Journal of Applied Mathematics and Physics, 2026, 14(5), 1845-1869
https://www.scirp.org/journal/jamp
ISSN Online: 2327-4379
ISSN Print: 2327-4352

Finite-Time Fast Synchronization of Chaotic Systems with Application to Secure Communications

Zhaowei Cai, Jiancheng Zhang, Xiaohua Wu, Ningbo Pang*

School of Mathematical Sciences & Center for Applied Mathematics of Guangxi, Guangxi Minzu University, Nanning, China
Email: ilzws222@163.com, jcz@gxmzu.edu.cn, 25430812@qq.com, *pangnb@gxmzu.edu.cn

How to cite this paper: Cai, Z.W., Zhang, J.C., Wu, X.H. and Pang, N.B. (2026) Finite-Time Fast Synchronization of Chaotic Systems with Application to Secure Communications. *Journal of Applied Mathematics and Physics*, **14**, 1845-1869.
https://doi.org/10.4236/jamp.2026.145090

Received: January 30, 2026
Accepted: May 16, 2026
Published: May 19, 2026

Abstract

This paper presents a fast chaotic synchronization and secure communication method by constructing a finite-time reduced-order observer under the drive-response configuration. As the output of the drive system, the drive signal usually contains a message signal which is inaccessible to the response system, which hinders the reduced-order observer design. To overcome this limitation, a new equivalent system is obtained by constructing a new output which does not contain the unknown message signal. Then, the observer matching condition and the strongly observable condition are developed, and they are proven to be maintained in the new equivalent system. Thereafter, a finite-time observer (FTO) with reduced-order as well as FTO-based message signal recovery methods are developed, which result in the simultaneous chaotic synchronization and secure communication. One of the advantages of the FTO-based chaotic synchronization and secure communication method over the conventional asymptotic convergence observer method is that both the chaotic synchronization and the message signal recovery can be reached in an arbitrarily pre-defined time, and thus it can achieve a faster as well as a higher quality secure communication. Finally, simulation results are given to illustrate the effectiveness of the proposed techniques.

Keywords

Fast Chaotic Synchronization, Secure Communication, Finite-Time Observer, Observer Matching Condition, Strongly Observable Condition

1. Introduction

Chaotic behavior is an interesting phenomenon in the nonlinearity community, and it has received a great amount of attention in the past decades. A chaotic system is a nonlinear deterministic system which exhibits complex and unpredictable behaviors. The sensitivity to initial values is the prominent characteristic of chaotic systems [1]. This unpredictable characteristic together with the noise-like property of a chaotic system makes chaotic synchronization a convenient carrier for secure communications.

The study on chaotic synchronization can track back to the 1990s [2]. Since then, various methods have been developed. One of the most widely used ideas is the so-called master-slave mode. The basic idea of the master-slave mode is to design an appropriate controller to guarantee that the slave system tracks the master system asymptotically. Following this idea, there are many different control methods that have been reported such as feedback control [3] [4], sliding mode control [5]-[7], impulsive control [8] [9], intermittent control [10] [11], adaptive control [12] [13] and so on. In addition to the master-slave mode, the well-known drive-response mode is also an effective and easily implemented idea for reaching chaotic synchronization. In 1990, Pecora and Carroll first discussed synchronization by proposing the so-called drive-response configuration in their pioneering work [14]. The mechanism of the drive-response mode is that the original chaotic system is used as a drive system, and an observer constructed based on some proper drive signals sent from the drive system is used as a response system to synchronize the drive system [15]. Following the drive-response idea or the observer-based synchronization scheme, much literature has been reported in the field of chaotic synchronization [14]-[23].

One of the important applications of chaotic synchronization is that it can be used for secure communication [15]. In literature, several kinds of chaotic secure communication techniques have been proposed including chaotic shift keying [24], chaotic parameter modulation [25] and chaotic masking [15] [26] [27]. In 1993, Dedieu *et al.* proposed the chaotic shift keying method, in which two chaotic systems with similar attractors are placed in the transmitter representing the bits 0 and 1. Meanwhile, at the receiver end, one of the transmitter systems is placed, and when the synchronization is achieved, the message signal is recovered [28]. This method is robust against noise, but the security level is low, especially when the transmitter and the receiver have different attractors. Subsequently, Yang *et al.* developed the chaotic parameter modulation approach [29]. In this approach, the message signal is employed to modulate the parameters of chaotic systems, and then, it can be retrieved based on the adaptation synchronization. This approach, compared with the chaotic shift keying method, has a higher security level [30]. In addition, for the chaotic masking secure communication, the message signal is hidden in a stronger chaotic signal in the transmitter, and the overall signal is then transmitted to the receiver end. And, based on the drive-response mode, when the synchronization is reached, the message signal can be recovered [15]. Note that although with a lower noise immunity, the chaotic masking method is simpler and easier to be implemented, and thus it, together with the observer-based chaotic synchronization, has become a predominant technique in the field of chaotic secure communication [9] [15] [21] [23] [26] [27] [31]-[36].

For chaotic masking secure communication schemes, the chaotic system is used as a message signal carrier. And only when the chaotic synchronization is achieved precisely, can the message signal recovery be done well at the receiver end. Therefore, there is no doubt that both the speed and the accuracy of the synchronization are of great significance. In fact, an accurate and fast chaotic synchronization can not only ensure the timely and accurate restoration of the message signal, but also effectively reduce the risk of information leakage during the long-time communication process. Unfortunately, for the conventional methods (see, for instance, [26] [27] [31] [34]), the asymptotic convergence observer (ACO) based synchronization schemes are merely able to achieve the asymptotic synchronization between the drive system and the response system. In this sense, for the ACO-based method neither the synchronization accuracy nor the synchronization achievement time could be guaranteed. This is because, for the ACO method, both the accuracy and the speed of the synchronization are usually not only dependent on the observer convergence rate but also dependent on the initial value differences between the drive system and the response system (ACO). At the same time, most existing fast synchronization approaches, such as sliding mode observers and super-twisting observers, achieve finite-time convergence but suffer from two critical drawbacks: they introduce undesirable chattering caused by discontinuous control terms, and their convergence time depends heavily on system parameters and initial conditions rather than being truly user-assignable. Therefore, it is desirable to find a feasible and easy-to-implement fast synchronization method. In contrast, the finite-time observer (FTO) scheme eliminates chattering completely and achieves a genuinely pre-defined convergence time independent of initial states and observer gains. This observation motivates our research.

In the present paper, we are dedicated to proposing a kind of finite-time fast chaotic synchronization and secure communication scheme via a finite-time observer (FTO) method which can reach the chaotic synchronization and the secure communication in an arbitrarily pre-defined time. The major contributions are summarized as follows: 1) In order to deal with the observer design in the case of the message signal being embedded into the output signal (drive signal), a new equivalent system is constructed in which the unknown signal is removed in the corresponding output. 2) The observer matching condition and the strongly observable condition are developed, and they are proven to be maintained in the new equivalent system. 3) Based on (1) and (2), a finite-time observer is constructed for the implementation of chaotic synchronization and secure communication. In addition, the features and the advantages of the proposed FTO-based method can be concluded as:

1) The FTO-based method can achieve accurate synchronization of chaotic systems, and thus, compared with other methods such as the conventional ACO-based methods [26] [27] [31] [34] or the fast synchronization methods [22] [32] [37], it can realize a higher quality of secure communication.

2) As a secure communication carrier, the FTO-based fast synchronization enables the message signal transmission to be implemented at any pre-defined time, which implies that provided that the safe communication beginning time is appropriately chosen, it can avoid some unexpected attacks in the signal channels. However, neither the conventional ACO-based methods [26] [27] [31] [34] nor the fast synchronization methods [22] [32] [37] have this advantage.

3) Unlike the conventional ACO-based methods [26] [27] [31] [34] or the fast synchronization methods [22] [32] [37], for the FTO-based secure communication method, both the communication speed and the communication quality are independent with the choices of the gain matrices and the initial values of FTO, which leaves much more freedom for the implementation of the FTO design.

The rest of the paper is organized as follows. In Section 2, a general model which represents a class of chaotic systems is given, and meanwhile, some background statements are presented. In Section 3, a reduced-order finite-time observer is constructed to reach the chaotic synchronization. In Section 4, for the implementation of secure communication, an algebraic message signal recovery method is proposed. Simulation results are given to illustrate the effectiveness of the proposed methods in Section 5. Finally, some conclusions are summarized in Section 6.

2. Problem Statement and Preliminaries

Consider a class of chaotic systems used as the transmitters in chaotic synchronization based secure communication as follows:

$$\begin{cases} \dot{x}(t) = Ax(t) + Bf(x) + Es(t) \\ y(t) = Cx(t) + Gs(t) \end{cases} \tag{1}$$

where $x \in \mathbb{R}^n$, $y \in \mathbb{R}^p$ and $s \in \mathbb{R}^r$ are the state, drive signal and message signal vectors, respectively. A, B, C, E and G are constant matrices with appropriate dimensions. $f(\cdot): \mathbb{R}^n \to \mathbb{R}^m$ represents the nonlinear item in the chaotic system.

