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The Time-Dependent Attractors for the Wave Equation with Fading Memory and Structural Damping

Xuan Wang, Luyi Yan*

College of Mathematics and Statistics, Northwest Normal University, Lanzhou, China
Email: wangxuan@nwnu.edu.cn, *yanly2025@163.com

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Abstract

In this article, the asymptotic behavior of the solutions to the wave equation with fading memory and structural damping $\partial_t^2 u-\Delta u+\gamma(-\Delta)^\theta \partial_t u-\int_0^\infty \mu(s)\Delta\eta^t(s)\mathrm{d}s+f(u)=g(x)$ is considered. First of all, when the growth exponent of nonlinear terms f satisfies $2\le p\le 3+2\theta$, the well-posedness of solutions is obtained by applying Faedo-Galerkin approximation method and energy estimation; secondly, the asymptotic compactness of the solution process is proved via the method of contraction function; finally, the existence of time-dependent global attractor is obtained in the natural energy space $H_0^1(\Omega)\times L^2(\Omega)\times L_\mu^2(\mathbb{R}^+;H_0^1(\Omega))$.

Keywords

Fading Memory, Structural Damping, Time-Dependent Global Attractors, Wave Equation

1. Introduction

In this paper, we investigate the long-time behavior of solutions to the wave equation with fading memory and structural damping:

$$\begin{aligned}&\partial_t^2 u-\Delta u+\gamma(-\Delta)^\theta \partial_t u-\int_0^\infty \mu(s)\Delta\eta^t(s)\mathrm{d}s+f(u)\\&=g(x),\ (x,t)\in\Omega\times[\tau,+\infty),\end{aligned} \tag{1.1}$$

$$u|_{\partial\Omega}=0,\ u(x,\tau)=u_\tau(x),\ \partial_t u(x,\tau)=\partial_t u_\tau(x),\ x\in\Omega, \tag{1.2}$$

where $\theta\in\left(\frac{1}{2},1\right)$, $\gamma>0$ is the damping coefficient, and $\Omega\subset\mathbb{R}^3$ is a bounded domain with smooth boundary.

Suppose that the nonlinear function f satisfies the following conditions:

(M_1) $f \in C^2(\mathbb{R})$, $f(0)=0$. For any $s \in \mathbb{R}$, f satisfies

$$\liminf_{|s|\to+\infty} \frac{f(s)}{s} > -\lambda_1, \tag{1.3}$$

as well as the growth condition:

$$\left|f''(s)\right| \le C_0\left(1+|s|^{p-2}\right),\ 2 \le p \le p_\theta = 3+2\theta, \tag{1.4}$$

where the constant $C_0 > 0$, and λ_1 is the first eigenvalue of the operator $-\Delta$ under Dirichlet boundary conditions.

Following the ideas in [1] [2], the memory kernel function μ and the forcing term g satisfy the following conditions:

(M_2) $\mu \in C^1(\mathbb{R}^+) \cap L^1(\mathbb{R}^+)$ and

$$\int_0^\infty \mu(s)\,\mathrm{d}s = k_0 < \infty. \tag{1.5}$$

And there exists a positive constant k such that

$$\mu'(s) \le -k\mu(s) \le 0,\ \forall s \ge 0. \tag{1.6}$$

(M_3) $g \in L^2(\Omega)$.

Remark 1. From (1.3), there exists a constant β_0 satisfying $0 < \beta_0 < 1$ such that

$$\langle F(s),1\rangle \ge -\frac{(1-\beta_0)\lambda_1}{2}\|s\|^2 - C_{\beta_0},$$

$$\langle f(s),s\rangle \ge -(1-\beta_0)\lambda_1\|s\|^2 - C_{\beta_0},\ \forall s \in \mathbb{R},$$

holds, where $F(s)=\int_0^s f(r)\,\mathrm{d}r$.

In recent years, wave equations with damping and memory terms have extensive applications in the viscoelastic dynamical systems while viscoelastic material serves as the medium for energy transmission, as can be found in [1] [2].

When Equation (1.1) does not contain a memory term (that is, when μ is a Dirac measure at a fixed time or μ takes the zero value), and the damping dissipation exponent satisfies $\theta = 0$ or $\theta = 1$, Equation (1.1) reduces to a weakly damped or strongly damped wave equation. For this model, when the nonlinear term satisfies subcritical or critical growth conditions, Pata *et al.* discussed the well-posedness of solutions and long-time dynamical behaviors in [3] [4]. In addition to strong damping and weak damping, there exists another type of damping called structural damping, which is usually expressed in the form of a fractional power of the operator $-\Delta$, namely $(-\Delta)^\theta u_t$ with $0 < \theta < 1$. Its dissipation strength lies between weak damping and strong damping, and it can more essentially reflect the internal friction effect of materials. For wave equations with structural damping, when $0 < \theta < \frac{1}{2}$, Savostianov investigated the existence of attractors and exponential attractors for wave equations with structural damping under the subcritical growth of the nonlinear term in [5]. When $\frac{1}{2} < \theta < 1$, Wang Xuan *et al.* studied a Kirchhoff wave equation model with structural damping in [6]:

$$\epsilon(t)\partial_t^2 u - M\left(\|\nabla u\|^2\right)\Delta u + (-\Delta)^\gamma \partial_t u + f(u) = g(x),\ \gamma \in \left(\frac{1}{2},1\right).$$

They gained the well-posedness and regularity of solutions to Kirchhoff-type wave equations with time-dependent coefficients and structural damping, and proved the existence of time-dependent global attractors by using the method of contraction functions. In addition, many scholars have carried out extensive research on equations with structural damping, leading to a large number of research results on this model, as shown in references (see [5]-[9] and related literature).

For dissipative evolution equations with fading memory, Dafermos systematically established the theoretical framework of memory kernels for viscoelastic models in [2], established the well-posedness of linear viscoelastic wave equations with exponentially fading memory kernels within this framework. In [10], Lions combined the Faedo-Galerkin method with compactness theorems to solve the existence problem of weak solutions for nonlinear wave equations with fading memory terms. In [11], Chueshov systematically expounded the global attractor theory of wave equations with fading memory terms and established a unified framework for analyzing the long-term dynamical behaviors of such systems. On the basis of this theory, Ma Qiaozhen *et al.* discussed the asymptotic behaviors of solutions to wave equations with linear memory in time-dependent spaces in [12].

Inspired by the aforementioned research findings, this paper investigates the wave equation with fading memory terms and structural damping. To the best of our knowledge, the long-time dynamical behavior of solutions to wave equations with damping dissipation exponent $\frac{1}{2} < \theta < 1$ and fading memory terms in time-dependent spaces has not yet been explored. Meanwhile, the structural damping term, nonlinear term, and fading memory term in the equation bring essential difficulties to the derivation of dissipative estimates for solutions, the verification of the existence of bounded absorbing sets, and the proof of asymptotic compactness of the solution process. When the damping dissipation exponent satisfies $\frac{1}{2} < \theta < 1$, classical Sobolev embeddings fail to guarantee compactness; in addition, the fading memory term introduces a history-dependent component, making it difficult for traditional energy functionals to simultaneously characterize the instantaneous dissipation of fractional damping and the cumulative effect of memory terms. To address these challenges, we construct a memory-coupled energy functional that can simultaneously describe the effects of structural damping and fading memory. By combining energy estimation techniques, asymptotic regularity estimates, as well as the contraction function tailored to the memory-damping coupling and the relevant theory of time-dependent attractors, we overcome these technical obstacles. Furthermore, we establish the well-posedness of solutions to problem (1.1) - (1.2) and verify their Lipschitz continuity in the space $H_0^1(\Omega)\times L^2(\Omega)\times L_\mu^2(\mathbb{R}^+;H_0^1(\Omega))$. Subsequently, we confirm the asymptotic compactness of the solution process, and finally prove the existence of time-dependent global attractors for problem (1.1) - (1.2) in the space $H_0^1(\Omega)\times L^2(\Omega)\times L_\mu^2(\mathbb{R}^+;H_0^1(\Omega))$.

The content and structure of this paper are arranged as follows: In Section 2, we review the preliminary results; in Section 3, we discuss the well-posedness of weak solutions; in Section 4, we prove the asymptotic compactness of the process by using the method of contraction functions, and then obtain the existence of the time-dependent attractor.

In this paper, the symbol C denotes a positive constant. Each occurrence of C in different formulas represents the corresponding positive constant. We also use $C_i, i\in\mathbb{N}$ to denote different positive constants, and $C(\cdot,\cdot)$ denotes a constant depending on the parameters in the parentheses.

2. Notations and Preliminary Results

Following the ideas in [2] [13] [14], we introduce the history function $\eta=\eta^t(x,s)$ of u, which satisfies

$$\partial_t\eta^t=-\partial_s\eta^t+\partial_t u \tag{2.1}$$

with the corresponding initial value conditions

$$\begin{cases} u(x,t)=0, & x\in\partial\Omega,\ t>\tau,\\ \eta^t(x,s)=0, & (x,s)\in\partial\Omega\times\mathbb{R}^+,\ t>\tau,\\ u(x,t,\tau)=u_\tau(x), & x\in\Omega,\ t\le\tau,\\ \partial_t u(x,t,\tau)=u_\tau(x), & x\in\Omega,\ t\le\tau,\\ \eta^\tau(x,s)=\eta_\tau(x,s), & (x,s)\in\Omega\times\mathbb{R}^+. \end{cases}$$

We set $A=-\Delta$ with domain $D(A)=H_0^1(\Omega)\cap H^2(\Omega)$.

Consider the family of Hilbert spaces $D\left(A^{\frac{s}{2}}\right)$ for $s\in\mathbb{R}$, equipped with their respective inner products and norms:

$$\langle\cdot,\cdot\rangle_{D\left(A^{\frac{s}{2}}\right)}=\left\langle A^{\frac{s}{2}}\cdot,A^{\frac{s}{2}}\cdot\right\rangle,\ \|u\|^2_{D\left(A^{\frac{s}{2}}\right)}=\left\|A^{\frac{s}{2}}u\right\|^2,$$

where $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ denote the inner product and norm in $L^2(\Omega)$, respectively.

For the sake of convenience, we introduce the notation $V_s=D\left(A^{\frac{s}{2}}\right)$ for $s\in\mathbb{R}$, with the inner product and norm expressed as follows:

$$\langle u,v\rangle_s=\int_\Omega A^{\frac{s}{2}}u(x)A^{\frac{s}{2}}v(x)\mathrm{d}x,\ \|u\|_s^2=\int_\Omega\left|A^{\frac{s}{2}}u(x)\right|^2\mathrm{d}x,\ \forall u,v\in V_s.$$

Then, $V_0=L^2(\Omega)$, $D\left(A^{\frac{s}{2}}\right)=V_s$ and $D\left(A^{-\frac{s}{2}}\right)=V_{-s}$.

By virtue of the Sobolev embedding theorem, we obtain the compact embedding

$$V_{s_1} \hookrightarrow\hookrightarrow V_{s_2}, \text{ for } s_1 > s_2, \tag{2.2}$$

as well as the continuous embedding

$$V_s \hookrightarrow L^{\frac{2n}{n-2s}}. \tag{2.3}$$

Therefore, the problem (1.1) - (1.2) can be rewritten in the following form:

$$\begin{aligned} &\partial_t^2 u + Au + \gamma(-\Delta)^{\theta}\partial_t u + \int_0^{\infty}\mu(s)A\eta^t(s)\mathrm{d}s + f(u) \\ &= g(x),\ (x,t)\in\Omega\times[\tau,+\infty), \end{aligned} \tag{2.4}$$

$$u(x,t) = 0,\ x\in\partial\Omega,\ t>\tau, \tag{2.5}$$

$$\eta^{\tau}(x,s) = 0,\ (x,s)\in\partial\Omega\times\mathbb{R}^+,\ t>\tau, \tag{2.6}$$

$$u(x,t,\tau) = u_{\tau}(x),\ \partial_t u(x,t,\tau) = \partial_t u_{\tau}(x),\ \eta^{\tau}(x,s) = \eta_{\tau}(x,s),\ x\in\Omega,\ t\le\tau. \tag{2.7}$$

Define the family of Hilbert spaces:

$$\mathcal{H}_t^{1+\vartheta} = V_{1+\vartheta}\times V_{\vartheta}\times L_{\mu}^2\left(\mathbb{R}^+;V_{1+\vartheta}\right),$$

equipped with the corresponding inner product and norm

$$\|z(t)\|_{\mathcal{H}_t^{1+\vartheta}}^2 = \left\|\left(u(t),\partial_t u(t),\eta^t(s)\right)\right\|_{\mathcal{H}_t^{1+\vartheta}}^2 = \|u(t)\|_{1+\vartheta}^2 + \|\partial_t u(t)\|_{\vartheta}^2 + \|\eta^t(s)\|_{\mu,1+\vartheta}^2, \tag{2.8}$$

In particular, for $\vartheta = 0$, the family of Hilbert spaces $\mathcal{H}_t^1$ is defined as follows:

$$\mathcal{H}_t^1 = V_1\times L^2(\Omega)\times L_{\mu}^2\left(\mathbb{R}^+;V_1\right), \tag{2.9}$$

with its norm given by

$$\|z(t)\|_{\mathcal{H}_t^1}^2 = \left\|\left(u(t),\partial_t u(t),\eta^t(s)\right)\right\|_{\mathcal{H}_t^1}^2 = \|u(t)\|_1^2 + \|\partial_t u(t)\|^2 + \|\eta^t(s)\|_{\mu,1}^2. \tag{2.10}$$

Furthermore, for $\vartheta > 0$, we have the compact embedding

$$\mathcal{H}_t^{1+\vartheta} \hookrightarrow\hookrightarrow \mathcal{H}_t^1.$$

Next, we will review the following concepts and some abstract results, which will be used to study the long-time dynamical behavior of solutions in time-dependent spaces.

