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Existence of Solutions for $\left(p_1(x), p_2(x)\right)$-Triharmonic Problem with Navier Boundary Conditions

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Abstract

In this paper, we use the Mountain Pass theorem and the Fountain theorem to study the existence of solutions for the following $\left(p_1(x), p_2(x)\right)$-triharmonic equations: $\begin{cases} -\Delta^3_{p_1(x)}u - \Delta^3_{p_2(x)}u = f(x,u), & \text{in } \Omega, \\ u = \Delta u = \Delta^2 u = 0, & \text{on } \partial\Omega, \end{cases}$. where the nonlinear term satisfying growth conditions weaker than Ambrosetti-Rabinowitz condition. We establish the existence of weak solutions using critical point theory and variational methods.

Keywords

Variable Exponent Space, $p(x)$-Triharmonic Operator, Navier Boundary Condition, Mountain Pass Theorem, Fountain Theorem

1. Introduction

In recent years, the study of partial differential equations with variable exponent growth conditions has received significant focus. These equations are widely used in many fields, such as electrorheological fluids [1]-[3], nonlinear elasticity [4], slow rotational flows [5], phase-field crystal growth [6], image processing [7]-[9], and geometric design [10]-[12]. In [13] [14], the authors studied the basic theory of variable exponent function spaces.

Fan and Zhang [15] studied the Dirichlet problem for variable-exponent elliptic $p(x)$-Laplacian equations:

$$\begin{cases} -\Delta_{p(x)}u = f(x,u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

and established the existence of solutions.

El Amrouss *et al.* [16] studied the $p(x)$-biharmonic problem under Navier boundary conditions:

$$\begin{cases} \Delta_{p(x)}^2 u = \lambda |u|^{p(x)-2} u + f(x,u), & \text{in } \Omega, \\ u = \Delta u = 0, & \text{on } \partial\Omega, \end{cases}$$

using the Mountain Pass Theorem and the Fountain Theorem, they obtain the existence and multiplicity of solutions. Research pertaining to the $p(x)$-biharmonic problem with Navier boundary conditions can be found in reference [17]-[20].

In [21], Zhao and Miao studied the following $p(x)$-triharmonic problem with Navier boundary conditions:

$$\begin{cases} -\Delta_{p(x)}^3 u = \lambda f(x,u), & \text{in } \Omega, \\ u = \Delta u = \Delta^2 u = 0, & \text{on } \partial\Omega, \end{cases}$$

where $\lambda > 0$, and $f(x,u)$ satisfies Ambrosetti-Rabinowitz-type growth condition. Using the Mountain Pass Theorem and the Fountain Theorem, they proved the existence of nontrivial and infinitely many solutions. In regard to investigations into the $p(x)$-triharmonic problem, the details are available in [22]-[24].

In recent years, there have been many research results on $(p(x),q(x))$-Laplace equations. This variable-exponent model has strong nonlinearity and spatial dependence. In [25], Vetro used critical point theory to prove the existence of nontrivial weak solutions.

Zhong and Wu [26] studied the $(p_1(x),p_2(x))$-biharmonic equation:

$$\begin{cases} \Delta_{p_1(x)}^2 u + \Delta_{p_2(x)}^2 u = f(x,u), & \text{in } \Omega, \\ u = \Delta u = 0, & \text{on } \partial\Omega, \end{cases}$$

using the Fountain Theorem, they obtained the existence results of solutions.

However, there are just a few results about the problems involving $(p_1(x),p_2(x))$-triharmonic operators. In this paper, we study the following nonlinear problem of $(p_1(x),p_2(x))$-triharmonic type with Navier boundary conditions:

$$\begin{cases} -\Delta_{p_1(x)}^3 u - \Delta_{p_2(x)}^3 u = f(x,u), & \text{in } \Omega, \\ u = \Delta u = \Delta^2 u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $\Omega \subset R^N (N \geq 3)$ is a bounded domain with smooth boundary $\partial\Omega$. $\Delta_{p_i(x)}^3 u$ is the $p(x)$-triharmonic operator which is not homogeneous and is related to the variable exponent Lebesgue space $L^{p(x)}$ and the variable exponent Sobolev space $W^{k,p(x)}(\Omega)$. It is also worth mentioning that the problems with the growth conditions $p(x)$-triharmonic have more complicated nonlinearities than the constant cases. Indeed, firstly the problem is not homogeneous, and secondly, the Lagrange multiplier theorem is not useful in such a case because $p(x)$ is variable.

Here, $p_i(x)\in C(\bar{\Omega})$ satisfies $p_i(x)>1$ for $i=1,2$ and all $x\in\bar{\Omega}$. We define:

$$p_M(x)=\max_{i=1,2}\{p_i(x)\},\ p_m(x)=\min_{i=1,2}\{p_i(x)\}$$

Denote

$$C_+(\bar{\Omega})=\{h \mid h\in C(\bar{\Omega}), h(x)>1, \forall x\in\bar{\Omega}\},$$

and

$$p^*(x)=\begin{cases}\dfrac{Np(x)}{N-3p(x)}, & 3p(x)<N,\\ +\infty, & 3p(x)\ge N.\end{cases}$$

The function f satisfies the Caratheodory condition. We assume the following hypotheses on $f(x,t)$:

(F0) $f:\Omega\times R\to R$ satisfies the Caratheodory condition, and there exists a constant $r>0$ such that

$$|f(x,t)|\le r\left(1+|t|^{\alpha(x)-1}\right), \forall (x,t)\in\Omega\times R,$$

where $\alpha(x)\in C_+(\bar{\Omega})$, and for any $x\in\bar{\Omega}$, there holds $\alpha(x)<p_m^*(x)$.

(F1) $f(x,t)=o\left(|t|^{p_M^+-1}\right)$ as $t\to 0$ for $x\in\Omega$ uniformly.

(F2) Suppose that $\lim_{|t|\to\infty}\dfrac{F(x,t)}{|t|^{2\beta}}=+\infty$ holds uniformly for $x\in\Omega$, where $\beta>\max_{i=1,2}\{p_i^+\}$.

(F3) $\limsup_{|t|\to\infty}\dfrac{F(x,t)}{|t|^{\theta(x)}}\le a(x)$ such that $\theta\in C_+(\bar{\Omega})$ with $\theta^+=\sup_{x\in\Omega}\theta(x)<p_m^-$, where $a\in L^\infty(\Omega)$.

