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Isothermal Limit of Entropy Solutions for the Euler Equations with Source Term

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Abstract

We investigate the isothermal limit of entropy solutions for isentropic Euler equations with a special source term. It is shown that the L^{∞} entropy solutions of the isentropic Euler equations converge strongly to the corresponding entropy solutions of the isothermal Euler equations as the adiabatic exponent $\gamma \to 1$. The proof is based on the homogeneous case which is achieved by combining the entropy analysis and refined kinetic formulation with compensated compactness framework. We apply a maximum principle to obtain the uniform estimates with respect to γ for both density and momentum.

Keywords

Entropy Solutions, Isothermal Limit, Isentropic Euler Equations, Isothermal Euler Equations, Strong Convergence

1. Introduction

In this paper, we consider the one-dimensional compressible Euler equations for isentropic gas dynamics with a special source term as follows:

$$\begin{cases} \partial_t \rho + \partial_x m = 0, \\ \partial_t m + \partial_x \left(\dfrac{m^2}{\rho} + p(\rho) \right) = F(t,x)\rho + H(t,x)m, \end{cases} \tag{1.1}$$

where $\rho, m = \rho u$ represent the density and momentum respectively, u is the velocity and $p(\rho) = \dfrac{\rho^{\gamma}}{\gamma}$ is the pressure of the gas with $\gamma \ge 1$ denoting the adiabatic exponent. The source terms $F(t,x)$ and $H(t,x)$ are C^1 functions on $\mathbb{R}_+^2 = \mathbb{R}^+ \times \mathbb{R}$, satisfying $|F(t,x)|, |H(t,x)| \le M_0$ for some positive constant M_0 for all (t,x).

System (1.1) is equipped with the following initial data:

$$(\rho, m)_{t=0} = (\rho_0(x), m_0(x)) \in L^\infty. \tag{1.2}$$

In [1], Chen, Huang and Wang first obtained that the weak entropy solution of isentropic Euler equations converges strongly to the corresponding isothermal Euler equations for homogeneous case. And there have been many results for the existence of entropy solutions of Euler equations with source terms. Cao, Huang, Li and Yu in [2] acquired the global existence of entropy solutions independent of time to isentropic compressible Euler equations with the source term same as in this context. Cao, Huang and Yuan [3] obtained the uniform bound (independent of time) of approximate solutions both for isentropic ($1<\gamma<3$) and isothermal Euler equations in a general nozzle. Chen and Luo in [4] obtained a convergence theorem of the fractional step Lax-Friedrichs scheme and Godunov scheme for a general source term $(U(\rho,u,x,t), V(\rho,u,x,t))$ and $1<\gamma\le\frac{5}{3}$ by using compensated compactness framework. When $H(t,x)=0$, Naoki Tsuge [5] proved the existence of a global solution without any boundary condition on the external force by employing an invariant region through a modified difference scheme. When $F(t,x)=E(t,x)$, where $E(t,x)$ stands for electric field, Li, Cheng and Yu in [6] got the global existence and large time behavior of entropy solutions. Fang and Yu in [7] proved that the L^∞ weak solutions derived by Lax-Friedrichs scheme are uniformly bounded in time when $F(t,x)=E(t,x)$ and $H(t,x)=-1$.

For isentropic Euler equations with source terms, existing results have primarily focused on establishing the existence of solutions, while investigations concerning the isothermal limit remain limited. This article is devoted to extending the findings from [1] to a class of Euler equations with special source terms. We show that the L^∞ entropy solutions of the Euler equations with such a source term converge strongly to the corresponding entropy solutions of the isothermal Euler equations as the adiabatic exponent $\gamma\to1$. This is achieved by combining entropy analysis, a refined kinetic formulation and the compensated compactness argument to obtain the necessary uniform estimates for the limit. For the case considered here, the point is to obtain a uniform bound (independent of γ) for the weak entropy solutions when $\gamma>1$. Inspired by the approach in [2], we set a new control function, using the maximum principle, and finally get the uniform bound independent of γ for the solution.

2. Preliminary Knowledge and Main Results

In this section, we provide some known results about hyperbolic conservation laws and present two main theorems of this article.

Firstly, system (1.1)-(1.2) can be written as a hyperbolic system of balance laws of the form:

$$\begin{cases} \partial_t U + \partial_x F^{(\theta)}(U) = G(U), \\ U_0(x) = (\rho_0(x), m_0(x)), \end{cases} \tag{2.1}$$

where $U=(\rho,m)^\top$, $F^{(\theta)}(U)=\left(m,\dfrac{m^2}{\rho}+p^{(\theta)}(\rho)\right)^\top$, $G(U)=(0,F(t,x)\rho+H(t,x)m)^\top$ and $\theta=\dfrac{\gamma-1}{2}$, $p^{(\theta)}(\rho)=\dfrac{\rho^{2\theta+1}}{2\theta+1}$. System (2.1) represents the isentropic gas when $\theta>0$ and the isothermal case when $\theta=0$.

