrm{d}x + \int_{\mathbb{R}^3} (\nabla + iA) f \cdot \overline{(\nabla + iA) g} \mathrm{d}x$$

is a Hilbert space.

Lemma 2.3 (Diamagnetic inequality [23]) Let $A \in L^2_{loc}\left(\mathbb{R}^3, \mathbb{R}^3\right)$ and $f \in H^1_A\left(\mathbb{R}^3\right)$. then $|f| \in H^1_A\left(\mathbb{R}^3\right)$ in particular, we have

$$\big|\nabla |f|(x)\big| < \left|(\nabla + iA) f(x)\right|, \ a.e.\ x \in \mathbb{R}^3. \tag{2.3}$$

3. Global Existence and Finite Time Blow-Up

In this section, we investigate the existence of solutions to (1.1) that are global in time as well as those exhibiting finite time blow-up. We begin with the subsequent virial identity associated with (1.1), which is of great significance in establishing the existence of finite time blow-up solutions.

Lemma 3.1 [24] Let $0 < p < 4$ and $\psi_0 \in \Sigma_A\left(\mathbb{R}^3\right)$. Let $\psi : \left[0, T^*\right) \times \mathbb{R}^3 \to \mathbb{C}$ be the corresponding solution to (1.1). Set

$$F(\psi(t)) := \int_{\mathbb{R}^3} |x|^2 |\psi(t,x)|^2 \mathrm{d}x. \tag{3.1}$$

Then the function $\left[0, T^*\right) \ni t \mapsto F(\psi(t))$ is in $C^2\left(\left[0, T^*\right)\right)$ and

$$F'(\psi(t)) = 4\,\mathrm{Im}\int_{\mathbb{R}^3} x \cdot \nabla\psi(t,x)\bar{\psi}(t,x)\mathrm{d}x,$$

$$F''(\psi(t)) = 16E(\psi(t)) - \frac{12p-6}{p+2}\|\psi(t)\|_{p+2}^{p+2} - 2\int_{\mathbb{R}^3}\left(\frac{1}{|x|} * |\psi(t)|^2\right)|\psi(t)|^2 \mathrm{d}x$$

for all $t \in \left[0, T^*\right)$.

Now we prove the sharp threshold for global existence versus blow-up for (1.1) in the mass-critical case given in Proposition 1.2.

Proof of Theorem1.2. 1) By the Gagliardo-Nirenberg inequality and the diamagnetic inequality (2.3), we have

$$\|f\|_{L^{\frac{10}{3}}}^{\frac{10}{3}} \le \frac{5}{3}\left(\frac{\|f\|_{L^2}}{\|R\|_{L^2}}\right)^{\frac{4}{3}}\|\nabla f\|_{L^2}^2 \le \frac{5}{3}\left(\frac{\|f\|_{L^2}}{\|R\|_{L^2}}\right)^{\frac{4}{3}}\|(\nabla+iA)f\|_{L^2}^2, \tag{3.2}$$

where R is a unique positive radial solution to (2.2) with $p=\frac{4}{3}$. From this inequality and the conservation laws of mass and energy, we infer that

$$\begin{aligned}E(\psi_0) &= E(\psi(t))\\ &= \frac{1}{2}\|(\nabla+iA)\psi(t)\|_{L^2}^2 + \frac{1}{4}\int_{\mathbb{R}^3}\left(\frac{1}{|x|}*|\psi(t)|^2\right)|\psi(t)|^2\,\mathrm{d}x - \frac{3}{10}\|\psi(t)\|_{L^{\frac{10}{3}}}^{\frac{10}{3}}\\ &\ge \frac{1}{2}\|(\nabla+iA)\psi(t)\|_{L^2}^2 - \frac{3}{10}\|\psi(t)\|_{L^{\frac{10}{3}}}^{\frac{10}{3}}\\ &\ge \frac{1}{2}\|(\nabla+iA)\psi(t)\|_{L^2}^2 - \frac{1}{2}\left(\frac{\|\psi(t)\|_{L^2}}{\|Q\|_{L^2}}\right)^{\frac{4}{3}}\|(\nabla+iA)\psi(t)\|_{L^2}^2\\ &= \frac{1}{2}\left(1-\left(\frac{\|\psi(t)\|_{L^2}}{\|Q\|_{L^2}}\right)^{\frac{4}{3}}\right)\|(\nabla+iA)\psi(t)\|_{L^2}^2\end{aligned}$$

for all $t\in[0,T^*)$. As $\|\psi_0\|_{L^2}<\|Q\|_{L^2}$, we have $\sup_{t\in[0,T^*)}\|(\nabla+iA)\psi(t)\|_{L^2}\le C$, which by the blow-up alternative, implies that $T^*=\infty$.