(F4) There exists $M>0, \beta>\max_{i=1,2}\{p_i^+\}$ such that for all $x\in\Omega$ and all $t\in R$ with $|t|\ge M$, $f(x,t)t-2\beta F(x,t)\ge 0$.

(F5) $f(x,-t)=-f(x,t)$ for all $(x,t)\in\Omega\times R$.

Our main results are given by the following theorems:

Theorem 1. Assume (F3) hold. Then problem (1) has a weak solution.

Theorem 2. Assume $\alpha^+>p_M^+$, (F0) - (F2), and (F4) hold, then problem (1) has a nontrivial weak solution.

Theorem 3. Assume $\alpha^+>p_M^+$, (F0), (F2), (F4) and (F5) hold, then problem (1) has infinitely many weak solutions.

Remark. In reference [21], where the $f(x,u)$ satisfies Ambrosetti-Rabinowitz condition, the $f(x,u)$ studied in this paper does not satisfy this condition. An example of the $f(x,u)$ that does not satisfy the Ambrosetti-Rabinowitz condition, see [27]. (F2), (F4) are weaker than (Ambrosetti-Rabinowitz).

Set

$$f(x,u)=\begin{cases}-u^{2\beta-1}|u|+\beta u^{2\beta-1}, & |u|\le 1,\\ 2\beta u^{2\beta-1}\ln|u|+u^{2\beta-1}, & |u|>1,\end{cases}$$

$$uf(x,u)-2\beta F(x,u)=u^{2\beta}-2\beta\left(\frac{1}{2}-\frac{1}{2\beta-1}\right),|u|>1,\forall x\in\Omega,$$

if we take $|u|>\max\left\{1,\left[2\beta\left(\frac{1}{2}-\frac{1}{2\beta-1}\right)\right]^{\frac{1}{2\beta}}\right\}$, then f satisfies the conditions (F2) and (F4), but does not satisfy the Ambrosetti-Rabinowitz condition.

2. Preliminaries

For the reader's convenience, we recall some background facts concerning Lebesgue-Sobolev spaces with variable exponent and introduce some notation; Problem (1) is analyzed in variable exponent Lebesgue and Sobolev spaces $L^{p(x)}(\Omega)$ and $W^{k,p(x)}(\Omega)$. We first present their fundamental properties; for $p(x)\in C_+(\bar{\Omega})$, the Lebesgue space is defined:

$$L^{p(x)}(\Omega)=\left\{u:u \text{ is a measurable real-valued function in } \Omega,\int_\Omega|u(x)|^{p(x)}\mathrm{d}x<\infty\right\}.$$

This space is endowed with the Luxemburg norm, specified by

$$\|u\|_{L^{p(x)}(\Omega)}=\inf\left\{\lambda>0:\int_\Omega\left|\frac{u(x)}{\lambda}\right|^{p(x)}\mathrm{d}x\le 1\right\},$$

when $p(x)\equiv p, p\ge 1$, this norm is equivalent to the classical L^p-norm, $\|u\|_{L^p(\Omega)}=\left(\int_\Omega|u(x)|^p\mathrm{d}x\right)^{\frac{1}{p}}$.

Proposition 4. [13] [14] If $p^->1$, the space $\left(L^{p(x)}(\Omega),\|\cdot\|_{L^{p(x)}(\Omega)}\right)$ is a separable, uniformly convex, reflexive Banach space. Its conjugate space is $L^{p'(x)}(\Omega)$, where $p'(x)$ is the conjugate function of $p(x)$, namely,

$$\frac{1}{p(x)}+\frac{1}{p'(x)}=1.$$

For any $u\in L^{p(x)}(\Omega)$ d $v\in L^{p'(x)}(\Omega)$, we have

$$\left|\int_\Omega uv\mathrm{d}x\right|\le\left(\frac{1}{p^-}+\frac{1}{(p')^-}\right)\|u\|_{L^{p(x)}(\Omega)}\|v\|_{L^{p'(x)}(\Omega)}\le 2\|u\|_{L^{p(x)}(\Omega)}\|v\|_{L^{p'(x)}(\Omega)}.$$

Proposition 5. [28] We assume that the variable exponents $p_1(x),p_2(x)$ satisfy the condition:

$$1<p_i^-\le p_i(x)<p_i^+<+\infty \quad \text{for} \quad i=1,2$$

and are log-Holder continuous on Ω, *i.e.*,

$$|p_i(x)-p_i(y)|\le\frac{C}{-\log|x-y|},$$

for all $x,y\in\Omega$, with $|x-y|<\frac{1}{2}$.

Proposition 6. [13] Let $\rho_{p(x)}(u)=\int_\Omega|u(x)|^{p(x)}\mathrm{d}x$, for all $u\in L^{p(x)}(\Omega)$. We have

1) $|u|_{p(x)}^{p^-} \le \rho_{p(x)}(u) \le |u|_{p(x)}^{p^+}$ if $|u|_{p(x)} > 1$.

2) $|u|_{p(x)}^{p^+} \le \rho_{p(x)}(u) \le |u|_{p(x)}^{p^-}$ if $|u|_{p(x)} \le 1$.

Next we introduce the variable exponent Sobolev space:

$$W^{k,p(x)}(\Omega) = \left\{u \in L^{p(x)}(\Omega) : D^\alpha u \in L^{p(x)}(\Omega), |\alpha| \le k\right\},$$

with the norm:

$$\|u\|_{k,p(x)} = \sum_{|\alpha|\le k} \left\|D^\alpha u\right\|_{L^{p(x)}(\Omega)},$$

where

$$D^\alpha u = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_N^{\alpha_N}} u,$$

$\alpha = (\alpha_1, \cdots, \alpha_N)$ is a multi-index, and $|\alpha| = \sum_{i=1}^N \alpha_i$.

In addition, $W_0^{k,p(x)}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $W^{k,p(x)}(\Omega)$.

Proposition 7. [13] [24] Let $p(x) \in C_+(\bar{\Omega})$. Then the space $\left(W^{k,p(x)}(\Omega), \|\cdot\|_{k,p(x)}\right)$ is a reflexive and separable Banach space.

Proposition 8. [13] Let $p(x), q(x) \in C_+(\bar{\Omega})$ such that $q(x) \le p_k^*(x)$. Then there is a continuous embedding:

$$W^{k,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega).$$

If $\le$ is replaced by $<$, then the embedding is compact.