Definition 2.1. ([15]) Let $\{X_t\}_{t\in\mathbb{R}}$ be a family of normed spaces. A two-parameter family of operators $\{U(t,\tau): X_{\tau}\to X_t, t\ge\tau, \tau\in\mathbb{R}\}$ is said to be a process, if for any $\tau\in\mathbb{R}$,

1) $U(\tau,\tau) = \mathrm{Id}$ is the identity operator on X_{τ};

2) $U(t,s)U(s,\tau) = U(t,\tau),\ \forall\tau\le s\le t$.

Let $\{X_t\}_{t\in\mathbb{R}}$ be a family of normed spaces. For every $t\in\mathbb{R}$, the R-ball of X_t is defined by:

$$\mathbb{B}_t(R) = \left\{z\in X_t \mid \|z\|_{X_t}\le R\right\}.$$

Definition 2.2. ([15]) A family $\mathfrak{C} = \{C_t\}_{t\in\mathbb{R}}$ of bounded sets $C_t\subset X_t$ is called uniformly bounded, if there exists a constant $R>0$ such that $C_t\subset\mathbb{B}_t(R)$, for all $t\in\mathbb{R}$.

Definition 2.3. ([15]) A uniformly bounded family $\mathfrak{B}_t=\left\{\mathbb{B}_t\left(R_0\right)\right\}_{t\in\mathbb{R}}$ is called a time-dependent absorbing set for the process $U\left(t,\tau\right)$, if for any $R>0$, there exist a $t_0=t_0\left(R\right)\le t$ and $R_0>0$ such that

$$\tau\le t-t_0\Rightarrow U\left(t,\tau\right)\mathbb{B}_\tau\left(R\right)\subset\mathbb{B}_t\left(R_0\right).$$

A process is said to be dissipative if it possesses a time-dependent absorbing set.

Lemma 2.4. ([6]) Let $\left\{x_n\right\}$ be a bounded sequence and also let $\psi\in \mathrm{C}\left(\mathbb{R}\right)$ be a monotonic function. Then

$$\psi\left(\liminf_{n\to\infty}x_n\right)\le\liminf_{n\to\infty}\psi\left(x_n\right).$$

Lemma 2.5. ([16] [17]) Let X, B and Y be three Banach spaces. For any $T>0$, if $X\hookrightarrow\hookrightarrow B\hookrightarrow Y$, and

$$\mathrm{W}_1=\left\{u\in L^p\left(\left[0,T\right];X\right)\mid\partial_t u\in L^r\left(\left[0,T\right];Y\right),r>1,1\le p<\infty\right\},$$

$$\mathrm{W}_2=\left\{u\in L^\infty\left(\left[0,T\right];X\right)\mid\partial_t u\in L^r\left(\left[0,T\right];Y\right),r>1\right\}.$$

Then the following compact embeddings hold:

$$W_1\hookrightarrow\hookrightarrow L^p\left(\left[0,T\right];B\right),\ W_2\hookrightarrow\hookrightarrow C\left(\left[0,T\right];B\right).$$

Theorem 2.6. ([15]) If $U\left(t,\tau\right)$ is asymptotically compact, that is, the set

$$\mathbb{K}=\left\{\mathfrak{K}=\left\{K_t\right\}_{t\in\mathbb{R}}\mid K_t\subset X_t\text{ is compact and }\mathfrak{K}\text{ is attracting}\right\}$$

is nonempty, then the time-dependent attractor $\mathfrak{A}$ exists and is unique.

Definition 2.7. ([15] [18] [19]) A time-dependent attractor $\mathfrak{A}=\left\{A_t\right\}_{t\in\mathbb{R}}$ is invariant, if for all $\tau\le t$,

$$U\left(t,\tau\right)A_\tau=A_t.$$

Theorem 2.8. ([20]) Let $U\left(\cdot,\cdot\right)$ be a process acting on a family of Banach spaces $\left\{X_t\right\}_{t\in\mathbb{R}}$. Then $U\left(\cdot,\cdot\right)$ has a time-dependent global attractor $\mathfrak{A}^*=\left\{A_t^*\right\}_{t\in\mathbb{R}}$ satisfying

$$A_t^*=\bigcap_{s\le t}\overline{\bigcup_{\tau\le s}U\left(t,\tau\right)\mathbb{B}_\tau\left(R\right)}$$

if and only if

1) $U\left(\cdot,\cdot\right)$ has a time-dependent absorbing family $\mathfrak{B}=\left\{\mathbb{B}_t\left(R_0\right)\right\}_{t\in\mathbb{R}}$;

2) $U\left(\cdot,\cdot\right)$ is asymptotically compact.

Definition 2.9. ([21] [22]) Let $\left\{X_t\right\}_{t\in\mathbb{R}}$ be a family of Banach spaces and let $\mathfrak{C}=\left\{C_t\right\}_{t\in\mathbb{R}}$ be a uniformly bounded family of subsets of $\left\{X_t\right\}_{t\in\mathbb{R}}$. A function $\Phi_\tau^t\left(\cdot,\cdot\right)$ defined on $X_t\times X_t$ is called a contraction function on $C_\tau\times C_\tau$, if for any fixed $t\in\mathbb{R}$ and any sequence $\left\{x_n\right\}_{n=1}^\infty\subset C_\tau$, there exists a subsequence $\left\{x_{n_k}\right\}_{k=1}^\infty\subset\left\{x_n\right\}_{n=1}^\infty$ such that

$$\lim_{k\to\infty}\lim_{l\to\infty}\Phi_\tau^t\left(x_{n_k},x_{n_l}\right)=0,$$

where $\tau \le t$.

We denote by $\mathfrak{C}(C_t)$ the set of all contraction functions on $C_t \times C_t$.

Theorem 2.10. ([20]) Let $U(\cdot,\cdot)$ be a process on $\{X_t\}_{t\in\mathbb{R}}$ that possesses a time-dependent absorbing family of sets $\mathfrak{B}_t = \{\mathbb{B}_t(R_1)\}_{t\in\mathbb{R}}$. If for any $\varepsilon > 0$, there exists a subsequence $T(\varepsilon) \le t$, $\Phi_T^t \in \mathfrak{C}(\mathbb{B}_T(R))$ such that

$$\|U(t,T)x - U(t,T)y\| \le \varepsilon + \Phi_T^t(x,y), \ \forall x, y \in \mathbb{B}_T(R)$$

for any fixed $t \in \mathbb{R}$, then $U(\cdot,\cdot)$ is asymptotically compact.

3. Well-Posedness of Solutions

First, we give the following definition of the solution to the problem (2.4) - (2.7).

Definition 3.1. For any $\tau \in \mathbb{R}$, a triple $z = (u, \partial_t u, \eta^t)$ is called a weak solution to the problem (2.4) - (2.7) on the interval $[\tau, T]$, if

$$u \in L^\infty([\tau,T];V_1), \ \partial_t u \in L^\infty([\tau,T];L^2(\Omega)) \cap L^2([\tau,T];V_\theta),$$
$$\eta^t \in L^\infty([\tau,t];L^2_\mu(\mathbb{R}^+;V_1))$$

and it satisfies

$$\langle \partial_t^2 u, \omega \rangle + \langle Au, \omega \rangle + \gamma \langle A^\theta \partial_t u, \omega \rangle + \left\langle \int_0^\infty \mu(s) A\eta^t(s) \mathrm{d}s, \omega \right\rangle$$
$$+ \langle f(u), \omega \rangle = \langle g(x), \omega \rangle$$

for all $\tau \le t$ and all $\omega \in V_1$.

Theorem 3.2. Suppose that (M_1) - (M_3) hold. Then for each $T > \tau$ and $\theta \in \left(\frac{1}{2}, 1\right)$, the problem (2.4) - (2.7) admits a weak solution $y = (u, \partial_t u, \eta^t) \in C([\tau,T];\mathcal{H}_t^1)$ with $\partial_t^2 u \in L^\infty([\tau,T];V_{-2\theta})$, which satisfies

$$\begin{aligned} &\|u(t)\|_1^2 + \|\partial_t u(t)\|^2 + \|\eta^t\|_{\mu,1}^2 + \|\partial_t^2 u(t)\|_{-2\theta}^2 + \int_\tau^T \|\partial_t u(t)\|_\theta \mathrm{d}t \\ &\le C(R, \beta_0, \|g\|, \lambda_1, C_2), \ t \ge \tau. \end{aligned} \tag{3.1}$$

Furthermore, the weak solution satisfies the following properties:

1) (Dissipativity) There exists a positive constant R independent of $\theta \in \left(\frac{1}{2}, 1\right)$ such that

$$\|(u, \partial_t u, \eta^t)\|_{\mathcal{H}_t^1} \le R_0, \ \forall t \ge t_0(R), \tag{3.2}$$

where $\tau \le t - t_0(R)$ and $t_0(R)$ is a moment depending on the positive constant R.

2) (Energy equality) For each $\tau \le t \le T$, the following energy equality holds

$$\begin{aligned} &E(u(t), \partial_t u(t), \eta^t(s)) + 2\gamma \int_\tau^T \|\partial_t u(t)\|_\theta^2 \mathrm{d}t + \frac{1}{2}\int_\tau^T \int_0^\infty \mu(s) \frac{\mathrm{d}}{\mathrm{d}s} \|\eta^t(s)\|_1^2 \mathrm{d}s\mathrm{d}t \\ &= E(u_\tau, \partial_t u_\tau, \eta_\tau(s)). \end{aligned} \tag{3.3}$$

Here,

$$E\left(u(t),\partial_t u(t),\eta^t(s)\right) = \|u(t)\|_1^2 + \|\partial_t u(t)\|^2 + \|\eta^t(s)\|_{\mu,1}^2 + 2\langle F(u(t)),1\rangle - 2\langle g,u(t)\rangle. \tag{3.4}$$

3) (Weak lipschitz stability) The solution $y=\left(u,\partial_t u,\eta^t\right)$ is Lipschitz continuous in the space $V_{1-\theta}\times V_{-\theta}\times L_\mu^2\left(\mathbb{R}^+;V_{1-\theta}\right)$, that is,

$$\|u(t)\|_{1-\theta}^2 + \|\partial_t u(t)\|_{-\theta}^2 + \|\eta^t(s)\|_{\mu,1-\theta}^2 \le \frac{m_3}{m_2}\mathrm{e}^{\tilde{C}(t-\tau)}\left(\|u_\tau\|_{1-\theta}^2 + \|\partial_t u_\tau\|_{-\theta}^2 + \|\eta_\tau(s)\|_{\mu,1-\theta}^2\right) + \frac{\tilde{C}_0(t-\tau)}{m_2}\mathrm{e}^{\tilde{C}(t-\tau)}. \tag{3.5}$$

where $\tilde{z}=\left(\tilde{u},\partial_t\tilde{u},\tilde{\eta}_t\right)=z_1-z_2$, and $z_i=\left(u_i,\partial_t u_i,\eta_i^t\right)(i=1,2)$ are two weak solutions to the problem (2.4) - (2.7) corresponding to the initial values $\left(u_{\tau_i},\partial_t u_{\tau_i},\eta_{\tau_i}\right)(i=1,2)$, respectively. Moreover

$$\tilde{C}=C\left(R,\beta_0,\|g\|,\lambda_1,C_1,C_2,\gamma,k\right),$$

$$\tilde{C}_0=C\left(R,\beta_0,\|g\|,\lambda_1,C_2,\gamma\right).$$

Proof.

1) (Existence of Weak Solutions) First, we establish some a priori estimates for the solutions to the problem (2.4) - (2.7). Taking the inner product of Equation (2.4) with $\partial_t u$ in $L^2(\Omega)$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}E\left(u(t),\partial_t u(t),\eta^t(s)\right) + 2\gamma\|\partial_t u\|_\theta^2 + \int_0^\infty \mu(s)\frac{\mathrm{d}}{\mathrm{d}s}\|\eta^t(s)\|_1^2\,\mathrm{d}s = 0,$$

where

$$E\left(u(t),\partial_t u(t),\eta^t(s)\right) = \|u(t)\|_1^2 + \|\partial_t u(t)\|^2 + \|\eta^t(s)\|_{\mu,1}^2 + 2\langle F(u(t)),1\rangle - 2\langle g,u(t)\rangle. \tag{3.6}$$

Integrating the above identity over the interval $[s,t]$, we deduce that (3.3) holds.