2) Let $c>\|Q\|_{L^2}$, then $c>\|Q\|_{L^2}$. We define

$$\psi_0(x) := a\lambda^{\frac{3}{2}}Q(\lambda x),$$

where $a:=\frac{c}{\|Q\|_{L^2}}>1$ and the parameter $\lambda>0$ will be chosen subsequently. Give that Q decays exponentially at infinity, it is evident that $Q\in\Sigma_A(\mathbb{R}^3)$. Furthermore, we get

$$\|\psi_0\|_{L^2}^2 = a^2\|Q\|_{L^2}^2,\ \|\nabla\psi_0\|_{L^2}^2 = a^2\lambda^2\|\nabla Q\|_{L^2}^2,$$

$$\|\psi_0\|_{L^{\frac{10}{3}}}^{\frac{10}{3}} = a^{\frac{10}{3}}\lambda^2\|Q\|_{L^{\frac{10}{3}}}^{\frac{10}{3}},\ \|\rho\psi_0\|_{L^2}^2 = a^2\lambda^{-2}\|\rho Q\|_{L^2}^2,$$

$$\int_{\mathbb{R}^3}\left(\frac{1}{|x|}*|\psi_0|^2\right)|\psi_0|^2\,\mathrm{d}x = a^4\lambda^4\int_{\mathbb{R}^3}\left(\frac{1}{|x|}*|Q|^2\right)|Q|^2\,\mathrm{d}x.$$

It follows that

$$\begin{aligned}E_0(\psi_0) &= \frac{1}{2}\|\nabla\psi_0\|_{L^2}^2 + \frac{b^2}{8}\|\rho\psi_0\|_{L^2}^2 + \frac{1}{4}\int_{\mathbb{R}^3}\left(\frac{1}{|x|}*|\psi_0|^2\right)|\psi_0|^2\,\mathrm{d}x - \frac{3}{10}\|\psi_0\|_{L^{\frac{10}{3}}}^{\frac{10}{3}}\\ &= \frac{1}{2}a^2\lambda^2\|\nabla Q\|_{L^2}^2 + \frac{b^2}{8}a^2\lambda^{-2}\|\rho Q\|_{L^2}^2 + \frac{1}{4}a^4\lambda^4\int_{\mathbb{R}^3}\left(\frac{1}{|x|}*|Q|^2\right)|Q|^2\,\mathrm{d}x\\ &\quad - \frac{3}{10}a^{\frac{10}{3}}\lambda^2\|Q\|_{L^{\frac{10}{3}}}^{\frac{10}{3}}\end{aligned}$$

$$
\begin{aligned}
&= a^2\lambda^2\left(\frac{1}{2}\|\nabla Q\|_{L^2}^2+\frac{b^2}{8}\lambda^{-4}\|\rho Q\|_{L^2}^2+\frac{1}{4}a^2\lambda^2\int_{\mathbb{R}^3}\left(\frac{1}{|x|}*|Q|^2\right)|Q|^2\,\mathrm{d}x\right.\\
&\left.\quad-\frac{3}{10}a^{\frac{10}{3}}\|Q\|_{L^{\frac{10}{3}}}^{\frac{10}{3}}\right).
\end{aligned}
$$