We set $X = X_1 \cap X_2$, where

$$X_i = W^{3,p_i(x)}(\Omega) \cap W_0^{1,p_i(x)}(\Omega), \ i = 1, 2,$$

with the norm

$$\|u\| = \|u\|_{X_1} + \|u\|_{X_2},$$

with the corresponding norm

$$\|u\|_{X_i} = \|u\|_{1,p_i(x)} + \|u\|_{2,p_i(x)} + \|u\|_{3,p_i(x)}, i = 1, 2,$$

$\|u\|_{X_i}$ and $\|\nabla \Delta u\|_{p_i(x)(\Omega)}$ are two equivalent norms in X_i, the detailed proof can be found in [29].

For the sake of convenience, we use

$$\|u\|_{X_i} = \inf\left\{\lambda > 0 : \int_\Omega \left|\frac{\nabla \Delta u(x)}{\lambda}\right|^{p_i(x)} \mathrm{d}x \le 1\right\},$$

as the norm of space X_i in the following text.

Assume $q(x) \in C_+(\bar{\Omega})$ and $q(x) < p^*(x)$. Then there is a continuous and compact embedding:

$$X_i \hookrightarrow L^{q(x)}(\Omega), \ i = 1, 2.$$

Proposition 9. [24] If $1 < p^- \le p^+ < \infty$, the space $\left(X_i, \|\cdot\|_{X_i}\right)$ is a reflexive and separable Banach space.

Proposition 10. Let $\zeta(u)=\int_\Omega |\nabla\Delta u|^{p(x)}\mathrm{d}x$ for all $u\in X_i$. We define

$$\|u\|_{X_i}^{\check{p}}:=\begin{cases}\|u\|_{X_i}^{p^+}, & 0<\|u\|_{X_i}\le 1,\\ \|u\|_{X_i}^{p^-}, & \|u\|_{X_i}>1,\end{cases}\quad \text{and}\quad \|u\|_{X_i}^{\hat{p}}:=\begin{cases}\|u\|_{X_i}^{p^-}, & 0<\|u\|_{X_i}\le 1,\\ \|u\|_{X_i}^{p^+}, & \|u\|_{X_i}>1,\end{cases}$$

then

$$\|u\|_{X_i}^{\check{p}}\le \zeta(u)\le \|u\|_{X_i}^{\hat{p}}$$

3. Existence of Solutions

Definition 11. If

$$\int_\Omega |\nabla\Delta u|^{p_1(x)-2}\nabla\Delta u\nabla\Delta v\mathrm{d}x+\int_\Omega |\nabla\Delta u|^{p_2(x)-2}\nabla\Delta u\nabla\Delta v\mathrm{d}x=\int_\Omega f(x,u)v\mathrm{d}x,$$

for all $v\in X$, then $u\in X$ is said to be a weak solution of the problem (1).

The functional associated to (1) is given by

$$I(u)=\int_\Omega \frac{1}{p_1(x)}|\nabla\Delta u|^{p_1(x)}\mathrm{d}x+\int_\Omega \frac{1}{p_2(x)}|\nabla\Delta u|^{p_2(x)}\mathrm{d}x-\int_\Omega F(x,u)\mathrm{d}x,$$

and

$$J(u)=\int_\Omega \frac{1}{p_1(x)}|\nabla\Delta u|^{p_1(x)}\mathrm{d}x+\int_\Omega \frac{1}{p_2(x)}|\nabla\Delta u|^{p_2(x)}\mathrm{d}x.$$

3.1. Proof of Theorem 1

Lemma 12. $J\in C^1(X,R)$ and its Frechet derivative is given by:

$$\langle J'(u),v\rangle=\int_\Omega |\nabla\Delta u|^{p_1(x)-2}\nabla\Delta u\nabla\Delta v\mathrm{d}x+\int_\Omega |\nabla\Delta u|^{p_2(x)-2}\nabla\Delta u\nabla\Delta v\mathrm{d}x.$$

Proof. The proof of this lemma follows a standard procedure for establishing the Gateaux differentiability of functionals involving variable exponents, which is analogous to the approach detailed in ([21], Lemma 12). We provide the key steps here for the sake of completeness.

Let $u(x),v(x)\in X, x\in\Omega$ and $0<|t|<1$. By the mean value theorem, there exists $s\in[0,1]$ such that

$$\begin{aligned}&\left|\frac{\nabla\Delta(u(x)+tv(x))^{p_i(x)}-|\nabla\Delta u(x)|^{p_i(x)}}{p_i(x)t}\right|\\&=|\nabla\Delta(u(x)+tsv(x))|^{p_i(x)-1}|\nabla\Delta v(x)|\\&\le(|\nabla\Delta u(x)|+|\nabla\Delta v(x)|)^{p_i(x)-1}|\nabla\Delta v(x)|.\end{aligned}$$

Using the inequality from [13]:

$$|u(x)+v(x)|^{p(x)}\le 2^{p(x)-1}\left(|u(x)|^{p(x)}+|v(x)|^{p(x)}\right).$$

Using the Proposition 4, and Proposition 9, and following the estimation procedure detailed in [21], we find that the integral is bounded. Consequently, we conclude that:

$$\left|\nabla\Delta u(x)+\nabla\Delta v(x)\right|^{p_1(x)-1}\left|\nabla\Delta v(x)\right|$$
$$+\left|\nabla\Delta u(x)+\nabla\Delta v(x)\right|^{p_2(x)-1}\left|\nabla\Delta v(x)\right|\in L^1(\Omega).$$

Following a procedure analogous to that in [21], we apply the Lebesgue dominated convergence theorem and compute the limit to obtain:

$$\left\langle J'(u(x)),v(x)\right\rangle=\int_\Omega\left|\nabla\Delta u(x)\right|^{p_1(x)-2}\nabla\Delta u(x)\nabla\Delta v(x)\mathrm{d}x$$
$$+\int_\Omega\left|\nabla\Delta u(x)\right|^{p_2(x)-2}\nabla\Delta u(x)\nabla\Delta v(x)\mathrm{d}x.$$