Many chaotic systems can be denoted in the form of (1). For example, the famous Lorenz system is a typical chaotic system which is described by

$$\begin{cases} \dot{x}_1(t) = -ax_1(t) + x_2(t) \cdot x_3(t) \\ \dot{x}_2(t) = -bx_2(t) + bx_3(t) \\ \dot{x}_3(t) = -x_1(t) \cdot x_2(t) + cx_2(t) - x_3(t) \end{cases}$$

It is known that when we choose parameters $a = 8/3$, $b = 10$ and $c = 28$, such the system owns chaotic behaviors. Obviously, this Lorenz system can be written in the form of (1) with

$$A = \begin{bmatrix} -a & 0 & 0 \\ 0 & -b & b \\ 0 & c & -1 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & -1 \end{bmatrix} \text{ and } f(x) = \begin{bmatrix} x_2(t) \cdot x_3(t) \\ x_1(t) \cdot x_2(t) \end{bmatrix}.$$

Besides, the Lur'e system is also a well-known chaotic system which is depicted by

$$\begin{cases} \dot{x}_1(t) = x_2(t) \\ \dot{x}_2(t) = x_3(t) \\ \dot{x}_3(t) = bx_2(t) + ax_3(t) + N(x_1(t)) \end{cases}$$

where

$$N(x_1(t)) = \begin{cases} 1.8x_1(t), & |x_1(t)| \le 1 \\ -2.6x_1(t) + 5.4\text{sign}(x_1(t)), & 1 < |x_1(t)| \le 3 \\ -5.4\text{sign}(x_1(t)), & |x_1(t)| > 3. \end{cases}$$

When the parameters are chosen as $a = -1$ and $b = -1.25$, the system will exhibit chaotic behaviors. Also, this system can be written in the form of (1) with

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & b & a \end{bmatrix}, B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \text{ and } f(x) = N(x_1(t)).$$

Remark 1. From (1) it can be seen that the message signal $s(t)$ is embedded into both the drive signal and the chaotic system. In [3] [15] [17] [26] [27] [34] [35], the message signal is merely embedded into chaotic systems. Therefore, the models in [3] [15] [17] [26] [27] [34] [35] can be considered as special cases of system (1) when $G = 0$. The signal masking method given by (1) has the following two advantages. 1) if the message signal is embedded into the drive signal, the message signal recovery in the secure communication will become easy at the receiver end [31]. Otherwise, some differentiation signals or equivalent control methods are needed, which may reduce the accuracy of the message recovery accuracy [15] [17] [26] [27] [34]. 2) if message signal is also embedded into the chaotic system at the same time, the structure of the chaotic system will become more complex and, for the unsuspecting receivers, the observer design will become more difficult, which enhances the security of the communication.

Remark 2. In chaotic masking secure communication schemes, it is usually required that the message signal must be smaller enough such that chaotic behaviors can still be kept even if the message signal is masked to the dynamic chaotic system [30]. However, this limitation would not increase the conservativeness of the chaotic synchronization implementation. In fact, if the message signal $s(t)$ is not small enough, a more smaller signal $\bar{s}(t) = s(t)/10^n$ (n is a positive integer) can be introduced as an intermediate message signal to replace $s(t)$ to participate in secure communications. Then, after the intermediate signal $\bar{s}(t)$ is recovered at the receiver end, the original message signal $s(t)$ can be obtained by $s(t) = 10^n \bar{s}(t)$ according to the prior agreed rules.

In system (1), let $D = [B \quad E]$, $F = [\mathbf{0}_{p\times m} \quad G]$ and $\omega(t) = [f^{\mathrm{T}}(x) \quad s^{\mathrm{T}}(t)]^{\mathrm{T}}$, then system (1) can be rewritten as

$$\begin{cases} \dot{x}(t) = Ax(t) + D\omega(t) \\ y(t) = Cx(t) + F\omega(t) \end{cases} \tag{2}$$

In order to construct a reduced-order observer (response system) to reach the fast chaotic synchronization and the message signal recovery, one definition, one lemma and two assumptions are presented firstly.

Definition 1. System (2) characterized by $\Sigma_2(A,D,C,F)$ is strongly observable, if for any initial value $x(0)\in\mathbb{R}^n$, $y(t)=0$ implies $x(t)=0$, for any $t\geq 0$.

Lemma 1. System (2) characterized by $\Sigma_2(A,D,C,F)$ is strongly observable, if and only if

$$\operatorname{rank}\begin{bmatrix} sI_n-A & -D \\ C & F \end{bmatrix}=n+\operatorname{rank}\begin{bmatrix} D \\ F \end{bmatrix},\ \text{for any } s\in\mathbb{C}.$$

Assumption 1. (**Observer matching condition**) For system (2), matrices D,C,F satisfy the following condition

$$\operatorname{rank}\begin{bmatrix} CD & F \\ F & 0 \end{bmatrix}=\operatorname{rank}(F)+\operatorname{rank}\begin{bmatrix} D \\ F \end{bmatrix}.$$

Assumption 2. (**Strongly observable condition**) System (2) characterized by $\Sigma_2(A,D,C,F)$ is strongly observable.

Remark 3. Generally speaking, the classical reduced-order observer design methods contain two steps. The first step is using state transformations to obtain some partial states by separating them from the measurable output. Then, in the second step, based on the known states, the rest of the unknown states can be estimated by constructing a dynamic system. Therefore, for the observer design for drive system (2), once the drive signal y (measurable output of drive system (2)) is tainted by an unknown message signal $s(t)$, neither the first step nor the second step can be implemented. On the other hand, as we know, if the output equation does not contain an unknown signal, it is easy to design a reduced-order observer provided that the observer matching condition and the strongly observable condition are satisfied [15] [17] [34]. Thus, in order to design a reduced-order observer for system (2), it is desirable to produce a new equivalent system whose output does not contain the unknown signal. In this scenario, if the new system still satisfies the observer matching condition and the strongly observable condition, the reduced-order observer design can follow.

3. Fast Chaotic Synchronization

In this section, we first construct a new equivalent system whose output does not contain an unknown signal $\omega(t)$. And, based on such convenience, a reduced-order finite-time observer as a response system is constructed to reach fast synchronization.

3.1. Equivalent System and Two Important Lemmas

According to matrix equation solution theory, the solution of $\omega(t)$ to the output equation of system (2) is

$$\omega(t)=F^{-}\left(y(t)-Cx(t)\right)+\left(I_q-F^{-}F\right)\bar{\omega}(t) \tag{3}$$

where F^{-} is an arbitrary generalized inverse of F satisfying $FF^{-}F = F$, and ϖ is a deterministic but unknown vector. Substituting (3) into (2) and pre-multiplying matrix $I_p - FF^{-}$ in the output equation of system (2) leads to a new equivalent system

$$\begin{cases} \dot{x}(t) = \bar{A}x(t) + DF^{-}y(t) + \bar{D}\varpi(t) \\ \bar{y}(t) = \bar{C}x(t) \end{cases} \tag{4}$$

where $\bar{y}(t) = \left(I_p - FF^{-}\right)y(t)$, $\bar{A} = A - DF^{-}C$, $\bar{D} = D\left(I_q - F^{-}F\right)$ and $\bar{C} = \left(I_p - FF^{-}\right)C$.

Assumption 3. Matrices $\bar{C}$ and $\bar{D}$ have full ranks, i.e., $\operatorname{rank}\left(\bar{C}\right) = p$ and $\operatorname{rank}\left(\bar{D}\right) = q$.

Remark 4. The condition of Assumption 3 does not increase the design conservatism. In fact, if matrix $\bar{C}$ is not full row rank, we always can find a full row rank matrix $C^{\ddagger}$, where all the row vectors of $C^{\ddagger}$ are selected from $\bar{C}$ such that $\operatorname{rank}\left(C^{\ddagger}\right) = \operatorname{rank}\left(\bar{C}\right)$. Then, a new output $y^{\ddagger}(t) = C^{\ddagger}x(t)$ with $C^{\ddagger}$ being full row rank is obtained. Then, redefine $\bar{y} := y^{\ddagger}$ and $\bar{C} = C^{\ddagger}$ and Assumption 3 is satisfied. The same operation can be applied to $\bar{D}$.

Remark 5. In the equivalent system (4), the output equation does not contain an unknown signal $\omega(t)$. By using this convenience, a reduced-order observer can be designed provided that both the strongly observable condition and the observer matching condition are satisfied for the equivalent system (4). It will be shown in the following that for the new equivalent system (4), both the strongly observable condition and the observer matching condition are maintained.

Lemma 2. Under Assumption 1, the observer matching condition for system (4) holds, *i.e.*,

$$\operatorname{rank}\left(\bar{C}\bar{D}\right) = \operatorname{rank}\left(\bar{D}\right). \tag{5}$$

Proof. On the one hand, we have

$$\begin{aligned}
\operatorname{rank}\begin{bmatrix} CD & F \\ F & 0 \end{bmatrix} &= \operatorname{rank}\left(\begin{bmatrix} CD & F \\ F & 0 \end{bmatrix}\begin{bmatrix} I_q - F^{-}F & F^{-}F & 0 \\ 0 & 0 & I_q \end{bmatrix}\right) \\
&= \operatorname{rank}\begin{bmatrix} C\bar{D} & CDF^{-}F & F \\ 0 & F & 0 \end{bmatrix} \\
&= \operatorname{rank}\left(\begin{bmatrix} I_p & -CDF^{-} \\ 0 & I_p \end{bmatrix}\begin{bmatrix} C\bar{D} & CDF^{-}F & F \\ 0 & F & 0 \end{bmatrix}\right) \\
&= \operatorname{rank}\begin{bmatrix} C\bar{D} & 0 & F \\ 0 & F & 0 \end{bmatrix} = \operatorname{rank}(F) + \operatorname{rank}\begin{bmatrix} C\bar{D} & F \end{bmatrix} \\
&= \operatorname{rank}(F) + \operatorname{rank}\left(\begin{bmatrix} I_p - FF^{-} \\ FF^{-} \end{bmatrix}\begin{bmatrix} C\bar{D} & F \end{bmatrix}\right) \\
&= \operatorname{rank}(F) + \operatorname{rank}\begin{bmatrix} \bar{C}\bar{D} & 0 \\ FF^{-}C\bar{D} & F \end{bmatrix}
\end{aligned}$$

$$= \operatorname{rank}(F) + \operatorname{rank}\left(\begin{bmatrix} \bar{C}\bar{D} & 0 \\ FF^{-}\bar{C}\bar{D} & F \end{bmatrix}\begin{bmatrix} I & 0 \\ -F^{-}\bar{C}\bar{D} & I \end{bmatrix}\right)$$

$$= \operatorname{rank}(F) + \operatorname{rank}\begin{bmatrix} \bar{C}\bar{D} & 0 \\ 0 & F \end{bmatrix} = 2\operatorname{rank}(F) + \operatorname{rank}(\bar{C}\bar{D}).$$

On the other hand,

$$\operatorname{rank}(F) + \operatorname{rank}\begin{bmatrix} D \\ F \end{bmatrix} = \operatorname{rank}(F) + \operatorname{rank}\left(\begin{bmatrix} D \\ F \end{bmatrix}\begin{bmatrix} I_q - F^{-}F & F^{-}F \end{bmatrix}\right)$$

$$= \operatorname{rank}(F) + \operatorname{rank}\begin{bmatrix} \bar{D} & DF^{-}F \\ 0 & F \end{bmatrix}$$

$$= \operatorname{rank}(F) + \operatorname{rank}\left(\begin{bmatrix} I & -DF^{-} \\ 0 & I \end{bmatrix}\begin{bmatrix} \bar{D} & DF^{-}F \\ 0 & F \end{bmatrix}\right)$$

$$= \operatorname{rank}(F) + \operatorname{rank}\begin{bmatrix} \bar{D} & 0 \\ 0 & F \end{bmatrix}$$

$$= 2\operatorname{rank}(F) + \operatorname{rank}(\bar{D}).$$

Therefore, under Assumption 1, we have $\operatorname{rank}(\bar{C}\bar{D}) = \operatorname{rank}(\bar{D})$. The proof is completed.

