

# Gaussian Curvature: From Definition of Measure of Curvature to Discrete Gaussian Curvature

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## Abstract

This work traces the historical and mathematical evolution of the Gaussian curvature, from Gauss's original integral formulation to contemporary discrete computational approaches. Understanding this historical progression provides essential insights for implementing geometric applications and advancing research in geometry. Gaussian curvature plays a fundamental role in feature analysis of two-dimensional structures and surfaces bounding three-dimensional solids. This research provides foundational knowledge in surface differential geometry and discrete computational geometry, equipping researchers with the conceptual framework necessary for both theoretical investigation and practical implementation. This rigorous examination of the theoretical foundations and computational methods of the Gaussian curvature can facilitate the development of robust geometric applications across multiple scientific domains.

## Keywords

Gaussian Curvature, Discrete Gaussian Curvature, Gauss Map, Theorem of Gauss-Bonnet, Euler-Poincaré Characteristic

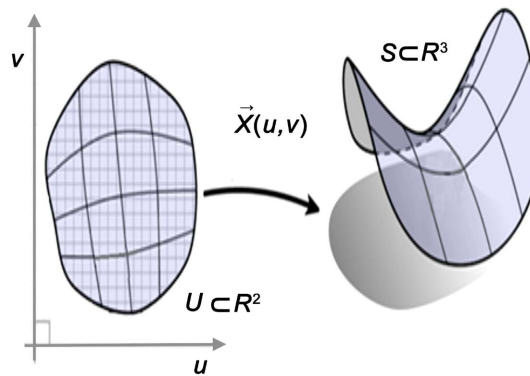
## 1. Introduction

Three-dimensional structures are bounded by surfaces, so what we perceive first is the external casing of an object, that is, its surface. Therefore, the study of the most important characteristics of a surface becomes of fundamental relevance for analysis, classification and recognition of objects in the real world. In the following sections, we focus on some basic concepts and definitions relating to the geometry of smooth surfaces. This summary tries to deepen some aspects of the intrinsic

and extrinsic geometry of regular surfaces, making particularly reference to Gaussian curvature. The main aim of this paper is to reconstruct the path that led from the original definition of *measure of curvature* by Gauss K. F. (1827) to the computational calculation of Gaussian curvature for discrete surfaces that are generally approximated by planar triangles or polygons. This work is suitable for providing the basics of differential geometry and topology to those who need to develop computational applications.

## 2. Gauss Map and Gaussian Curvature

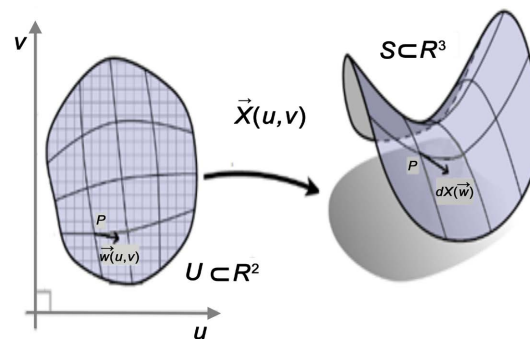
An immediate and direct way to interpret a surface is imaging that a flat rubber sheet  $U \subset R^2$  is stretched and deformed into a surface  $S$  embedded into the three-dimensional space  $R^3$ , as illustrated in **Figure 1**. Its geometry is described by a map  $\vec{X}(u, v): U \rightarrow S$  from a region  $U \subset R^2$  in the Euclidean space  $R^2$  to a subset  $S \subset R^3$  of  $R^3$  [1].



**Figure 1.** Surface representation.

The differential (see **Appendix 1**) of such a map, denoted by  $d\vec{X}(\vec{w})$ , indicates how to transform a vector  $\vec{w}$  applied at a point  $P \in R^2$  to the corresponding vector  $d\vec{X}(\vec{w})$  at  $p \in S$ , after having deformed and stretched  $U$  into  $S$  (**Figure 2**).

The vector  $d\vec{X}(\vec{w})$  belongs to the bundle of complanar vectors coming out from  $p$  and forming the tangent plane  $T_p(S)$  of  $S$  at  $p$ .



**Figure 2.** Differential map.

So far, we have been talking about tangent plane, let us introduce a great simplification by associating the vector  $\vec{N}(p)$  orthogonal to it and with length equal to one, called *unit normal vector*. Practically, from now on, instead of tangent plane we deal with its normal vector. If we are able to define a specific orientation of the normal vector  $\vec{N}(p)$  the surface is said to be *orientable*. For orientable surfaces, we can actually define a continuous map  $\vec{N}(p): S \rightarrow R^3$  which associates each point  $p \in S$  with its unit normal  $\vec{N}(p)$ . Through the differential of this map  $d\vec{N}(p)$ , we can explore changes of the normal direction while moving from one point of the surface to another. By investigating the variations of the normal direction along a particular tangent direction, we are able to carry out information about the geometry of the surface, especially on the changes of its curvatures.

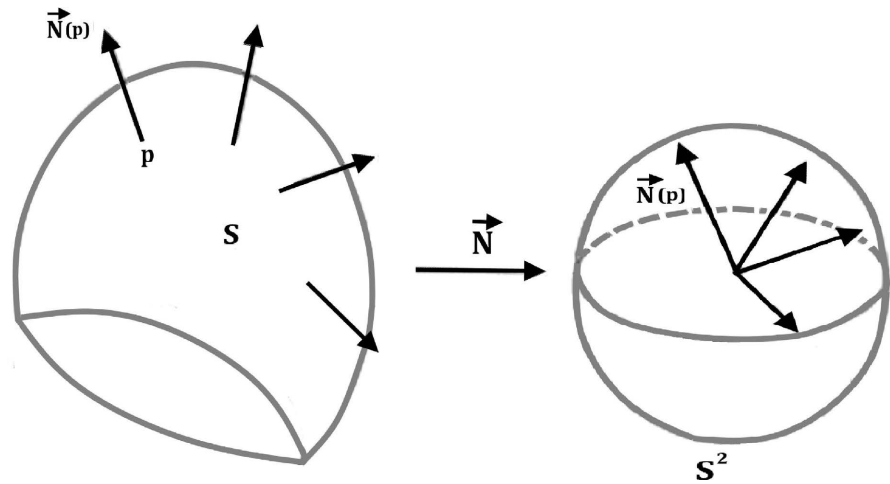
Historically, K. F. Gauss was the first to use the mapping between each point  $p$  of a surface and the associated unit normal vector  $\vec{N}(p)$ , introducing its representation as a point on a unit sphere in  $R^3$ , *i.e.* a sphere with radius equal to one.

The need of coming back to the original definitions included in mathematical texts is now widely recognized and considered of the utmost importance. Since the introduction of the treatise entitled “*Disquisitiones Generales Circa Superficies Curvas*” (1827), Gauss suggested investigating the characteristics of a surface by resorting to the so-called *auxiliary sphere*. For this reason, it will be the starting point of this summary as well, despite the use of the modern symbolic representation of differential geometry. Basically, aim of this work is to explain how, from the initial definition of **measure of curvature** introduced by Gauss (see Equation (15)), we arrive at the current definition used in differential geometry, as shown with Equation (19) and at the end, at the discrete Gaussian curvature. Hereinafter, we proceed with a synthetic analysis of some fundamental concepts of differential geometry, for all those familiar with them referred to the final part of this paragraph, for a more in-depth reading refer to classic textbooks [2]-[4].

Let  $S \subset R^3$  be a regular surface, given a parametrization  $\vec{X}(u, v): U \subset R^2 \rightarrow S$ , we can define a unit normal vector  $\vec{N}(p)$  at each point  $p \in \vec{X}(U) = (x(u, v), y(u, v), z(u, v))$  by the formula:

$$\vec{N}(p) = \frac{\vec{X}_u \times \vec{X}_v}{|\vec{X}_u \times \vec{X}_v|} = (N_x(u, v), N_y(u, v), N_z(u, v))_p \quad (1)$$

where  $\vec{X}_u$  and  $\vec{X}_v$  are the partial derivative with respect the parametric coordinates  $u, v$ . Thus, if the map  $\vec{N}: \vec{X}(U) \rightarrow R^3$ , that associates to each  $p \in \vec{X}(U)$  a unit normal vector  $\vec{N}(p)$  is differentiable, we shall say that the surface is orientable. Let  $S \subset R^3$  be a surface with an orientation  $\vec{N}$ . The map  $\vec{N}: S \rightarrow R^3$ , called the *Gauss Map* of  $S$ , takes its values in the unit sphere  $S^2 = \{(x, y, z) \in R^3 : x^2 + y^2 + z^2 = 1\}$ , the auxiliary sphere. Using the Gauss Map, we may bring back to the unit sphere the study of variations of the normal to the surface  $S$ , in this way we are able to explore the shape of a surface and to measure how much a surface varies and curves in the space (**Figure 3**).



**Figure 3.** Gauss map.

The Gauss map uniquely determines the tangent planes to the surface, since  $T_p(S)$  is the orthogonal of  $\vec{N}(p)$ . Therefore, the variation of  $\vec{N}$  measures how the tangent planes change, that is, how far the surface is from being a plane. Basically, the tangent plane  $T_p(S)$  represents the closest linear approximation to  $S$  at  $p$ , so it is reasonable to expect that the curvature of a surface is related to the differential of the Gauss map. Consequently, from now on, we focus on the differential of  $\vec{N}$  at  $p \in S$ . Let  $d\vec{N}_p : T_p(S) \rightarrow T_{N(p)}(S^2)$  be the differential of  $\vec{N}$  at  $p \in S$ , where  $T_p(S)$  and  $T_{N(p)}(S^2)$  are the tangent plane at  $p \in S$  and  $N(p) \in S^2$ , respectively. Strictly speaking,  $T_p(S)$  and  $T_{N(p)}(S^2)$  are not the same vector spaces, although they both are 2-dimensional and oriented in a parallel way. Despite these proper considerations, we may reasonably identify these two spaces  $T_p(S) \approx T_{N(p)}(S^2)$ . So, we consider the differential  $d\vec{N}_p : T_p(S) \rightarrow T_p(S)$  that can be thought as a linear map on  $T_p(S)$ , able to map tangent vectors to tangent vectors. The differential  $d\vec{N}_p$  quantifies the measure of how the normal  $\vec{N}$  differs by  $\vec{N}_p$  in a neighborhood of  $p$ . Unlike what happens in the one-dimensional case, in this context we do not have a preferred direction, for example the horizontal one. Thus, the rate of change of the normal vector  $\vec{N}$  have to be investigated in a neighborhood of  $p$  recurring to parameterized curves passing at  $p$  and lying on the surface  $S$ . To interpret and compute, the differential of the Gauss map, let us consider a parameterized and differentiable curve  $\vec{\alpha}(t)$  (Figure 4), where  $t$  is the arc length, we get:

$$\vec{\alpha}(t) = \vec{X}(u(t), v(t)) = (x(u(t), v(t)), y(u(t), v(t)), z(u(t), v(t))) \in S \subset \mathbb{R}^3$$

The tangent vector of the curve  $\vec{\alpha}(t)$  at  $t$  is given by:

$$\vec{\alpha}'(t) = \frac{\partial \vec{X}}{\partial u} \frac{du}{dt} + \frac{\partial \vec{X}}{\partial v} \frac{dv}{dt} = \vec{X}_u u' + \vec{X}_v v' \in T_p(S) \quad \text{with} \quad u' = \frac{du}{dt}, \quad v' = \frac{dv}{dt}$$