Using Pohozaev's identity (see [21])

$$
-\frac{1}{2}\|\nabla Q\|_{L^2}^2+\frac{3}{2}\|Q\|_{L^2}^2+\frac{3}{4}\int_{\mathbb{R}^3}\left(\frac{1}{|x|}*|Q|^2\right)|Q|^2\,\mathrm{d}x+\frac{3}{p+2}\|Q\|_{L^{p+2}}^{p+2}=0,
$$

then

$$
\begin{aligned}
E_0(\psi_0)=a^2\lambda^2&\left(-\left(\frac{1}{2}+\frac{a^2\lambda^2}{6}\right)\|\nabla Q\|_{L^2}^2+\frac{b^2}{8}\lambda^{-4}\|\rho Q\|_{L^2}^2-\frac{1}{2}\|Q\|_{L^2}^2\right.\\
&\left.-\frac{3}{10}\left(a^2\lambda^2+a^{\frac{10}{3}}\right)\|Q\|_{L^{\frac{10}{3}}}^{\frac{10}{3}}\right).
\end{aligned}
$$

Taking $\lambda>0$ sufficiently large, we have $E_0(\psi_0)<0$. According to Theorem 1.1 the solution corresponding to Equation (1.1), with initial condition $\psi|_{t=0}=\psi_0$ exhibits finite-time blow-up. The proof is complete.

Proof of Theorem1.3. 1) Let us consider an initial condition $\psi_0\in\Sigma_A(\mathbb{R}^3)$ that fulfills conditions (1.7) and (1.9). Suppose $\psi:[0,T^*)\times\mathbb{R}^3\to\mathbb{C}$ represents the corresponding solution to Equation (1.1). According to the Gagliardo-Nirenberg inequality, we obtain

$$
\begin{aligned}
&E_0(\psi(t))\left[M(\psi(t))\right]^{\text{œ}_{cr}}\\
&\geq\frac{1}{2}\left(\|(\nabla+iA)(\psi(t))\|_{L^2}\|\psi(t)\|_{L^2}^{\text{œ}_{cr}}\right)^2-\frac{\mathcal{C}_{GN}}{4}\|(\nabla+iA)\psi(t)\|_{L^2}\|\psi(t)\|_{L^2}^{3+2\text{œ}_{cr}}\\
&\quad+\frac{b^2}{8}\|\rho\psi(t)\|_{L^2}\|\psi(t)\|_{L^2}^{2\text{œ}_{cr}}-\frac{\mathcal{C}_{opt}}{p+2}\|(\nabla+iA)\psi(t)\|_{L^2}^{\frac{3p}{2}}\|\psi(t)\|_{L^2}^{\frac{4-p}{2}+2\text{œ}_{cr}}\\
&\geq G\left(\|\nabla(\psi(t))\|_{L^2}\|\psi(t)\|_{L^2}^{\text{œ}_{cr}}\right)
\end{aligned}
$$

for all $t\in[0,T^*)$. Using (1.5) and (1.7), we have

$$
\begin{aligned}
G\left(\|\nabla(\psi(t))\|_{L^2}\|\psi(t)\|_{L^2}^{\text{œ}_{cr}}\right)&\leq E_0(\psi_0)\left[M(\psi_0)\right]^{\text{œ}_{cr}}\\
&<E^0(Q)\left[M(Q)\right]^{\text{œ}_{cr}}\\
&\leq G\left(\|\nabla Q\|_{L^2}\|Q\|_{L^2}^{\text{œ}_{cr}}\right)
\end{aligned}
\tag{3.3}
$$

for all $t\in[0,T^*)$. By (1.9), the continuity argument implies

$$
\|\nabla(\psi(t))\|_{L^2}\|\psi(t)\|_{L^2}^{\text{œ}_{cr}}<\|\nabla Q\|_{L^2}\|Q\|_{L^2}^{\text{œ}_{cr}}
$$

for all $t\in[0,T^*)$. By the conservation of mass, we infer that

$$
\sup_{t\in[0,T^*)}\|(\nabla+iA)\psi(t)\|_{L^2}\leq C\left(\|\psi\|_{L^2},\|Q\|_{L^2},\|\nabla Q\|_{L^2}\right)
$$