Since

$$\int_0^\infty \mu(s)\frac{\mathrm{d}}{\mathrm{d}s}\|\eta^t(s)\|_1^2\,\mathrm{d}s = -\int_0^\infty \mu'(s)\|\eta^t(s)\|_1^2\,\mathrm{d}s \ge k\int_0^\infty \mu(s)\|\eta^t(s)\|_1^2\,\mathrm{d}s = k\|\eta^t(s)\|_{\mu,1}^2,$$

we have

$$E\left(u(t),\partial_t u(t),\eta^t(s)\right) + 2\gamma\int_\tau^T \|\partial_t u(t)\|_\theta^2\,\mathrm{d}t \le E\left(u_\tau,\partial_t u_\tau,\eta_\tau\right). \tag{3.7}$$

By virtue of (1.4) and the compact embedding $V_1 \hookrightarrow\hookrightarrow L^{p+1}(\Omega)$, it follows that

$$2\langle F(u),1\rangle \le 2C_1\left(\|u\|^2 + \|u\|_{L^{p+1}(\Omega)}^{p+1}\right) \le C_2\left(\|u\|_1^2 + \|u\|_1^{p+1}\right). \tag{3.8}$$

From (M_3), we know that

$$2\left|\langle g,u\rangle\right| \le \frac{\beta_0}{4}\|u\|_1^2 + \frac{4}{\beta_0\lambda_1}\|g\|^2. \tag{3.9}$$

Therefore,

$$
\begin{aligned}
&E(u_\tau,\partial_t u_\tau,\eta_\tau)\\
&=\|u_\tau\|_1^2+\|\partial_t u_\tau\|^2+\|\eta_\tau\|_{\mu,1}^2+2\langle F(u_\tau),1\rangle-2\langle g,u_\tau\rangle\\
&\le \|u_\tau\|_1^2+\|\partial_t u_\tau\|^2+\|\eta_\tau\|_{\mu,1}^2+C_2\left(\|u_\tau\|_1^2+\|u_\tau\|_1^{p+1}\right)+\frac{\beta_0}{4}\|u_\tau\|_1^2+\frac{4}{\beta_0\lambda_1}\|g\|^2\\
&\le m_0\left(\|u_\tau\|_1^2+\|\partial_t u_\tau\|^2+\|\eta_\tau\|_{\mu,1}^2\right)+C_2\left(\|u_\tau\|_1^{p+1}\right)+C\\
&\le C(R,\beta_0,\|g\|,\lambda_1,C_2),
\end{aligned}
$$

where $m_0=\max\left\{1,1+C_2+\frac{\beta_0}{4}\right\}$ and $C=\frac{4}{\beta_0\lambda_1}\|g\|^2$.

From Remark 1.1, we have

$$
2\int_\Omega F(u)\,\mathrm{d}x\ge-(1-\beta_0)\|u(t)\|_1^2-2C_{\beta_0}\ge(\beta_0-1)\|u(t)\|_1^2-2C_{\beta_0}. \tag{3.10}
$$

By virtue of estimates (3.8) - (3.9), we obtain

$$
\begin{aligned}
&m_1\left(\|u(t)\|_1^2+\|\partial_t u(t)\|^2+\|\eta^t\|_{\mu,1}^2\right)-C\\
&\le E\left(u(t),\partial_t u(t),\eta^t(s)\right)\le C(R,\beta_0,\|g\|,\lambda_1,C_2).
\end{aligned} \tag{3.11}
$$

where $m_1=\min\left\{1,\frac{3\beta_0}{4}\right\}$ and $C=\frac{4}{\beta_0\lambda_1}\|g\|^2+2C(\beta_0)$.

According to (3.7) and (3.11), we conclude that

$$
\int_\tau^T\|\partial_t u(t)\|_\theta^2\,\mathrm{d}t\le C(R,\beta_0,\|g\|,\lambda_1,C_2). \tag{3.12}
$$

Using the embeddings $L^{1+\frac{1}{p}}(\Omega)\hookrightarrow\hookrightarrow V_{-1}\hookrightarrow\hookrightarrow V_{-2\theta}$ and Equation (2.3), we get

$$
\begin{aligned}
&\|\partial_t^2 u(t)\|_{-2\theta}^2\\
&\le\|u(t)\|_{2-2\theta}^2+\gamma\|\partial_t u(t)\|^2+\|\eta^t(s)\|_{\mu,2-2\theta}^2+\|f(u)\|_{-2\theta}^2+\|g\|_{-2\theta}^2\\
&\le C(R,\beta_0,\|g\|,\lambda_1,C_2)\left(\|u(t)\|_{2-2\theta}^2+\gamma\|\partial_t u(t)\|^2+\|\eta^t(s)\|_{\mu,2-2\theta}^2+\|f(u)\|_{L^{1+\frac{1}{p}}}^2+\|g\|^2\right)\\
&\le C(R,\beta_0,\|g\|,\lambda_1,C_2)\left(\|u(t)\|_1^2+\|u(t)\|_1^{2p}+\gamma\|\partial_t u(t)\|^2+\|\eta^t(s)\|_{\mu,1}^2+\|g\|^2\right)\\
&\le C(R,\beta_0,\|g\|,\lambda_1,C_2).
\end{aligned} \tag{3.13}
$$

Combining (3.7), (3.11) - (3.13), we deduce that (3.1) holds.

Let us prove the existence of solutions for the problem (2.4) - (2.7) in the space $C([\tau,T];\mathcal{H}_t^1)$. Suppose that $\{w_j\}_{j=1}^\infty$ is an orthonormal basis of V_1 with $Aw_j=\lambda_j w_j$ for $j=1,2,\cdots$. Let $\{\varsigma_j\}_{j=1}^\infty$ be an orthonormal basis of $L_{\mu_t}^2(\mathbb{R}^+;V_1)$ satisfying $A\varsigma_j=\lambda_j\varsigma_j$ for $j=1,2,\cdots$. For each $n\in\mathbb{N}$, there exist finite-dimensional subspaces

$$
V_n=\operatorname{span}\{w_1,\cdots,w_n\}\subset V_1,\ M_n=\operatorname{span}\{\varsigma_1,\cdots,\varsigma_n\}\subset L_\mu^2(\mathbb{R}^+;V_1).
$$

Define $P_n:V_1\to V_n$ as the orthogonal projection onto V_n and

$Q_n : L^2_\mu\left(\mathbb{R}^+;V_1\right)\to M_n$ as the orthogonal projection onto M_n.

For each $n\in\mathbb{N}$, let $z_n(t)=\left(u_n,\partial_t u_n,\eta_n^t\right)$ be an approximate solution to the problem, where $u_n=\sum_{j=1}^n T_j^n(t)w_j$ with $T_j^n\in C^1([\tau,T])$, and $\eta_n^t=\sum_{j=1}^n \Lambda_j^n(t)\varsigma_j$ with $\Lambda_j^n\in C^1([\tau,T])$. Then for any $\varphi\in V_n$ and each $t\in[\tau,T]$, $z_n(t)=\left(u_n,\partial_t u_n,\eta_n^t\right)$ satisfies

$$\begin{aligned}&\left\langle\partial_t^2 u,\varphi\right\rangle+\left\langle Au,\varphi\right\rangle+\gamma\left\langle A^\theta\partial_t u,\varphi\right\rangle+\left\langle\int_0^\infty\mu(s)A\eta^t(s)\,\mathrm{d}s,\varphi\right\rangle\\&+\left\langle f(u),\varphi\right\rangle=\left\langle g(x),\varphi\right\rangle,\end{aligned}\tag{3.14}$$

together with

$$\eta_n^t(s)=\begin{cases}u_n(t)-u_n(t-s), & 0\le s\le t-\tau,\\ \eta_{\tau_n}(s-t+\tau)+u_n(t)-u_{\tau_n}, & s>t-\tau.\end{cases}$$

Multiplying Equation (3.14) by $\psi\in C_0^\infty([\tau,T])$ and integrating over $[\tau,T]$, we obtain

$$\begin{aligned}&\int_\tau^T\psi\Big(\left\langle\partial_t^2 u,\varphi\right\rangle+\left\langle Au,\varphi\right\rangle+\gamma\left\langle A^\theta\partial_t u,\varphi\right\rangle+\left\langle\int_0^\infty\mu(s)A\eta^t(s)\,\mathrm{d}s,\varphi\right\rangle\\&+\left\langle f(u),\varphi\right\rangle-\left\langle g(x),\varphi\right\rangle\Big)\mathrm{d}t=0.\end{aligned}\tag{3.15}$$

Then we have the following results:

$$u_n \text{ is bounded in } L^\infty\left([\tau,T];V_1\right),$$

$$\partial_t u_n \text{ is bounded in } L^\infty\left([\tau,T];L^2(\Omega)\right)\cap L^2\left([\tau,T];V_\theta\right),$$

$$\eta_n^t \text{ is bounded in } L^\infty\left([\tau,T];L^2_\mu\left(\mathbb{R}^+,V_1\right)\right),$$

$$\partial_t^2 u_n \text{ is bounded in } L^\infty\left([\tau,T];V_{-2\theta}\right).$$

By applying the Galerkin approximation method, there exists $z=\left(u,\partial_t u,\eta^t\right)\in L^\infty\left([\tau,T];\mathcal{H}_t^1\right)$ such that

$$u_n\to u \text{ weakly}^* \text{ in } L^\infty\left([\tau,T];V_1\right),$$

$$\partial_t u_n\to\partial_t u \text{ weakly}^* \text{ in } L^\infty\left([\tau,T];L^2(\Omega)\right),$$

$$\partial_t u_n\to\partial_t u \text{ weakly in } L^2\left([\tau,T];V_\theta\right),$$

$$\eta_n^t\to\eta^t \text{ weakly}^* \text{ in } L^\infty\left([\tau,T];L^2_\mu\left(\mathbb{R}^+,V_1\right)\right),$$

$$\partial_t^2 u_n\to\partial_t^2 u \text{ weakly}^* \text{ in } L^\infty\left([\tau,T];V_{-2\theta}\right).$$

Applying Lemma 2.2 and Alaoglu Theorem, for $0<\alpha\ll 1$, we deduce that

$$u_n\to u \text{ in } C\left([\tau,T];V_{1-\alpha}\right),$$

$$\partial_t u_n\to\partial_t u \text{ in } C\left([\tau,T];V_{-\alpha}\right),$$

$$u_n\to u \text{ in } L^2\left([\tau,T];V_1\right) \text{ and } u_n(x,t)\to u(x,t), \text{ a.e. } (x,t)\in\Omega\times[\tau,T],$$

$$\partial_t u_n\to\partial_t u \text{ in } L^2\left([\tau,T];L^2(\Omega)\right),$$

$$f(u_n) \to f(u) \text{ weakly in } L^{1+\frac{1}{p}}\left([\tau,T];L^{1+\frac{1}{p}}(\Omega)\right).$$

Since for arbitrary $\varphi \in V_n$, we have

$$\begin{aligned}
&\int_\tau^T \langle f(u_n)-f(u),\varphi\rangle \mathrm{d}t \\
&\le C_1\int_\tau^T \left(1+|u_n|^{p-1}+|u|^{p-1}\right)|u_n-u||\varphi|\mathrm{d}t \\
&\le C_1\left(\int_\tau^T \left(1+|u_n|^{p-1}+|u|^{p-1}\right)^{\frac{p+1}{p-1}}\mathrm{d}t\right)^{\frac{p-1}{p+1}}\left(\int_\Omega |u_n-u|^{p+1}\mathrm{d}t\right)^{\frac{1}{p+1}}\left(\int_\Omega|\varphi|^p\mathrm{d}t\right)^{\frac{1}{p+1}} \\
&\le C_1\left(1+\|u_n\|_{L^{p+1}}^{p-1}+\|u\|_{L^{p+1}}^{p-1}\right)\|u_n-u\|_{L^{p+1}}\|\varphi\|_{L^{p+1}} \\
&\le C_2\left(1+\|u_n\|_1^{p-1}+\|u\|_1^{p-1}\right)\|u_n-u\|_1\|\varphi\|_1 \\
&\le C\left(R,\beta_0,\|g\|,\lambda_1,C_2\right)\|u_n-u\|_{L^2([\tau,T];V_1)} \to 0,
\end{aligned}$$

Since $f(u_n)$ and $f(u)$ are bounded in $L^{1+\frac{1}{p}}\left([\tau,T];L^{1+\frac{1}{p}}(\Omega)\right)$, by the Lebesgue Dominated Convergence Theorem, we obtain

$$\int_\tau^T \psi\langle f(u_n)-f(u),\varphi\rangle \mathrm{d}y \to 0.$$

Furthermore, for arbitrary $\varphi \in V_n$, we have

$$\begin{aligned}
\int_\tau^T \langle Au_n-Au,\varphi\rangle \mathrm{d}t &\le \int_\tau^T \left\|A^{\frac{1}{2}}\left(u_n(t)-u(t)\right)\right\|\left\|A^{\frac{1}{2}}\varphi\right\|\mathrm{d}t \\
&\le \int_\tau^T \|u_n(t)-u(t)\|_1\|\varphi\|_1\mathrm{d}t \to 0.
\end{aligned}$$

In addition, since u_n and u are bounded in V_1, an application of the Lebesgue Dominated Convergence Theorem yields

$$\int_\tau^T \psi\langle Au_n-Au,\varphi\rangle \mathrm{d}t \to 0.$$

We further set

$$\bar\eta_n^t = \eta_n^t-\eta^t,\ \bar u_n = u_n-u,\ \bar\eta_{\tau_n} = \eta_{\tau_n}-\eta_\tau,\ \bar u_{\tau_n} = u_{\tau_n}-u_\tau.$$

Consider the mapping $p_n:[\tau,T]\to\mathbb{R}$ defined by $t\mapsto p_n(t)=\langle \bar u_n(t),\varphi\rangle_1$, where $p_n\to 0$ in $L^2([\tau,T];V_1)$.