The natural inner product of  $\mathbb{R}^3$  induces on the tangent plane  $T_p(S)$  an inner product, a bilinear symmetric form that is equal to the inner product of  $\vec{v}$  and  $\vec{w}$  seen as 3-dimensional vectors of  $\mathbb{R}^3$ . So, given two vectors  $\vec{v}$  and

$\vec{w} \in T_p(S) \subset R^3$ , we have:

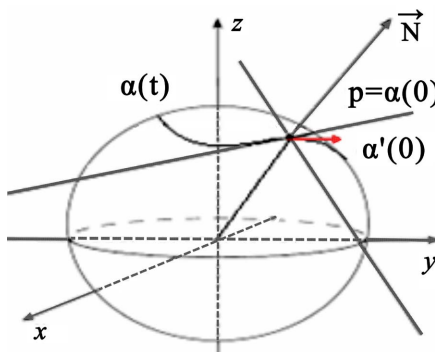


Figure 4. Differentiable curve on surface  $S$ .

$$\langle \vec{v}, \vec{w} \rangle \in R$$

To this inner product, it is associated a quadratic form  $I_p : T_p(S) \rightarrow R$  called the *first fundamental form* of the surface  $S$  :

$$I_p(\vec{v}) = \langle \vec{v}, \vec{v} \rangle = \|\vec{v}\|^2$$

that allows us to make measures on it, such as angles between tangent vectors, length of arcs, areas of regions etc., without referring to the 3-dimensional space  $R^3$  on which the surface is embedded. Without loss of generality, we suppose that for  $t = 0$   $\vec{\alpha}(0) = p \in S$ , so we have  $\vec{\alpha}'(0) \in T_p(S)$ , we say that the parameterization is centered at  $p$ . Thus, we may express the first fundamental form in the basis  $\{\vec{X}_u, \vec{X}_v\}$  of the plane  $T_p(S)$  considering the tangent vector to the parametrized curve  $\vec{\alpha}(t) = \vec{X}(u(t), v(t))$  at  $p = \vec{\alpha}(0)$ , we obtain:

$$\begin{aligned} I_p(\vec{\alpha}'(0)) &= \langle \vec{\alpha}'(0), \vec{\alpha}'(0) \rangle_p = \langle \vec{X}_u u' + \vec{X}_v v', \vec{X}_u u' + \vec{X}_v v' \rangle_p \\ &= \langle \vec{X}_u u', \vec{X}_u u' \rangle_p + 2 \langle \vec{X}_u u', \vec{X}_v v' \rangle_p + \langle \vec{X}_v v', \vec{X}_v v' \rangle_p \\ &= E(u')^2 + 2Fu'v' + G(v')^2 \end{aligned} \tag{2}$$

where  $E = \langle \vec{X}_u, \vec{X}_u \rangle_p$ ,  $F = \langle \vec{X}_u, \vec{X}_v \rangle_p$ ,  $G = \langle \vec{X}_v, \vec{X}_v \rangle_p$ , these numbers are named *metric coefficients* of  $S$  or *coefficients of the first fundamental form*. In matrix form Equation (2) will be:

$$I_p(\vec{\alpha}'(0)) = (u', v') \begin{bmatrix} E & F \\ F & G \end{bmatrix} \begin{pmatrix} u' \\ v' \end{pmatrix}$$

The differential of Gauss map  $d\vec{N}_p : T_p(S) \rightarrow T_p(S)$  is a self-adjoint linear map satisfying, by definition, the relation:

$$\langle d\vec{N}_p(\vec{v}), \vec{w} \rangle = \langle \vec{v}, d\vec{N}_p(\vec{w}) \rangle \quad \forall \vec{v}, \vec{w} \in T_p(S)$$

Coming back to the Gauss map, the composite function  $\vec{N} \circ \vec{\alpha} = \vec{N}(\vec{\alpha}(t)) : R \rightarrow R^3$  can be interpreted as a normal vector field restricted to the curve  $\vec{\alpha}(t)$ , more explicitly we have:

$$\vec{N}(\vec{\alpha}(t)) = \vec{N}(t) = (N_x(u(t), v(t)), N_y(u(t), v(t)), N_z(u(t), v(t)))$$

Using the chain rule, its differential is defined as:

$$\begin{aligned} d\vec{N}(\vec{\alpha}'(t)) = \vec{N}'(t) &= \begin{bmatrix} \frac{\partial N_x}{\partial u} & \frac{\partial N_x}{\partial v} \\ \frac{\partial N_y}{\partial u} & \frac{\partial N_y}{\partial v} \\ \frac{\partial N_z}{\partial u} & \frac{\partial N_z}{\partial v} \end{bmatrix} \begin{bmatrix} du \\ dv \\ dt \end{bmatrix} \\ &= \left[ \frac{\partial N_x}{\partial u} \frac{du}{dt} + \frac{\partial N_x}{\partial v} \frac{dv}{dt}, \frac{\partial N_y}{\partial u} \frac{du}{dt} + \frac{\partial N_y}{\partial v} \frac{dv}{dt}, \frac{\partial N_z}{\partial u} \frac{du}{dt} + \frac{\partial N_z}{\partial v} \frac{dv}{dt} \right] \\ &= \left[ \frac{\partial N_x}{\partial u}, \frac{\partial N_y}{\partial u}, \frac{\partial N_z}{\partial u} \right] \frac{du}{dt} + \left[ \frac{\partial N_x}{\partial v}, \frac{\partial N_y}{\partial v}, \frac{\partial N_z}{\partial v} \right] \frac{dv}{dt} \\ &= \vec{N}_u u' + \vec{N}_v v' \end{aligned} \tag{3}$$

with  $\vec{N}_u$  and  $\vec{N}_v$  belonging to  $T_p(S)$ . Based on our preliminary assumption  $\vec{\alpha}(0) = p \in S$ , we have  $d\vec{N}_p(\vec{\alpha}'(0)) = \vec{N}'(0)$ , so the vector  $d\vec{N}_p(\vec{\alpha}'(0))$  belonging to  $T_p(S)$  measures the rate of change of the normal vector  $\vec{N}(t)$  at  $p = \vec{\alpha}(0)$ . In summary,  $d\vec{N}_p$  measures how the normal vector at  $p$  varies when moving along the surface  $S$  in the direction of the tangent vector  $\vec{\alpha}'(0) \in T_p(S)$ .

The main goal of this summary is to arrive at a meaningful definition of the concept of curvature of a surface. There are, at least, two ways to proceed, one more geometric and the other more analytical, which however will lead to the same result. Let us start with the geometrical approach, considering a more explicit representation of vectors lying on the tangent plane.

Let  $\vec{\alpha}(t)$  be a parametrized curve lying in  $S$ , as asserted above, the tangent vector is:

$$\vec{\alpha}'(t) = \vec{X}_u u' + \vec{X}_v v'$$

Since  $\vec{\alpha}(t)$  is a regular curve on  $S$  parametrized by the arc length, we can set  $\vec{\tau}(t) = \vec{\alpha}'(t)$ , where  $\vec{\tau}$  is the tangent vector belonging to the orthonormal basis of the Frenet-Serret coordinate system.

Given that  $\langle \vec{\alpha}'(t), \vec{\alpha}'(t) \rangle = 1$ , by differentiating we obtain  $2\langle \vec{\alpha}''(t), \vec{\alpha}'(t) \rangle = 0$

This result implies that the second derivative  $\vec{\alpha}''(t)$  is normal to  $\vec{\alpha}'(t)$ , it measures how rapidly the curve moves away from the tangent line in a neighborhood of  $p$ . Let  $\vec{n}$  be the unit vector perpendicular to  $\vec{\tau}$ , we can introduce the relationship:

$$\vec{\alpha}''(t) = k(t) \cdot \vec{n}(t)$$

where  $k(t)$  is termed curvature of  $\vec{\alpha}(t)$  at  $p$ . The above vector  $k(t) \cdot \vec{n}(t)$  can be decomposed in two components: the first one  $k_n = k \cos(\theta)$  defines the *normal curvature* at  $p$  parallel to  $\vec{N}(p)$  and the second one  $k_g = k \sin(\theta)$  is

the *geodesic curvature*, where  $\theta$  is the angle between  $\vec{N}(p)$  and  $\vec{n}(p)$  (Figure 5).

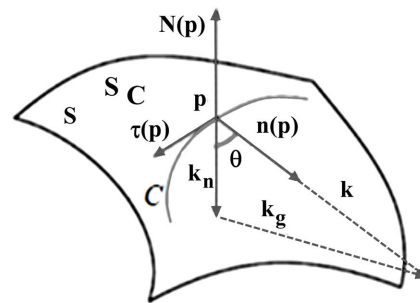


Figure 5. Normal and geodesic curvatures.

In other words, the normal curvature  $k_n$  is the signed length of the projection of the acceleration vector  $\vec{\alpha}''(t)$  on the normal direction to the surface. The normal curvature has also another interpretation. In differential geometry a *normal section* is the regular plane curve formed by slicing a surface  $S$  with a plane that contains  $\vec{N}(p)$ ,  $\vec{\alpha}'(0)$  and  $p$ , the curvature of the normal section at  $p$  is equal to the absolute value of  $k_n$ .

Given a point on the surface, there is only one vector normal to it. There are infinitely many planes containing this vector and passing through a given point, these normal planes intersect the surface  $S$  in curves, *i.e.* the normal sections, for which the geodesic curvature is  $k_g = 0$ . The curves having null geodesic curvature are called *geodesics*, they are curves of shortest length between two points belonging to the surface. Essentially, on a curved surface the concept of geodesic extends the concept of straight line in the Euclidean space. Intuitively, geodesic curvature measures how far a curve is from being a geodesic. The normal section is a curve that depends only on the geometry of the surface  $S$  in the direction of a given tangent vector. Practically, the normal section represents the behavior of  $S$  in the direction of  $\vec{\alpha}'(0)$  reasonably well. There are infinitely many planes normal to the surface and therefore infinitely many normal sections with different normal curvatures. The minimum and maximum values of these curvatures are called *principal curvatures* (Figure 6).

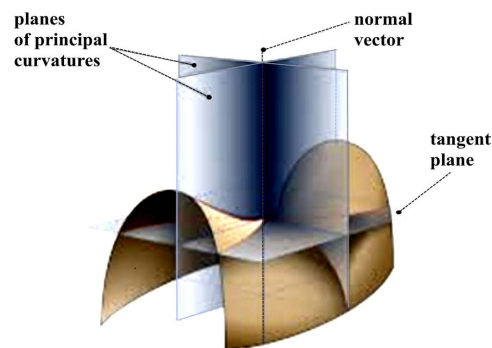


Figure 6. Principal curvatures.

Euler was the first to introduce the concept of normal curvature as the curvature of the section obtained from a plane perpendicular to a surface and passing through a given point on it. By rotating the plane, one obtains curves with different curvatures. Among these normal curvatures, there is one, indicated by  $k_1$ , that assumes a minimum value and one  $k_2$  that has the maximum value, *i.e.* the principal curvatures. In this framework, the Gaussian curvature at the surface point in consideration was defined as:

$$K = k_1 \cdot k_2$$

As we see later, this definition derives straightforward from the initial geometric definition of curvature introduced by Gauss.