Lemma 3. Under Assumption 2, the following strongly observable condition for system (4) holds, *i.e.*,

$$\operatorname{rank}\begin{bmatrix} sI - \bar{A} & -\bar{D} \\ \bar{C} & 0 \end{bmatrix} = n + \operatorname{rank}(\bar{D}) \text{ for any } s \in \mathbb{C}. \tag{6}$$

Proof. On the one hand, for any $s \in \mathbb{C}$ we have

$$\operatorname{rank}\begin{bmatrix} sI - A & -D \\ C & F \end{bmatrix} = \operatorname{rank}\left(\begin{bmatrix} sI - A & -D \\ C & F \end{bmatrix}\begin{bmatrix} I_n & 0 & 0 \\ 0 & I_q - F^{-}F & F^{-}F \end{bmatrix}\right)$$

$$= \operatorname{rank}\begin{bmatrix} sI - A & -\bar{D} & -DF^{-}F \\ C & 0 & F \end{bmatrix}$$

$$= \operatorname{rank}\left(\begin{bmatrix} I_n & DF^{-} \\ 0 & I_p \end{bmatrix}\begin{bmatrix} sI - A & -\bar{D} & -DF^{-}F \\ C & 0 & F \end{bmatrix}\right)$$

$$= \operatorname{rank}\begin{bmatrix} sI - \bar{A} & -\bar{D} & 0 \\ C & 0 & F \end{bmatrix}$$

$$= \operatorname{rank}\left(\begin{bmatrix} I_n & 0 \\ 0 & I_p - FF^{-} \\ 0 & FF^{-} \end{bmatrix}\begin{bmatrix} sI - \bar{A} & -\bar{D} & 0 \\ C & 0 & F \end{bmatrix}\right)$$

$$= \operatorname{rank}\begin{bmatrix} sI - \bar{A} & -\bar{D} & 0 \\ \bar{C} & 0 & 0 \\ FF^{-}C & 0 & F \end{bmatrix}$$

$$= \operatorname{rank}\left(\begin{bmatrix} sI - \bar{A} & -\bar{D} & 0 \\ \bar{C} & 0 & 0 \\ FF^{-}C & 0 & F \end{bmatrix}\begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ -F^{-}C & 0 & I \end{bmatrix}\right)$$

$$\begin{aligned}
&= \operatorname{rank}\begin{bmatrix} sI-\bar{A} & -\bar{D} & 0 \\ \bar{C} & 0 & 0 \\ 0 & 0 & F \end{bmatrix} \\
&= \operatorname{rank}(F) + \operatorname{rank}\begin{bmatrix} sI-\bar{A} & -\bar{D} \\ \bar{C} & 0 \end{bmatrix}.
\end{aligned} \tag{7}$$

On the other hand,

$$\begin{aligned}
n+\operatorname{rank}\begin{bmatrix} D \\ F \end{bmatrix} &= n+\operatorname{rank}\left(\begin{bmatrix} D \\ F \end{bmatrix}\begin{bmatrix} I_q - F^- F & F^- F \end{bmatrix}\right) \\
&= n+\operatorname{rank}\begin{bmatrix} \bar{D} & DF^- F \\ 0 & F \end{bmatrix} \\
&= n+\operatorname{rank}\left(\begin{bmatrix} I & -DF^- \\ 0 & I \end{bmatrix}\begin{bmatrix} \bar{D} & DF^- F \\ 0 & F \end{bmatrix}\right) \\
&= n+\operatorname{rank}\begin{bmatrix} \bar{D} & 0 \\ 0 & F \end{bmatrix} \\
&= n+\operatorname{rank}(F)+\operatorname{rank}(\bar{D}).
\end{aligned} \tag{8}$$

Therefore, it follows from (7) and (8) that (6) holds, this ends the proof. □

Remark 6. It has been shown that the properties of the observer matching condition (5) and the strongly observability condition (6) of the system (A, D, C, F) can be maintained in the new system $(\bar{A}, \bar{D}, \bar{C})$. Such properties will play key roles in the following design process.

3.2. Reduced-Order Finite-Time Observer Design

Since $\operatorname{rank}(\bar{D}) < n$, there exists a matrix $\bar{U} \in \mathbb{R}^{n\times(n-q)}$ such that $T = \begin{bmatrix} \bar{D} & \bar{U} \end{bmatrix}$ is non-singular. Now, performing state transformation $\breve{x}(t) = T^{-1}x(t)$ yields

$$\begin{cases} \dot{\breve{x}}(t) = \breve{A}\breve{x}(t) + \breve{R}y(t) + \breve{D}\varpi(t) \\ \bar{y}(t) = \breve{C}\breve{x}(t) \end{cases} \tag{9}$$

where $\breve{A} = T^{-1}\bar{A}T$, $\breve{R} = T^{-1}DF^-$, $\breve{D} = T^{-1}\bar{D}$ and $\breve{C} = \bar{C}T = \begin{bmatrix} \bar{C}\bar{D} & \bar{C}\bar{U} \end{bmatrix}$. Decompose state $\breve{x}$, matrices $\breve{A}, \breve{R}, \breve{D}$ into block vector or matrices as follows:

$$\breve{x} = \begin{bmatrix} \breve{x}_1 \\ \breve{x}_2 \end{bmatrix}, \breve{A} = \begin{bmatrix} \breve{A}_{11} & \breve{A}_{12} \\ \breve{A}_{21} & \breve{A}_{22} \end{bmatrix}, \breve{R} = \begin{bmatrix} \breve{R}_1 \\ \breve{R}_2 \end{bmatrix}, \breve{D} = \begin{bmatrix} \breve{D}_1 \\ \breve{D}_2 \end{bmatrix} = \begin{bmatrix} I_q \\ 0 \end{bmatrix}$$

where $\breve{x}_1 \in \mathbb{R}^q$, $\breve{A}_{11} \in \mathbb{R}^{q\times q}$ and $\breve{R}_1 \in \mathbb{R}^{q\times p}$. Subsequently, in (9) we drop the state equation of $\breve{x}_1$ and deduce that

$$\begin{cases} \dot{\breve{x}}_2(t) = \breve{A}_{21}\breve{x}_1(t) + \breve{A}_{22}\breve{x}_2(t) + \breve{R}_2 y(t) \\ \bar{y}(t) = \breve{C}\breve{x}(t) = \bar{C}\bar{D}\breve{x}_1(t) + \bar{C}\bar{U}\breve{x}_2(t) \end{cases} \tag{10}$$

Note that $\operatorname{rank}(\bar{C}\bar{D}) = q < p$, thus we extend $\bar{C}\bar{D}$ to be a non-singular matrix $V = \begin{bmatrix} H & \bar{C}\bar{D} \end{bmatrix}$ with $H \in \mathbb{R}^{p\times(p-q)}$. Let $V^{-1} = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$ with $V_1 \in \mathbb{R}^{(p-q)\times p}$ and $V_2 \in \mathbb{R}^{q\times p}$, thus we have

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \begin{bmatrix} H & \bar{C}\bar{D} \end{bmatrix} = \begin{bmatrix} V_1 H & V_1 \bar{C}\bar{D} \\ V_2 H & V_2 \bar{C}\bar{D} \end{bmatrix} = \begin{bmatrix} I_{p-q} & 0 \\ 0 & I_q \end{bmatrix},$$

which implies that $V_1\bar{C}\bar{D} = 0$ and $V_2\bar{C}\bar{D} = I_q$. Pre-multiplying matrix V^{-1} in both sides of the output of (10), and one obtains

$$V_1\bar{y}(t) = V_1\bar{C}\bar{U}\breve{x}_2(t) \tag{11}$$

$$V_2\bar{y}(t) = \breve{x}_1(t) + V_2\bar{C}\bar{U}\breve{x}_2(t). \tag{12}$$

Then, substituting (12) into (10) and considering (11), we have

$$\begin{cases} \dot{\breve{x}}_2(t) = \left(\breve{A}_{22} - \breve{A}_{21}V_2\bar{C}\bar{U}\right)\breve{x}_2(t) + \breve{A}_{21}V_2\bar{y}(t) + \breve{R}_2 y(t) \\ V_1\bar{y}(t) = V_1\bar{C}\bar{U}\breve{x}_2(t) \end{cases} \tag{13}$$

Based on system (13) and according to the finite-time observer theory [38], a finite-time observer exists provided that $\left(\breve{A}_{22} - \breve{A}_{21}V_2\bar{C}\bar{U}, V_1\bar{C}\bar{U}\right)$ is observable. To this end, we give Lemma 4.

Lemma 4. Matrix pair $\left(\breve{A}_{22} - \breve{A}_{21}V_2\bar{C}\bar{U}, V_1\bar{C}\bar{U}\right)$ is observable, if and only if matrix triple $\left(\bar{A}, \bar{C}, \bar{D}\right)$ is strongly observable.