From Equation (2.2), we have

$$\begin{aligned}
\|\bar\eta_n^t\|_{\mu,1}^2 &= \int_0^{t-\tau}\mu(s)\|\bar u_n(t-s)-\bar u(t-s)\|_1^2\mathrm{d}s \\
&\quad+\int_{t-\tau}^\infty \mu(s)\left\|\bar\eta_{\tau_n}(s-t+\tau)+\bar u_n(t)-\bar u_{\tau_n}\right\|_1^2\mathrm{d}s \\
&\le \int_0^{t-\tau}\mu(s)\|\bar u_n(t-s)-\bar u(t-s)\|_1^2\mathrm{d}s+2\int_{t-\tau}^\infty\mu(s)\|\bar\eta_{\tau_n}(s-t+\tau)\|_1^2\mathrm{d}s \\
&\quad+2\int_{t-\tau}^\infty \mu(s)\|\bar u_n(t)-\bar u_{\tau_n}\|_1^2\mathrm{d}s,
\end{aligned}$$

where

$$\int_{t-\tau}^{\infty}\mu(s)\left\|\bar{\eta}_{\tau_n}(s-t+\tau)\right\|_1^2 \mathrm{d}s$$
$$=\int_0^{\infty}\mu(s+t-\tau)\left\|\bar{\eta}_{\tau_n}(s)\right\|_1^2 \mathrm{d}s \to 0 \le \int_0^{\infty}\mu(s)\left\|\bar{\eta}_{\tau_n}(s)\right\|_1^2 \mathrm{d}s \to 0.$$

As $n\to\infty$, $\bar{u}_n \to 0$ in $L^2([\tau,T];V_1)$, hence

$$\left\|\bar{u}_n(t)-\bar{u}(t-s)\right\| \le \left\|\bar{u}_n(t)\right\|+\left\|\bar{u}_n(t-s)\right\| \to 0.$$

Combining with (1.5) in (M_1), we obtain

$$\int_0^{t-\tau}\mu(s)\left\|\bar{u}_n(t)-\bar{u}(t-s)\right\|_1^2 \mathrm{d}s \to 0.$$

Similarly,

$$\int_{t-\tau}^{\infty}\mu(s)\left\|\bar{u}_n(t)-\bar{u}_{\tau_n}\right\|_1^2 \mathrm{d}s \to 0.$$

Therefore,

$$\left\|\bar{\eta}^t(s)\right\|_{\mu,1}^2 \to 0,\ \forall t\in[\tau,T].$$

Since

$$\eta_n^t(s)=\begin{cases} u_n(t)-u_n(t-s), & 0\le s\le t-\tau,\\ \eta_{\tau_n}(s-t+\tau)+u_n(t)-u_{\tau_n}, & s>t-\tau,\end{cases}$$

we have

$$\int_0^{\infty}\mu(s)\left\langle \bar{\eta}_n^t(s),\varphi\right\rangle_1 \mathrm{d}s$$
$$=\int_0^{t-\tau}\mu(s)\left\langle \bar{u}_n(t),\varphi\right\rangle_1 \mathrm{d}s-\int_0^{t-\tau}\mu(s)\left\langle \bar{u}_n(t-s),\varphi\right\rangle_1 \mathrm{d}s$$
$$+\int_{t-\tau}^{\infty}\mu(s)\left\langle \bar{u}_n(t),\varphi\right\rangle_1 \mathrm{d}s+\int_{t-\tau}^{\infty}\mu(s)\left\langle \bar{\eta}_{\tau_n}(s-t+\tau)-\bar{u}_{\tau_n},\varphi\right\rangle_1 \mathrm{d}s.$$

By applying (1.5) in (M_1) again, we obtain

$$\int_0^{\infty}\mu(s)\left\langle \bar{u}_n(t),\varphi\right\rangle_1 \mathrm{d}s=\int_0^{\infty}\mu(s)p_n(t)\mathrm{d}s=k_0p_n(t)\to 0,$$

$$\int_0^{t-\tau}\mu(s)\left\langle \bar{u}_n(t-s),\varphi\right\rangle_1 \mathrm{d}s=\int_{\tau}^{t}\mu(t-s)p_n(s)\mathrm{d}s\to 0,$$

$$\int_{t-\tau}^{\infty}\mu(s)\left\langle \bar{\eta}_{\tau_n}(s-t+\tau)-\bar{u}_{\tau_n},\varphi\right\rangle_1 \mathrm{d}s$$
$$\le\|\varphi\|_1\int_0^{\infty}\mu(s+t-\tau)\left(\left\|\bar{\eta}_{\tau_n}(s)\right\|_1+\left\|\bar{u}_{\tau_n}\right\|_1\right)\mathrm{d}s$$
$$\le\|\varphi\|_1\left\|\bar{\eta}_{\tau_n}(s)\right\|_{L_\mu^2(\mathbb{R}^+;V_1)}+\|\varphi\|_1\left\|\bar{u}_{\tau_n}\right\|_{L^2([\tau,T];V_1)}\to 0,$$

Therefore, we have

$$\lim_{n\to\infty}\int_0^{\infty}\mu(s)\left\langle \bar{\eta}_n^t(s),\varphi\right\rangle_1 \mathrm{d}s=0,\ \forall t\in[\tau,T].$$

By applying the Lebesgue Dominated Convergence Theorem, we deduce that

$$\lim_{n\to\infty}\int_{\tau}^{T}\psi(t)\int_0^{\infty}\mu(s)\left\langle \bar{\eta}_n^t(s),\varphi\right\rangle_1 \mathrm{d}s\mathrm{d}t=0.$$

As a result, letting $n\to\infty$ in Equation (3.15), we conclude that $z=(u,\partial_t u,\eta^t)$ is a weak solution to the problem and satisfies the estimate (3.1).

Next, we shall prove that the solution $z=(u,\partial_t u,\eta^t)$ to the problems (2.4) -

(2.7) belongs to $C\left(\left[\tau,T\right];\mathcal{H}_t^1\right)$.

Since $\left(u,\partial_t u\right)\in C\left(\left[\tau,T\right];V_{1-\alpha}\times V_{-\alpha}\right)$ and $\left(u,\partial_t u,\eta^t\right)\in L^\infty\left(\tau,T;\mathcal{H}_t^1\right)$, it follows that $\left(u,\partial_t u,\eta^t\right)\in C_w\left(\left[\tau,T\right];\mathcal{H}_t^1\right)$ and

$$\left\|\left(u,\partial_t u,\eta^t\right)\right\|_{\mathcal{H}_t^1}\le\liminf_{\tau\to t}\left\|\left(u_\tau,\partial_t u_\tau,\eta_\tau\right)\right\|_{\mathcal{H}_\tau^1}.$$

For any $t\in\left[\tau,T\right]$, by (3.3) we have

$$\lim_{\tau\to t}E\left(u_\tau,\partial_t u_\tau,\eta_\tau\right)=E\left(u\left(t\right),\partial_t u\left(t\right),\eta^t\left(s\right)\right). \tag{3.16}$$

From (3.16), we deduce that $u\left(x,\tau\right)\to u\left(x,t\right)$ a.e. $x\in\Omega$ as $\tau\to t$. Applying Lemma 2.4, Remark 1.1 and Fatou Lemma, we have

$$\lim_{\tau\to t}2\left\langle g,u_\tau\right\rangle=2\left\langle g,u\left(t\right)\right\rangle,$$

$$\left\|\left(u\left(t\right),\partial_t u\left(t\right),\eta^t\right)\right\|_{\mathcal{H}_t^1}^2\le\liminf_{\tau\to t}\left\|\left(u_\tau,\partial_t u_\tau,\eta_\tau\right)\right\|_{\mathcal{H}_\tau^1}^2,$$

$$\begin{aligned}&\int_\Omega\left(2F\left(u\left(t\right)\right)+\left(1-2C_{\beta_0}\right)\lambda_1\left|u\left(t\right)\right|^2+2C_{\beta_0}\right)\mathrm{d}x\\&\le\liminf_{\tau\to t}\int_\Omega\left(2F\left(u_\tau\right)+\left(1-2C_{\beta_0}\right)\lambda_1\left|u_\tau\right|^2+2C_{\beta_0}\right)\mathrm{d}x\\&\le\liminf_{\tau\to t}\int_\Omega 2F\left(u_\tau\right)\mathrm{d}x+\left(1-2C_{\beta_0}\right)\lambda_1\left\|u\right\|^2+2C_{\beta_0}\left|\Omega\right|,\end{aligned}$$

That is,

$$\int_\Omega 2F\left(u\left(t\right)\right)\mathrm{d}x\le\liminf_{\tau\to t}\int_\Omega 2F\left(u_\tau\right)\mathrm{d}x.$$

From the above estimates and (3.16), we get

$$\begin{aligned}&\liminf_{\tau\to t}\left\|\partial_t u_\tau\right\|^2+\liminf_{\tau\to t}\left\|u_\tau\right\|_1^2+\liminf_{\tau\to t}\left\|\eta_\tau\right\|_{\mu,1}^2+\liminf_{\tau\to t}2\left\langle F\left(u_\tau\right),1\right\rangle\\&\le\lim_{\tau\to t}\left[\left\|\partial_t u_\tau\right\|^2+\left\|u_\tau\right\|^2+\left\|\eta_\tau\right\|_{\mu,1}^2+2\left\langle F\left(u_\tau\right),1\right\rangle\right]\\&=\left\|\partial_t u\left(t\right)\right\|^2+\left\|u\left(t\right)\right\|_1^2+\left\|\eta^t\right\|_{\mu,1}^2+2\left\langle F\left(u\left(t\right)\right),1\right\rangle\\&\le\liminf_{\tau\to t}\left\|\partial_t u_\tau\right\|^2+\liminf_{\tau\to t}\left\|u_\tau\right\|_1^2+\liminf_{\tau\to t}\left\|\eta_\tau\right\|_{\mu,1}^2+\liminf_{\tau\to t}2\left\langle F\left(u_\tau\right),1\right\rangle,\end{aligned}$$

Hence

$$\left\|\partial_t u\left(t\right)\right\|^2=\lim_{\tau\to t}\left\|\partial_t u_\tau\right\|^2. \tag{3.17}$$

Similarly, we obtain

$$\left\|u\left(t\right)\right\|_1^2=\lim_{\tau\to t}\left\|u_\tau\right\|_1^2, \tag{3.18}$$

$$\left\|\eta^t\right\|_{\mu,1}^2=\lim_{\tau\to t}\left\|\eta_\tau\right\|_{\mu,1}^2. \tag{3.19}$$

By the uniform continuity of the space $\mathcal{H}_t^1$, combining (3.17) - (3.19) and $\left(u,\partial_t u,\eta^t\right)\in C_w\left(\left[\tau,T\right];\mathcal{H}_t^1\right)$, we conclude that $\left(u,\partial_t u,\eta^t\right)\in C\left(\left[\tau,T\right];\mathcal{H}_t^1\right)$.

2) (Weak lipschitz continuity) Let $z_i\left(t\right)$ ($i=1,2$) be solutions to the problems (2.4) - (2.7) satisfying $\left\|z_i\left(\tau\right)\right\|_{\mathcal{H}_\tau^1}\le R$ ($i=1,2$). Then $\tilde{z}=\left(\tilde{u},\partial_t\tilde{u},\tilde{\eta}^t\right)=z_1-z_2$ satisfies

$$\partial_t^2\tilde{u}+A\tilde{u}+\gamma A^\theta\partial_t\tilde{u}+\int_0^\infty\mu\left(s\right)A\tilde{\eta}^t\left(s\right)\mathrm{d}s+f_1-f_2=0,\ \left(x,t\right)\in\Omega\times\left[\tau,\infty\right), \tag{3.20}$$

$$\tilde{u}(x,t)=0,\ x\in\partial\Omega,\ t>\tau, \tag{3.21}$$

$$\tilde{\eta}^{\tau}(x,s)=0,\ (x,s)\in\partial\Omega\times\mathbb{R}^{+},\ t>\tau, \tag{3.22}$$

$$\tilde{u}(x,t,\tau)=\tilde{u}_{\tau_1}(x)-\tilde{u}_{\tau_2}(x),\ \partial_t\tilde{u}(x,t,\tau)=\partial_t\tilde{u}_{\tau_1}(x)-\partial_t\tilde{u}_{\tau_2}(x),$$
$$\tilde{\eta}^{\tau}(x,s)=\tilde{\eta}_{\tau_1}(x,s)-\tilde{\eta}_{\tau_2}(x,s),\ x\in\Omega,\ t\le\tau. \tag{3.23}$$

where $f_i=f(u_i)$ for $i=1,2$.