Let us introduce now the second approach, the analytical one. Let us come back to the linear map  $d\vec{N}$ , since  $\vec{N}_u$  and  $\vec{N}_v$  belong to  $T_p(S)$ , we may write:

$$\begin{aligned}\vec{N}_u &= a_{11}\vec{X}_u + a_{21}\vec{X}_v \\ \vec{N}_v &= a_{12}\vec{X}_u + a_{22}\vec{X}_v\end{aligned}\quad (4)$$

hence by Equation (3),

$$d\vec{N}(\vec{\alpha}'(t)) = \vec{N}'(t) = \vec{N}_u u' + \vec{N}_v v' = (a_{11}\vec{X}_u + a_{21}\vec{X}_v)u' + (a_{12}\vec{X}_u + a_{22}\vec{X}_v)v' \quad (5)$$

rearranging Equation (5) we have:

$$d\vec{N}(\vec{\alpha}'(t)) = (a_{11}u' + a_{12}v')\vec{X}_u + (a_{21}u' + a_{22}v')\vec{X}_v$$

It shows that the linear map  $d\vec{N}$  can be represented in the basis  $\{\vec{X}_u, \vec{X}_v\}$  of  $T_p(S)$  by the matrix:

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

If the basis  $\{\vec{X}_u, \vec{X}_v\}$  is orthonormal, the matrix  $A$  is symmetric. The determinant of the matrix  $A$   $\det(A) = a_{11}a_{22} - a_{12}a_{21}$  and its trace  $\text{trace}(A) = a_{11} + a_{22}$  do not depend on the choice of the basis but on the linear map. Furthermore, given that the differential of the Gauss map  $d\vec{N}$  is a self-adjoint linear map, it is possible to associate to it a bilinear map  $II : T_p(S) \times T_p(S) \rightarrow R$  defined by:

$$II(\vec{v}, \vec{w}) = \langle d\vec{N}(\vec{v}), \vec{w} \rangle$$

Moreover, the fact that  $d\vec{N}$  is a self-adjoint linear map makes the bilinear map  $II$  symmetric. It can be proved that, given a self-adjoint linear map, there exists a orthonormal basis such that the associated matrix is a diagonal matrix, whose elements on the diagonal are the maximum and the minimum of the corresponding quadratic form obtained with  $\vec{w} = \vec{v}$  and restricted to the unit circle of  $T_p(S)$ , in other words to unit vectors of  $T_p(S)$  (**Figure 7**):

$$II(\vec{v}) = -\langle d\vec{N}(\vec{v}), \vec{v} \rangle$$

If we consider the parametrized curve  $\vec{\alpha}(t)$  and denote with  $\vec{N}(t)$  the restriction of the normal vectors of the surface  $S$  to the curve  $\vec{\alpha}(t)$ , we have:

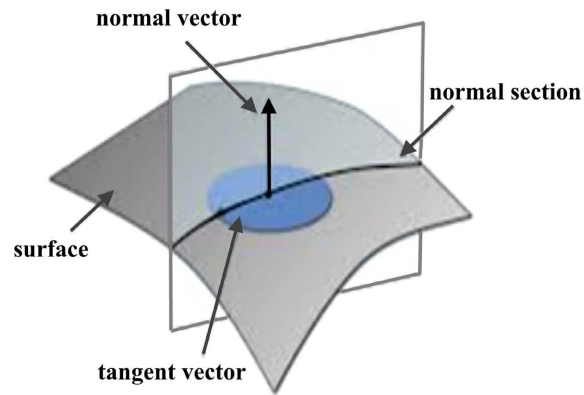


Figure 7. Unit circle of  $T_p(S)$ .

$$\langle \vec{N}(t), \vec{\alpha}'(t) \rangle = 0 \tag{6}$$

because the vectors are perpendicular. Then, by differentiating Equation (6), we get:

$$\begin{aligned} \langle \vec{N}'(t), \vec{\alpha}'(t) \rangle + \langle \vec{N}(t), \vec{\alpha}''(t) \rangle &= 0 \Rightarrow \\ \langle \vec{N}'(t), \vec{\alpha}'(t) \rangle &= -\langle \vec{N}(t), \vec{\alpha}''(t) \rangle \end{aligned} \tag{7}$$

Therefore, analyzing the quadratic form associated to the bilinear map  $II$  with  $\vec{v} = \vec{\alpha}'(0) \in T_p(S)$ , we obtain by Equation (3) and Equation (7):

$$\begin{aligned} II(\vec{\alpha}'(0)) &= -\langle d\vec{N}(\vec{\alpha}'(0)), \vec{\alpha}'(0) \rangle = -\langle \vec{N}'(0), \vec{\alpha}'(0) \rangle \\ &= \langle \vec{N}(0), \vec{\alpha}''(0) \rangle = k_n(p) \end{aligned} \tag{8}$$

The expression of Equation (8) is called the *second fundamental form* of  $S$  at  $p$ .

The second fundamental form, like the normal curvature  $k_n$  defined by Euler, allows us to associate a number to each unit tangent vector of a surface. Evidently, it is closely related to how much a surface curves in the space. From the geometrical point of view, the meaning of the second fundamental form  $II_p$  for a unit vector  $\vec{v} \in T_p(S)$  is the normal curvature of a regular curve passing through  $p$  and tangent to  $\vec{v}$ . Moreover, starting from a local parametrization, the second fundamental form is very simple to compute unlike the geometrical evaluation of  $k_n$ . Lastly, we can state that the normal curvature can be computed using the second fundamental form.

From now on, the quadratic form at  $p = \vec{\alpha}(0)$  will be indicated as:

$$II_p(\vec{v}) = -\langle d\vec{N}(\vec{v}), \vec{v} \rangle \text{ with } \vec{v} \in T_p(S)$$

Given that  $d\vec{N} : T_p(S) \rightarrow T_p(S)$  is a self-adjoint linear map, there exists an orthonormal basis  $\vec{v}$  of  $T_p(S)$ , such that  $d\vec{N}(\vec{v}_1) = k_1\vec{v}_1$  and  $d\vec{N}(\vec{v}_2) = k_2\vec{v}_2$ , which implies that  $\vec{v}_1$ ,  $\vec{v}_2$  and  $k_1$ ,  $k_2$  are eigenvectors and eigenvalues of the matrix  $A$ . In the basis  $\{\vec{v}_1, \vec{v}_2\}$  the matrix  $A$  is clearly diagonal and the elements  $k_1 \geq k_2$  are the maximum and the minimum, respectively of the quadratic

form  $\langle \vec{v} | -d\vec{N}(\vec{v}), \vec{v} \rangle$ . Therefore, we can conclude that the eigenvalues of  $A$  are the principal curvatures of the surface  $S$  and its determinant is:

$$\det(A) = a_{11}a_{22} - a_{12}a_{21} = k_1 \cdot k_2 \tag{9}$$

More explicitly, referring to a parametrization of the surface  $S$ , we are able to express the quadratic form at  $p$  as:

$$\begin{aligned} II_p(\vec{\alpha}'(0)) &= -\langle d\vec{N}(\vec{\alpha}'(0)), \vec{\alpha}'(0) \rangle = -\langle \vec{N}_u u' + \vec{N}_v v', \vec{X}_u u' + \vec{X}_v v' \rangle_p \\ &= \langle \vec{N}_u, \vec{X}_u \rangle_p u'^2 + \langle \vec{N}_u, \vec{X}_v \rangle_p u'v' + \langle \vec{N}_v, \vec{X}_u \rangle_p u'v' + \langle \vec{N}_v, \vec{X}_v \rangle_p v'^2 \\ &= eu'^2 + 2fu'v' + gv'^2 \end{aligned}$$

where:

$$\begin{aligned} e &= -\langle \vec{N}_u, \vec{X}_u \rangle_p & g &= -\langle \vec{N}_v, \vec{X}_v \rangle_p \\ f &= -\langle \vec{N}_u, \vec{X}_v \rangle_p = \langle \vec{N}, \vec{X}_{uv} \rangle_p = \langle \vec{N}, \vec{X}_{vu} \rangle_p = -\langle \vec{N}_v, \vec{X}_u \rangle_p \end{aligned}$$

where  $e, f, g$  are called *coefficients of the second fundamental form*. The perpendicular condition implies that:

$$\langle \vec{N}, \vec{X}_u \rangle_p = 0 \tag{10}$$

differentiating respect  $v$  Equation (10), we have:

$$\langle \vec{N}_v, \vec{X}_u \rangle_p + \langle \vec{N}, \vec{X}_{uv} \rangle_p = 0 \Rightarrow \langle \vec{N}_v, \vec{X}_u \rangle_p = -\langle \vec{N}, \vec{X}_{uv} \rangle_p$$

analogously differentiating respect  $u$  Equation (11)

$$\langle \vec{N}, \vec{X}_v \rangle_p = 0 \tag{11}$$

$$\langle \vec{N}_u, \vec{X}_v \rangle_p + \langle \vec{N}, \vec{X}_{vu} \rangle_p = 0 \Rightarrow -\langle \vec{N}, \vec{X}_{vu} \rangle_p = -\langle \vec{N}, \vec{X}_{uv} \rangle_p = \langle \vec{N}_v, \vec{X}_u \rangle_p$$

Recurring to the first fundamental form Equation (2):

$$\begin{aligned} -e &= \langle \vec{N}_u, \vec{X}_u \rangle_p = \langle a_{11}\vec{X}_u + a_{21}\vec{X}_v, \vec{X}_u \rangle_p = a_{11}\langle \vec{X}_u, \vec{X}_u \rangle_p + a_{21}\langle \vec{X}_v, \vec{X}_u \rangle_p = a_{11}E + a_{21}F \\ -f &= \langle \vec{N}_u, \vec{X}_v \rangle_p = \langle a_{11}\vec{X}_u + a_{21}\vec{X}_v, \vec{X}_v \rangle_p = a_{11}\langle \vec{X}_u, \vec{X}_v \rangle_p + a_{21}\langle \vec{X}_v, \vec{X}_v \rangle_p = a_{11}F + a_{21}G \\ -f &= \langle \vec{N}_v, \vec{X}_u \rangle_p = \langle a_{12}\vec{X}_u + a_{22}\vec{X}_v, \vec{X}_u \rangle_p = a_{12}\langle \vec{X}_u, \vec{X}_u \rangle_p + a_{22}\langle \vec{X}_v, \vec{X}_u \rangle_p = a_{12}E + a_{22}F \\ -g &= \langle \vec{N}_v, \vec{X}_v \rangle_p = \langle a_{12}\vec{X}_u + a_{22}\vec{X}_v, \vec{X}_v \rangle_p = a_{12}\langle \vec{X}_u, \vec{X}_v \rangle_p + a_{22}\langle \vec{X}_v, \vec{X}_v \rangle_p = a_{12}F + a_{22}G \end{aligned}$$

Rewriting these equations in matrix form:

$$-\begin{bmatrix} e & f \\ f & g \end{bmatrix} = \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix} \begin{bmatrix} E & F \\ F & G \end{bmatrix}$$

Hence, the matrix associated to the first fundamental form will be:

$$\begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix} = -\begin{bmatrix} e & f \\ f & g \end{bmatrix} \begin{bmatrix} E & F \\ F & G \end{bmatrix}^{-1} \tag{12}$$

where  $\begin{bmatrix} E & F \\ F & G \end{bmatrix}^{-1}$  is the inverse matrix of  $\begin{bmatrix} E & F \\ F & G \end{bmatrix}$  that can be immediately evaluated:

$$\begin{bmatrix} E & F \\ F & G \end{bmatrix}^{-1} = \frac{1}{EG - F^2} \begin{bmatrix} G & -F \\ -F & E \end{bmatrix}$$

From Equation (12) we have:

$$\begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix} = -\frac{1}{EG - F^2} \begin{bmatrix} e & f \\ f & g \end{bmatrix} \begin{bmatrix} G & -F \\ -F & E \end{bmatrix} = \frac{1}{EG - F^2} \begin{bmatrix} fF - eG & eF - fE \\ gF - fG & fF - gE \end{bmatrix}$$