Let $u_n,u\in X$ with $u_n(x)\to u(x)$ in X_i, *i.e.*, $\nabla\Delta u_n(x)\to\nabla\Delta u(x)$ in $L^{p_i(x)}(\Omega)$, $i=1,2$. Then,

$$\begin{aligned}&\left|\left\langle J'(u_n)-J'(u),v\right\rangle\right|\\&=\left|\int_\Omega\left(\left|\nabla\Delta u_n\right|^{p_1(x)-2}\nabla\Delta u_n-\left|\nabla\Delta u\right|^{p_1(x)-2}\nabla\Delta u\right)\nabla\Delta v\right.\\&\quad\left.+\left(\left|\nabla\Delta u_n\right|^{p_1(x)-2}\nabla\Delta u_n-\left|\nabla\Delta u\right|^{p_1(x)-2}\nabla\Delta u\right)\nabla\Delta v\mathrm{d}x\right|\\&\le 2\left\|\left|\nabla\Delta u_n\right|^{p_1(x)-2}\nabla\Delta u_n-\left|\nabla\Delta u\right|^{p_1(x)-2}\nabla\Delta u\right\|_{L^{\frac{p_1(x)}{p_1(x)-1}}(\Omega)}\left\|v\right\|_{X_1}\\&\quad+2\left\|\left|\nabla\Delta u_n\right|^{p_2(x)-2}\nabla\Delta u_n-\left|\nabla\Delta u\right|^{p_2(x)-2}\nabla\Delta u\right\|_{L^{\frac{p_2(x)}{p_2(x)-1}}(\Omega)}\left\|v\right\|_{X_2}.\end{aligned}$$

Let $P_i(x,\nabla\Delta u)=\left|\nabla\Delta u\right|^{p_i(x)-2}\nabla\Delta u$. We deduce from theorem 1 of [30] that $P_i(x,\cdot):L^{p_i(x)}(\Omega)\to L^{\frac{p_i(x)}{p_i(x)-1}}(\Omega)$ is continuous, which shows that

$$P_i(x,\nabla\Delta u_n)\to P_i(x,\nabla\Delta u)\ \text{in}\ L^{\frac{p_i(x)}{p_i(x)-1}}(\Omega).$$

Therefore,

$$\begin{aligned}\left\|J'(u_n)-J'(u)\right\|&=\sup_{0\ne v\in X}\frac{\left|\left\langle J'(u_n)-J'(u),v\right\rangle\right|}{\left\|v\right\|}\\&\le 2\left\|P_1(x,\nabla\Delta u_n)-P_1(x,\nabla\Delta u_n)\right\|_{L^{\frac{p_1(x)}{p_1(x)-1}}(\Omega)}\\&\quad+2\left\|P_2(x,\nabla\Delta u_n)-P_2(x,\nabla\Delta u_n)\right\|_{L^{\frac{p_2(x)}{p_2(x)-1}}(\Omega)}\\&\to 0,n\to\infty.\end{aligned}$$

To sum up, we can conclude that $J\in C^1(X,R)$. □

Lemma 13.

1) J' is continuous, bounded and strictly monotone.

2) J' is of (S_+) type, namely: $u_n\to u$ and $\limsup_{n\to\infty}\left\langle J'(u_n),u_n-u\right\rangle\le 0$ implies $u_n\to u$.

Proof.

1) Since J' is the Frechet derivative of J, it follows that J' is continuous and bounded. To prove monotonicity, we use a classic argument for variable-exponent operators, similar to the proof of Theorem 3.4 [16]. Using the elementary inequalities

$$|x-y|^{\gamma} \le 2^{\gamma}\left(|x|^{\gamma-2}x-|y|^{\gamma-2}y\right)\cdot(x-y), \text{ if } \gamma \ge 2,$$

$$|x-y|^{2} \le \frac{1}{\gamma-1}(|x|+|y|)^{2-\gamma}\left(|x|^{\gamma-2}x-|y|^{\gamma-2}y\right)\cdot(x-y), \text{ if } 1<\gamma<2,$$

the equality holds if and only if $x=y$, for all $(x,y)\in R^N\times R^N$, where $\langle x\cdot y\rangle$ denotes the usual inner product in R^N, we obtain for all $u,v\in X$ such that $u\neq v$, we abtain

$$\langle J'(u)-J'(v),u-v\rangle \ge 0,$$

this implies that J' is strictly monotone.

2) Assume $u_n \to u$ in X and $\limsup_{n\to\infty}\langle J'(u_n),u_n-u\rangle \le 0$. From the strict monotonicity, we have

$$\langle J'(u_n)-J'(u),u_n-u\rangle \ge 0,$$

from proposition 5, it suffices to show that

$$\int_\Omega |\nabla\Delta u_n-\nabla\Delta u|^{p_1(x)}+|\nabla\Delta u_n-\nabla\Delta u|^{p_2(x)}\,\mathrm{d}x \to 0, \tag{2}$$

and the weak convergence of u together with the hypothesis implies

$$\limsup_{n\to\infty}\langle J'(u_n)-J'(u),u_n-u\rangle = 0, n\to+\infty, \tag{3}$$

put

$$\varphi_n = \left(|\nabla\Delta u_n|^{p_i(x)-2}\nabla\Delta u_n-|\nabla\Delta u|^{p_i(x)-2}\nabla\Delta u\right)\nabla\Delta(u_n-u),$$

following the approach of ([16], Theorem 3.4), let us define the sets

$$U_p=\{x\in\Omega: p_i(x)\ge 2\},\ V_p=\{x\in\Omega: 1<p_i(x)<2\},$$

on U_p, the elementary inequality for $\gamma\ge 2$ yields

$$\int_{U_p}|\nabla\Delta u_n-\nabla\Delta u|^{p_i(x)}\,\mathrm{d}x \le 2^{p_i^+}\int_{U_p}\varphi_n(x)\,\mathrm{d}x, i=1,2,$$

from (3) we have $\limsup_{n\to\infty}\int_\Omega \varphi_n(x)\,\mathrm{d}x=0$; hence

$$\int_{U_p}|\nabla\Delta u_n-\nabla\Delta u|^{p_1(x)}+|\nabla\Delta u_n-\nabla\Delta u|^{p_2(x)}\,\mathrm{d}x \to 0, n\to\infty, \tag{4}$$

on V_p, using the inequality valid for $1<\gamma<2$ together with Holder's and Young's inequalities, we have

$$\int_{V_p}|\nabla\Delta u_n-\nabla\Delta u|^{p_i(x)}\,\mathrm{d}x \to 0 \text{ as } n\to+\infty, \tag{5}$$

we conclude that

$$\int_{V_p}|\nabla\Delta u_n-\nabla\Delta u|^{p_1(x)}+|\nabla\Delta u_n-\nabla\Delta u|^{p_2(x)}\,\mathrm{d}x \to 0, n\to\infty, \tag{6}$$

finally, (2) is given by combining (4) and (6). □

Proof of Theorem 1. Under the assumptions (F3), I is sequentially weakly lower semi-continuous and coercive.