In the following estimates, we choose δ to be an arbitrarily small positive constant. Taking the inner product of Equation (3.20) with $2A^{-\theta}\partial_t\tilde{u}+2\delta A^{-\theta}\tilde{u}$, we obtain

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}K(\tilde{u},\partial_t\tilde{u},\tilde{\eta}^t)+2\gamma\|\partial_t\tilde{u}\|^2-2\delta\|\partial_t\tilde{u}\|_{-\theta}^2+2\delta\|\tilde{u}\|_{1-\theta}^2\\&+2\int_0^\infty\mu(s)\frac{\mathrm{d}}{\mathrm{d}s}\|\tilde{\eta}^t\|_{1-\theta}^2\,\mathrm{d}s+2\delta\int_0^\infty\mu(s)\langle\tilde{\eta}^t,\tilde{u}\rangle_{1-\theta}\,\mathrm{d}s=\sum_{j=1}^{2}\Pi_j.\end{aligned} \tag{3.24}$$

where

$$K(\tilde{u},\partial_t\tilde{u},\tilde{\eta}^t)=2\delta\langle\partial_t\tilde{u},A^{-\theta}\tilde{u}\rangle+\|\partial_t\tilde{u}\|_{-\theta}^2+\|\tilde{u}\|_{1-\theta}^2+\delta\gamma\|\tilde{u}\|^2+\|\tilde{\eta}^t\|_{\mu,1-\theta}^2,$$

$$\Pi_1=-2\langle f(u_1)-f(u_2),A^{-\theta}\partial_t\tilde{u}\rangle,$$

$$\Pi_2=-2\delta\langle f(u_1)-f(u_2),A^{-\theta}\tilde{u}\rangle.$$

Since

$$2\left|\delta\langle\partial_t\tilde{u},A^{-\theta}\tilde{u}\rangle\right|\le2\delta^2\lambda_1^{-1}\|\tilde{u}\|_{1-\theta}^2+\frac{1}{2}\|\partial_t\tilde{u}\|_{-\theta}^2,$$

there exist constants m_2,m_3 such that

$$\begin{aligned}&m_2\left(\|\tilde{u}(t)\|_{1-\theta}^2+\|\partial_t\tilde{u}(t)\|_{-\theta}^2+\|\tilde{\eta}^t\|_{\mu,1-\theta}^2\right)\le K(\tilde{u},\partial_t\tilde{u},\tilde{\eta}^t)\\&\le m_3\left(\|\tilde{u}(t)\|_{1-\theta}^2+\|\partial_t\tilde{u}(t)\|_{-\theta}^2+\|\tilde{\eta}^t\|_{\mu,1-\theta}^2\right),\end{aligned} \tag{3.25}$$

where $m_2=\min\left\{\frac{1}{2},1-2\delta^2\lambda_1^{-1}\right\}$ and $m_3=\max\left\{\frac{3}{2},1+2\delta^2\lambda_1^{-1}+\delta\gamma\lambda_1^{\theta-1}\right\}$.

By the Interpolation Theorem, we deduce that

$$\begin{aligned}|\Pi_1|&\le2\int_\Omega|f(u_1)-f(u_2)||A^{-\theta}\partial_t\tilde{u}|\mathrm{d}x\\&\le2C_1\int_\Omega\left(1+|u_1|^{p-1}+|u_2|^{p-1}\right)|\tilde{u}||A^{-\theta}\partial_t\tilde{u}|\mathrm{d}x\\&\le2C_1\left(\int_\Omega\left(1+|u_1|^{p-1}+|u_2|^{p-1}\right)^{\frac{p+1}{p-1}}\mathrm{d}x\right)^{\frac{p-1}{p+1}}\left(\int_\Omega|\tilde{u}|^{p+1}\mathrm{d}x\right)^{\frac{1}{p+1}}\left(\int_\Omega|A^{-\theta}\partial_t\tilde{u}|^{p+1}\mathrm{d}x\right)^{\frac{1}{p+1}}\\&\le C_1\left(1+\|u_1\|_{L^{p+1}(\Omega)}^{p-1}+\|u_2\|_{L^{p+1}(\Omega)}^{p-1}\right)\left(\|\tilde{u}\|_{L^{p+1}(\Omega)}^2+\|A^{-\theta}\partial_t\tilde{u}\|_{L^{p+1}(\Omega)}^2\right)\\&\le C(R,\beta_0,\|g\|,\lambda_1,C_1,C_2)\left(\|\tilde{u}\|_{1-\alpha}^2+\|\partial_t\tilde{u}\|_{1-2\theta-\alpha}^2\right)\\&\le2\gamma\left(\|\tilde{u}\|_1^2+\|\partial_t\tilde{u}\|^2\right)+C(R,\beta_0,\|g\|,\lambda_1,C_1,C_2,\gamma)\left(\|\tilde{u}\|_{1-\theta}^2+\|\partial_t\tilde{u}\|_{-\theta}^2\right),\end{aligned}$$

$$
\begin{aligned}
|\Pi_2| &\le 2\delta\int_\Omega |f(u_1)-f(u_2)||A^{-\theta}\tilde{u}|\mathrm{d}x \\
&\le 2C_1\delta\int_\Omega \left(1+|u_1|^{p-1}+|u_2|^{p-1}\right)|\tilde{u}||A^{-\theta}\tilde{u}|\mathrm{d}x \\
&\le 2C_1\delta\left(\int_\Omega \left(1+|u_1|^{p-1}+|u_2|^{p-1}\right)^{\frac{p+1}{p-1}}\mathrm{d}x\right)^{\frac{p-1}{p+1}}\left(\int_\Omega|\tilde{u}|^{p+1}\mathrm{d}x\right)^{\frac{1}{p+1}}\left(\int_\Omega\left|A^{-\theta}\tilde{u}\right|^{p+1}\mathrm{d}x\right)^{\frac{1}{p+1}} \\
&\le C_1\delta\left(1+\|u_1\|_{L^{p+1}(\Omega)}^{p-1}+\|u_2\|_{L^{p+1}(\Omega)}^{p-1}\right)\left(\|\tilde{u}\|_{L^{p+1}}^2+\left\|A^{-\theta}\tilde{u}\right\|_{L^{p+1}}^2\right) \\
&\le C\delta\left(1+\|u_1\|_{L^{p+1}(\Omega)}^{p-1}+\|u_2\|_{L^{p+1}(\Omega)}^{p-1}\right)\left(\|\tilde{u}\|_{1-\alpha}^2+\left\|A^{-\theta}\tilde{u}\right\|_{1-\alpha-2\theta}^2\right) \\
&\le \gamma\left(\|\tilde{u}\|_1^2+\|\tilde{u}\|^2\right)+C\left(R,\beta_0,\|g\|,\lambda_1,C_1,C_2,\delta,\gamma\right)\|\tilde{u}\|_{1-\theta}^2 \\
&\le \gamma\left(1+\lambda_1^{-1}\right)\|\tilde{u}\|_1^2+C\left(R,\beta_0,\|g\|,\lambda_1,C_1,C_2,\delta,\gamma\right)\|\tilde{u}\|_{1-\theta}^2,
\end{aligned}
$$

where we have used the Sobolev embedding $V_{1-\alpha}\hookrightarrow L^{p+1}(\Omega)$ for $0<\alpha\ll 1$.

Substituting the above estimates into Equation (3.24), we get

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}K\left(\tilde{u},\partial_t\tilde{u},\tilde{\eta}^t\right) \\
&\le C\left(R,\beta_0,\|g\|,\lambda_1,C_2,\gamma\right)+C\left(R,\beta_0,\|g\|,\lambda_1,C_1,C_2,\delta,\gamma,k\right)K\left(\tilde{u},\partial_t\tilde{u},\tilde{\eta}^t\right),
\end{aligned}
$$

where

$$
\tilde{C}=C\left(R,\beta_0,\|g\|,\lambda_1,C_1,C_2,\delta,\gamma,k\right)
$$

$$
\tilde{C}_0=C\left(R,\beta_0,\|g\|,\lambda_1,C_2,\gamma\right).
$$

Furthermore, we can derive Equation (3.5).

3) (Dissipativity) Let $K_1\left(u,\partial_t u,\eta^t\right)=E\left(u,\partial_t u,\eta^t\right)+2\delta\langle\partial_t u,u\rangle$.

By virtue of the estimate

$$
2\left|\delta\left\langle u,\partial_t u\right\rangle\right|\le\frac{2\delta^2}{\lambda_1}\|u\|_1^2+\frac{1}{2}\|\partial_t u\|^2
$$

and estimate (3.1), there exist constants m_4,m_5 such that

$$
m_4\left\|\left(u,\partial_t u,\eta^t\right)\right\|_{\mathcal{H}_t^1}^2-C_3\le K_1\left(u,\partial_t u,\eta^t\right)\le m_5\left\|\left(u,\partial_t u,\eta^t\right)\right\|_{\mathcal{H}_t^1}^2+C\left(R,\beta_0,\|g\|,\lambda_1,C_2\right), \tag{3.26}
$$

where $m_4=\min\left\{\frac{1}{2},\frac{3\beta_0}{4}-\frac{2\delta^2}{\lambda_1}\right\}$, $m_5=\max\left\{\frac{1}{2},\frac{2\delta^2}{\lambda_1}\right\}$,

$C_3=\frac{4}{\beta_0\lambda_1}\|g\|^2+2C\left(\beta_0\right)$.

Multiplying Equation (2.4) by $2\partial_t u+2\delta u$ and integrating over Ω, we obtain

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(K_1\left(u,\partial_t u,\eta^t\right)+C_3\right)+\delta\left(K_1\left(u,\partial_t u,\eta^t\right)+C_3\right)+2\delta\gamma\left\langle A^\theta\partial_t u,u\right\rangle \\
&+2\delta\left\langle f(u),u\right\rangle+2\gamma\|\partial_t u\|_\theta^2-2\delta^2\left\langle u,\partial_t u\right\rangle-3\delta\|\partial_t u\|^2+\delta\|u\|_1^2 \\
&+2k\left\|\eta^t(s)\right\|_{\mu,1}^2+2\delta\int_0^\infty\mu(s)\left\langle A\eta^t(s),u\right\rangle\mathrm{d}s \\
&=\delta\left\|\eta^t(s)\right\|_{\mu,1}^2+2\delta\left\langle F(u),1\right\rangle+\delta C_3.
\end{aligned} \tag{3.27}
$$

From estimates (3.1) and (3.8) - (3.9), it follows that

$$\delta\left\|\eta^t(s)\right\|_{\mu,1}^2+2\delta\left\langle F(u),1\right\rangle+\delta C_3\le C\left(R,\beta_0,\|g\|,\lambda_1,C_2,C_3,\delta\right).$$

It is easy to see that

$$\left|2\delta\gamma\left\langle A^{\theta}\partial_t u,u\right\rangle\right|\le\gamma\left\|\partial_t u\right\|_{\theta}^2+\frac{\delta^2\gamma}{\lambda_1^{1-\theta}}\|u\|_1^2,$$

$$\left\|\partial_t u\right\|_{\theta}^2\ge\lambda_1^{\theta}\left\|\partial_t u\right\|^2,$$

$$\begin{aligned}2\delta\left\langle f(u),u\right\rangle&\ge2\delta\left(-\left(1-3\beta_0\right)\right)\|u\|_1^2-2\delta C_{\beta_0}\\&\ge6\delta\beta_0\|u\|_1^2-2\delta C_{\beta_0}-2\delta\|u\|_1^2,\end{aligned}$$

$$\left|2\delta\int_0^{\infty}\mu(s)\left\langle A\eta^t(s),u\right\rangle\mathrm{d}s\right|\le\frac{\delta}{4}\|u\|_1^2+4\delta\left\|\eta^t(s)\right\|_{\mu,1}^2.$$

Substituting the above estimates into Equation (3.27), we get

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left(K_1\left(u,\partial_t u,\eta^t\right)+C_3\right)+\delta\left(K_1\left(u,\partial_t u,\eta^t\right)+C_3\right)+\Upsilon\left(u,\partial_t u,\eta^t\right)\\&\le C\left(R,\beta_0,\|g\|,\lambda_1,C_2,C_3,\delta\right),\end{aligned}\tag{3.28}$$

where

$$\begin{aligned}\Upsilon\left(u,\partial_t u,\eta^t\right)&=\left(6\delta\beta_0-\frac{2\delta^3}{\lambda_1}-\frac{\delta^2\gamma}{\lambda_1^{1-\theta}}\right)\|u\|_1^2+\left(\gamma\lambda_1^{\theta}-\frac{7\delta}{2}\right)\left\|\partial_t u\right\|^2\\&\quad+\left(2k-4\delta\right)\left\|\eta^t(s)\right\|_{\mu,1}^2\\&\ge0.\end{aligned}$$

Thus, the dissipativity of the solutions to the problems (2.4) - (2.7) can be achieved.