Applying the determinant's rules to Equation (12) and keeping in mind Equation (9), [2]-[4] we deduce that:

$$\det(A) = a_{11}a_{22} - a_{12}a_{21} = \frac{eg - f^2}{EG - F^2} = k_1 \cdot k_2 \quad (13)$$

Let us carry out another important result that will be very useful in the following formulas, using Equation (4) we obtain:

$$\begin{aligned} \bar{N}_u \times \bar{N}_v &= (a_{11}\bar{X}_u + a_{21}\bar{X}_v) \times (a_{12}\bar{X}_u + a_{22}\bar{X}_v) \\ &= a_{11}a_{12}\bar{X}_u \times \bar{X}_u + a_{11}a_{22}\bar{X}_u \times \bar{X}_v + a_{21}a_{12}\bar{X}_v \times \bar{X}_u + a_{21}a_{22}\bar{X}_v \times \bar{X}_v \\ &= (a_{11}a_{22} - a_{21}a_{12})\bar{X}_u \times \bar{X}_v = \det(A)\bar{X}_u \times \bar{X}_v = (k_1 \cdot k_2)\bar{X}_u \times \bar{X}_v \end{aligned} \quad (14)$$

Now, we would like to go back to the original definition of the Gaussian curvature given by Gauss [5]:

*“To each part of a curved surface inclosed within definite limits, we assign a **total or integral curvature**, which is represented by the area of the figure on the sphere corresponding to it. From this integral curvature must be distinguished the somewhat more specific curvature which we shall call the **measure of curvature**. The latter refers to a point of the surface, and shall denote the quotient obtained when the integral curvature of the surface element about a point is divided by the area of the element itself; hence it denotes the ratio of the infinitely small areas which corresponds to one another on the curved surface and on the sphere”.*

Essentially, Gauss began with the introduction of the concept of integral curvature referring to the curvature of a limited region of the surface under examination, while he indicated with the term measure of curvature the curvature associated to a specific point, nowadays simply termed Gaussian curvature. In short, the integral curvature of a compact region  $M \subset S$  is defined as:

$$\iint_M K d\sigma$$

From the geometrical point of view, Gauss defined the measure of curvature in the following way:

**Definition:** Let  $p$  be a point of a surface  $S$  and let  $B$  a connected neighborhood of  $p$  where the measure of curvature  $K(p)$  does not change the sign, we have:

$$K(p) = \lim_{A \rightarrow 0} \frac{A'}{A} \quad (15)$$

where  $A = \text{Area}(B)$  is the area of a region  $B$  containing  $p$ ,  
 $A' = \text{Area}(\bar{N}(B))$  is the area of the image of  $\bar{N}(B)$  by the Gauss Map (**Figure**

8), and the limit is taken through a sequence of open sets  $B_n$  that converges to  $p$  for  $n$  sufficiently large.

We would now like to highlight that this definition can be traced back to the one given by Euler.

Given a parametrization  $\vec{X}(u, v)$  of the surface  $S$  at  $p$ , the area  $A$  of  $B$  that can be assumed being sufficiently small, is given by [2]:

$$A = \iint_R |\vec{X}_u \times \vec{X}_v| \, du \, dv = \iint_R \sqrt{EG - F^2} \, du \, dv \tag{16}$$

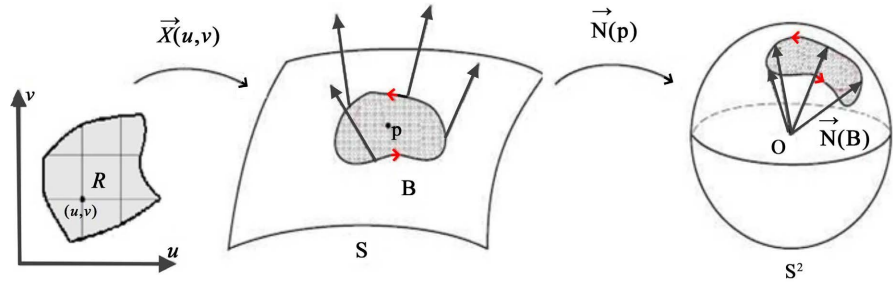


Figure 8. Parametrization  $\vec{X}(u, v)$  of a surface and Gauss Map.

where  $R$  is the area of the region in the  $uv$  plane corresponding to  $B$ . The area  $A'$  of  $\vec{N}(B)$  is:

$$A' = \iint_R |\vec{N}_u \times \vec{N}_v| \, du \, dv \tag{17}$$

Using Equation (14) we recall that:  $|\vec{N}_u \times \vec{N}_v| = (k_1 \cdot k_2) |\vec{X}_u \times \vec{X}_v|$

$$A' = \iint_R |\vec{N}_u \times \vec{N}_v| \, du \, dv = \iint_R k_1 \cdot k_2 |\vec{X}_u \times \vec{X}_v| \, du \, dv \tag{18}$$

Going to the limit for  $R \rightarrow 0$ , Equation (15) can be written as:

$$\begin{aligned} K(p) &= \lim_{R \rightarrow 0} \frac{A'}{A} = \lim_{R \rightarrow 0} \frac{\iint_R k_1 \cdot k_2 |\vec{X}_u \times \vec{X}_v| \, du \, dv}{\iint_R |\vec{X}_u \times \vec{X}_v| \, du \, dv} \\ &= \frac{\lim_{R \rightarrow 0} \iint_R k_1 \cdot k_2 |\vec{X}_u \times \vec{X}_v| \, du \, dv}{\lim_{R \rightarrow 0} \iint_R |\vec{X}_u \times \vec{X}_v| \, du \, dv} \\ &= \frac{k_1 \cdot k_2 |\vec{X}_u \times \vec{X}_v|}{|\vec{X}_u \times \vec{X}_v|} = k_1 \cdot k_2 \end{aligned} \tag{19}$$

The previous formulas put in evidence that the Gaussian curvature  $K$ , defined by Euler as product of the principal curvatures  $k_1$  and  $k_2$  can be directly carried out from the original Gauss's definition of measure of curvature Equation (15). In turn, the principal curvatures are evaluated as eigenvalues of the matrix  $A$ , associated to the differential of the Gauss Map:

$$d\vec{N}_p : T_p(S) \rightarrow T_p(S)$$

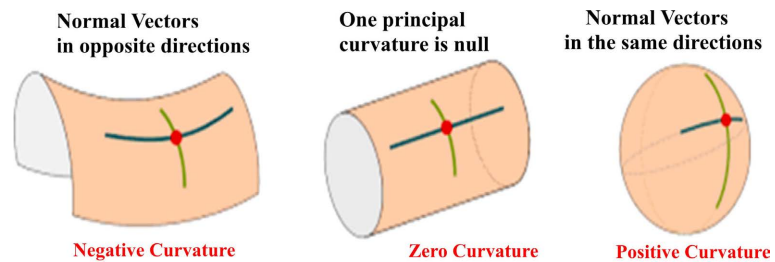
**Definition:** The sign of the Gaussian curvature can be used to characterize a point  $p$  of a surface  $S$ , that it is called (Figure 9):

*Elliptic* if  $\det(d\vec{N}_p) = (a_{11}a_{22} - a_{21}a_{12}) > 0$ , namely if  $k_1$  and  $k_2$  have the same sign

*Hyperbolic* if  $\det(d\vec{N}_p) = (a_{11}a_{22} - a_{21}a_{12}) < 0$ , namely if  $k_1$  and  $k_2$  have opposite signs

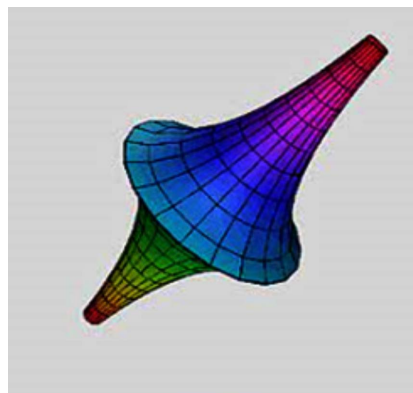
*Parabolic* if  $\det(d\vec{N}_p) = (a_{11}a_{22} - a_{21}a_{12}) = 0$ , namely if  $k_1$  or  $k_2$  is null

*Planar* if  $\det(d\vec{N}_p) = (a_{11}a_{22} - a_{21}a_{12}) = 0$ , namely if  $k_1$  and  $k_2$  are null.



**Figure 9.** Classification of geometric points.

At elliptic points, the surface will be dome like, locally lying on one side of its tangent plane. At hyperbolic or *saddle* points the principal curvatures have opposite signs, so there are normal sections through  $p$  whose normal vectors point toward one side or the other side of the tangent plane. Most surfaces will contain regions of positive Gaussian curvature and regions of negative Gaussian curvature separated by a curve of points with zero Gaussian curvature called a *parabolic line*. When a surface has everywhere *null* Gaussian curvature, the geometry of the surface is the *Euclidean geometry*. When the Gaussian curvature of a surface is *positive* and *constant* at every point, like a sphere of radius  $R$  whose curvature is positive, constant, and equal to  $R^2$ , then the geometry of the surface is *spherical geometry*. If the curvature is *negative* and *constant* at every points, like a pseudo-sphere of radius  $R$ , having curvature  $-1/R^2$  in each point and generated by rotating a tractrix around a central axis, the geometry of the surface is *hyperbolic geometry* (**Figure 10**).



**Figure 10.** Pseudo-Sphere.

The Gaussian curvature is carried out from the first fundamental form and can be expressed through the partial derivatives of first and second order of the surface parametrization. Therefore, it is an *intrinsic* measure, depending only on distances that are computed within or along the surface, not on the way it is embedded in Euclidean space. In this regard, we can make an illustrative consideration that may seem bizarre: any human being can be able to evaluate the curvature of the surface that hosts him remaining on it, without referring to any external coordinate system. Moreover, the definition of Gauss curvature, based on the first fundamental form, allows us to state the *Remarkable Theorem (Egregium Theorem)* of Gauss [5]:

“*The Gaussian curvature  $K$  of a surface is invariant by local isometry.*”

We recall that an isometry of a metric space onto another or onto itself is a mapping that preserves the distances between any two points. Therefore, regardless the embedding of a surface in the space, the Gaussian curvature does not change. For example, if a sheet of paper, with zero Gaussian curvature, is rolled up into a cylinder, or is crumpled or wrinkled, its Gaussian curvature remains unchanged (**Figure 11**). On the contrary, the *mean curvature*, defined as:

$$H = \frac{\text{trace}(A)}{2} = \frac{a_{11} + a_{22}}{2} = \frac{k_1 + k_2}{2}$$

is an *extrinsic* curvature, namely the Gaussian curvature of a cylindrical tube is zero, as for the unrolled tube, but its mean curvature is not zero, so the former surface is extrinsically different from the latter.



**Figure 11.** Different embedding of the same sheet in  $\mathbb{R}^3$  space.

The *Remarkable Theorem* entails some particularly important consequences. A surface can be superimposed on another only if the two surfaces have the same curvature. Hence, it solves the age-old question of the representation of one surface on another, especially the cartographic representation of the Earth’s surface on a flat map. This theorem states that is impossible to represent Earth’s surface having positive curvature everywhere on a flat sheet of paper having null curvature. From the theoretical point of view, the *Remarkable Theorem* implies that a piece of surface, for example a triangle or another geometric figure, can be trans-

ported from one part of the surface to another only if it has the same curvature at all points, that is, if the curvature of the surface is constant. It is well known that the congruence of geometric figures can be verified and detected by moving one figure onto another until they overlap. Accordingly, the congruence can be ascertained if a perfect superposition is achieved by moving one figure over the other: if the figures overlap perfectly then they are equal. It follows that a geometry can be built only on surfaces satisfying these requirements, more precisely, not only on the plane but on all surfaces having constant positive or negative curvature.