Proof. Matrix triple $\left(\bar{A}, \bar{C}, \bar{D}\right)$ is strongly observable, if and only if for any $s \in \mathbb{C}$ we have

$$\begin{aligned}
n + \operatorname{rank}\left(\bar{D}\right) &= \operatorname{rank}\begin{bmatrix} sI - \bar{A} & -\bar{D} \\ \bar{C} & 0 \end{bmatrix} \\
&= \operatorname{rank}\left(\begin{bmatrix} T^{-1} & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} sI - \bar{A} & -\bar{D} \\ \bar{C} & 0 \end{bmatrix}\begin{bmatrix} T & 0 \\ 0 & I \end{bmatrix}\right) \\
&= \operatorname{rank}\begin{bmatrix} sI - \breve{A}_{11} & -\breve{A}_{12} & -I_q \\ -\breve{A}_{21} & sI - \breve{A}_{22} & 0 \\ \bar{C}\bar{D} & \bar{C}\bar{U} & 0 \end{bmatrix} \\
&= \operatorname{rank}\left(\begin{bmatrix} I_q & 0 & 0 \\ 0 & I_{n-q} & 0 \\ 0 & 0 & \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \end{bmatrix}\begin{bmatrix} sI - \breve{A}_{11} & -\breve{A}_{12} & -I_q \\ -\breve{A}_{21} & sI - \breve{A}_{22} & 0 \\ \bar{C}\bar{D} & \bar{C}\bar{U} & 0 \end{bmatrix}\right) \\
&= \operatorname{rank}\begin{bmatrix} sI - \breve{A}_{11} & -\breve{A}_{12} & -I_q \\ -\breve{A}_{21} & sI - \breve{A}_{22} & 0 \\ 0 & V_1\bar{C}\bar{U} & 0 \\ I_q & V_2\bar{C}\bar{U} & 0 \end{bmatrix} \\
&= \operatorname{rank}\left(\begin{bmatrix} sI - \breve{A}_{11} & -\breve{A}_{12} & -I_q \\ -\breve{A}_{21} & sI - \breve{A}_{22} & 0 \\ 0 & V_1\bar{C}\bar{U} & 0 \\ I_q & V_2\bar{C}\bar{U} & 0 \end{bmatrix}\begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ sI - \breve{A}_{11} & -\breve{A}_{12} & I_q \end{bmatrix}\right)
\end{aligned}$$

$$= \operatorname{rank}\begin{bmatrix} 0 & 0 & -I_q \\ -\breve{A}_{21} & sI-\breve{A}_{22} & 0 \\ 0 & V_1\bar{C}\bar{U} & 0 \\ I_q & V_2\bar{C}\bar{U} & 0 \end{bmatrix} = q + \operatorname{rank}\begin{bmatrix} sI-\breve{A}_{22} & -\breve{A}_{21} \\ V_1\bar{C}\bar{U} & 0 \\ V_2\bar{C}\bar{U} & I_q \end{bmatrix}$$

$$= q + \operatorname{rank}\left(\begin{bmatrix} I_{n-q} & 0 & \breve{A}_{21} \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}\begin{bmatrix} sI-\breve{A}_{22} & -\breve{A}_{21} \\ V_1\bar{C}\bar{U} & 0 \\ V_2\bar{C}\bar{U} & I_q \end{bmatrix}\right)$$

$$= q + \operatorname{rank}\begin{bmatrix} sI_{n-q}-\left(\breve{A}_{22}-\breve{A}_{21}V_2\bar{C}\bar{U}\right) & 0 \\ V_1\bar{C}\bar{U} & 0 \\ V_2\bar{C}\bar{U} & I_q \end{bmatrix}$$

$$= 2q + \operatorname{rank}\begin{bmatrix} sI_{n-q}-\left(\breve{A}_{22}-\breve{A}_{21}V_2\bar{C}\bar{U}\right) \\ V_1\bar{C}\bar{U} \end{bmatrix}.$$

Based on $\operatorname{rank}\left(\bar{D}\right)=q$, we conclude that the above equation holds if and only if $\operatorname{rank}\begin{bmatrix} sI_{n-q}-\left(\breve{A}_{22}-\breve{A}_{21}V_2\bar{C}\bar{U}\right) \\ V_1\bar{C}\bar{U} \end{bmatrix}=n-q$ holds for any $s\in\mathbb{C}$. This completes the proof. □

Based on Lemma 4, for any $\rho_2>\rho_1>0$ we can find gain matrices L_1 and L_2 such that both the matrix $N_1=\breve{A}_{22}-\breve{A}_{21}V_2\bar{C}\bar{U}-L_1V_1\bar{C}\bar{U}$ and $N_2=\breve{A}_{22}-\breve{A}_{21}V_2\bar{C}\bar{U}-L_2V_1\bar{C}\bar{U}$ are Hurwitz, and they satisfy $-\rho_2<\lambda_i\left(N_1\right)<-\rho_1<\lambda_j\left(N_2\right)<0$, where $\lambda_i\left(N_1\right)$ denotes the i-th eigenvalue of matrix N_1. Let $N=\operatorname{diag}\left\{N_1,N_2\right\}$, $L=\begin{bmatrix} L_1 \\ L_2 \end{bmatrix}$ and $S=\begin{bmatrix} I_{n-q} \\ I_{n-q} \end{bmatrix}$. Then, a finite-time observer is constructed in the form of as follows

$$\begin{cases} \dot{z}(t)=Nz(t)+LV_1\bar{y}(t)+S\left(\breve{A}_{21}V_2\bar{y}(t)+\breve{R}_2y(t)\right) \\ \hat{\breve{x}}_2(t)=K\left[z(t)-e^{N\tau}z(t-\tau)\right] \\ \hat{x}(t)=T\begin{bmatrix} V_2\bar{y}(t)-V_2\bar{C}\bar{U}\hat{\breve{x}}_2(t) \\ \hat{\breve{x}}_2(t) \end{bmatrix} \end{cases} \tag{14}$$

where $K=\begin{bmatrix} I_{n-q} & 0_{(n-q)\times(n-q)} \end{bmatrix}\begin{bmatrix} S & e^{N\tau}S \end{bmatrix}^{-1}$ with constant $\tau>0$ and $z(t)\equiv 0$ when $-\tau\le t<0$.

Theorem 5. Under Assumptions 1 - 2, (14) is a finite-time observer of the drive system (2), *i.e.*, for an arbitrarily pre-defined time $\tau>0$ equation $\hat{x}(t)\equiv x(t)$ holds when $t\ge t_0+\tau$.

Proof. If $\hat{\breve{x}}_2(t)\equiv\breve{x}_2(t)$ holds, according to the state transformation $\breve{x}(t)=T^{-1}x(t)$ and equation (12), it can be concluded that $\hat{x}(t)\equiv x(t)$. Thus, in the following we only need to show $\hat{\breve{x}}_2(t)\equiv\breve{x}_2(t)$ when $t\ge t_0+\tau$.

Since the eigenvalues of N_1 and N_2 satisfy $-\rho_2<\lambda_i\left(N_1\right)<-\rho_1<\lambda_j\left(N_2\right)<0$, according to the discussion in [38]

$\det\begin{bmatrix} S & e^{N\tau}S \end{bmatrix} \neq 0$ holds for any $\tau > 0$, and thus matrix K exists. In this way, for $t \geq t_0 + \tau$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left[z(t) - S\breve{x}_2(t)\right] &= Nz(t) + LV_1\bar{y}(t) + S\left(\breve{A}_{21}V_2\bar{y}(t) + \breve{R}_2 y(t)\right) \\ &\quad - S\left[\left(\breve{A}_{22} - \breve{A}_{21}V_2\bar{C}\bar{U}\right)\breve{x}_2(t) + \breve{A}_{21}V_2\bar{y}(t) + \breve{R}_2 y(t)\right] \\ &= N\left[z(t) - S\breve{x}_2(t)\right].\end{aligned}$$

This implies that for $t \geq t_0 + \tau$

$$z(t) - S\breve{x}_2(t) = \mathrm{e}^{N\tau}\left[z(t-\tau) - S\breve{x}_2(t-\tau)\right]. \tag{15}$$

On the other hand, it is easy to verify that $KS = I_{n-q}$ and $K\mathrm{e}^{N\tau}S = 0$. This together with (15) gives

$$\breve{x}_2(t) = KS\breve{x}_2(t) = K\left[z(t) - \mathrm{e}^{N\tau}z(t-\tau) + \mathrm{e}^{N\tau}S\breve{x}_2(t-\tau)\right] = K\left[z(t) - \mathrm{e}^{N\tau}z(t-\tau)\right].$$

Therefore, based on the finite-time observer (14), we have $\hat{\breve{x}}_2(t) \equiv \breve{x}_2(t)$ when $t \geq t_0 + \tau$, and this ends the proof. □

4. Chaotic Synchronization-Based Secure Communication

The secure communication is one of the most important applications of chaotic synchronization. In general, in the drive-response based secure communication configuration, the drive system acts as a transmitter and sends drive signal with message signal hidden in it. Meanwhile, at the receiver end, the response system synchronizes the transmitter. During this process, the message signal can be recovered. Based on the mechanism, in the present paper the drive system (2) is used as the transmitter, and its output y is the drive signal. The message signal s is embedded into both the chaotic signal and the drive signal. Since in the present paper the response system at the receiver end can synchronize the drive system within any pre-defined time τ, accordingly the message signal s can also be recovered within τ. The mechanism of the FTO-based chaotic synchronization and secure communication in the present paper can be illustrated in Figure 1.

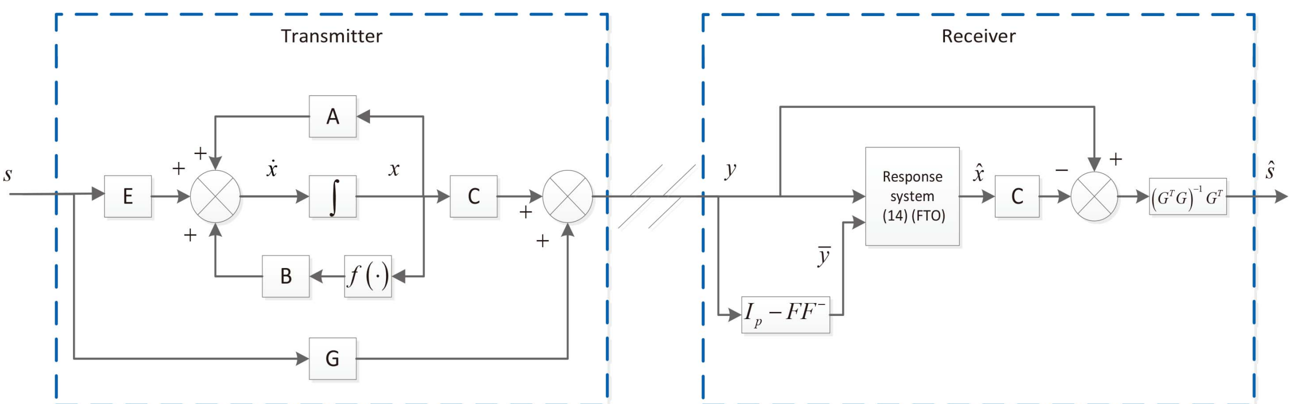

Figure 1. FTO-based chaotic synchronization and secure communication flow diagram.