Proof. From the continuity of F and assumption (F3) we deduce that

$$F(x,u) \le a(x)|u|^{\theta(x)}+c,$$

$\forall u \in R$ and $\forall x \in \Omega$. We have

$$\begin{aligned}
I(u) &= \int_\Omega \frac{1}{p_1(x)} |\nabla \Delta u|^{p_1(x)} \mathrm{d}x + \int_\Omega \frac{1}{p_2(x)} |\nabla \Delta u|^{p_2(x)} \mathrm{d}x - \int_\Omega F(x,u) \mathrm{d}x \\
&\ge \frac{1}{p_1^+} \int_\Omega |\nabla \Delta u|^{p_1(x)} \mathrm{d}x + \frac{1}{p_2^+} \int_\Omega |\nabla \Delta u|^{p_2(x)} \mathrm{d}x - \int_\Omega \left(a(x) |u|^{\theta(x)} + c \right) \mathrm{d}x \\
&\ge \frac{1}{p_1^+} \int_\Omega |\nabla \Delta u|^{p_1(x)} \mathrm{d}x + \frac{1}{p_2^+} \int_\Omega |\nabla \Delta u|^{p_2(x)} \mathrm{d}x - |a|_\infty \int_\Omega |u|^{\theta(x)} \mathrm{d}x - c|\Omega| \\
&\ge \frac{1}{p_M^+} \|u\|_{X_1}^{p_1^-} + \frac{1}{p_M^+} \|u\|_{X_2}^{p_2^-} - |a|_\infty \|u\|_{L^{\theta(x)}(\Omega)}^{\theta^+} - c|\Omega| \\
&\ge \frac{1}{p_M^+} \left(\|u\|_{X_1}^{p_1^-} + \|u\|_{X_2}^{p_2^-} \right) - |a|_\infty \left(c_1 \|u\|_{X_1} + c_2 \|u\|_{X_2} \right)^{\theta^+} - c|\Omega| \\
&\ge \frac{c_5}{p_M^+} \|u\|^{p_m^-} - |a|_\infty \left(\max(c_1, c_2) \|u\| \right)^{\theta^+} - c|\Omega|,
\end{aligned}$$

since $\theta^+ < \min_{i=1,2} p_i^-$, then I is coercive. As the function $u \mapsto \int_\Omega F(x,u) \mathrm{d}x$ is weakly lower semi-continuous and I is convex uniformly, we deduce that I is weakly lower semi-continuous. Therefore I has a global minimum point $u \in X$, which is a weak solution to problem (1). □

3.2. Proof of Theorem 2

Lemma 14. If (F0) - (F2), (F4) hold, then I satisfies the (C) condition in X, namely, if any sequence $\{u_n\} \subset X$ such that $\{I(u_n)\}$ is bounded and $(1 + \|u_n\|)\|I'(u_n)\| \to 0$ as $n \to \infty$, has convergent subsequence.

Proof. Let $\{u_n\}$ be a sequence in *X* for the functional *I*, which is a (C)-sequence. First, use proof by contradiction to demonstrate that the sequence $\{u_n\}$ is bounded in *X*. Suppose that as $n \to \infty$,

$$\|u_n\|_X \to \infty, I(u_n) \to c \text{ and } (1 + \|u_n\|)\|I'(u_n)\| \to 0, \tag{7}$$

let $\omega_n = \frac{u_n}{\|u_n\|}$. Using the reflexivity of *X*, we can extract a subsequence such that the sequence $\{\omega_n\}$ weakly converges to ω in *X*, and $\{\omega_n\}$ strongly converges to ω in $L^r(\Omega)$, where $1 \le r \le p^*(x)$. Moreover, $\omega_n(x) \to \omega(x)$, a.e. $x \in \Omega$.

Case 1: If $\omega = 0$, then by condition (F0), we have

$$\begin{aligned}
|F(x,t)| &\le \int_0^1 |f(x,\theta t)||t| \mathrm{d}\theta \\
&\le \int_0^1 r\left(|t| + \theta^{\alpha(x)-1} |t|^{\alpha(x)} \right) \mathrm{d}\theta \\
&\le r|t| + \frac{r}{\alpha(x)} |t|^\alpha \\
&\le r|t| + \frac{r}{\alpha^-} |t|^{\alpha(x)},
\end{aligned} \tag{8}$$

which holds for all $(x,t) \in \Omega \times R$, where $\alpha^- = \inf_\Omega \alpha(x)$. Therefore, for all $x \in \Omega$.

For $|t| \le M$, from (8), we have:

$$\left|tf(x,t)-2\beta F(x,t)\right|\le r(2\beta+1)\left(1+|t|^{\alpha(x)-1}\right)|t|\le c_3|t|, \tag{9}$$

since $|t|\le M$, $|t|^{\alpha(x)-1}\le M^{\alpha^+-1}$, where $c_3=r(2\beta+1)\left(1+|M|^{\alpha^+-1}\right)$.

For $|t|\ge M$, by condition (F4), we get

$$tf(x,t)-2\beta F(x,t)\ge -C|t|,$$

choose M large enough so that $C\le c_3$, we get

$$tf(x,t)-2\beta F(x,t)\ge -C|t|\ge -c_3|t|,$$

combining conditions (F4) and (8), we have:

$$tf(x,t)-2\beta F(x,t)\ge -c_3|t|, \tag{10}$$

it holds for all $(x,t)\in\Omega\times R$. Using Equation (7) and Equation (10), we have:

$$\begin{aligned}
c_4 &\ge I(u_n)+\frac{1}{2\beta}\|I'(u_n)\|(1+\|u_n\|)\ge I(u_n)-\frac{1}{2\beta}\langle I'(u_n),u_n\rangle\\
&=\int_\Omega\frac{1}{p_1(x)}|\nabla\Delta u_n|^{p_1(x)}\mathrm{d}x+\int_\Omega\frac{1}{p_2(x)}|\nabla\Delta u_n|^{p_2(x)}\mathrm{d}x-\int_\Omega F(x,u_n)\mathrm{d}x\\
&\quad-\frac{1}{2\beta}\left[\int_\Omega|\nabla\Delta u_n|^{p_1(x)}\mathrm{d}x+\int_\Omega|\nabla\Delta u_n|^{p_2(x)}\mathrm{d}x-\int_\Omega f(x,u_n)u_n\mathrm{d}x\right]\\
&\ge\int_\Omega\frac{1}{p_M^+}|\nabla\Delta u_n|^{p_1(x)}\mathrm{d}x+\int_\Omega\frac{1}{p_M^+}|\nabla\Delta u_n|^{p_2(x)}\mathrm{d}x-\frac{1}{2\beta}\int_\Omega|\nabla\Delta u_n|^{p_1(x)}\mathrm{d}x\\
&\quad-\frac{1}{2\beta}\int_\Omega|\nabla\Delta u_n|^{p_2(x)}\mathrm{d}x+\frac{1}{2\beta}\int_\Omega\left(f(x,u_n)u_n-2\beta F(x,u_n)\right)\mathrm{d}x\\
&=\left(\frac{1}{p_M^+}-\frac{1}{2\beta}\right)\left[\int_\Omega|\nabla\Delta u_n|^{p_1(x)}\mathrm{d}x+\int_\Omega|\nabla\Delta u_n|^{p_2(x)}\mathrm{d}x\right]\\
&\quad+\frac{1}{2\beta}\int_\Omega\left(f(x,u_n)u_n-2\beta F(x,u_n)\right)\mathrm{d}x\\
&\ge\left(\frac{1}{p_M^+}-\frac{1}{2\beta}\right)\left(\|u_n\|_{X_1}^{p_1^-}+\|u_n\|_{X_2}^{p_2^-}\right)-\frac{c_3}{2\beta}\int_\Omega|u_n|\mathrm{d}x\\
&\ge c_6\left(\frac{1}{p_M^+}-\frac{1}{2\beta}\right)\|u_n\|^{p_m^-}-\frac{c_3}{2\beta}\int_\Omega|\omega_n|\mathrm{d}x.
\end{aligned} \tag{11}$$

Dividing both sides of Equation (11) by $\|u_n\|^{p_m^-}$, we get

$$\frac{c_4}{\|u_n\|^{p_m^-}}\ge c_6\left(\frac{1}{p_M^+}-\frac{1}{2\beta}\right)-\frac{c_3}{2\beta\|u_n\|^{p_m^--1}}\int_\Omega|\omega_n|\mathrm{d}x, \tag{12}$$

using Equation (7) and noting that $\omega=0$, the above equation implies $0\ge c_5\left(\frac{1}{p_M^+}-\frac{1}{2\beta}\right)$, which contradicts the conditions $\beta>\max p_i^+$, $p_M(x)=\max\{p_1(x),p_2(x)\}$.

Case 2: If $\omega\ne0$, define $\Omega_1=\{x\in\Omega:\omega(x)\ne0\}$, then $|\Omega_1|>0$, where $|\Omega_1|$ denotes the measure of Ω_1. According to condition (F2) and (7) as $n\to\infty$, $\frac{F(x,u_n(x))}{|u_n(x)|^{2\beta}}|\omega_n(x)|^{2\beta}\to+\infty$, Using Fatou's Lemma, as $n\to\infty$, we have

$$\int_{\omega \neq 0} \frac{F\left(x, u_n\left(x\right)\right)}{\left|u_n\left(x\right)\right|^{2\beta}} \left|\omega_n\left(x\right)\right|^{2\beta} \mathrm{d}x \to +\infty, \tag{13}$$

on the other hand, from (F2), there exists a constant $M_1 > 0$ such that $F\left(x,t\right) \geq 0$, $\forall x \in \Omega$, $|t| \geq M_1$. From Equation (8), we have $\left|F\left(x,t\right)\right| \leq c_6 |t|$, $\forall x \in \Omega$, $|t| \leq M_1$, where $c_7 = r + \frac{r M_1^{\alpha^+ - 1}}{\alpha^-}$. Thus,

$$F\left(x,t\right) \geq -c_7 |t|, \forall \left(x,t\right) \in \Omega \times R,$$

if $r \in \left[1, p^*\right)$, using the Sobolev embedding theorem, we have, $\|u\|_{L^r(\Omega)} \leq \tau_r \|u\|$, $\forall u \in X$. Then

$$\int_{\omega=0} \frac{F\left(x,u_n\right)}{\|u_n\|^{2\beta}} \mathrm{d}x \geq -\frac{c_7 \int_{\omega=0} |u_n| \mathrm{d}x}{\|u_n\|^{2\beta}} \geq -\frac{c_7 \|u\|_{L^1(\Omega)}}{\|u_n\|^{2\beta}} \geq -\frac{c_7 \tau_1 \|u_n\|}{\|u_n\|^{2\beta}},$$

this shows that

$$\liminf_{n\to\infty} \int_{\omega=0} \frac{F\left(x,u_n\right)}{\|u_n\|^{2\beta}} \mathrm{d}x \geq 0, \tag{14}$$

from Equation (7), assume $\|u_n\| \geq 1$, we have

$$\begin{aligned}
\int_\Omega F\left(x,u_n\right)\mathrm{d}x + I\left(u_n\right) &= \int_\Omega \frac{1}{p_1\left(x\right)} \left|\nabla \Delta u_n\right|^{p_1(x)} \mathrm{d}x + \int_\Omega \frac{1}{p_2\left(x\right)} \left|\nabla \Delta u_n\right|^{p_2(x)} \mathrm{d}x \\
&\leq \frac{1}{p_m^-} \int_\Omega \left|\nabla \Delta u_n\right|^{p_1(x)} \mathrm{d}x + \frac{1}{p_m^-} \int_\Omega \left|\nabla \Delta u_n\right|^{p_2(x)} \mathrm{d}x \\
&= \frac{1}{p_m^-} \left(\|u_n\|_{X_1}^{p_1^+} + \|u_n\|_{X_2}^{p_2^+} \right) \\
&\leq \frac{c_8}{p_m^-} \|u_n\|^{p_M^+} \leq \frac{c_8}{p_m^-} \|u_n\|^{2\beta},
\end{aligned}$$

thus

$$\int_\Omega \frac{F\left(x,u_n\right)}{\|u_n\|^{2\beta}} \mathrm{d}x + \frac{I\left(u_n\right)}{\|u_n\|^{2\beta}} \leq \frac{c_8}{p_m^-}. \tag{15}$$