### 3. Local Gauss-Bonnet Theorem

In Art.20 of *General Investigations of Curved Surfaces*, Gauss began to investigate “triangles formed by shortest lines”, that are, in synthesis, *geodesic triangles*. The original version of the Gauss-Bonnet Theorem deals with geodesic triangles on a surface, Gauss in 1827 termed this theorem “the most elegant theorem”. The extension of it to a region bounded by a non-geodesic simple curve is due to Bonnet O. [6], in this case we refer to the *Gauss-Bonnet Local Theorem*. A further extension of this theorem to compact surfaces gives rise to the Gauss-Bonnet Global Theorem that asserts an important relationship between the topology of a surface and the integral of the Gaussian curvature on it.

**Theorem on Geodesic Triangles:** Given a geodesic triangle  $T$ , *i.e.* formed by the arcs of three geodesics lying on a smooth surface  $S$  (Figure 12), let  $\varphi_1, \varphi_2, \varphi_3$  be its interior angles,  $K$  the Gaussian curvature, the integral curvature is given by:

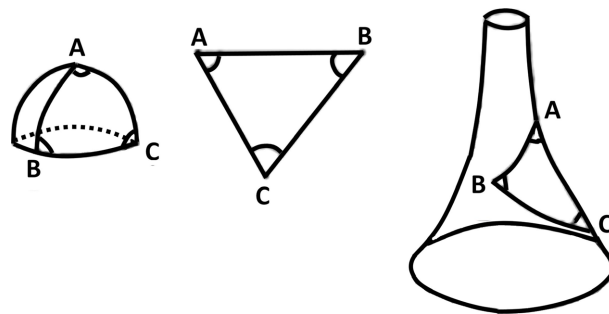


Figure 12. Geodesic triangles.

$$\iint_T K d\sigma = \sum_{i=1}^3 \varphi_i - \pi \quad (20)$$

Equivalently, if  $\theta_i$  are the exterior angles  $\theta_i = \pi - \varphi_i$ :

$$\begin{aligned} \iint_T K d\sigma &= \sum_{i=1}^3 \varphi_i - \pi = \sum_{i=1}^3 (\pi - \theta_i) - \pi = 2\pi - \sum_{i=1}^3 \theta_i \\ \iint_T K d\sigma + \sum_{i=1}^3 \theta_i &= 2\pi \end{aligned} \quad (21)$$

As a consequence, we have:

If  $K = K_0$  is constant, we get

$$K_0 \text{Area}(T) = \sum_{i=1}^3 \varphi_i - \pi \Rightarrow \text{Area}(T) = \frac{\sum_{i=1}^3 \varphi_i - \pi}{K_0}$$

this formula implies that the area of a triangle does not depend on the length of its sides but on the size of its internal angles.

If the surface is locally isometric to a Euclidean plane, then  $K_0 = 0$

$$\sum_{i=1}^3 \varphi_i - \pi = 0$$

we recognize the familiar result of Euclidean geometry.

If  $K_0 > 0$  the surface is locally isometric to a sphere of ray  $1/K_0$ , we have:

$$\sum_{i=1}^3 \varphi_i > \pi$$

If  $K_0 < 0$  the surface is locally isometric to a pseudo-sphere, we have:

$$\sum_{i=1}^3 \varphi_i < \pi$$

Gauss's discovery revealed the existence of other geometries than the Euclidean and of intrinsic geometrical properties whose validity is independent by the modality of surface embedding on the space. For proving the theorem, Gauss examined a triangle delimited by three geodesic curves and a surface parametrization with orthogonal coordinate curves, so  $F = \langle \vec{X}_u, \vec{X}_v \rangle = 0$  [5]. The extension of this Gauss Theorem to the case of a region bounded by a simple closed curve, not necessarily a geodesic, is due to Bonnet O. (1848). Before stating this extension, we give the following definitions:

Let  $\vec{\alpha}(t): [a, b] \rightarrow M$  be a continuous map from the closed interval  $[a, b]$  onto a regular region  $M \subset S$ , we state that  $\vec{\alpha}$  is a *simple, closed and piecewise regular curve* if:

1) for  $t_1 \neq t_2 \in ]a, b[ \Rightarrow \vec{\alpha}(t_1) \neq \vec{\alpha}(t_2)$  no self-intersections

2)  $\vec{\alpha}(a) = \vec{\alpha}(b)$  closed curve

3) there exists a partition of  $[a, b]$  such that  $a = t_0 < t_1 < \dots < t_{L-1} < t_L = b$  for which  $\vec{\alpha}$  is differentiable and regular in every interval  $[t_i, t_{i+1}]$   $i = 0, \dots, L$ . The points  $\vec{\alpha}(t_i)$  are called vertices of  $\vec{\alpha}$ . The curve segments  $\vec{\alpha}([t_i, t_{i+1}])$  are called the *edges* or *sides*. From piecewise regularity it follows that, for every vertex  $\vec{\alpha}(t_i)$  there exists the left limit:  $\lim_{t \rightarrow t_i^-} \vec{\alpha}'(t) = \vec{\alpha}'_-(t_i)$  and the right limit:

$\lim_{t \rightarrow t_i^+} \vec{\alpha}'(t) = \vec{\alpha}'_+(t_i)$  that implies the vertices have different tangent lines and are not cusp points, so:

$$\lim_{t \rightarrow t_i^-} \vec{\alpha}'(t) = \vec{\alpha}'_-(t_i) \neq \lim_{t \rightarrow t_i^+} \vec{\alpha}'(t) = \vec{\alpha}'_+(t_i) \neq \pm \infty$$

In order to define unambiguously exterior and interior angles, let us consider an oriented surface  $S$  with  $\vec{N}$  as the unit vector field normal to it. We indicate with  $|\theta_i| \in [0, \pi]$  the angle formed by  $\vec{\alpha}'_-(t_i)$  and  $\vec{\alpha}'_+(t_i)$ , with sign equal to the sign of the determinant of the vectors  $\{\vec{\alpha}'_-(t_i), \vec{\alpha}'_+(t_i), \vec{N}\}$ , as can be directly ver-

ified using the right-hand rule (Figure 13).

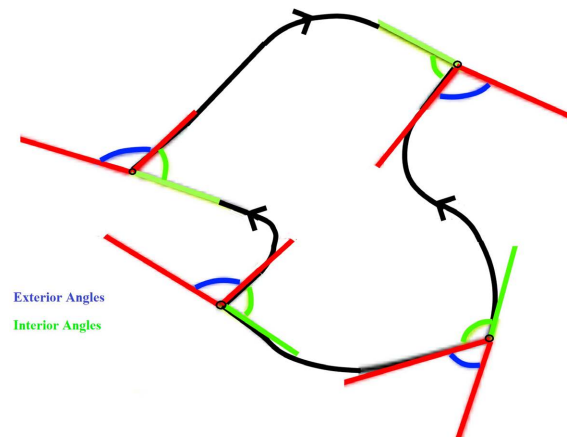


Figure 13. Exterior and interior angles of an oriented curve.

The angle  $\theta_i$  with sign such that  $-\pi < \theta_i < +\pi$ , is the exterior angle, while  $\varphi_i = \pi - \theta_i$  is the interior angle. We say that the curve is *positively oriented* if an observer, with direction and orientation of  $\vec{N}$ , who travels along this curve, leaves at every point the region delimited by it to his left (Figure 14).

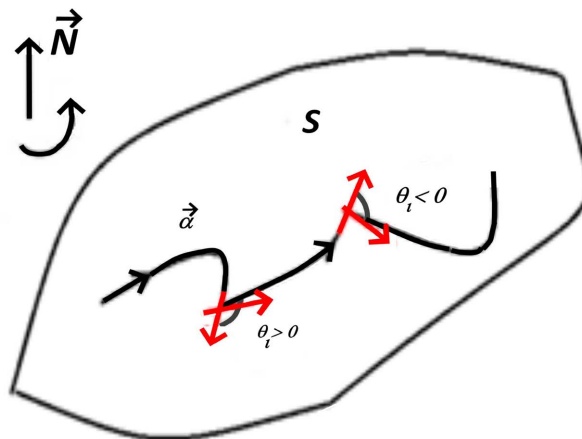


Figure 14. Sign of exterior angles.

**Local Gauss-Bonnet Theorem.** Let  $M \subset S$  be a regular region of  $S$ , that is a compact subset of  $S$  obtained as a closure of its interior and with a boundary parameterized by a finite number of curvilinear polygons. Let us choose for the curvilinear polygon a parametrization  $\vec{\alpha}(t): [a, b] \rightarrow S$  with  $\vec{\alpha}([a, b]) = \partial M$  the boundary of  $M$ . Assume that  $\vec{\alpha}$  is a simple, closed, piecewise regular curve, positively oriented and parametrized by arc-length  $t$ . Furthermore, let  $\vec{\alpha}(t_i)$  and  $\theta_i$  with  $i = 0, \dots, L$  the vertices and the exterior angles at them of the curve, we have:

$$\iint_M K d\sigma + \int_{\partial M} k_g dt + \sum_{i=0}^L \theta_i = \iint_M K d\sigma + \sum_{i=0}^L \int_{t_i}^{t_{i+1}} k_g dt + \sum_{i=0}^L \theta_i = 2\pi \quad (22)$$

If the boundary  $\partial M$  is piecewise smooth, the integral  $\int_{\partial M} k_g dt$  corresponds to the sum of integrals along the smooth components of the boundary, while the sum of the external angles  $\theta_i$  evaluate how much the smooth sides turn at boundary corners, for proof see [2]. As we can observe, Equation (22) represents a generalization of the Gauss Theorem for Geodesic Triangles Equation (21) to a region  $M$  with boundary formed by piecewise smooth sides that are not geodesics. Intuitively, the geodesic curvature  $k_g$  is a measure of the amount of deviation of the curve from the shortest arc between two points on a surface. In short, geodesic curvature measures how far a curve is from being a geodesic. It is certainly noteworthy that the local Gauss-Bonnet theorem relies exclusively on intrinsic quantities: the Gaussian curvature, the geodesic curvature, and angles on the surface [6] [7].

**Examples:** Let us apply the Local Gauss-Bonnet Theorem examining two simple regions: a square  $S_1$  and a semi-sphere  $S_2$  (Figure 15).

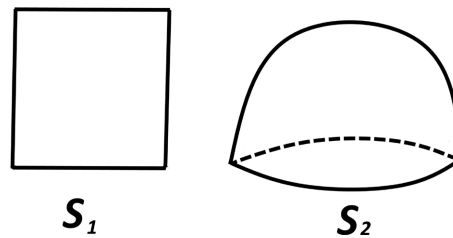


Figure 15. Examples of simple regions.