Theorem 3.3. Assume that Conditions (M_1) - (M_3) hold. If $z_1=\left(u_1,\partial_t u_1,\eta_1^t\right)$ and $z_2=\left(u_2,\partial_t u_2,\eta_2^t\right)$ are two solutions to the problems (2.4) - (2.7) corresponding to the initial values $z_1(\tau)$ and $z_2(\tau)$ respectively, then for any $\tau<T$, we have

$$\left\|z_1(t)-z_2(t)\right\|_{\mathcal{H}_t^1}^2\le\frac{\tilde{C}_2}{\tilde{C}_1}\mathrm{e}^{\frac{2\tilde{C}_3}{\tilde{C}_1}\sqrt{\tilde{C}}(t-\tau)}\left\|z_1(\tau)-z_2(\tau)\right\|_{\mathcal{H}_\tau^1}^2,\ \forall t\in[\tau,T].\tag{3.29}$$

Proof. Let $\tilde{z}=z_1-z_2$, then $\tilde{z}=\left(\tilde{u},\partial_t\tilde{u},\tilde{\eta}^t\right)$ satisfies

$$\partial_t^2\tilde{u}+A\tilde{u}+\gamma A^{\theta}\partial_t\tilde{u}+\int_0^{\infty}\mu(s)A\tilde{\eta}^t(s)\mathrm{d}s+f_1-f_2=0,\ (x,t)\in\Omega\times[\tau,\infty),\tag{3.30}$$

$$\tilde{u}(x,t)=0,\ x\in\partial\Omega,\ t>\tau,\tag{3.31}$$

$$\tilde{\eta}^{\tau}(x,s)=0,\ (x,s)\in\partial\Omega\times\mathbb{R}^+,\ t>\tau,\tag{3.32}$$

$$\begin{aligned}&\tilde{u}(x,t,\tau)=\tilde{u}_{\tau_1}(x)-\tilde{u}_{\tau_2}(x),\ \partial_t\tilde{u}(x,t,\tau)=\partial_t\tilde{u}_{\tau_1}(x)-\partial_t\tilde{u}_{\tau_2}(x),\\&\tilde{\eta}^{\tau}(x,s)=\tilde{\eta}_{\tau_1}(x,s)-\tilde{\eta}_{\tau_2}(x,s),\ x\in\Omega,\ t\le\tau.\end{aligned}\tag{3.33}$$

where $f_i=f(u_i)$ for $i=1,2$.

Taking the inner product of Equation (3.57) with $\partial_t\tilde{u}$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}K_2\left(\tilde{u},\partial_t\tilde{u},\tilde{\eta}^t\right)+2\gamma\left\|\partial_t\tilde{u}\right\|_\theta^2+\int_0^\infty\mu(s)\frac{\mathrm{d}}{\mathrm{d}s}\left\|\tilde{\eta}^t(s)\right\|_1^2\mathrm{d}s=-2\left\langle f(u_1)-f(u_2),\partial_t\tilde{u}\right\rangle. \tag{3.34}$$

where

$$K_2\left(\tilde{u},\partial_t\tilde{u},\tilde{\eta}^t\right)=\|\tilde{u}\|_1^2+\left\|\partial_t\tilde{u}\right\|^2+\left\|\tilde{\eta}^t\right\|_{\mu,1}^2.$$

Integrating Equation (3.34) over the interval $[\tau,t]$, we get

$$\begin{aligned}&K_2\left(\tilde{u}(t),\partial_t\tilde{u}(t),\tilde{\eta}^t(s)\right)+2\gamma\int_\tau^t\left\|\partial_t\tilde{u}(r)\right\|_\theta^2\mathrm{d}r+k\int_\tau^t\left\|\tilde{\eta}^r(s)\right\|_{\mu,1}^2\mathrm{d}r\\&\le K_2\left(\tilde{u}_\tau,\partial_t\tilde{u}_\tau,\tilde{\eta}_\tau(s)\right)-\int_\tau^t2\left\langle f(u_1)-f(u_2),\partial_t\tilde{u}\right\rangle\mathrm{d}r.\end{aligned} \tag{3.35}$$

For $2\le p\le p_\theta=3+2\theta$, by integration by parts, we have

$$\begin{aligned}-2\left\langle f(u_1)-f(u_2),\partial_t\tilde{u}\right\rangle&=-2\int_\Omega\left(f(u_1)-f(u_2)\right)\partial_t\tilde{u}\mathrm{d}x\\&=-2\int_\Omega f'\left(u_1+c(u_2-u_1)\right)\tilde{u}\partial_t\tilde{u}\mathrm{d}x\\&=-\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega f'\left(u_1+c(u_2-u_1)\right)\tilde{u}^2\mathrm{d}x\\&\quad+\int_\Omega f_t'\left(u_1+c(u_2-u_1)\right)\tilde{u}^2\mathrm{d}x,\end{aligned}$$

where $c\in[0,1]$. Hence

$$\begin{aligned}&-2\int_\tau^t\int_\Omega\left(f(u_1)-f(u_2)\right)\partial_t\tilde{u}\mathrm{d}x\mathrm{d}r\\&=-\int_\Omega f'\left(u_1+c(u_2-u_1)\right)\tilde{u}^2\mathrm{d}x\Big|_\tau^t+\int_\tau^t\int_\Omega f_t'\left(u_1+c(u_2-u_1)\right)\tilde{u}^2\mathrm{d}x\mathrm{d}r.\end{aligned} \tag{3.36}$$

By virtue of Hölder inequality, we obtain

$$\begin{aligned}&-\int_\Omega f'\left(u_1(t)+c\left(u_2(t)-u_1(t)\right)\right)\tilde{u}^2\mathrm{d}x\\&\le C_1\int_\Omega\left(1+|u_1(t)|^{p-1}+|u_2(t)|^{p-1}\right)|\tilde{u}(t)|^2\mathrm{d}x\\&\le C_1\left(1+\|u_1(t)\|_{L^{p+1}}^{p-1}+\|u_2(t)\|_{L^{p+1}}^{p-1}\right)\|\tilde{u}(t)\|_{L^{p+1}}^2\\&\le C_2\left(1+\|u_1(t)\|_1^{p-1}+\|u_2(t)\|_1^{p-1}\right)\|\tilde{u}(t)\|_{1-\alpha}^2\\&\le\delta\|\tilde{u}\|_1^2+C\left(R,\beta_0,\|g\|,\lambda_1,C_2,\delta\right)\|\tilde{u}(t)\|_{1-\theta}^2\\&\le\delta\|\tilde{u}\|_1^2+\frac{m_3}{m_2}C\left(R,\beta_0,\|g\|,\lambda_1,C_2,\delta\right)\mathrm{e}^{\tilde{C}(t-\tau)}\left(\|\tilde{u}_\tau\|_{1-\theta}^2+\|\partial_t\tilde{u}_\tau\|_{-\theta}^2+\|\tilde{\eta}_\tau(s)\|_{\mu,1-\theta}^2\right),\end{aligned}$$

where we have used the Sobolev embedding $V_{1-\alpha}\hookrightarrow L^{p+1}(\Omega)$ for $0<\alpha\ll1$ and chosen δ as an arbitrarily small positive constant. Similarly, we have

$$\int_\Omega f'\left(u_{\tau_1}+c\left(u_{\tau_2}-u_{\tau_1}\right)\right)\tilde{u}_\tau^2\mathrm{d}x\le C(R,C_2)\|\tilde{u}_\tau\|_1^2.$$

Again by Hölder inequality, we deduce that

$$\begin{aligned}&\int_\tau^t\int_\Omega f_t'\left(u_1+c(u_2-u_1)\right)\tilde{u}^2\mathrm{d}x\mathrm{d}r\\&\le C_0\int_\tau^t\int_\Omega\left(1+|u_1|^{p-2}+|u_2|^{p-2}\right)\left(|\partial_tu_1|+|\partial_tu_2|\right)|\tilde{u}|^2\mathrm{d}x\mathrm{d}r\\&\le C_0\int_\tau^t\left(1+\|u_1\|_{L^6}^{p-2}+\|u_2\|_{L^6}^{p-2}\right)\left(\|\partial_tu_1\|_{L^{\frac{6}{6-p}}}+\|\partial_tu_2\|_{L^{\frac{6}{6-p}}}\right)\|\tilde{u}\|_{L^6}^2\mathrm{d}r\\&\le C_2\int_\tau^t\left(1+\|u_1\|_1^{p-2}+\|u_2\|_1^{p-2}\right)\left(\|\partial_tu_1\|_{L^{\frac{6}{3-2\theta}}}+\|\partial_tu_2\|_{L^{\frac{6}{3-2\theta}}}\right)\|\tilde{u}\|_1^2\mathrm{d}r\\&\le C\left(R,\beta_0,\|g\|,\lambda_1,C_1,C_2\right)\int_\tau^t\left(\sqrt{\|\partial_tu_1\|_\theta^2}+\sqrt{\|\partial_tu_2\|_\theta^2}\right)\|\tilde{u}(r)\|_1^2\mathrm{d}r,\end{aligned}$$

where $\frac{p-2}{6}+\frac{6-p}{6}+\frac{1}{3}=1$ and $\frac{6}{6-p}\le\frac{6}{3-2\theta}$ for $p\le 3+2\theta$.

Substituting the above estimates into Equation (3.36), we get

$$\begin{aligned}
&K_2\left(\tilde{u}(t),\partial_t\tilde{u}(t),\tilde{\eta}^t(s)\right)+2\gamma\int_\tau^t\left\|\partial_t\tilde{u}(r)\right\|_\theta^2\mathrm{d}r\\
&\le K_2\left(\tilde{u}_\tau,\partial_t\tilde{u}_\tau,\tilde{\eta}_\tau(s)\right)+\delta\|\tilde{u}\|_1^2\\
&\quad+\frac{m_3}{m_2}C\left(R,\beta_0,\|g\|,\lambda_1,C_2,\delta\right)\mathrm{e}^{\tilde{C}(t-\tau)}\left(\|\tilde{u}_\tau\|_{1-\theta}^2+\|\partial_t\tilde{u}_\tau\|_{-\theta}^2+\|\tilde{\eta}_\tau(s)\|_{\mu,1-\theta}^2\right)\\
&\quad+C\left(R,C_2\right)\|\tilde{u}(\tau)\|_1^2+C\left(R,\beta_0,\|g\|,\lambda_1,C_1,C_2\right)\int_\tau^t\left(\sqrt{\|\partial_t u_1\|_\theta^2}+\sqrt{\|\partial_t u_2\|_\theta^2}\right)\|\tilde{u}(r)\|_1^2\mathrm{d}r,
\end{aligned}$$

which implies that

$$\begin{aligned}
&(m_4-\delta)\left(\|\tilde{u}(t)\|_1^2+\|\partial_t\tilde{u}(t)\|^2+\|\tilde{\eta}^t(s)\|_{\mu,1}^2\right)\\
&\le\frac{m_3}{m_2}\lambda_1^{-\theta}C\left(R,\beta_0,\|g\|,\lambda_1,C_2,\delta\right)\mathrm{e}^{\tilde{C}(t-\tau)}\left(\|\tilde{u}_\tau\|_1^2+\|\partial_t\tilde{u}_\tau\|^2+\|\tilde{\eta}_\tau(s)\|_{\mu,1}^2\right)\\
&\quad+C\left(R,C_2\right)\left(\|\tilde{u}_\tau\|_1^2+\|\partial_t\tilde{u}_\tau\|^2+\|\tilde{\eta}_\tau(s)\|_{\mu,1}^2\right)+m_5\left(\|\tilde{u}_\tau\|_1^2+\|\partial_t\tilde{u}_\tau\|^2+\|\tilde{\eta}_\tau(s)\|_{\mu,1}^2\right)\\
&\quad+C\left(R,\beta_0,\|g\|,\lambda_1,C_1,C_2\right),
\end{aligned}$$

$$\begin{aligned}
&\int_\tau^t\left(\sqrt{\|\partial_t u_1(r)\|_\theta^2}+\sqrt{\|\partial_t u_2(r)\|_\theta^2}\right)\left(\|\tilde{u}(r)\|_1^2+\|\partial_t\tilde{u}(r)\|^2+\|\tilde{\eta}^r(s)\|_{\mu,1}^2\right)\mathrm{d}r\\
&\le\tilde{C}_2\left(\|\tilde{u}_\tau\|_1^2+\|\partial_t\tilde{u}_\tau\|^2+\|\tilde{\eta}_\tau(s)\|_{\mu,1}^2\right)\\
&\quad+\tilde{C}_3\int_\tau^t\left(\sqrt{\|\partial_t u_1(r)\|_\theta^2}+\sqrt{\|\partial_t u_2(r)\|_\theta^2}\right)\left(\|\tilde{u}(r)\|_1^2+\|\partial_t\tilde{u}(r)\|^2+\|\tilde{\eta}^r(s)\|_{\mu,1}^2\right)\mathrm{d}r,
\end{aligned}\tag{3.37}$$

where

$$\tilde{C}_1=m_4-\delta,$$

$$\tilde{C}_2=C\left(R,C_2\right)+\frac{m_3}{m_2}\lambda_1^{-\theta}C\left(R,\beta_0,\|g\|,\lambda_1,C_2,\delta\right)\mathrm{e}^{\tilde{C}(t-\tau)}+m_5,$$

$$\tilde{C}_3=C\left(R,\beta_0,\|g\|,\lambda_1,C_1,C_2\right).$$

By applying Gronwall lemma, we can prove inequality (3.29). Meanwhile, we also obtain the uniqueness of the solutions to the problems (2.4) - (2.7) in the space $\mathcal{H}_t^1$.