Theorem 6. Under Assumptions 1 - 2, and assuming that the message signal $s(t)$ is uniformly bounded and piecewise continuously differentiable, the message signal can be recovered within an arbitrarily pre-defined time τ based on the fast chaotic synchronization achieved by the finite-time observer (14), and it is given by

$$\hat{s}(t) = \left(G^{\mathrm{T}}G\right)^{-1} G^{\mathrm{T}}\left[y(t) - C\hat{x}(t)\right]. \tag{16}$$

Proof. Since G has full column rank, then $\left(G^{\mathrm{T}}G\right)^{-1} G^{\mathrm{T}}$ exists, and the solution of $s(t)$ to equation $y(t) = Cx(t) + Gs(t)$ exists and is unique, thus we have

$$s(t) = \left(G^{\mathrm{T}}G\right)^{-1} G^{\mathrm{T}}\left[y(t) - Cx(t)\right]. \tag{17}$$

The uniform boundedness and piecewise continuous differentiability of $s(t)$ ensure the finite-time observer (14) achieves chaotic synchronization within τ, *i.e.*, $\hat{x}(t) \equiv x(t)$, $t \geq t_0 + \tau$. From (16) and (17), it follows that $\hat{s}(t) \equiv s(t)$. The proof is complete. □

Finally, we conclude the finite-time observer design procedure in the following algorithm.

Algorithm 1

Step 1 Check if Assumptions 1 - 2 hold, if so, go to next step; Otherwise, chaotic synchronization design fails.

Step 2 Compute matrix F^{-}, and obtain parameters $\bar{A}, \bar{D}, \bar{C}$ in the equivalent system (4).

Step 3 Compute matrices T and V, and further obtain $\breve{A}_{22}, \breve{A}_{21}, V_1, V_2$ and $\bar{U}$.

Step 4 Choose L_1 and L_2 such that for any $\rho_2 > \rho_1 > 0$, $-\rho_2 < \lambda_i(N_1) < \lambda_j(N_2) < -\rho_1 < 0$.

Step 5 Set the finite-time observer convergence time τ.

Step 6 Check if $\det\left[S \quad \mathrm{e}^{N\tau}S\right] \neq 0$ holds, if so, design observer (14); Otherwise, turn back to Step 5 to choose a different τ.

Remark 7. In the secure communication scenario, an eavesdropper can only capture the public output $y(t)$. Since the observer (14) converges in finite time and reconstructs the message via an algebraic formula (16), the attacker cannot extract the signal $s(t)$ without solving the observer error dynamics. This ensures security beyond basic masking.

Remark 8. The chaotic synchronization and secure communication can also be reached by using the traditional ACO. For example, if we choose L^* such that $N^* = \breve{A}_{22} - \breve{A}_{21}V_2\bar{C}\bar{U} - L^*V_1\bar{C}\bar{U}$ is Hurwitz stable, then

$$\begin{cases} \dot{\hat{\bar{x}}}_2(t) = \left(\breve{A}_{22} - \breve{A}_{21}V_2\bar{C}\bar{U}\right)\hat{\bar{x}}_2(t) + \breve{A}_{21}V_2\bar{y}(t) + \breve{R}_2 y(t) + L^*\left(V_1\bar{y}(t) - V_1\bar{C}\bar{U}\hat{\bar{x}}_2(t)\right) \\ \hat{x}(t) = T\begin{bmatrix} V_2\bar{y} - V_2\bar{C}\bar{U}\hat{\bar{x}}_2(t) \\ \hat{\bar{x}}_2(t) \end{bmatrix} \end{cases} \tag{18}$$

is just an ACO. Then, the message signal recovery can also be achieved by (16). It should be pointed out that such the ACO method usually cannot guarantee the estimation accuracy or the estimation speed within a short time because the effect of the ACO-based asymptotic synchronization depends on not only the observer gains but also the initial value differences between the drive system and the response system. By comparison, the proposed FTO method neither depends on the observer gains nor depends on observer initial value differences. And thus, both the drive system states and the message signal can be exactly recovered within an arbitrarily pre-defined time. This point will be further illustrated in simulation part.

Remark 9. There are some existing results which are also focused on investigating fast synchronization as well as the secure communication problems [22] [32] [37]. By comparison, both the mechanism of the synchronization achievement and the synchronization effect in the present paper are different from those in [22] [32] and [37]. Firstly, in [32] and [37] the fast synchronization is achieved via sliding mode control [37] or super-twisting observer [32]. In their methods, however, switching functions are used, which may bring unexpected chattering and thus destroy synchronization performance. While, in the present paper the fast synchronization is reached via a finite-time observer technique, which would not bring any chattering. Secondly, in [22] [32] and [37] although the fast synchronization can be achieved, the synchronization convergence time is usually unable to be controlled. While, in the present paper, the synchronization time can be arbitrarily pre-defined according to our demand.

Remark 10. It should be pointed out that in the present paper, in order to design a finite-time observer, the nonlinear chaotic system is written in the form of a linear system with the known nonlinear item being taken as an unknown input vector, which actually increases the conservative. On the other hand, for the FTO design the strongly observable condition is also a harsh condition for many chaotic systems. How to reduce such conservativeness need to be considered. One possible approach is to design a finite-time observer for a nonlinear system directly and the other is to reduce the strongly observable condition to a weaker strongly detectable condition. However, both of the tasks are not trivial work, and they will be discussed in our future work.

5. Simulation Example

In this section, some simulation results are given to illustrate the effectiveness of the proposed finite-time fast synchronization-based secure communication method.

5.1. Synchronization and Message Recovery Performance under Ideal Conditions

Consider the well-known Lorenz-Stenflo hyper-chaotic system [39] described by

$$\begin{cases} \dot{x}_1(t) = ax_1(t) - 1.2x_2(t) \\ \dot{x}_2(t) = 1.1x_1(t) - 0.1x_2(t)x_3^2(t) \\ \dot{x}_3(t) = -0.05x_2(t) - 1.2x_3(t) - 5x_4(t) + 0.65x_1(t) \\ \dot{x}_4(t) = 0.1x_1(t) + 1.62x_3(t) + 0.8x_4(t) \end{cases} \tag{19}$$

where $a = 0.58$, and the nonlinear item is denoted as $f(x) = -0.1x_2(t)x_3^2(t)$. For the secure communication purpose, consider (19) as a drive system represented in the form of (1) or (2) with

$$A = \begin{bmatrix} 0.5800 & -1.2000 & 0 & 0 \\ 1.1000 & 0 & 0 & 0 \\ 0.6500 & -0.0500 & -1.2000 & -5.0000 \\ 0.1000 & 0 & 1.6200 & 0.8000 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, E = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, G = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

Furthermore, we have

$$D = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, F = \begin{bmatrix} 0 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \text{ and } \omega(t) = \begin{bmatrix} f(t) \\ s(t) \end{bmatrix}.$$

It is easy to check that for such the drive system, both the observer matching condition and the strongly observable condition are satisfied. In the following, a finite-time observer will be constructed, and it will be used as a response system to reach a fast synchronization as well as a message signal recovery within an arbitrarily pre-defined time.

One generalized inverse matrix of F is chosen as

$F^{\dagger} = \begin{bmatrix} 0 & 0 & 0 \\ 0.5000 & 0.5000 & 0 \end{bmatrix}$. Then, matrices $\bar{A}, \bar{D}, \bar{C}$ can be computed as

$$\bar{A} = \begin{bmatrix} 0.0800 & -1.7000 & 0 & 0 \\ 1.1000 & 0 & 0 & 0 \\ 0.6500 & -0.0500 & -1.2000 & -5.0000 \\ 0.1000 & 0 & 1.6200 & 0.8000 \end{bmatrix}, \bar{D} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \text{ and}$$

$$\bar{C} = \begin{bmatrix} 0.5000 & -0.5000 & 0 & 0 \\ -0.5000 & 0.5000 & 0 & 0 \\ 0 & 0 & 1.0000 & 0 \end{bmatrix}.$$

Note that neither $\bar{D}$ nor $\bar{C}$ is full rank matrix, then we define two new matrices $\bar{D} = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}^{\mathrm{T}}$ and $\bar{C} = \begin{bmatrix} -0.5000 & 0.5000 & 0 & 0 \\ 0 & 0 & 1.0000 & 0 \end{bmatrix}$ with full ranks to replace the original matrices $\bar{D}$ and $\bar{C}$. Correspondingly, a newly defined output

$$\bar{y} = \begin{bmatrix} -0.5000 & 0.5000 & 0 & 0 \\ 0 & 0 & 1.0000 & 0 \end{bmatrix} x := \bar{C}x$$

and a new unknown input vector $\begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}^{\mathrm{T}} \bar{\omega}_1 := \bar{D}\bar{\omega}$ are obtained which are used to replace the original ones in (4). Next, choose $\bar{U} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, and it is computed that

$$T=\begin{bmatrix}0&1&0&0\\1&0&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix},\ \breve{A}=\begin{bmatrix}0&1.1000&0&0\\-1.7000&0.0800&0&0\\-0.0500&0.6500&-1.2000&-5.0000\\0&0.1000&1.6200&0.8000\end{bmatrix}\text{ and}$$

$$\breve{R}=\begin{bmatrix}0&0&0\\0.5000&0.5000&0\\0&0&0\\0&0&0\end{bmatrix}.$$

Then, choose $H=\begin{bmatrix}0\\0.5000\end{bmatrix}$. Next, it is obtained that $V_1=\begin{bmatrix}0&2\end{bmatrix}$ and $V_2=\begin{bmatrix}2&0\end{bmatrix}$. Based on them, choose gain matrices $L_1=\begin{bmatrix}-0.0004\\1.2900\\0.2943\end{bmatrix}$ and $L_2=\begin{bmatrix}0.7298\\-0.7100\\0.7389\end{bmatrix}$ such that the eigenvalues of matrices

$$N_1=\breve{A}_{22}-\breve{A}_{21}V_2\bar{C}\bar{U}-L_1V_1\bar{C}\bar{U}=\begin{bmatrix}-1.6200&0.0008&0\\0.6000&-3.7800&-5.0000\\0.1000&1.0314&0.8000\end{bmatrix}$$

and

$$N_2=\breve{A}_{22}-\breve{A}_{21}V_2\bar{C}\bar{U}-L_2V_1\bar{C}\bar{U}=\begin{bmatrix}-1.6200&-1.4596&0\\0.6000&0.2200&-5.0000\\0.1000&0.1421&0.8000\end{bmatrix}$$

are $\{-1.2,-1.6,-1.8\}$ and $\{-0.1,-0.2,-0.3\}$, respectively. In the following, square wave signal is introduced as the message signal to test the effectiveness of the proposed method.