In the first case, the boundary  $\partial S_1$  is a closed piecewise regular curve formed by four geodesics. The Gaussian curvature is  $K = 0$  because the surface is on the plane,  $k_g = 0$  because the geodesic curvature is a measure of how much a curve deviates from a geodesic, since a segment is a geodesic, its geodesic curvature is null, the exterior angles are  $\frac{\pi}{2}$ , then we can verify that:

$$\iint_{S_1} K d\sigma + \int_{\partial S_1} k_g dt + \sum_{i=1}^4 \theta_i = \sum_{i=1}^4 \theta_i = 2\pi \Rightarrow \sum_{i=1}^4 \frac{\pi}{2} = 2\pi$$

In the second surface the Gaussian curvature is constant  $K = 1/R^2$ , the boundary is a simple regular close curve without corners, defined by a maximum circle that is a geodesic with  $k_g = 0$ , even in this case we have:

$$\iint_{S_2} K d\sigma + \int_{\partial S_2} k_g dt + \sum_{i=0}^L \theta_i = \iint_{S_2} K d\sigma = 2\pi \Rightarrow \frac{1}{R^2} 2\pi R^2 = 2\pi$$

### 4. Global Gauss-Bonnet Theorem

As already asserted, the Global Gauss-Bonnet Theorem is an extension of the Local Gauss-Bonnet Theorem to compact surfaces that, by definition, are bounded and closed. Before considering the theorem, we need to introduce some preliminary topological concepts. Actually, this theorem creates a bridge between differ-

ential geometry and topology. Its relevance lies in having related geometrical information of a compact surface to its purely topological characteristics, which has resulted of strong impact in applications, especially nowadays.

**Definition:** A *Topological Space* is an ordered pair  $(X, T)$ , where  $X$  is a set and  $T$  is a collection of subsets of  $X$ , satisfying the following axioms:

The empty set and  $X$  itself belong to  $T$ .

Any arbitrary (finite or infinite) union of members of  $T$  still belongs to  $T$ .

The intersection of any finite number of members of  $T$  still belongs to  $T$ .

The elements of  $T$  are called open sets and the collection  $T$  is called a topology on  $X$ . A topological space is the most general and basic type of space for which the concept of proximity, continuity are formalized through the notion of neighborhoods without the need to introduce a metric structure in it [8].

A *homeomorphism* between two topological spaces is a continuous, bijective mapping that has a continuous inverse. In short, a homeomorphism can continuously transform one shape into another without tearing, gluing parts, or creating holes. So, recurring to a continuous deformation the original shape can be remodeled in a new one topologically equivalent to it. A *topological property* or *topological invariant* is a property of a topological space that is invariant under homeomorphisms, having been expressed using open sets.

A region  $M \subset S$  is said *regular* if it is compact and its boundary  $\partial M$  is a union of a finite number of piecewise, closed curves, with no intersections. A regular region of a surface  $S$  is said to be *simple* if it is homeomorphic to a closed disk, for which the boundary is constituted by a single curvilinear polygon (Figure 16). If  $M = S$  is a compact surface of  $R^3$ ,  $\partial M = \emptyset$ .

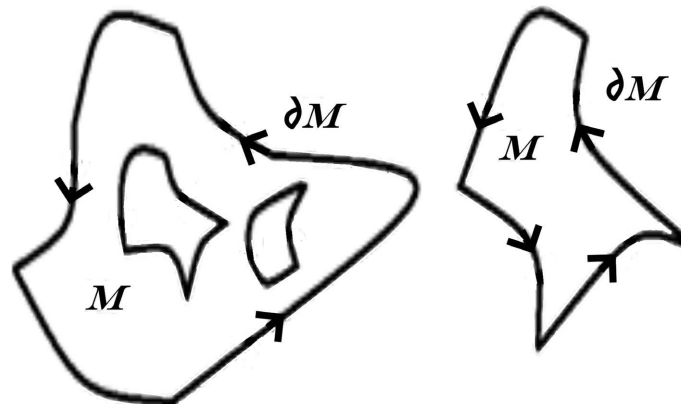


Figure 16. Regular region and regular simple region.

Consider first the local form of the Gauss-Bonnet theorem, it relies exclusively on intrinsic quantities. For the formulation and the proof of the Gauss-Bonnet theorem in the so-called global version, the key point is to perform a procedure for partitioning a regular region of a surface  $S$  into smaller pieces, in such a way the local version can be applied to each of them summing up the obtained results. In order to extend the local formulation to the global Gauss-Bonnet theorem, we

need to introduce a triangulation of a surface.

**Definition:** A smooth triangle of a surface is a simple regular region  $T$  bounded by a curvilinear polygon with three vertices.

Let  $M \subset S$  be a regular, simple region of  $S$ . The triangulation of a compact surface consists of a network of a finite number of regular curve segments on it, such that any point on the surface either lies on one of the smooth segment or belongs to the interior of a region delimited by three sides. Now divide  $M$  into triangles  $T_i$  such that each side of a triangle is the side of precisely one other triangle. In this subdivision, each edge is the side of precisely two triangles. Let us note that the triangulation can be realized without referring to a metric structure of the surface. Every regular region  $M$  of a regular surface  $S$  admits a triangulation (Figure 17).

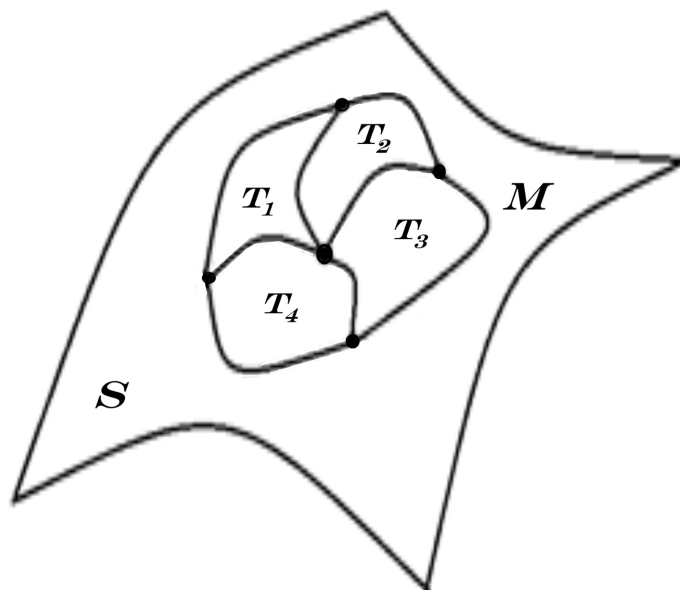


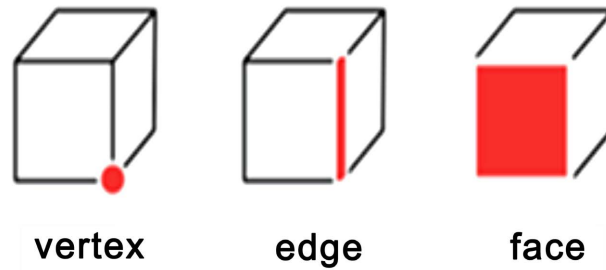
Figure 17. A triangulation.

The global Gauss–Bonnet theorem provides an important link between local geometric properties and global topological properties. More specifically, it concerns with a very important topological invariant of compact surfaces the *Euler-Poincaré characteristic*. The Euler-Poincaré characteristic  $\chi$  was classically defined for the surfaces of polyhedral, according to the formula (Figure 18):

$$\chi = V - E + F$$

where  $V$ ,  $E$ , and  $F$  are respectively the numbers of vertices (corners), edges and faces in the given polyhedron. A simple regular region is homeomorphic to a closed disk which, in turn, is homeomorphic to a triangle, whose Euler-Poincaré characteristic is clearly 1:  $\chi = V - E + F = 3 - 3 + 1$ . Any convex polyhedron has Euler-Poincaré characteristic (Figure 19):

$$\chi = V - E + F = 2$$

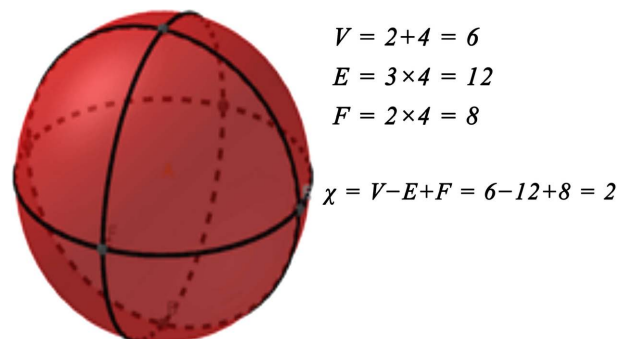


**Figure 18.** Vertex, edge, face.

Name	Image	Vertices $V$	Edges $E$	Faces $F$	Euler characteristic: $\chi = V - E + F$
Tetrahedron		4	6	4	2
Hexahedron or cube		8	12	6	2
Octahedron		6	12	8	2
Dodecahedron		20	30	12	2
Icosahedron		12	30	20	2

**Figure 19.** Euler-Poincarè of convex polyhedrons.

The Euler-Poincarè characteristic does not change by stretching or shrinking any side or face, even shrinking a side or face to zero. This means that it is a topological invariant because it remains unaltered by any continuous mapping. Moreover, the Euler-Poincarè characteristic of the sphere is 2, that implies all convex polyhedra are homeomorphic to a bi-dimensional sphere. In order to verify this statement, among all the possible triangulations of a sphere, we may consider the most intuitive that is the one generated by 3 maximum circles (**Figure 20**).



**Figure 20.** Euler-Poincarè characteristic of a sphere.

We have already asserted that a topological property is defined to be a property that is preserved under a homeomorphism. For example, connectedness, compactness and the number of components of the boundary for a plane domain are topological properties. Therefore, any simple curvilinear polygon of  $R^2$  is homeomorphic to a circle and any compact surface of  $R^3$  is homeomorphic to a sphere (Figure 21).

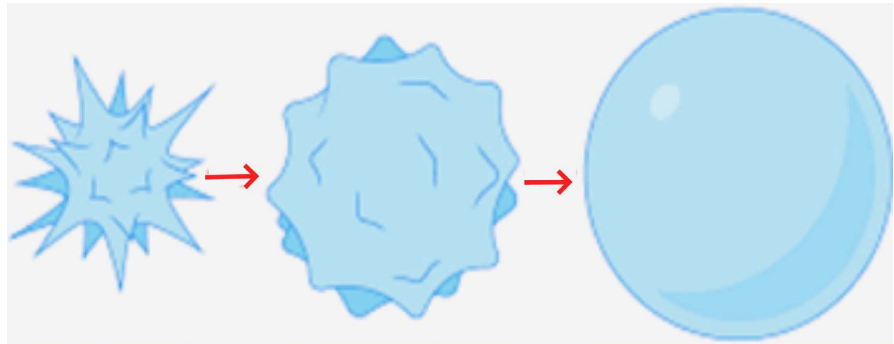


Figure 21. A spiky ball is homeomorphic to a sphere.

Let us consider a triangulation of a regular region  $M \subset S$ , the Euler-Poincaré characteristic of  $M$  is defined as:

$$\chi(M) = V(M) - E(M) + F(M) = \#(\text{vertices}) - \#(\text{edges}) + \#(\text{triangles})$$

where  $V(M)$  specifies the number of vertices of the triangulation,  $F(M)$  the number of triangles,  $E(M)$  the number of edges and the symbol  $\#$  represents the cardinality of a set.

**Theorem:** The Euler-Poincaré characteristic is independent by the triangulation.

In summary, it can be proved that:

The Euler-Poincaré characteristic of a regular, simple region is  $\chi = 1$ .

The Euler-Poincaré characteristic of a sphere  $S^2$  is  $\chi(S^2) = 2$ .

The Euler-Poincaré characteristic of a torus  $T$  is  $\chi(T) = 0$ .