Based on Theorem 3.2 and Theorem 3.3, we can define the process $U(t,\tau)$ for the problems (2.4) - (2.7) as follows:

$$z(t)=U(t,\tau)z(\tau):\mathcal{H}_\tau^1\to\mathcal{H}_t^1,$$

and this process is continuous from $\mathcal{H}_\tau^1$ to $\mathcal{H}_t^1$.

4. The Existence of Time Dependent Attractor

4.1. Time-Dependent Absorbing Set in $\mathcal{H}_t^1$

Based on Theorem 3.2, we obtain the following result.

Theorem 4.1. Under the assumptions of Theorem 3.2, if for any initial value

$z(\tau)\in\mathbb{B}_\tau(R)\subset\mathcal{H}_\tau^1$, then there exists $R_0>0$ such that the process $U(t,\tau)$ corresponding to the problems (2.4) - (2.7) possesses a time-dependent absorbing set, namely the family of sets $\mathfrak{B}_t=\{\mathbb{B}_t(R_0)\}_{t\in\mathbb{R}}$.

4.2. A Priori Estimates

Next, we verify the compactness of the solution process $U(t,\tau)$ to the problems (2.4) - (2.7). To this end, we establish the following a priori estimates.

Let $z_i(t)=(u_i(t),\partial_t u_i(t),\eta_i^t(s))(i=1,2)$ be solutions to the problems (2.4) - (2.7) corresponding to the initial values $(u_{\tau_i},\partial_t u_{\tau_i},\eta_{\tau_i})\in\{\mathbb{B}_\tau(R)\}_{\tau\in\mathbb{R}}$ respectively. The difference between the two solutions

$\tilde{z}(t)=z_1(t)-z_2(t)=(\omega(t),\partial_t\omega(t),\zeta^t(s))$ satisfies the following equations:

$$\partial_t^2\omega+A\omega+\gamma A^\theta\partial_t\omega+\int_0^\infty\mu(s)A\zeta^t(s)\mathrm{d}s+f_1-f_2=0,\ (x,t)\in\Omega\times[\tau,\infty),\tag{4.1}$$

$$\omega(x,t)=0,\ x\in\partial\Omega,\ t>\tau,\tag{4.2}$$

$$\zeta^\tau(x,s)=0,\ (x,s)\in\partial\Omega\times\mathbb{R}^+,\ t>\tau,\tag{4.3}$$

$$\begin{gathered}\omega(x,t,\tau)=u_{\tau_1}(x)-u_{\tau_2}(x),\ \partial_t\omega(x,t,\tau)=\partial_t u_{\tau_1}(x)-\partial_t u_{\tau_2}(x),\\ \zeta^\tau(x,s)=\eta_{\tau_1}(x,s)-\eta_{\tau_2}(x,s),\ x\in\Omega,\ t\le\tau.\end{gathered}\tag{4.4}$$

where $f_i=f(u_i)$ for $i=1,2$.

Define

$$H(t)=\|\omega(t)\|_1^2+\|\partial_t\omega(t)\|^2+\|\zeta^t(s)\|_{\mu,1}^2.$$

We shall carry out the a priori estimates in the following four steps.

Step 1. Multiply Equation (4.1) by $2\partial_t\omega$ and integrate over $[s,t]\times\Omega$, we obtain

$$\begin{aligned}&H(t)-H(s)+2\gamma\int_s^t\int_\Omega\left|A^{\frac{\theta}{2}}\partial_t\omega(r)\right|^2\mathrm{d}x\mathrm{d}r+\int_s^t\int_0^\infty\mu(s)\frac{\mathrm{d}}{\mathrm{d}s}\|\zeta^r(s)\|_1^2\mathrm{d}s\mathrm{d}r\\&=-2\int_s^t\int_\Omega(f(u_1)-f(u_2))\partial_t\omega(r)\mathrm{d}x\mathrm{d}r,\end{aligned}\tag{4.5}$$

where $T\le s\le t$.

From

$$\int_0^\infty\mu(s)\frac{\mathrm{d}}{\mathrm{d}s}\|\eta^t(s)\|_1^2\mathrm{d}s\ge k\|\eta^t(s)\|_{\mu,1}^2,$$

$$\int_s^t\int_\Omega\left|A^{\frac{\theta}{2}}\partial_t\omega(r)\right|^2\mathrm{d}x\mathrm{d}r\ge\lambda_1^\theta\int_s^t\int_\Omega|\partial_t\omega(r)|^2\mathrm{d}x\mathrm{d}r,$$

there exists a constant m_8 such that

$$\begin{aligned}&H(t)-H(s)+m_6\left(\int_s^t\|\partial_t\omega(r)\|^2\mathrm{d}r+\int_s^t\|\zeta^r(s)\|_{\mu,1}^2\mathrm{d}r\right)\\&\le-2\int_s^t\int_\Omega(f(u_1)-f(u_2))\partial_t\omega(r)\mathrm{d}x\mathrm{d}r,\end{aligned}\tag{4.6}$$

where $m_6=\min\{2\gamma\lambda_1^\theta,k\}$.

Then

$$\begin{aligned}&\int_s^t\left\|\partial_t\omega(r)\right\|^2\mathrm{d}r+\int_s^t\left\|\zeta^r(s)\right\|_{\mu,1}^2\mathrm{d}r\\&\le\frac{1}{m_6}H(T)-\frac{2}{m_6}\int_s^t\int_\Omega\left(f(u_1)-f(u_2)\right)\partial_t\omega(r)\mathrm{d}x\mathrm{d}r.\end{aligned}\tag{4.7}$$

Step 2. Multiply Equation (4.1) by ω and integrate over $[T,t]\times\Omega$, we get

$$\begin{aligned}&\int_\Omega\partial_t\omega(t)\omega(t)\mathrm{d}x-\int_\Omega\partial_t\omega(T)\omega(T)\mathrm{d}x+\frac{\gamma}{2}\left\|\omega(t)\right\|_\theta^2-\frac{\gamma}{2}\left\|\omega(T)\right\|_\theta^2\\&+\int_T^t\left\|\omega(r)\right\|_1^2\mathrm{d}r+\int_T^t\int_0^\infty\mu(s)\left\langle A\zeta^r,\omega(r)\right\rangle\mathrm{d}s\mathrm{d}r\\&+\int_T^t\int_\Omega\left(f(u_1)-f(u_2)\right)\omega(r)\mathrm{d}x\mathrm{d}r\\&=\int_T^t\left\|\partial_t\omega(r)\right\|^2\mathrm{d}r.\end{aligned}\tag{4.8}$$

By virtue of (4.4) and (4.5), we have

$$\begin{aligned}\int_T^t H(r)\mathrm{d}r&=\int_T^t\left(\left\|\omega(r)\right\|_1^2+\left\|\partial_t\omega(r)\right\|^2+\left\|\zeta^r\right\|_{\mu,1}^2\right)\mathrm{d}r\\&\le\frac{1}{m_6}H(T)-\frac{2}{m_6}\int_s^t\int_\Omega\left(f(u_1)-f(u_2)\right)\partial_t\omega(r)\mathrm{d}x\mathrm{d}r\\&\quad+\int_\Omega\partial_t\omega(T)\omega(T)\mathrm{d}x-\int_\Omega\partial_t\omega(t)\omega(t)\mathrm{d}x\\&\quad+\frac{\gamma}{2}\left\|\omega(T)\right\|_\theta^2+\int_T^t\left\|\partial_t\omega(r)\right\|^2\mathrm{d}r-\int_T^t\left\langle\zeta^r,\omega(r)\right\rangle_{\mu,1}\mathrm{d}r\\&\quad-\int_T^t\int_\Omega\left(f(u_1)-f(u_2)\right)\omega(r)\mathrm{d}x\mathrm{d}r.\end{aligned}$$

Step 3. Integrate Equation (4.6) with respect to s over $[T,t]$, we obtain

$$\begin{aligned}H(t)(t-T)&\le\int_T^t H(s)\mathrm{d}s-2\int_T^t\int_s^t\int_\Omega\left(f(u_1)-f(u_2)\right)\partial_t\omega(r)\mathrm{d}x\mathrm{d}r\mathrm{d}s\\&\le\frac{1}{m_6}H(T)+\frac{\gamma}{2}\left\|\omega(T)\right\|_\theta^2+\int_\Omega\partial_t\omega(T)\omega(T)\mathrm{d}x\\&\quad+\int_T^t\left\|\partial_t\omega(s)\right\|^2\mathrm{d}s-\int_\Omega\partial_t\omega(t)\omega(t)\mathrm{d}x-\int_T^t\left\langle\zeta^s,\omega(s)\right\rangle_{\mu,1}\mathrm{d}s\\&\quad-2\int_T^t\int_s^t\int_\Omega\left(f(u_1)-f(u_2)\right)\partial_t\omega(r)\mathrm{d}x\mathrm{d}r\mathrm{d}s\\&\quad-\frac{2}{m_6}\int_s^t\int_\Omega\left(f(u_1)-f(u_2)\right)\partial_t\omega(s)\mathrm{d}x\mathrm{d}s\\&\quad-\int_T^t\int_\Omega\left(f(u_1)-f(u_2)\right)\omega(s)\mathrm{d}x\mathrm{d}s.\end{aligned}$$

Step 4. Denote

$$C(M)=\frac{1}{m_6}H(T)+\frac{\gamma}{2}\left\|\omega(T)\right\|_\theta^2+\int_\Omega\partial_t\omega(T)\omega(T)\mathrm{d}x,\tag{4.9}$$

and

$$\Phi_T^t\left(\left(u_1(T),\partial_t u_1(T),\zeta_1^T(s)\right),\left(u_2(T),\partial_t u_2(T),\zeta_2^T(s)\right)\right)=\Psi_1+\Psi_2,\tag{4.10}$$

where

$$\Psi_1=\frac{1}{t-T}\left(\int_T^t\left\|\partial_t\omega(s)\right\|^2\mathrm{d}s-\int_\Omega\partial_t\omega(t)\omega(t)\mathrm{d}x-\int_T^t\left\langle\zeta^s,\omega(s)\right\rangle_{\mu,1}\mathrm{d}s\right),$$

$$\Psi_2 = -\frac{1}{t-T}\left(\frac{2}{m_6}\int_T^t\int_\Omega\left(f(u_1)-f(u_2)\right)\partial_t\omega(s)\mathrm{d}x\mathrm{d}s\right.$$
$$+\int_T^t\int_\Omega\left(f(u_1)-f(u_2)\right)\omega(s)\mathrm{d}x\mathrm{d}s$$
$$\left.+2\int_T^t\int_s^t\int_\Omega\left(f(u_1)-f(u_2)\right)\partial_t\omega(r)\mathrm{d}x\mathrm{d}r\mathrm{d}s\right).$$

Thus

$$H(t)\le\frac{1}{t-T}C_M+\Phi_T^t\left(\left(u_1(T),\partial_t u_1(T),\eta_1^T(s)\right),\left(u_2(T),\partial_t u_2(T),\eta_2^T(s)\right)\right). \quad (4.11)$$

Next, we shall prove the asymptotic compactness of the solution process to the problems (2.4) - (2.7) by using the method of contraction functions.

4.3. Asymptotic Compactness

Theorem 4.2. If the assumptions (M_1) - (M_3) hold, for any fixed $t\in\mathbb{R}$, any bounded sequence $\{\tau_n\}_{n=1}^{\infty}\subset(-\infty,t]$ with $\tau_n\to-\infty$ as $n\to\infty$, and any sequence $\{x_n\}_{n=1}^{\infty}\subset\mathcal{H}_{\tau_n}^{\alpha}$, the sequence $\{U(t,\tau_n)x_n\}_{n=1}^{\infty}$ has a convergent subsequence.

Proof. For any $\varepsilon>0$ and fixed t, there exists $T<t$ such that $\frac{C_M}{t-T}<\varepsilon$. Thanks to Theorem 2.10, we also need to show that $\Phi_T^t\in\mathfrak{C}\left(\mathbb{B}_T(R)\right)$, for every fixed t.

Let $\left(u_n,\partial_t u_n,\eta_n^t\right)$ be the solutions to the problems (2.4) - (2.7) corresponding to the initial values $\left(u_{n_0},u_{n_1},\eta_n^0\right)\in\mathbb{B}_T(R)$. From Theorem 3.2, we know that $\|u_n\|_1^2+\|\partial_t u_n\|^2+\|\eta_n^t\|_{\mu,1}^2$ is bounded.