on the flip side, utilizing (1.5) and Gagliardo-Nirenberg we obtain

$$\begin{aligned}\frac{b^2}{8}\left\|\rho\psi(t)\right\|_{L^2}^2 &= E_0(\psi(t)) - \frac{1}{2}\left\|\nabla\psi(t)\right\|_{L^2}^2 - \frac{1}{4}\int_{\mathbb{R}^3}\left(\frac{1}{|x|}*|\psi(t)|^2\right)|\psi(t)|^2\,\mathrm{d}x \\ &\quad + \frac{1}{p+2}\left\|\psi(t)\right\|_{L^{p+2}}^{p+2} \\ &\le E_0(\psi(t)) - \frac{1}{4}\int_{\mathbb{R}^3}\left(\frac{1}{|x|}*|\psi(t)|^2\right)|\psi(t)|^2\,\mathrm{d}x + \frac{1}{p+2}\left\|\psi(t)\right\|_{L^{p+2}}^{p+2} \\ &\le E_0(\psi_0) - \frac{\mathcal{C}_{GN}}{4}\left\|\nabla\psi(t)\right\|_{L^2}\left\|\psi(t)\right\|_{L^2}^3 + \frac{\mathcal{C}_{opt}}{p+2}\left\|\nabla\psi(t)\right\|_{L^2}^{\frac{3p}{2}}\left\|\psi(t)\right\|_{L^2}^{\frac{4-p}{2}} \\ &\le C\left(E_0(\psi_0), M(\psi_0), \left\|\nabla Q\right\|_{L^2}, \left\|Q\right\|_{L^2}\right)\end{aligned}$$

for all $t\in\left[0,T^*\right)$. By (1.3) and the conservation of angular momentum, we have

$$\sup_{t\in\left[0,T^*\right)}\left\|(\nabla+iA)\psi(t)\right\|_{L^2} \le C\left(E_0(\psi_0), M(\psi_0), \left\|\nabla Q\right\|_{L^2}, \left\|Q\right\|_{L^2}\right),$$

which by the blow-up alternative, implies that $T^* = \infty$.

2) Let us now examine $\psi_0\in\Sigma_A\left(\mathbb{R}^3\right)$, which satisfies both conditions (1.7) and (1.10).

By employing the identical logical approach as used before, we can deduce that

$$\left\|\nabla\psi(t)\right\|_{L^2}\left\|\psi(t)\right\|_{L^2}^{\text{œ}_{cr}} > \left\|\nabla Q\right\|_{L^2}\left\|Q\right\|_{L^2}^{\text{œ}_{cr}} \tag{3.4}$$

for all $t\in\left[0,T^*\right)$. Next, we will demonstrate that the solution experiences blow-up within a finite timeframe. By drawing from Equation (1.7), we select a parameter $\vartheta = \vartheta(\psi_0, Q) > 0$, such that

$$E(\psi_0)\left[M(\psi_0)\right]^{\text{œ}_{cr}} \le (1-\vartheta)E^0(Q)\left[M(Q)\right]^{\text{œ}_{cr}}.$$

We also denote

$$\begin{aligned}H(f) &:= \left\|\nabla f\right\|_{L^2}^2 + \frac{b^2}{4}\left\|\rho f\right\|_{L^2}^2 + \frac{1}{4}\int_{\mathbb{R}^3}\left(\frac{1}{|x|}*|f|^2\right)|f|^2\,\mathrm{d}x - \frac{3p}{2(p+2)}\left\|f\right\|_{L^{p+2}}^{p+2} \\ &= \frac{3p}{2}E_0(f) - \frac{3p-4}{4}\left\|\nabla f\right\|_{L^2}^2 - \frac{3p-2}{8}\int_{\mathbb{R}^3}\left(\frac{1}{|x|}*|f|^2\right)|f|^2\,\mathrm{d}x \\ &\quad - \frac{(3p-4)b^2}{16}\left\|\rho f\right\|_{L^2}^2\end{aligned} \tag{3.5}$$