**Proposition:** Let  $M \subset S$  and  $M' \subset S'$  two regular homeomorphic regions, then  $\chi(M) = \chi(M')$

**Global Gauss-Bonnet Theorem.** Let  $M$  be a compact region of an oriented surface  $S$  with a boundary  $\partial M$  parameterized by a curvilinear polygon or a finite number of them, such that  $\vec{\alpha}(t) : [a, b] \rightarrow S$  with  $\vec{\alpha}([a, b]) = \partial M$ . Let  $\vec{\alpha}(t_i)$  be a vertex of the piecewise, simple boundary curve and  $\theta_i$  with  $i = 0, \dots, L$  the exterior angles, formed between the tangent vectors at  $\vec{\alpha}(t_{i-1})$  and  $\vec{\alpha}(t_i)$  at the vertex  $\vec{\alpha}(t_i)$ , we have:

$$\iint_M K d\sigma + \int_{\partial M} k_g dt + \sum_{i=0}^L \theta_i = 2\pi\chi(M) \tag{23}$$

for proof see [2].

**Proposition:** Let  $M$  be a connected, compact and orientable surface of  $R^3$  (meaning a single piece, closed surface contained in a ball of finite size, having

two distinct sides), so  $\partial M = \emptyset$ , for the Gauss-Bonnet Global Theorem we have:

$$\iint_M K d\sigma = 2\pi\chi(M)$$

The Gauss-Bonnet Theorem is surprising because relates closely two very distant a priori invariants: the Euler-Poincarè characteristic, a topological invariant and the Gaussian curvature a metric invariant, as a consequence the total curvature does not change when we deform the surface.

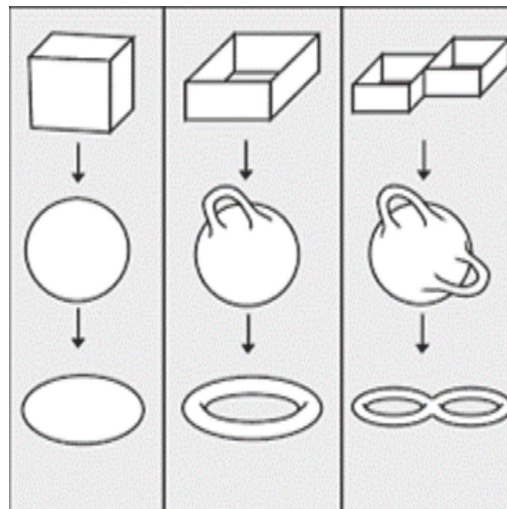
The Gauss-Bonnet theorem states that, even if the Gaussian curvature would locally change at different points on the surface, the total Gaussian curvature still remains the same under deformations.

To put in evidence this assertion, we may consider an alternative definition of the Euler-Poincarè characteristic of an oriented compact surface  $S$ :

$$\chi(S) = 2 - 2g \quad (24)$$

where  $g$  is the *genus* of  $S$ , roughly speaking it represents the number “handles” or “holes” of  $S$ .

**Definition:** A handle in a surface  $S$  is a regular region  $M$  in  $S$  homeomorphic to a finite closed circular cylinder, such that  $S \setminus M$  is connected. Consequently, the surface  $S$  can be characterized topologically by their Euler-Poincarè characteristic or equivalently by the number of holes or handles. **Figure 22** shows graphically surfaces that are topologically equivalent, given that they have equal genus.



**Figure 22.** Examples of surfaces topologically equivalent.

The alternative definition of the Euler-Poincarè characteristic with Equation (24), allows us to introduce a general classification of orientable, compact and connected surface  $S$ .

**Theorem:** Every orientable compact surface is homeomorphic to a sphere with  $g \geq 0$  handles, having the Euler-Poincarè characteristic equal to  $2 - 2g$  [8].

Two orientable compact surfaces are homeomorphic if and only if they have

the same Euler-Poincaré characteristic;

The sphere is the only compact orientable surface with positive Euler-Poincaré characteristic;

The torus is the only compact orientable surface with zero Euler-Poincaré characteristic.

The torus is not the only surface with null Euler-Poincaré characteristic, as an example a coffee cup has one hole, that being its handle, so a coffee cup and a doughnut are homeomorphic to one another, from the topological point of view they have the same structure (Figure 23).

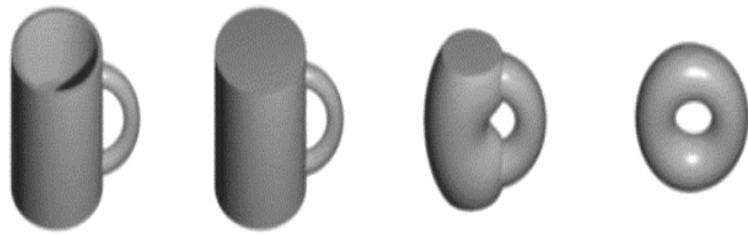


Figure 23. Surfaces topologically equivalent.

In summary, let  $S$  be any orientable compact surface of  $R^3$  :

If  $\iint_S K d\sigma > 0 \Leftrightarrow$  is homeomorphic to a sphere.

If  $\iint_S K d\sigma = 0 \Leftrightarrow$  is homeomorphic to a torus.

If  $\iint_S K d\sigma < 0 \Leftrightarrow$  is homeomorphic to a sphere with  $g > 1$  handles.

## 5. Discrete Gaussian Curvature

Triangle meshes are commonly used for representing surfaces embedded in three-dimensional space. We can consider a mesh either as the limit of a family of smooth surfaces, or as a linear approximation of an arbitrary surface. Given a discrete set of vertices sampled from a smooth surface forming a point cloud, the triangle mesh generates an approximation of the smooth surface with these vertices, obtaining a discrete surface representation.

A triangle mesh  $M$  consists of a set of vertices  $V = \{\vec{v}_i \in R^3\}$  connected by a set of edges:

$$E = \{\vec{e}_j = (\vec{v}_{j_1}, \vec{v}_{j_2})\}$$

and a set of triangles:

$$T = \{t_k = (\vec{v}_{k_1}, \vec{v}_{k_2}, \vec{v}_{k_3})\}$$

usually with the restriction that each edge is entirely shared by two adjacent triangles. The mesh's edges that are owned by only one triangle are called mesh borders. A mesh is closed if there are no borders.

From a theoretical point of view, triangle meshes do not have any curvature at all, since all faces are flat. Therefore, the theoretical definition of Gaussian curva-

ture based on the first fundamental form cannot be translated to discrete surfaces. On a discrete surface represented by a triangle mesh, curvature certainly cannot be assigned to the flat faces, neither along their edges, because we can always develop the triangles on either side of an edge flattening them. This suggests that discrete Gaussian curvature, like other curvatures, had to be assigned at vertices. With this assumption, we can define geometric properties of the surface at each vertex as spatial averages around this vertex, limiting the average within the immediately neighboring triangles, often referred as the 1-ring or star neighborhood.

Let  $\bar{v} \in V$  a vertex of a triangle mesh  $M$  and  $\bar{v}_1, \bar{v}_2, \dots, \bar{v}_n$  be the ordered neighboring vertices of  $\bar{v}$ , that compose the 1-ring associated with it. We define the edge  $\bar{e}_i = (\bar{v}, \bar{v}_i)$  and  $\alpha_i = \angle(\bar{e}_i, \bar{e}_{i+1})$  the angle between two successive edges, meeting at the vertex  $\bar{v}$ . The triangle between  $\bar{e}_i$  and  $\bar{e}_{i+1}$  is indicated as  $t_i = (\bar{v}, \bar{v}_i, \bar{v}_{i+1})$ , the unit normal corresponding to its face is:

$$\bar{n}_i = \frac{\bar{e}_i \times \bar{e}_{i+1}}{\|\bar{e}_i \times \bar{e}_{i+1}\|}$$

with  $\delta_i = \angle(\bar{n}_{i-1}, \bar{n}_i)$  the dihedral angle, namely the angle between the two normal vectors of adjacent triangles (Figure 24).

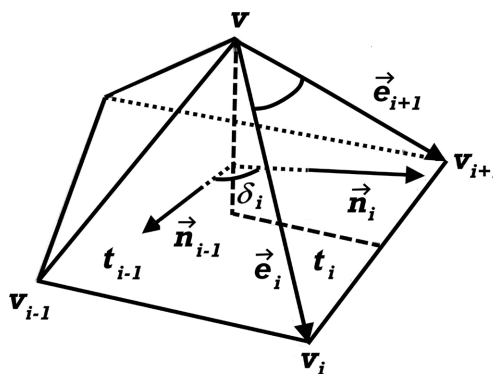


Figure 24. Triangles associated with vertex  $\bar{v}$ .

In order to compute spatial averages of geometry features around this vertex, we need to associate a surface  $S_{\bar{v}}$  to the vertex  $\bar{v}$  [9] [10]. There are many different ways of defining this area, which generally will result in different curvature values. We are interested in those methods for which summing up the areas around all vertices, we get the area of the whole triangle mesh  $M$ , that is  $Area(M) = \sum_{\bar{v} \in M} S_{\bar{v}}$ . We restrict our attention to the most common methods, represented by the barycentric and the Voronoi areas.

As an example, let us consider the barycentric area associated to a vertex, in the simple case of an equilateral triangle (Figure 25). A barycentric cell can be constructed by connecting the vertex in consideration with the midpoints of the two edges meeting at it and the intersection point of the triangle's medians. Note that the area  $S_{\bar{v}}^B$  is one third of the area of the triangles adjacent to  $\bar{v}$  and summing up the areas associated to the three vertices, we get the triangle total area. Alter-

natively, we may consider the Voronoi area  $S_v^V$ , which is the area of the subset of the mesh that contains all points that are closer to the vertex  $\bar{v}$  than to any other vertex.



Figure 25. Barycentric and Voronoi cells.

By definition, the discrete Gaussian curvature at a vertex  $\bar{v}$  is the area bounded by a spherical polygon on the unit sphere, whose vertices are the unit normals of the faces formed by adjacent triangles and normalized to the area of  $S_v$ , as shown in the example of Figure 26.

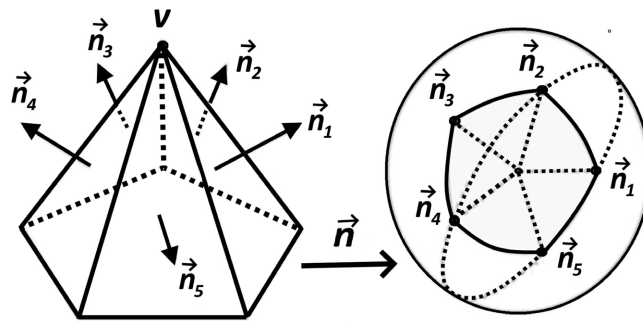


Figure 26. Triangles adjacent to vertex  $\bar{v}$  and spherical polygon.

The area of a spherical polygon, on a sphere of radius  $R$ , is derived by the *Girard's Theorem*, which relates the area of a spherical triangle to its interior angles, as following:

$$A = R^2 (\alpha + \beta + \gamma - \pi)$$

where  $\alpha, \beta, \gamma$  are interior angles, for proof see [11]. The quantity  $(\alpha + \beta + \gamma - \pi)$  is called the *spherical excess*, representing how much the angle sum exceeds that of a flat triangle [12]. Given that any polygon with  $n$  sides for  $n \geq 4$  can be divided into  $n - 2$  triangles, we can extend Girad's Theorem to a more general case of a spherical polygon with  $n$  vertices,  $n$  sides and angles  $\beta_1, \beta_2, \dots, \beta_n$  at its vertices, in this case the area is equal to:

$$A = R^2 (\beta_1 + \beta_2 + \dots + \beta_n - (n - 2)\pi)$$

Therefore, considering a unit sphere, we get:

$$A = \beta_1 + \beta_2 + \dots + \beta_n - (n - 2)\pi \tag{25}$$

As an illustrative example, let us consider a spherical polygon with  $n = 5$  (Figure 27).