For system (2.1), a general entropy pair (η,q) satisfies the following hyperbolic system:

$$\nabla q(\rho,m)=\nabla\eta(\rho,m)\nabla F^{(\theta)}(\rho,m).$$

A weak entropy is an entropy $\eta(\rho,m)$ that vanishes at $\rho=0$ (vacuum), the corresponding pair (η,q) is then called a weak entropy pair.

Next, we introduce the concept of weak entropy solutions for system (1.1)-(1.2).

Definition 2.1. *Let* $\theta\ge 0$. *If for any test function* $\varphi\in C_0^\infty(\mathbb{R}_+^2)$, *function* $U(t,x)=(\rho,m)(t,x)\in L^\infty(\mathbb{R}_+^2)$ *satisfies*

$$\int_{\mathbb{R}_+^2}\left(U\partial_t\varphi+F^{(\theta)}(U)\partial_x\varphi+G(U)\varphi\right)\mathrm{d}x\mathrm{d}t+\int_{\mathbb{R}}U_0(x)\varphi(0,x)\mathrm{d}x=0, \tag{2.2}$$

and if we further have for $\varphi\ge 0$, *for any weak entropy-entropy flux* η *and* q:

$$\begin{aligned}&\int_{\mathbb{R}_+^2}\left(\eta(U)\partial_t\varphi+q(U)\partial_x\varphi+\nabla\eta(U)G(U)\varphi\right)\mathrm{d}x\mathrm{d}t\\&+\int_{\mathbb{R}}\eta(U_0)\varphi(0,x)\mathrm{d}x\ge 0,\end{aligned} \tag{2.3}$$

then the function $U(t,x)=(\rho,m)(t,x)$ *is called an entropy solution of the Cauchy problem* (1.1)-(1.2).

We now discuss the isentropic and isothermal cases respectively.

2.1. Isentropic Case ($\theta>0$)

The Riemann invariants of system (2.1) are

$$w_j^{(\theta)}=\frac{m}{\rho}+(-1)^{j+1}\frac{\rho^\theta}{\theta}\ \text{ for } j=1,2.$$

Weak entropy-entropy flux pairs can be expressed as follows:

Lemma 2.1. *For* $\theta>0$, *the weak entropy pairs of* (*2.1*) *can be represented as the following form*:

$$\eta^{(\theta)}(\rho,m;\psi)=\int_{\mathbb{R}}\chi^{(\theta)}\left(\rho;s-\frac{m}{\rho}\right)\psi(s)\mathrm{d}s,$$

$$q^{(\theta)}(\rho,m;\psi)=\int_{\mathbb{R}}\left(\theta s+(1-\theta)\frac{m}{\rho}\right)\chi^{(\theta)}\left(\rho;s-\frac{m}{\rho}\right)\psi(s)\mathrm{d}s,$$

for any $\psi\in C^2(\mathbb{R})$, *where*

$$\chi^{(\theta)}(\rho;s-u)=a_\theta\left[\left(\frac{\rho^\theta}{\theta}\right)^2-(s-u)^2\right]_+^{\frac{1-\theta}{2\theta}}=a_\theta\left[(w_1-s)(s-w_2)\right]_+^{\frac{1-\theta}{2\theta}}, \tag{2.4}$$

is the weak entropy kernel of (*2.1*), $a_\theta = \theta^{\frac{1}{\theta}} \left(\int_{-1}^{1} \left[1-\tau^2\right]_+^{\frac{1-\theta}{2\theta}} \mathrm{d}\tau \right)^{-1}$ *and* $[x]_+ = \max(0, x)$ (*cf.* [8]).

Using the standard change of variable $s = u + \frac{\rho^\theta}{\theta}\tau$, we have

$$
\begin{aligned}
\eta^{(\theta)}(\rho, m; \psi) &= a_\theta \int_{\mathbb{R}} \left[\left(\frac{\rho^\theta}{\theta} \right)^2 - (u-s)^2 \right]_+^{\frac{1-\theta}{2\theta}} \psi(s)\,\mathrm{d}s \\
&= \frac{\rho \int_{\mathbb{R}} \psi\left(u + \frac{\rho^\theta}{\theta}\tau \right) \left[1-\tau^2\right]_+^{\frac{1-\theta}{2\theta}} \mathrm{d}\tau}{\int_{\mathbb{R}} \left[1-\tau^2\right]_+^{\frac{1-\theta}{2\theta}} \mathrm{d}\tau},
\end{aligned}
\tag{2.5}
$$