From (1.5) (3.6), the conservation of mass, we see that

$$\begin{aligned}&H(\psi(t))\left[M(\psi(t))\right]^{\text{œ}_{cr}} \\ &\le \frac{3p}{2}E(\psi(t))\left[M(\psi(t))\right]^{\text{œ}_{cr}} - \frac{3p-4}{4}\left(\left\|\nabla\psi(t)\right\|_{L^2}\left\|\psi(t)\right\|_{L^2}^{\text{œ}_{cr}}\right)^2 \\ &\le \frac{3p}{2}E(\psi(t))\left[M(\psi(t))\right]^{\text{œ}_{cr}} - \frac{3p-4}{4}\left(\left\|\nabla Q\right\|_{L^2}\left\|Q\right\|_{L^2}^{\text{œ}_{cr}}\right)^2 \\ &\le \frac{3p}{2}(1-\vartheta)E^0(Q)\left[M(Q)\right]^{\text{œ}_{cr}} - \frac{3p-4}{4}\left(\left\|\nabla Q\right\|_{L^2}\left\|Q\right\|_{L^2}^{\text{œ}_{cr}}\right)^2 \\ &= -\frac{3p-4}{4}\vartheta\left(\left\|\nabla Q\right\|_{L^2}\left\|Q\right\|_{L^2}^{\text{œ}_{cr}}\right)^2\end{aligned}$$

for all $t \in \left[0, T^*\right)$. The following can be derived from Lemma 3.1.

$$F''\left(\psi\left(t\right)\right) = 8H\left(\psi\left(t\right)\right) \le -2\left(3p-4\right)\vartheta\left(\frac{\|Q\|_{L^2}}{\|\psi_0\|_{L^2}}\right)^{2\text{œ}_{cr}}\|\nabla Q\|_{L^2}^2 < 0$$

for all $t \in \left[0, T^*\right)$. This proves that $T^* < \infty$. The proof is complete.

4. Conclusion

This paper studies the nonlinear Schrödinger equation with a constant magnetic field, and identifies the precise thresholds for the global existence and finite-time blow-up of its solutions in the mass-critical and mass-supercritical regimes where $\frac{4}{3} < p < 4$. In the mass-critical case, the L^2 norm of Q the unique positive radial solution to a certain elliptic equation serves as the mass threshold, solutions exist globally for initial data with mass below this threshold, while there exist initial data with mass above it that lead to finite-time blow-up, and the case of critical initial mass remains an open problem. For the mass-supercritical case, double thresholds based on the energy-mass and gradient-mass products associated with Q are established. Solutions exist globally when specific inequalities for these products are satisfied, and finite-time blow-up occurs when the inequalities are reversed. Meanwhile, several sufficient conditions for the finite-time blow-up of solutions are derived. In addition, the presence of a magnetic field makes the solutions of the equation lack scattering properties, whereas the scattering conclusion for the equation without a magnetic field has been proven.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

References

[1] Jin, R., Zeng, Z., You, C. and Yuan, C. (2024) Quantum Interferometers: Principles and Applications. *Progress in Quantum Electronics*, **96**, Article 100519. https://doi.org/10.1016/j.pquantelec.2024.100519

[2] Dinh, V.D. (2022) Remarks on Nonlinear Schrödinger Equations Arising on Rotational Bose-Einstein Condensates. *Nonlinear Analysis*, **214**, Article 112587. https://doi.org/10.1016/j.na.2021.112587

[3] Pitaevskii, L. and Stringari, S. (2016) Bose-Einstein Condensation and Superfluidity. Oxford Academic.

[4] Glassey, R.T. (1977) On the Blowing up of Solutions to the Cauchy Problem for Nonlinear Schrödinger Equations. *Journal of Mathematical Physics*, **18**, 1794-1797. https://doi.org/10.1063/1.523491

[5] Weinstein, M.I. (1983) Nonlinear Schrödinger Equations and Sharp Interpolation Estimates. *Communications in Mathematical Physics*, **87**, 567-576. https://doi.org/10.1007/bf01208265

[6] Feng, B. (2016) Sharp Threshold of Global Existence and Instability of Standing Wave for the Schrödinger-Hartree Equation with a Harmonic Potential. *Nonlinear Analysis: Real World Applications*, **31**, 132-145.

https://doi.org/10.1016/j.nonrwa.2016.01.012

[7] Cazenave, T. (2003) Semilinear Schrödinger Equations. American Mathematical Society. https://doi.org/10.1090/cln/010

[8] Sulem, C. and Sulem, P.L. (1999) The Nonlinear Schrodinger Equation: Self-Focusing and Wave Collapse. Springer.