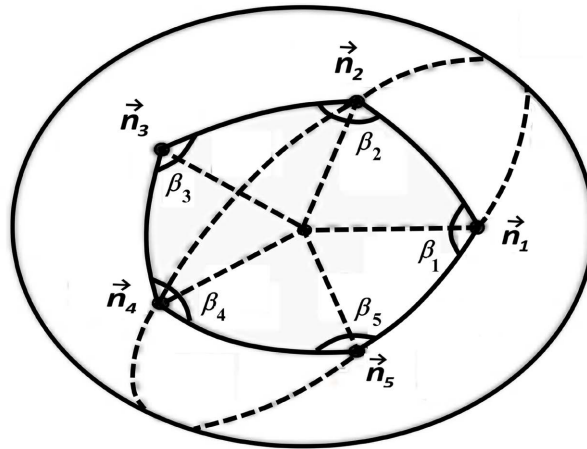


Figure 27. Spherical polygon and interior angles.

Let us focus our attention on angle  $\beta_5$ , formed between the tangent vectors  $\vec{t}_{51}$  and  $\vec{t}_{54}$ , represented respectively on the unit sphere in Figure 28(a) and on the original polyhedron in Figure 28(b). Just fixing ideas, let us analyze the tangent vector  $\vec{t}_{51}$ , it belongs to the plane containing the great circle defined by  $\vec{n}_5$  and  $\vec{n}_1$ . Moreover, the edge  $\vec{e}_5$ , shared by two adjacent triangles is perpendicular to both  $\vec{n}_5$  and  $\vec{n}_1$ , because it belongs to both adjacent faces. Given that the edge  $\vec{e}_5$  is normal to the plane defined by  $\vec{n}_5$  and  $\vec{n}_1$ , in turn the vector  $\vec{t}_{51}$  is perpendicular to the edge  $\vec{e}_5$ . Similarly, it can be proved that  $\vec{t}_{54}$  is normal to  $\vec{e}_4$ . Consequently, the sum of angles of quadrilateral ABCD (Figure 28(c)), satisfies the following equation [13]:

$$\beta_5 + \frac{\pi}{2} + \alpha_5 + \frac{\pi}{2} = 2\pi$$

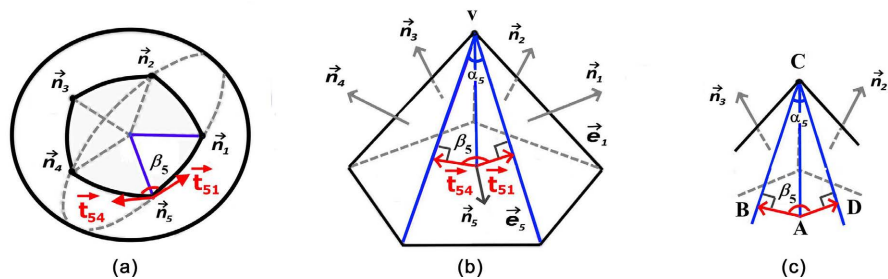


Figure 28. (a) Spherical polygon; (b) Original polyhedron; (c) Quadrilateral.

then, we have:

$$\beta_5 = \pi - \alpha_5 \tag{26}$$

Using Equation (26), we can replace  $\beta_n$  with  $\alpha_n$  in Equation (25) to compute the area of a general spherical polygon:

$$A = n\pi - \alpha_1 - \alpha_2 - \dots - \alpha_n - (n-2)\pi = 2\pi - \alpha_1 - \alpha_2 - \dots - \alpha_n \tag{27}$$

In conclusion, the discrete Gaussian curvature can be directly expressed through the input data deriving by the triangle mesh:

$$\bar{K} = \iint_S K d\sigma = 2\pi - \sum_{i=1}^n \alpha_i$$

To evaluate the curvature  $K$  at the vertex  $\bar{v}$  from its integral value, we assume the curvature to be uniformly distributed around the vertex, so keeping in mind the Gauss definition of Equation (15), we normalize by the area of the pre-defined surface  $S_{\bar{v}}$ :

$$K = \frac{\bar{K}}{Area(S_{\bar{v}})} \tag{28}$$

We have to point out that the discrete Gaussian curvature  $\bar{K}$  is dimensionless while the Gaussian curvature of a smooth surface  $S$  at a given point  $p$  is homogeneous to the inverse of a surface area, by normalizing with  $Area(S_{\bar{v}})$  we achieve the correct dimensionality.

By definition, the *angle defect*  $d(\bar{v})$  of a vertex  $\bar{v}$  is the amount by which the angles incident to  $\bar{v}$  fail to add up to  $2\pi$ . If  $F_{\bar{v}}$  is the set of faces containing  $\bar{v}$  with cardinality  $\#(F_{\bar{v}})$  and  $\alpha_n$  is the interior angle at  $\bar{v}$  of the  $n$ -th face, we get:

$$d(\bar{v}) = 2\pi - \sum_{n=1}^{\#(F_{\bar{v}})} \alpha_n \tag{29}$$

Thereby, we conclude that the discrete Gaussian curvature at a vertex  $\bar{v}$  can be determined by the angle defect, this method is commonly indicated as the *angular defect scheme* or *Gauss-Bonnet scheme*.

Hereinafter, we focus on how precisely we can estimate the curvature of a smooth surface at a point  $p$  using the angular defect of the triangles surrounding  $\bar{v}$ .

A preliminary consideration is that no proper results can be obtained by calculating the point-wise curvature at a resolution lower than that of the triangulation [14]. Indeed, we should keep in mind the distinction between point curvature, the measure of curvature as was named by Gauss, and the integral curvature over a region, upon which the angle defect method is based.

The discrete Gaussian curvature estimated by Equation (28) approximates the Gaussian curvature, so it is rather appropriate to consider the sensitivity of the angular defect scheme to many different factors, such as noise, mesh resolution, mesh regularity etc.

Meyer M. *et al.* [15] tested that, if the triangle mesh does not degenerate, the approximation quality gets better as the mesh is refined. However, the rate of sampling does not compromise significantly the level of accuracy, of course in the hypothesis of meshes dense enough so as to look like smooth surfaces. The efficiency of discrete operators is not strongly affected by the irregular sampling of surface points. The error analysis carried out by [16] shows that vertices of valence four or six are the only ones where the values computed with the angular defect

converge to analytical results.

We sincerely hope that this contribution, having a mere informative and didactic purpose, may be of help to all those who are involved in research activities on surface geometry, given its great relevance in a wide range of mathematical applications.

## Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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## Appendices

### Appendix 1: Differential of a Vector Function

Given a function  $f : U \subset \mathbb{R}^n \rightarrow \mathbb{R}^m$ , the Jacobian matrix of the function is the matrix of order  $m \times n$  whose elements are the first partial derivatives. The Jacobian is not a simple matrix representation of the partial derivatives, using it we can evaluate the differential of the function  $f$ . A function  $f$  is said to be differentiable at a point  $\bar{x}$  of the domain if there exists a linear application such that:

$$df = f(\bar{x} + \Delta\bar{x}) - f(\bar{x}) = L(\bar{x})\Delta\bar{x} + r(\Delta\bar{x}) = Jf(\bar{x}) \cdot \Delta\bar{x} + r(\Delta\bar{x})$$

where  $df$  is the **differential** of the function  $f$ ,  $\bar{x} = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$  is a vector variable,  $Jf$  is the Jacobian matrix of  $f$  and  $r(\Delta\bar{x}) \xrightarrow{\Delta\bar{x} \rightarrow 0} 0$ , more explicitly we have:

$$Jf = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \dots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \dots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}_{m \times n}$$

$$\Rightarrow df = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \dots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \dots & \frac{\partial f_m}{\partial x_n} \end{bmatrix} \cdot \begin{bmatrix} \Delta x_1 \\ \vdots \\ \Delta x_n \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} \Delta x_1 + \dots + \frac{\partial f_1}{\partial x_n} \Delta x_n \\ \vdots \\ \frac{\partial f_m}{\partial x_1} \Delta x_1 + \dots + \frac{\partial f_m}{\partial x_n} \Delta x_n \end{bmatrix}_{m \times 1}$$

The importance of the Jacobian matrix is due to the fact that we may obtain the best linear approximation of the function near a given point:

$$f(\bar{x} + \Delta\bar{x}) \cong f(\bar{x}) + L(\bar{x})\Delta\bar{x} \cong f(\bar{x}) + Jf(\bar{x})\Delta\bar{x}$$

In this sense, the Jacobian allows us to generalize the concept of derivative by extending this notion to vector functions of a vector variable. Depending on  $n$  and  $m$  dimension, we have some special cases with a specific geometric interpretations.

**Case  $m = 1$ :** given  $f : U \subset \mathbb{R}^n \rightarrow \mathbb{R}$  that is a scalar function of a vector variable  $\bar{x}$ , the Jacobian matrix is reduced to a  $n$ -dimensional vector, call the gradient of  $f$  at  $\bar{x}$ , represented as  $\nabla f$ , so we have  $L(\bar{x}) = \nabla f$ .

$$Jf = \begin{bmatrix} \frac{\partial f}{\partial x_1} & \dots & \frac{\partial f}{\partial x_n} \end{bmatrix}_{1 \times n}$$

$$\Rightarrow df = \begin{bmatrix} \frac{\partial f}{\partial x_1} & \dots & \frac{\partial f}{\partial x_n} \end{bmatrix} \begin{bmatrix} \Delta x_1 \\ \vdots \\ \Delta x_n \end{bmatrix} = \frac{\partial f}{\partial x_1} \Delta x_1 + \dots + \frac{\partial f}{\partial x_n} \Delta x_n$$

**Case  $n = 1$ :** given  $f : \mathbb{R} \rightarrow \mathbb{R}^m$ , the function is a parametric representation of a curve in the space  $\mathbb{R}^m$ , the Jacobian matrix is a  $m$ -dimensional vector that defines the direction of the tangent line to the curve at the point  $x$ :

$$Jf = \begin{bmatrix} \frac{df_1}{dx} \\ \dots \\ \frac{df_m}{dx} \end{bmatrix}_{m \times 1} \Rightarrow df = \begin{bmatrix} \frac{df_1}{dx} \\ \dots \\ \frac{df_m}{dx} \end{bmatrix} \Delta x$$

**Case  $n = m = 1$ :** given  $f : R \rightarrow R$ , the function is a real function of a real variable, the Jacobian matrix is reduced to a scalar quantity, the first derivative of  $f$ :

$$Jf = \frac{df}{dx} \Rightarrow df = \frac{df}{dx} \Delta x$$

**Case  $n = m$ :** given  $f : U \subset R^n \rightarrow R^n$ , the Jacobian matrix of the function is a square matrix, in this case, its determinant, known as the Jacobian, can be calculated:

$$Jf = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \dots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \dots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}_{n \times n} \Rightarrow df = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \dots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \dots & \frac{\partial f_n}{\partial x_n} \end{bmatrix} \begin{bmatrix} \Delta x_1 \\ \vdots \\ \Delta x_n \end{bmatrix}$$

$$df = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} \Delta x_1 + \dots + \frac{\partial f_1}{\partial x_n} \Delta x_n \\ \vdots \\ \frac{\partial f_n}{\partial x_1} \Delta x_1 + \dots + \frac{\partial f_n}{\partial x_n} \Delta x_n \end{bmatrix}$$