$$
\begin{aligned}
q^{(\theta)}(\rho, m; \psi) &= a_\theta \int_{\mathbb{R}} \left(\theta s + (1-\theta) u \right) \left[\left(\frac{\rho^\theta}{\theta} \right)^2 - (u-s)^2 \right]_+^{\frac{1-\theta}{2\theta}} \psi(s)\,\mathrm{d}s \\
&= \frac{\rho \int_{\mathbb{R}} \left(u + \rho^\theta \tau \right) \psi\left(u + \frac{\rho^\theta}{\theta}\tau \right) \left[1-\tau^2\right]_+^{\frac{1-\theta}{2\theta}} \mathrm{d}\tau}{\int_{\mathbb{R}} \left[1-\tau^2\right]_+^{\frac{1-\theta}{2\theta}} \mathrm{d}\tau}.
\end{aligned}
\tag{2.6}
$$

The weak entropy pairs $\left(\eta_\xi^{(\theta)}, q_\xi^{(\theta)} \right)$ of system (2.1) with $\theta > 0$ under consideration are

$$
\eta_\xi^{(\theta)}(\rho, m) = \frac{\rho \int_{\mathbb{R}} \mathrm{e}^{\frac{\xi}{1-\xi^2}\left(\frac{m}{\rho} + \frac{\rho^\theta}{\theta}\tau \right)} \left[1-\tau^2\right]_+^{\frac{1-\theta}{2\theta}} \mathrm{d}\tau}{\int_{\mathbb{R}} \mathrm{e}^{\frac{\xi\tau}{\theta\left(1-\xi^2\right)}} \left[1-\tau^2\right]_+^{\frac{1-\theta}{2\theta}} \mathrm{d}\tau} \geq 0,
\tag{2.7}
$$

$$
q_\xi^{(\theta)}(\rho, m) = \frac{\rho \int_{\mathbb{R}} \left(\frac{m}{\rho} + \rho^\theta \tau \right) \mathrm{e}^{\frac{\xi}{1-\xi^2}\left(\frac{m}{\rho} + \frac{\rho^\theta}{\theta}\tau \right)} \left[1-\tau^2\right]_+^{\frac{1-\theta}{2\theta}} \mathrm{d}\tau}{\int_{\mathbb{R}} \mathrm{e}^{\frac{\xi\tau}{\theta\left(1-\xi^2\right)}} \left[1-\tau^2\right]_+^{\frac{1-\theta}{2\theta}} \mathrm{d}\tau}.
\tag{2.8}
$$

These entropy pairs are obtained by choosing $\psi(s)$ in (2.5)-(2.6) as

$$
\psi_\xi^{(\theta)}(s) = \frac{\int_{\mathbb{R}} \left[1-\tau^2\right]_+^{\frac{1-\theta}{2\theta}} \mathrm{d}\tau}{\int_{\mathbb{R}} \mathrm{e}^{\frac{\xi\tau}{\theta\left(1-\xi^2\right)}} \left[1-\tau^2\right]_+^{\frac{1-\theta}{2\theta}} \mathrm{d}\tau} \mathrm{e}^{\frac{\xi}{1-\xi^2}s}.
$$

This choice is the same as in [1].

Lemma 2.2. (*Existence theorem for* $\theta > 0$). *For any* $\theta > 0$, *there exists a bounded entropy solution* (ρ, m) *of the Cauchy problem* (1.1)-(1.2) *on* $\mathbb{R}_+^2$ *satisfying*

$$
\frac{|m(t,x)|}{\rho(t,x)} + \frac{\rho(t,x)^\theta - 1}{\theta} \leq C^{(\theta)},
\tag{2.9}
$$

where $C^{(\theta)}$ *is a constant depending on* θ (*cf.* [2] [3] [8]-[12]).

2.2. Isothermal Case ($\theta = 0$)

The Riemann invariants are

$$w_j^{(0)} = \rho \mathrm{e}^{(-1)^{j+1}\frac{m}{\rho}} \quad \text{for } j = 1, 2.$$

The family of weak entropy pairs can be shown as:

Lemma 2.3. *When* $\theta = 0$, *the typical entropy pairs* (η_ξ, q_ξ) *are*:

$$\eta_\xi(\rho, m) = \rho^{\frac{1}{1-\xi^2}} \mathrm{e}^{\frac{\xi}{1-\xi^2}\frac{m}{\rho}}, \quad q_\xi(\rho, m) = \left(\frac{m}{\rho} + \xi\right)\eta_\xi \quad \text{for } \xi \in (-1, 1), \tag{2.10}$$