By virtue of Alaoglu theorem, Lemma 2.5 and Theorem 3.2, for any $\tau\le T\le t$, without loss of generality, we assume that

$$u_n\to u \text{ weakly}^* \text{ in } L^\infty\left([T,t];V_1\right), \quad (4.12)$$

$$\partial_t u_n\to\partial_t u \text{ weakly in } L^2\left([T,t];L^2(\Omega)\right), \quad (4.13)$$

$$\partial_t^2 u_n\to\partial_t^2 u \text{ weakly}^* \text{ in } L^\infty\left([T,t];V_{-2\theta}\right), \quad (4.14)$$

$$\eta_n^t\to\eta^t \text{ weakly}^* \text{ in } L^\infty\left([T,t];L_\mu^2\left(\mathbb{R}^+,V_1\right)\right), \quad (4.15)$$

$$\partial_t u_n\to\partial_t u \text{ weakly in } L^2\left([T,t];V_\theta\right), \quad (4.16)$$

$$u_n\to u \text{ in } L^{p+1}\left([T,t];L^{p+1}(\Omega)\right), \quad (4.17)$$

$$u_n\to u \text{ in } L^2\left([T,t];V_1\right), \quad (4.18)$$

$$u_n(t)\to u(t) \text{ and } u_n(T)\to u(T) \text{ in } L^{p+1}(\Omega), \quad (4.19)$$

$$\partial_t u_n\to\partial_t u \text{ in } L^2\left([T,t];L^2(\Omega)\right). \quad (4.20)$$

where we have used the compact Sobolev embedding $V_1\hookrightarrow\hookrightarrow L^{p+1}(\Omega)$ for $p\le 3+2\theta$.

From (3.29), we obtain that

$$\left\{\left(u_n(s),\partial_t u_n(s),\eta_n^s\right)\right\}\subset C\left([T,t];\mathcal{H}_s^1\right) \text{ is a Cauchy sequence,} \tag{4.21}$$

and there exists $\left(u(s),\partial_t u(s),\eta^s\right)\in C\left([T,t];\mathcal{H}_s^1\right)$ such that

$$\left(u_n(s),\partial_t u_n(s),\eta_n^s\right) \text{ converges to } \left(u(s),\partial_t u(s),\eta^s\right) \text{ in } C\left([T,t];\mathcal{H}_s^1\right). \tag{4.22}$$

Next, we deal with each term in (4.11).

First of all, by (4.15) and (4.20), we have

$$\lim_{n\to\infty}\lim_{m\to\infty}\int_T^t\left\|\partial_t u_n-\partial_t u_m\right\|^2\mathrm{d}s=0, \tag{4.23}$$

$$\begin{aligned}
&\lim_{n\to\infty}\lim_{m\to\infty}\int_\Omega\left(\partial_t u_n-\partial_t u_m\right)\left(u_n-u_m\right)\mathrm{d}x\\
&\le\lim_{n\to\infty}\lim_{m\to\infty}\left(\left\|\partial_t u_n-\partial_t u_m\right\|\left\|u_n-u_m\right\|\right)\\
&\le\lim_{n\to\infty}\lim_{m\to\infty}\left(\left(\left\|\partial_t u_n\right\|+\left\|\partial_t u_m\right\|\right)\left\|u_n-u_m\right\|\right)=0,
\end{aligned} \tag{4.24}$$

$$\begin{aligned}
&\lim_{n\to\infty}\lim_{m\to\infty}\int_T^t\left\langle\eta_n^s-\eta_m^s,u_n-u_m\right\rangle_{\mu,1}\mathrm{d}s\\
&=\lim_{n\to\infty}\lim_{m\to\infty}\int_T^t\int_0^\infty\mu(l)\left\langle\eta_n^s(l)-\eta_m^s(l),u_n-u_m\right\rangle_1\mathrm{d}l\mathrm{d}s\\
&\le\lim_{n\to\infty}\lim_{m\to\infty}\int_T^t\int_0^\infty\mu(l)\left\|\eta_n^s(l)-\eta_m^s(l)\right\|_1\left\|u_n-u_m\right\|_1\mathrm{d}l\mathrm{d}s\\
&\le\lim_{n\to\infty}\lim_{m\to\infty}\int_T^t\left\|u_n-u_m\right\|_1\int_0^\infty\mu(l)\left\|\eta_n^s(l)-\eta_m^s(l)\right\|_1\mathrm{d}l\mathrm{d}s\\
&\le\lim_{n\to\infty}\lim_{m\to\infty}\left(\int_T^t\left\|u_n-u_m\right\|_1^2\mathrm{d}s\right)^{\frac12}\left(\int_T^t\left(\int_0^\infty\mu(l)\mathrm{d}l\right)\left(\int_0^\infty\mu(l)\left\|\eta_n^s(l)-\eta_m^s(l)\right\|_1^2\mathrm{d}l\right)\mathrm{d}s\right)^{\frac12}\\
&\le\sqrt{k_0}\left(\int_T^t\left\|u_n-u_m\right\|_1^2\mathrm{d}s\right)^{\frac12}\left(\int_T^t\left\|\eta_n^s(l)-\eta_m^s(l)\right\|_{\mu,1}^2\mathrm{d}s\right)^{\frac12}=0.
\end{aligned} \tag{4.25}$$

Combining (4.23) - (4.25), we get

$$\lim_{n\to\infty}\lim_{m\to\infty}\Psi_1=0. \tag{4.26}$$

Secondly, from (1.4) and (4.18), we obtain

$$\begin{aligned}
&\lim_{n\to\infty}\lim_{m\to\infty}\int_T^t\int_\Omega\left(f\left(u_n\right)-f\left(u_m\right)\right)\left(u_n-u_m\right)\mathrm{d}x\mathrm{d}s\\
&\le C\lim_{n\to\infty}\lim_{m\to\infty}\int_T^t\int_\Omega\left(1+\left|u_n\right|^{p-1}+\left|u_m\right|^{p-1}\right)\left|u_n-u_m\right|^2\mathrm{d}x\mathrm{d}s\\
&\le C\lim_{n\to\infty}\lim_{m\to\infty}\int_T^t\left(1+\left\|u_n\right\|_{L^{p+1}}^{p-1}+\left\|u_m\right\|_{L^{p+1}}^{p-1}\right)\cdot\left\|u_n-u_m\right\|_{L^{p+1}}^2\mathrm{d}s\\
&\le C\lim_{n\to\infty}\lim_{m\to\infty}\int_T^t\left(1+\left\|u_n\right\|_1^{p-1}+\left\|u_m\right\|_1^{p-1}\right)\cdot\left\|u_n-u_m\right\|_1^2\mathrm{d}s\\
&\le C\lim_{n\to\infty}\lim_{m\to\infty}\int_T^t\left\|u_n-u_m\right\|_1^2\mathrm{d}s=0.
\end{aligned} \tag{4.27}$$

It is easy to see that

$$\begin{aligned}
&\int_T^t\int_\Omega\left(f\left(u_n\right)-f\left(u_m\right)\right)\left(\partial_t u_n-\partial_t u_m\right)\mathrm{d}x\mathrm{d}s\\
&=\int_T^t\int_\Omega f\left(u_n\right)\partial_t u_n\mathrm{d}x\mathrm{d}s+\int_T^t\int_\Omega f\left(u_m\right)\partial_t u_m\mathrm{d}x\mathrm{d}s\\
&\quad-\int_T^t\int_\Omega f\left(u_m\right)\partial_t u_n\mathrm{d}x\mathrm{d}s-\int_T^t\int_\Omega f\left(u_n\right)\partial_t u_m\mathrm{d}x\mathrm{d}s\\
&=\int_\Omega F\left(u_n(t)\right)\mathrm{d}x-\int_\Omega F\left(u_n(T)\right)\mathrm{d}x+\int_\Omega F\left(u_m(t)\right)\mathrm{d}x\\
&\quad-\int_\Omega F\left(u_m(T)\right)\mathrm{d}x-\int_T^t\int_\Omega f\left(u_m\right)\partial_t u_n\mathrm{d}x\mathrm{d}s-\int_T^t\int_\Omega f\left(u_n\right)\partial_t u_m\mathrm{d}x\mathrm{d}s.
\end{aligned} \tag{4.28}$$

By (1.4) and the compact embedding $V_1 \hookrightarrow\hookrightarrow L^{p+1}(\Omega)$, we have

$$\begin{aligned}
&\left|\int_\Omega \left(F\left(u_n(t)\right)-F\left(u(t)\right)\right)\mathrm{d}x\right| \\
&\le \int_\Omega \left|f\left(u(t)+\vartheta\left(u_n(t)-u(t)\right)\right)\right|\left|u_n(t)-u(t)\right|\mathrm{d}x \\
&\le C\int_\Omega \left(1+\left|u_n(t)\right|^p+\left|u(t)\right|^p\right)\left|u_n(t)-u(t)\right|\mathrm{d}x \\
&\le C\left(1+\left\|u_n(t)\right\|_{L^{p+1}}^p+\left\|u(t)\right\|_{L^{p+1}}^p\right)\left\|u_n(t)-u(t)\right\|_{L^{p+1}} \\
&\le C\varepsilon.
\end{aligned} \tag{4.29}$$

Because $f(u_n)\in L^2\left([\tau,T];V_{-\gamma}\right)$ and $\partial_t u_m \in L^2\left([\tau,T];V_\gamma\right)$ as $n\to\infty$, $m\to\infty$, we have

$$\begin{aligned}
&\lim_{n\to\infty}\lim_{m\to\infty}\int_T^t \left\langle f(u_n),\partial_t u_m\right\rangle \mathrm{d}s \\
&=\lim_{n\to\infty}\int_T^t \left\langle f(u_n),\partial_t u\right\rangle \mathrm{d}s \\
&=\int_T^t \left\langle f(u),\partial_t u\right\rangle \mathrm{d}s \\
&=\int_\Omega F\left(u(t)\right)\mathrm{d}x-\int_\Omega F\left(u(T)\right)\mathrm{d}x.
\end{aligned}$$

Similarly, we have

$$\lim_{n\to\infty}\lim_{m\to\infty}\int_T^t \left\langle f(u_m),\partial_t u_n\right\rangle \mathrm{d}s=\int_\Omega F\left(u(t)\right)\mathrm{d}x-\int_\Omega F\left(u(T)\right)\mathrm{d}x.$$

Therefore,

$$\lim_{n\to\infty}\lim_{m\to\infty}\int_T^t\int_\Omega \left(f(u_n)-f(u_m)\right)\left(\partial_t u_n-\partial_t u_m\right)\mathrm{d}x\mathrm{d}s=0. \tag{4.30}$$

For each fixed t, the term $\left|\int_s^t\int_\Omega \left(f(u_n)-f(u_m)\right)\left(\partial_t u_n-\partial_t u_m\right)\mathrm{d}x\mathrm{d}r\right|$ is bounded. Then, thanks to Lebesgue Dominated Convergence Theorem, we get

$$\begin{aligned}
&\lim_{n\to\infty}\lim_{m\to\infty}\int_T^t\int_s^t\int_\Omega \left(f(u_n)-f(u_m)\right)\left(\partial_t u_n-\partial_t u_m\right)\mathrm{d}x\mathrm{d}r\mathrm{d}s \\
&=\int_T^t \lim_{n\to\infty}\lim_{m\to\infty}\int_s^t\int_\Omega \left(f(u_n)-f(u_m)\right)\left(\partial_t u_n-\partial_t u_m\right)\mathrm{d}x\mathrm{d}r\mathrm{d}s \\
&=\int_T^t 0\mathrm{d}s=0.
\end{aligned} \tag{4.31}$$

Hence, from (4.28) - (4.31), we obtain

$$\lim_{n\to\infty}\lim_{m\to\infty}\Psi_2=0. \tag{4.32}$$

In conclusion, we have $\Phi_T^t\left(\left(u_1(T),\partial_t u_1(T)\right),\left(u_2(T),\partial_t u_2(T)\right)\right)\in\mathfrak{C}\left(\mathbb{B}_T(R)\right)$.

4.4. The Time-Dependent Attractors

Theorem 4.3. If the assumptions of Theorem 4.2 hold, then the dynamical system $\left(U(t,\tau),\mathcal{H}_t^1\right)$ corresponding to the problems (2.4) - (2.7) possesses an invariant time-dependent attractor $\mathfrak{A}^{\mathfrak{A}}=\{A_t\}_{t\in\mathbb{R}}$.

Proof. According to Theorems 3.2, Theorems 3.3, Theorems 4.1 and Theorems 4.2, we obtain that the conclusion of Theorem 4.3 is valid.

5. Conclusion

This paper focuses on the wave equation with fading memory terms and structural damping. When the nonlinear term satisfies the critical exponential growth condition $2 \le p \le 3+2\theta$ ($\frac{1}{2} < \theta < 1$), we systematically analyze the dynamical behavior of the equation solutions by employing the Faedo-Galerkin approximation method, energy estimation techniques and the contraction function method. We not only rigorously prove the well-posedness, Lipschitz continuity and asymptotic compactness of the solutions to the equation, but also successfully establish the existence of time-dependent attractor in the natural energy space $H_0^1(\Omega)\times L^2(\Omega)\times L_\mu^2\left(\mathbb{R}^+;H_0^1(\Omega)\right)$.

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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