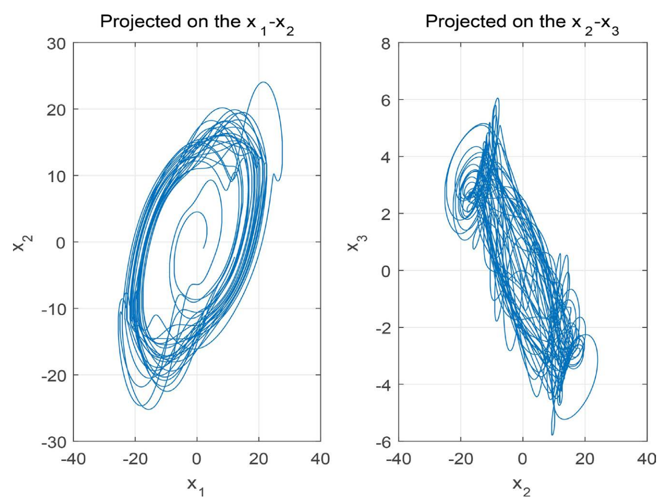

Figure 2. Simulated phase portraits of the system (19).

Choose a square wave signal $s(t) = 2\text{square}(0.5\pi t, 50)$ as the message signal. Set the initial value of the drive system as $x(0) = [2 \quad -1 \quad 0 \quad 2]^{\mathrm{T}}$. Form **Figure 2**, **Figure 3** it can be seen that although a message signal is added, the drive system still presents chaotic behaviors.

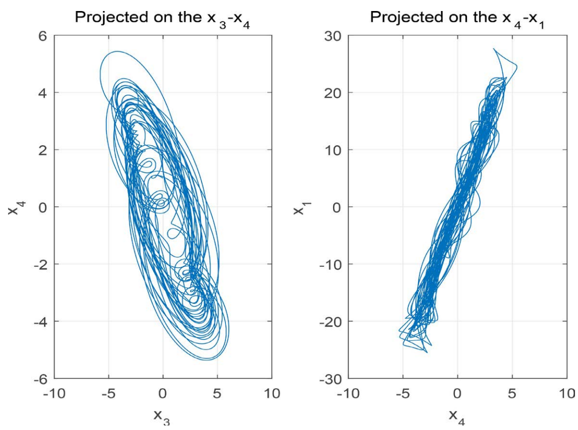

Figure 3. Simulated phase portraits of the system (19).

In order to compare the synchronization and the secure communication performance of the proposed FTO-based method with the conventional ACO-based method, we firstly give the results by using ACO method. According to Remark 7, choose gain matrix $L^* = L_1$ such that the eigenvalues of $N^* = \breve{A}_{22} - \breve{A}_{21}V_2\bar{C}\bar{U} - L^*V_1\bar{C}\bar{U}$ is $\{-1.2, -1.6, -1.8\}$. Based on ACO (18), the chaotic synchronization and the message recovery effect are presented in **Figures 4-8**, where the actual states or message signals are plotted in solid lines in blue, and the states or message signals estimated by ACO are plotted in dashed lines in black. It can be seen from the convergence time marked in blue lines (t_f and t_a represent the convergence time of FTO and ACO, respectively) that by using ACO the chaotic synchronization and the secure communication can be achieved after about 6 - 8 s.

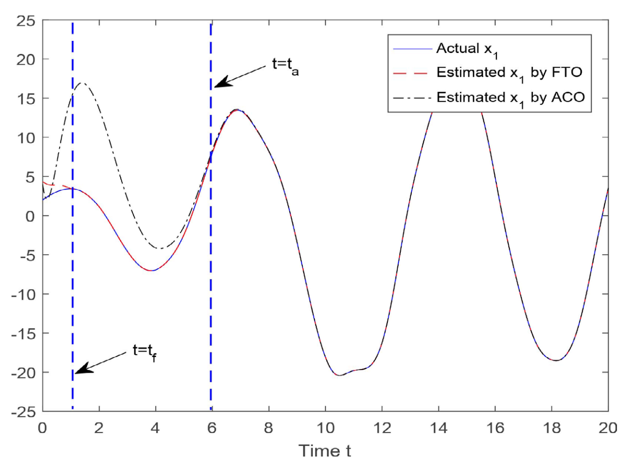

Figure 4. State estimation of x_1 via ACO and FTO.

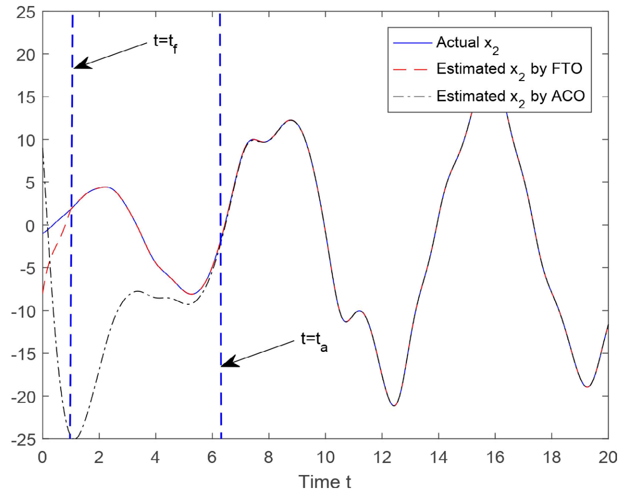

Figure 5. State estimation of x_2 via ACO and FTO.

In practical, it is desirable that the chaotic synchronization and the secure communication can be achieved at a faster speed. For this purpose, without loss of generality, in FTO we set the observer convergence time $\tau = 1\,\text{s}$, and then the observer parameter matrix K is

$$K = 10^3 \times \begin{bmatrix} 0.6038 & 0.6081 & 1.7675 & -0.6028 & -0.6081 & -1.7675 \\ -0.1403 & -0.1418 & -0.4201 & 0.1403 & 0.1428 & 0.4201 \\ 0.2722 & 0.2756 & 0.8058 & -0.2722 & -0.2756 & -0.8048 \end{bmatrix}.$$

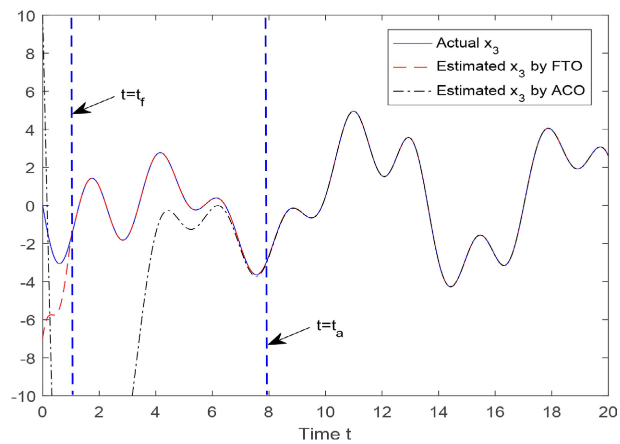

Figure 6. State estimation of x_3 via ACO and FTO.

The chaotic synchronization and secure communication performance obtained by using the FTO is also illustrated in **Figures 4-8** with the dashed lines in red. Meanwhile, we plot the synchronization errors in **Figure 9**. It can be seen from the **Figures 4-9** that by using the FTO both the estimations of the states $x_1 - x_4$ of the drive system and the message signal $s(t)$ recovery can be achieved well within the pre-defined time $\tau = 1\,\text{s}$. Also, by comparison, it can be concluded from the convergence time marked in blue lines in **Figures 4-8** that the convergence time of the FTO is much shorter than that of ACO.

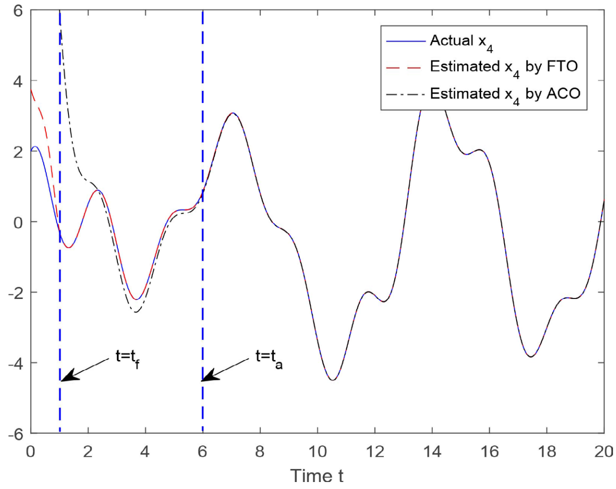

Figure 7. State estimation of x_4 via ACO and FTO.