[9] Lions, P.L. (1984) The Concentration-Compactness Principle in the Calculus of Variations. The Locally Compact Case, Part 1. *Annales de l'Institut Henri Poincaré C, Analyse Non linéaire*, **1**, 109-145. https://doi.org/10.1016/s0294-1449(16)30428-0

[10] Feng, B. (2018) On the Blow-Up Solutions for the Fractional Nonlinear Schrödinger Equation with Combined Power-Type Nonlinearities. *Communications on Pure and Applied Analysis*, **17**, 1785-1804. https://doi.org/10.3934/cpaa.2018085

[11] Guo, Q. and Zhu, S. (2018) Blow-Up for the Nonlinear Schrödinger Equation with Combined Nonlinearities.

[12] Zhang, J. (2020) Sharp Threshold of Global Existence for Nonlinear Schrödinger Equation with Partial Confinement. *Nonlinear Analysis*, **196**, Article 111832. https://doi.org/10.1016/j.na.2020.111832

[13] Arioli, G. and Szulkin, A. (2003) A Semilinear Schrödinger Equation in the Presence of a Magnetic Field. *Archive for Rational Mechanics and Analysis*, **170**, 277-295. https://doi.org/10.1007/s00205-003-0274-5

[14] Esteban, M.J. and Lions, P. (1989) Stationary Solutions of Nonlinear Schrödinger Equations with an External Magnetic Field. In: Colombini, F., Marino, A., Modica, L. and Spagnolo, S., Eds., *Partial Differential Equations and the Calculus of Variations*, Birkhäuser, 401-449.

[15] Kurata, K. (2000) Existence and Semi-Classical Limit of the Least Energy Solution to a Nonlinear Schrödinger Equation with Electromagnetic Fields. *Nonlinear Analysis: Theory, Methods & Applications*, **41**, 763-778. https://doi.org/10.1016/s0362-546x(98)00308-3

[16] Helffer, B. (2013) Spectral Theory and Its Applications. Cambridge University Press. https://doi.org/10.1017/cbo9781139505727

[17] Bouclet, J. and Mizutani, H. (2018) Uniform Resolvent and Strichartz Estimates for Schrödinger Equations with Critical Singularities. *Transactions of the American Mathematical Society*, **370**, 7293-7333. https://doi.org/10.1090/tran/7243

[18] Cazenave, T. and Esteban, M.J. (1988) On the Stability of Stationary States for Nonlinear Schrödinger Equations with an External Magnetic Field. *Computation and Applied Mathematics*, **7**, 155-168.

[19] Kieffer, T.F. and Loss, M. (2021) Non-Linear Schrödinger Equation in a Uniform Magnetic Field. In: Exner, P., Frank, R.L., Gesztesy, F., Holden, H. and Weidl, T., Eds., *Partial Differential Equations, Spectral Theory and Mathematical Physics*, EMS Press, 247-265. https://doi.org/10.4171/ecr/18-1/14

[20] Gonçalves Ribeiro, J.M. (1991) Finite Time Blow-Up for Some Nonlinear Schrödinger Equations with an External Magnetic Field. *Nonlinear Analysis: Theory, Methods & Applications*, **16**, 941-948. https://doi.org/10.1016/0362-546x(91)90098-l

[21] Ruiz, D. (2006) The Schrödinger-Poisson Equation under the Effect of a Nonlinear Local Term. *Journal of Functional Analysis*, **237**, 655-674. https://doi.org/10.1016/j.jfa.2006.04.005

[22] Tian, S., Yang, Y., Zhou, R. and Zhu, S. (2021) Energy Thresholds of Blow-Up for the Hartree Equation with a Focusing Subcritical Perturbation. *Studies in Applied Mathematics*, **146**, 658-676. https://doi.org/10.1111/sapm.12362

[23] Lieb, E.H. and Loss, M. (2001) Analysis: Graduate Studies in Mathematics. 2nd Edition, American Mathematical Society.

[24] Dinh, V.D. (2022) The 3D Nonlinear Schrödinger Equation with a Constant Magnetic Field Revisited. *Journal of Dynamics and Differential Equations*, **36**, 3643-3686. https://doi.org/10.1007/s10884-022-10235-1