By combining Equations (13) to (15), we conclude that

$$\begin{aligned}
\frac{c_8}{p_m^-} &\geq \liminf_{n\to\infty} \int_\Omega \frac{F\left(x,u_n\right)}{\|u_n\|^{2\beta}} \mathrm{d}x \\
&= \liminf_{n\to\infty} \left(\int_{\omega=0} + \int_{\omega\neq 0} \right) \frac{F\left(x,u_n\right)}{\left|u_n\right|^{2\beta}} \left|\omega_n\right|^{2\beta} \mathrm{d}x \\
&= +\infty,
\end{aligned}$$

this is a contradiction. Therefore, $\{u_n\}$ is bounded in *X*. Note that *X* is a reflexive space, so there exists $u \in X$ such that $\{u_n\}$ weakly converges to u in *X*, and $\{u_n\}$ strongly converges to u in $L^{\alpha(x)}\left(\Omega\right)$. By Holder's inequality, condition (F0), and $\frac{1}{\alpha\left(x\right)} + \frac{1}{\alpha'\left(x\right)} = 1$, as $n \to +\infty$, we have

$$\left|\int_\Omega f(x,u_n)(u_n-u)\,\mathrm{d}x\right| \le r\int_\Omega \left(1+|u_n|^{\alpha(x)-1}\right)|u_n-u|\,\mathrm{d}x$$
$$\le 2r\int_\Omega \left|1+|u_n|^{\alpha(x)-1}\right|_{(\alpha'(x))} |u_n-u|_{\alpha(x)} \to 0.$$

As $n\to\infty, I'(u_n)\to 0$, so we get

$$\int_\Omega |\nabla\Delta u_n|^{p_i(x)-2}\nabla\Delta u_n \nabla\Delta(u_n-u)\,\mathrm{d}x \to 0, i=1,2. \quad (16)$$

Define

$$\langle J'(u),v\rangle = \int_\Omega |\nabla\Delta u|^{p_1(x)-2}\nabla\Delta u\nabla\Delta v\mathrm{d}x + \int_\Omega |\nabla\Delta u|^{p_2(x)-2}\nabla\Delta u\nabla\Delta v\mathrm{d}x, \forall u,v\in X.$$

According to lemma 13, the continuous mapping $J': X\to X^*$ has the property (S_+), *i.e.*, if $\{u_n\}$ weakly converges to u in X and $\limsup\langle J'(u_n)-J'(u),u_n-u\rangle\le 0$, which implies that $\{u_n\}$ strongly converges to u in X. From Equation (16), $\limsup_{n\to\infty}\langle J'(u_n)-J'(u),u_n-u\rangle = 0$, so $\{u_n\}$ strongly converges to u in X. Therefore, the functional I satisfies condition (C). □

Proof of Theorem 2. We will prove that I satisfies the Mountain-Pass Lemma below.

1) It follows from Lemma 14 that J satisfies the condition in X. Since $p_M^+ \le \alpha^- \le \alpha(x) < p_i^*(x)$, and $X \hookrightarrow L^{p_M^+}(\Omega)$ there exists $C_0>0$ such that

$$|u|_{p_M^+} \le C_0\|u\|, \forall u\in X.$$

Let $\epsilon(>0)$ be small enough such that $\epsilon C_0^{p_M^+} \le \dfrac{C_8}{2p_M^+}$. By assumptions (F0) and (F1), we have

$$F(x,t)\le \epsilon|t|^{p_M^+} + C(\epsilon)|t|^{\alpha(x)}, \forall (x,t)\in\Omega\times R.$$

From $\|u\|\le 1$, we get

$$\begin{aligned}
J(u) &= \int_\Omega \frac{1}{p_1(x)}|\nabla\Delta u|^{p_1(x)}\mathrm{d}x + \int_\Omega \frac{1}{p_2(x)}|\nabla\Delta u|^{p_2(x)}\mathrm{d}x - \int_\Omega F(x,u)\mathrm{d}x \\
&\ge \frac{1}{p_M^+}\int|\nabla\Delta u|^{p_1(x)}\mathrm{d}x + \frac{1}{p_M^+}\int|\nabla\Delta u|^{p_2(x)}\mathrm{d}x \\
&\quad -\varepsilon\int_\Omega |u|^{p_M^+}\mathrm{d}x - C(\varepsilon)\int_\Omega |u|^{\alpha(x)}\mathrm{d}x \\
&\ge \frac{1}{p_M^+}\left(\|u\|_{X_1}^{p_M^+} + \|u\|_{X_2}^{p_M^+}\right) - \varepsilon C_0^{p_M^+}\|u\|^{p_M^+} - C(\varepsilon)\|u\|^{\alpha^-} \\
&\ge \frac{c_9}{p_M^+}\|u\|^{p_M^+} - \varepsilon C_0^{p_M^+}\|u\|^{p_M^+} - C(\varepsilon)\|u\|^{\alpha^-} \\
&\ge \frac{c_9}{2p_M^+}\|u\|^{p_M^+} - C(\varepsilon)\|u\|^{\alpha^-}.
\end{aligned}$$

This means that there exist $\rho\in(0,1)$ and $\delta>0$ such that $I(u)\ge\delta>0$ for each $u\in X$ satisfying $\|u\|=\rho$.

2) From (F4), there exist two positive constants C_{10}, C_{11}, such that

$$F(x,t)\ge C_{10}|t|^\beta - C_{11}, \forall x\in\Omega, t\ge M.$$

For any fixed $\omega \in X \setminus \{0\}$ and $t > 1$, we have

$$I(t\omega) = \int_\Omega \frac{1}{p_1(x)} |\nabla \Delta t\omega|^{p_1(x)} dx + \int_\Omega \frac{1}{p_2(x)} |\nabla \Delta t\omega|^{p_2(x)} dx - \int_\Omega F(x, t\omega) dx$$

$$I(t\omega) \le t^{p_M^+} \left(\int_\Omega \frac{1}{p_1(x)} |\nabla \Delta \omega|^{p_1(x)} dx + \int_\Omega \frac{1}{p_2(x)} |\nabla \Delta \omega|^{p_2(x)} dx \right) - C_{10} t^\beta \int_\Omega |\omega|^\beta dx - C_{11} |\Omega|.$$

Due to $\beta > p_M^+$, we have

$$I(u) \to -\infty, \text{ as } t \to +\infty.$$

3) Obviously, $I(0) = 0$.