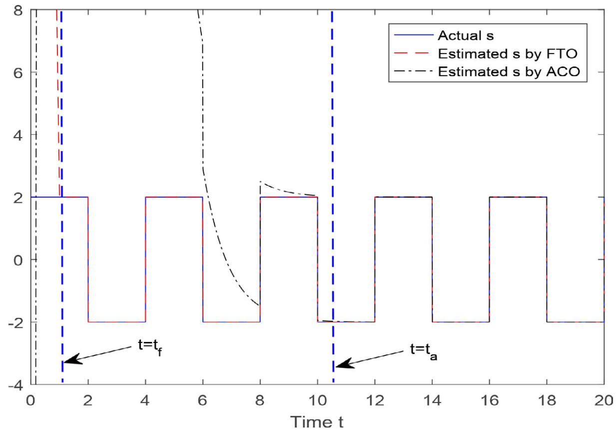

Figure 8. Message signal recovery via ACO and FTO.

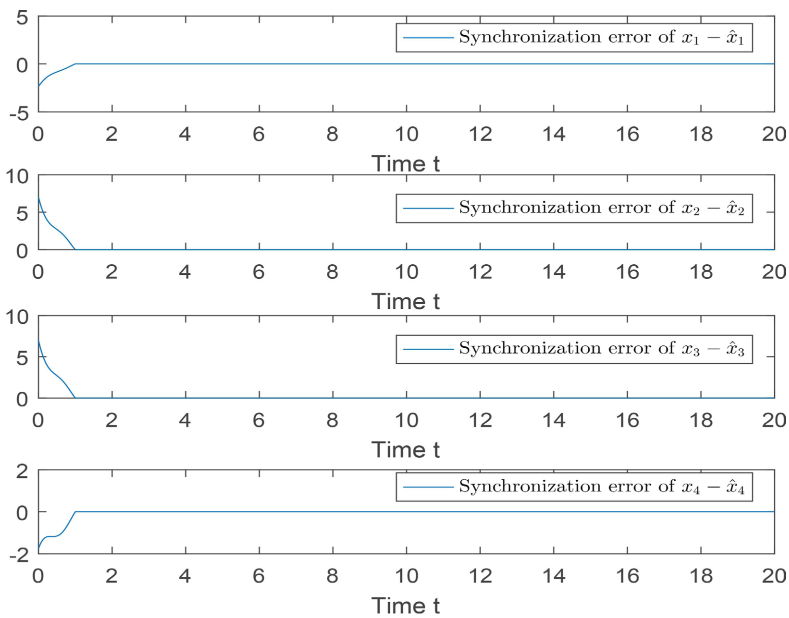

Figure 9. Synchronization errors via FTO.

5.2. Robustness Analysis and Simulation Verification under Perturbations

To verify the feasibility of the proposed scheme in practical communication environments, this section analyzes its robustness against disturbances such as channel noise, measurement noise, and parameter mismatch, and conducts verification through simulations. Theoretical analysis shows that in the presence of bounded disturbances, strict finite-time exact recovery degrades to finite-time stable recovery, *i.e.*, the state enters and remains within a bounded neighborhood of the origin within a finite time, achieving near-synchronization. The size of this neighborhood is related to the intensity of the disturbance.

As a typical disturbance, Gaussian white noise with an amplitude of 0.01 is injected into the output of the original system:

$$y_{\text{noisy}}(t) = y(t) + \delta(t),$$

where $\delta(t)$ denotes the measurement noise, and the message signal $s(t)$ still adopts the square-wave message signal as in the previous text.

The results of message signal recovery are shown in Figure 10 and Figure 11. The results indicate that despite the presence of noise, the proposed FTO can still quickly achieve and maintain high-precision tracking of the original message signal. In the flat-top phase, the estimated value exhibits bounded small ripple fluctuations, and the absolute error between the recovered signal and the original signal is kept within ±0.04, which is the near-synchronization state.

It is worth noting that the above simulation results reveal a fundamental difference in the anti-disturbance convergence properties between FTO and ACO: at the same noise level, the proposed FTO, with its fast finite-time convergence mechanism, can converge and stabilize the tracking error within a bounded neighborhood in an extremely short time; in contrast, the traditional ACO can

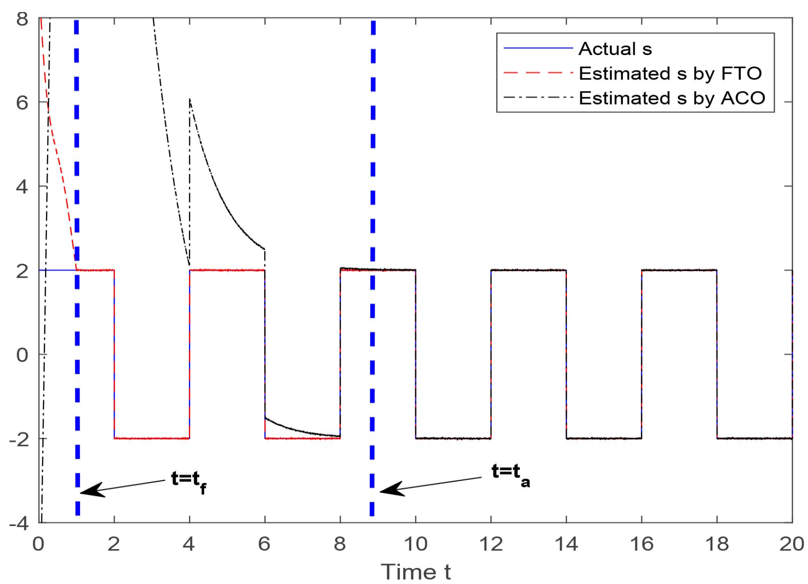

Figure 10. Information signal recovery under noisy environment.

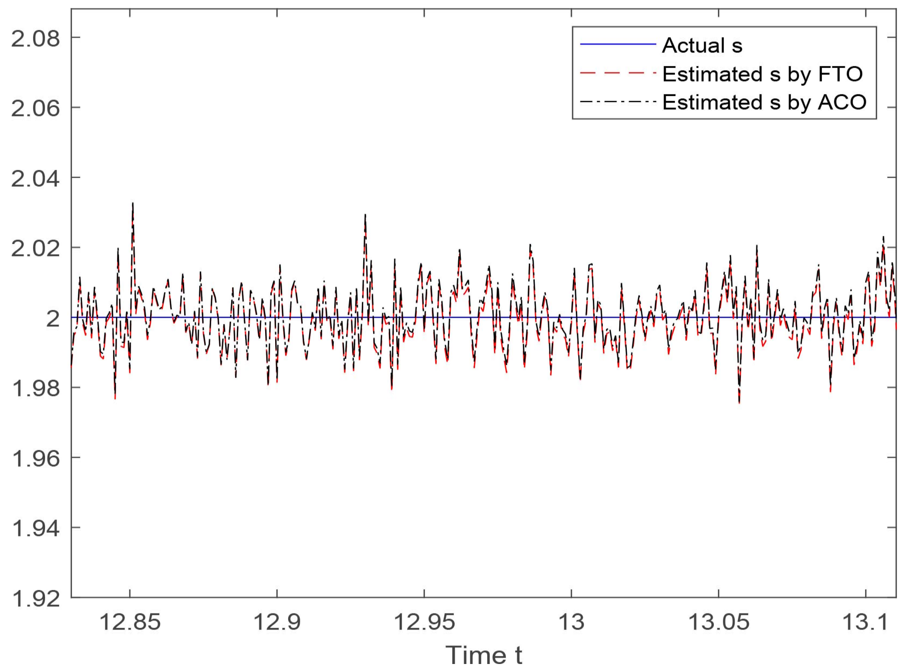

Figure 11. Magnified view of information signal recovery.

only achieve asymptotic convergence, and the time required to reach the same error level is significantly longer. This characteristic is crucial for the real-time performance of secure communication. Based on the same robustness mechanism, it can be expected that the proposed scheme also has a similar fast convergence advantage against channel noise and parameter mismatch.

6. Conclusion

In this paper, a fast chaotic synchronization method as well as a new chaotic masking secure communication method are developed by constructing a finite-time observer. In order to design such a finite-time observer, the drive system needs to be written in the form of a linear system, where the nonlinear item and the message signal are considered as unknown input vectors. Both the theoretical and simulation results validate that under the observer matching condition and the strongly observable condition, the response system constructed by designing a finite-time observer can reach an accurate synchronization before an arbitrarily pre-defined time. Subsequently, based on the fast and accurate synchronization, the message signal is recovered via an algebraic reconstruction method. Simulation results further demonstrate the method's robustness to measurement noise, where accurate near-synchronization is maintained under bounded perturbations. Note that in the present paper considering a nonlinear chaotic system as a linear system by treating a known nonlinear item as an unknown input vector actually increases the conservativeness. How to design a finite-time observer for a nonlinear system directly and apply it to chaotic synchronization as well as secure communication will be considered in our future work.

Funding

This work was supported in part by the National College Students' Innovation and Entrepreneurship Training Program of Guangxi Minzu University (Grant No. 202510608032), in part by the Guangxi Science and Technology Program (Grant No. AD23023001), and in part by the National Natural Science Foundation of China (Grant No. 62573142).

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

References

[1] Xu, C. and Xu, R. (2017) Chaos Control in a 3D Autonomous System. *International Journal of Innovative Computing, Information and Control*, **13**, 1095-1115.

[2] Carroll, T.L. and Pecora, L.M. (1994) Synchronizing Chaotic Circuits. *AIP Conference Proceedings*, **296**, 127-136. https://doi.org/10.1063/1.45294

[3] Chen, M. and Han, Z. (2003) Controlling and Synchronizing Chaotic Genesio System via Nonlinear Feedback Control. *Chaos, Solitons & Fractals*, **17**, 709-716. https://doi.org/10.1016/s0960-0779(02)00487-3

[4] Deng, Y.S., Hu, H., Xiong, W., Xiong, N.N. and Liu, L. (2015) Analysis and Design of Digital Chaotic Systems with Desirable Performance via Feedback Control. *IEEE Transactions on Systems, Man, and Cybernetics: Systems*, **45**, 1187-1200. https://doi.org/10.1109/tsmc.2015.2398836

[5] Wang, H., Han, Z.Z., Xie, Q.Y. and Zhang, W. (2008) Finite-Time Synchronization of Uncertain Unified Chaotic Systems Based on CLF. *Nonlinear Analysis: Real World Applications*, **10**, 2842-2849. https://doi.org/10.1016/j.nonrwa.2008.08.010

[6] Wu, X. and Yan, S. (2026) Adaptive Sliding Mode Synchronization of a Controllable Grid-Multi-Wing Chaotic System with Parameter Identification. *Applied Mathematical Modelling*, **150**, Article 116351. https://doi.org/10.1016/j.apm.2025.116351

[7] Wang, F.X., Cui, J.Q., Chen, R.D. and Liu, X.G. (2026) Predefined Time Synchronization of Fractional Order Chaotic System via Sliding Mode Control. *Communications in Nonlinear Science and Numerical Simulation*, **152**, Article 109332. https://doi.org/10.1016/j.cnsns.2025.109332

[8] Li, C.D., Chen, G.R., Liao, X.F. and Fan, Z.P. (2006) Chaos Quasisynchronization Induced by Impulses with Parameter Mismatches. *Chaos: An Interdisciplinary Journal of Nonlinear Science*, **16**, Article 023102. https://doi.org/10.1063/1.2179648

[9] Chen, G., Yang, X., Wang, Y. and Zhang, X. (2025) Impulsive Control for Synchronization of Chaotic Neural Networks with Multiple Time-Varying Delays and Its Applications to Secure Communications. *Information Sciences*, **694**, Article 121702. https://doi.org/10.1016/j.ins.2024.121702

[10] Wang, Y. and Yu, H. (2018) Fuzzy Synchronization of Chaotic Systems via Intermittent Control. *Chaos, Solitons & Fractals*, **106**, 154-160. https://doi.org/10.1016/j.chaos.2017.11.024

[11] Wen, G., Wan, Y., Cao, J., Huang, T. and Yu, W. (2018) Master-Slave Synchronization of Heterogeneous Systems under Scheduling Communication. *IEEE Transactions on Systems, Man, and Cybernetics: Systems*, **48**, 473-484. https://doi.org/10.1109/tsmc.2016.2599012

[12] Xiang, W., Sun, Y., Li, N. and Yang, C. (2017) Uncertain Chaotic Gyros Synchronization Using Adaptive Fuzzy Prescribed Performance Control with Unknown Dead-Zone Input. *International Joursnal of Innovative Computing, Information and Control*, **13**, 429-440.

[13] Derakhshannia, M., Moosapour, S.S. and Mobayen, S. (2025) Disturbance Observer Based Adaptive Control Scheme for Synchronization of Fractional Order Chaotic Systems with Input Delay. *IET Control Theory & Applications*, **19**, e70037. https://doi.org/10.1049/cth2.70037

[14] Pecora, L.M. and Carroll, T.L. (1990) Synchronization in Chaotic Systems. *Physical Review Letters*, **64**, 821-824. https://doi.org/10.1103/physrevlett.64.821

[15] Zhu, F., Xu, J. and Chen, M. (2012) The Combination of High-Gain Sliding Mode Observers Used as Receivers in Secure Communication. *IEEE Transactions on Circuits and Systems I: Regular Papers*, **59**, 2702-2712. https://doi.org/10.1109/tcsi.2012.2190570

[16] Nijmeijer, H. and Mareels, I.M.Y. (1997) An Observer Looks at Synchronization. *IEEE Transactions on Circuits and Systems I: Fundamental Theory and Applications*, **44**, 882-890. https://doi.org/10.1109/81.633877

[17] Zhu, F.L. and Chen, M.Y. (2008) Observer-Based Synchronization of Uncertain Chaotic Systems: Comparison between Reduced-Order and Full-Order Observers. *International Journal of Bifurcation and Chaos in Applied Sciences and Engineering*, **18**, 3129-3136. https://doi.org/10.1142/s0218127408022299

[18] Chadli, M., Zelinka, I. and Youssef, T. (2013) Unknown Inputs Observer Design for Fuzzy Systems with Application to Chaotic System Reconstruction. *Computers & Mathematics with Applications*, **66**, 147-154. https://doi.org/10.1016/j.camwa.2013.01.018

[19] Chadli, M. and Zelinka, I. (2014) Chaos Synchronization of Unknown Inputs Takagi-Sugeno Fuzzy: Application to Secure Communications. *Computers & Mathematics with Applications*, **68**, 2142-2147. https://doi.org/10.1016/j.camwa.2013.01.013

[20] Lee, S., Park, M. and Baek, J. (2017) Robust Adaptive Synchronization of a Class of Chaotic Systems via Fuzzy Bilinear Observer Using Projection Operator. *Information Sciences*, **402**, 182-198. https://doi.org/10.1016/j.ins.2017.03.004

[21] Babanli, K.M. and Ortac Kabaoglu, R. (2024) Synchronization of Fuzzy-Chaotic Systems with Z-Controller in Secure Communication. *Information Sciences*, **657**, Article 119988. https://doi.org/10.1016/j.ins.2023.119988

[22] Durdu, A. and Uyaroğlu, Y. (2017) The Shortest Synchronization Time with Optimal Fractional Order Value Using a Novel Chaotic Attractor Based on Secure Communication. *Chaos, Solitons & Fractals*, **104**, 98-106. https://doi.org/10.1016/j.chaos.2017.08.008

[23] Iqbal, S. and Wang, J. (2025) A Novel Fractional-Order 3-D Chaotic System and Its Application to Secure Communication Based on Chaos Synchronization. *Physica Scripta*, **100**, Article 025243. https://doi.org/10.1088/1402-4896/ad9cfe

[24] Kolumban, G., Kennedy, M.P. and Chua, L.O. (1998) The Role of Synchronization in Digital Communications Using Chaos. II. Chaotic Modulation and Chaotic Synchronization. *IEEE Transactions on Circuits and Systems I: Fundamental Theory and Applications*, **45**, 1129-1140. https://doi.org/10.1109/81.735435

[25] Wang, X.Y. and Wang, M.J. (2009) A Chaotic Secure Communication Scheme Based on Observer. *Communications in Nonlinear Science and Numerical Simulation*, **14**, 1502-1508. https://doi.org/10.1016/j.cnsns.2008.01.012

[26] Chen, M., Zhou, D. and Shang, Y. (2004) A New Observer-Based Synchronization

Scheme for Private Communication. *Chaos*, *Solitons & Fractals*, **24**, 1025-1030. https://doi.org/10.1016/j.chaos.2004.09.096

[27] Yang, J., Chen, Y. and Zhu, F. (2015) Associated Observer-Based Synchronization for Uncertain Chaotic Systems Subject to Channel Noise and Chaos-Based Secure Communication. *Neurocomputing*, **167**, 587-595. https://doi.org/10.1016/j.neucom.2015.04.030

[28] Dedieu, H., Kennedy, M.P. and Hasler, M. (1993) Chaos Shift Keying: Modulation and Demodulation of a Chaotic Carrier Using Self-Synchronizing Chua's Circuits. *IEEE Transactions on Circuits and Systems II*: *Analog and Digital Signal Processing*, **40**, 634-642. https://doi.org/10.1109/82.246164

[29] Yang, T. and Chua, L.O. (1996) Secure Communication via Chaotic Parameter Modulation. *IEEE Transactions on Circuits and Systems I*: *Fundamental Theory and Applications*, **43**, 817-819. https://doi.org/10.1109/81.536758

[30] Smaoui, N., Karouma, A. and Zribi, M. (2011) Secure Communications Based on the Synchronization of the Hyperchaotic Chen and the Unified Chaotic Systems. *Communications in Nonlinear Science and Numerical Simulation*, **16**, 3279-3293. https://doi.org/10.1016/j.cnsns.2010.10.023

[31] Liao, T.L. and Huang, N.S. (1999) An Observer-Based Approach for Chaotic Synchronization with Applications to Secure Communications. *IEEE Transactions on Circuits and Systems I*: *Fundamental Theory and Applications*, **46**, 1144-1150. https://doi.org/10.1109/81.788817

[32] Martínez-Guerra, R., Montesinos García, J.J. and Delfín Prieto, S.M. (2016) Secure Communications via Synchronization of Liouvillian Chaotic Systems. *Journal of the Franklin Institute*, **353**, 4384-4399. https://doi.org/10.1016/j.jfranklin.2016.08.011

[33] Muthukumar, P., Balasubramaniam, P. and Ratnavelu, K. (2017) Sliding Mode Control for Generalized Robust Synchronization of Mismatched Fractional Order Dynamical Systems and Its Application to Secure Transmission of Voice Messages. *ISA Transactions*, **82**, 51-61. https://doi.org/10.1016/j.isatra.2017.07.007

[34] Yang, J. and Zhu, F. (2013) Synchronization for Chaotic Systems and Chaos-Based Secure Communications via Both Reduced-Order and Step-by-Step Sliding Mode Observers. *Communications in Nonlinear Science and Numerical Simulation*, **18**, 926-937.

[35] Jiang, X., Yang, J., Zhu, F. and Xu, L. (2014) Observer-Based Synchronization of Chaotic Systems with Both Parameter Uncertainties and Channel Noise. *International Journal of Bifurcation and Chaos*, **24**, Article 1450095. https://doi.org/10.1142/s0218127414500953

[36] Wan, Z.L., Liao, T.L., Hou, Y.Y. *et al.* (2012) H_∞ Synchronization of Switched Chaotic Systems and Its Application to Secure Communications. *International Journal of Bifurcation and Chaos*, **22**, Article 1250058. https://doi.org/10.1142/s0218127412500587

[37] Kajbaf, A., Akhaee, M.A. and Sheikhan, M. (2016) Fast Synchronization of Non-Identical Chaotic Modulation-Based Secure Systems Using a Modified Sliding Mode Controller. *Chaos*, *Solitons & Fractals*, **84**, 49-57. https://doi.org/10.1016/j.chaos.2015.12.002

[38] Engel, R. and Kreisselmeier, G. (2002) A Continuous-Time Observer Which Converges in Finite Time. *IEEE Transactions on Automatic Control*, **47**, 1202-1204. https://doi.org/10.1109/tac.2002.800673

[39] Chen, D., Chen, H.T., Ma, X.Y., *et al.* (2010) Hyperchaos System with Only One Non-

linear Term and Comparative Study of Its Chaotic Control: Hyperchaos System with Only One Nonlinear Term and Comparative Study of Its Chaotic Control. *Journal of Computer Applications*, **30**, 2045-2048. https://doi.org/10.3724/sp.j.1087.2010.02045