From 1), 2) and 3), we deduce that I satisfies the conditions of the Mountain-Pass Theorem. Therefore, I has at least one nontrivial critical point. The proof is completed.

3.3. Proof of Theorem 3

Let X be a separable and reflexive Banach space, then there exist $\{e_j\} \subset X$ and $\{e_j^*\} \subset X^*$ such that

$$X = \overline{\text{span}} \{e_j : j = 1, 2, \cdots\}, \; X^* = \overline{\text{span}} \{e_j^* : j = 1, 2, \cdots\},$$

with

$$\langle e_i, e_j^* \rangle = \begin{cases} 1, & i = j, \\ 0, & i \ne j. \end{cases}$$

Define

$$X_j = \text{span}\{e_j\}, \; Y_k = \bigoplus_{j=1}^{k} X_j, \; Z_k = \overline{\bigoplus_{j=k}^{\infty} X_j}.$$

Lemma 15 (Fountain Theorem). If $I \in C^1(X, R)$, satisfying:
$I(0) = 0, I(-u) = I(u)$, and

1) For $k \in N$, there exists $r_k > 0$ and as $k \to +\infty, b_k := \inf_{u \in Z_k, \|u\| = r_k} I(u) \to +\infty$;

2) For $\rho_k > r_k > 0$, there is $a_k := \max_{u \in Y_k, \|u\| = \rho_k} I(u) \le 0$;

3) The functional I satisfies the (C) condition, that is, for any sequence $\{u_n\} \subset X$, from $\{I(u_n)\}$ being bounded, $(1 + \|u_n\|)\|I'(u_n)\| \to 0 (n \to +\infty)$, it implies that $\{u_n\}$ has a convergent subsequence.

Then the functional I has a sequence of critical values tending to $+\infty$.

Proof of Theorem 3

By using Lemma 14 and (F5), it is known that I satisfies condition (C), $I(-u) = I(u)$, and $I(0) = 0$. To prove that Theorem 3 holds, it is only necessary to verify the linking conditions (1) and (2) in the Fountain Theorem (Lemma 15) are satisfied.

First, prove that (1) holds. Denote $\beta_k = \sup\left\{ \|u\|_{\alpha(x)} : u \in Z_k, \|u\| = 1 \right\}$. Then as

$k \to \infty, \beta_k \to 0$.

For $u \in Z_k$ with $\|u\| = r_k > 1$, from (F0), we have

$$\begin{aligned}
I(u) &= \int_\Omega \frac{1}{p_1(x)} |\nabla \Delta u|^{p_1(x)} \mathrm{d}x + \int_\Omega \frac{1}{p_2(x)} |\nabla \Delta u|^{p_2(x)} \mathrm{d}x - \int_\Omega F(x,u) \mathrm{d}x \\
&\ge \frac{1}{p_M^+}\left(\|u\|_{X_1}^{p_1^-} + \|u\|_{X_2}^{p_2^-} \right) - c_{12} \int_\Omega |u|^{\alpha(x)} \mathrm{d}x - c_{13} \\
&\ge \frac{c_{14}}{p_M^+} \|u\|^{p_m^-} - c_{12} |u|_{\alpha(x)}^{\alpha(\xi)} - c_{15}, \xi \in \Omega \\
&\ge \begin{cases} \frac{c_{14}}{p_M^+} \|u\|^{p_m^-} - c_{12} - c_{15}, |u|_{\alpha(x)} \le 1 \\ \frac{c_{14}}{p_M^+} \|u\|^{p_m^-} - c_{12} \beta_k^{\alpha^+} \|u\|^{\alpha^+} - c_{15}, |u|_{\alpha(x)} > 1 \end{cases} \\
&\ge \frac{c_{14}}{p_M^+} \|u\|^{p_m^-} - c_{12} \beta_k^{\alpha^+} \|u\|^{\alpha^+} - c_{16} \\
&\ge c_{14} \left(\frac{1}{p_M^+} \|u\|^{p_m^-} - c_{17} \beta_k^{\alpha^+} \|u\|^{\alpha^+} \right) - c_{16}.
\end{aligned}$$

Take $\|u\| = r_k = \left(c_{17} \alpha^+ \beta_k^{\alpha^+} \right)^{\frac{1}{p_m^- - \alpha^+}}$, then, as $k \to +\infty$, $r_k \to +\infty$. thus, we have

$$\begin{aligned}
I(u) &\ge c_{14} \left(\frac{1}{p_M^+} \|u\|^{p_m^-} - c_{17} \beta_k^{\alpha^+} \|u\|^{\alpha^+} \right) - c_{16} \\
&= c_{14} \left(\frac{1}{p_M^+} \left(c_{17} \alpha^+ \beta_k^{\alpha^+} \right)^{\frac{p_m^-}{p_m^- - \alpha^+}} - c_{17} \beta_k^{\alpha^+} \left(c_{17} \alpha^+ \beta_k^{\alpha^+} \right)^{\frac{\alpha^+}{p_m^- - \alpha^+}} \right) - c_{16} \\
&= c_{14} \left(\left(\frac{1}{p_M^+} - \frac{1}{\alpha^+} \right) c_{17} \alpha^+ \beta_k^{\alpha^+} \right)^{\frac{p_m^-}{p_m^- - \alpha^+}} - c_{16} \to \infty.
\end{aligned}$$

As $k \to +\infty$, which shows that (1) holds.

Next, verify that (2) holds. From (F2), (F4), we have

$$F(x,t) \ge c_{18} |t|^\theta - c_{19}.$$

Since $\theta > p_M^+$ and $Y_k = k$, it is obvious that for $u \in Y_k$, as $\|u\| \to \infty$, $I(u) \to -\infty$. Therefore, (2) is correct, and Theorem 3 is proved.

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All authors read and approved the final manuscript.

Authors' Contributions

Conceptualization: LZH and MQ; methodology: LZH; formal analysis: LZH; writing-original draft: LZH; writing-review & editing: MQ; supervision: MQ; funding acquisition: MQ.

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Conflicts of Interest

The authors declare no conflicts of interest.

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