Lemma 2.4. (*Compactness Framework for* $\theta = 0$). *Let* $(\rho^\varepsilon, m^\varepsilon)$ *be a sequence of approximate solutions of* (*2.1*) *with* $\theta = 0$ *satisfying*

$$0 \le \rho^\varepsilon(t, x) \le C, \left|m^\varepsilon(t, x)\right| \le \rho^\varepsilon(t, x)\left(\left|\ln \rho^\varepsilon(t, x)\right| + C\right) \quad \text{a.e. } (t, x) \in \mathbb{R}_+^2, \tag{2.11}$$

where $C > 0$ *is a constant independent of* θ. *Assume there exists a small constant* $\delta > 0$ *such that, for any* $\xi \in (-\delta, \delta)$,

$$\partial_t \eta_\xi(\rho^\varepsilon, m^\varepsilon) + \partial_x q_\xi(\rho^\varepsilon, m^\varepsilon) \text{ is compact in } H_{loc}^{-1}, \tag{2.12}$$

where (η_ξ, q_ξ) *are the weak entropy pairs defined in* (*2.10*). *Then there exists both a sequence* (*still denoted*) $(\rho^\varepsilon, m^\varepsilon)$ *and a vector function* $(\rho, m)(t, x)$ *such that*

$$(\rho^\varepsilon, m^\varepsilon)(t, x) \to (\rho, m)(t, x) \quad \text{in } L_{loc}^p(\mathbb{R}_+^2) \quad \text{for all } p \in [1, \infty) \quad \text{as } \varepsilon \to 0.$$

See Theorem 2.1 and Theorem 2.2 in [13] for details.

2.3. Main Theorem

Since we focus on the limit $\theta \to 0$ in the isentropic case, we take $\theta \in \left(0, \frac{1}{2}\right]$ throughout this paper. Two main results of this paper are as follows:

Theorem 2.1. (*uniform bound with respect to* θ). *Let* $0 < \theta \le \frac{1}{2}$. *Assume that there exists a positive constant C such that the initial data* $\left(\rho_0^{(\theta)}(t, x), m_0^{(\theta)}(t, x)\right)$ *satisfies*

$$\frac{\left|m_0^{(\theta)}(t, x)\right|}{\rho_0^{(\theta)}(t, x)} + \frac{\left(\rho_0^{(\theta)}(t, x)\right)^\theta - 1}{\theta} \le C, \tag{2.13}$$

then there exists a constant C' *depending only on the initial data* (*independent of* θ), *such that the weak entropy solutions* $\left(\rho^{(\theta)}(t, x), m^{(\theta)}(t, x)\right)$ *satisfies*

$$0 \le \rho^{(\theta)}(t, x) \le \mathrm{e}^{C'}, \left|m^{(\theta)}(t, x)\right| \le \rho^{(\theta)}(t, x)\left(\left|\ln \rho^{(\theta)}(t, x)\right| + C'\right). \tag{2.14}$$

Theorem 2.2 (*The strong convergence*). *For any* $\theta > 0$, $\left(\rho^{(\theta)}, m^{(\theta)}\right)$ *is a weak entropy solution of* (*2.1*) *satisfying condition* (*2.9*), *and initial data* $\left(\rho_0^{(\theta)}, m_0^{(\theta)}\right)$ *satisfies* (*2.13*), *then there exists a sequence* (*still denoted*) $\left(\rho^{(\theta)}, m^{(\theta)}\right)$ *and a vector function* (ρ, m) *such that*

$$\left(\rho^{(\theta)}, m^{(\theta)}\right)(t,x) \to (\rho, m)(t,x) \text{ in } L^p_{loc}\left(\mathbb{R}^2_+\right) \text{ for all } p \in [1,\infty) \text{ as } \theta \to 0.$$

Moreover, (ρ, m) *is an entropy solution of (2.1) when* $\theta = 0$*, satisfying*

$$0 \le \rho(t,x) \le C, \left|m(t,x)\right| \le \rho(t,x)\left(\left|\ln \rho(t,x)\right| + C\right) \text{ a.e. } (t,x) \in \mathbb{R}^2_+,$$

where $C > 0$ *is a constant independent of* θ*. Furthermore, for any entropy pair* (η_ξ, q_ξ)*,* $\xi \in \left(-\sqrt{2}+1, \sqrt{2}-1\right)$*,* $U = (\rho, m)$ *satisfies the entropy inequality*

$$\partial_t \eta_\xi(U) + \partial_x q_\xi(U) + \nabla \eta_\xi(U) G(U) \le 0,$$

in the sense of distributions.

3. The Uniform Estimate Independent of θ

In this section, we focus on proving Theorem 2.1.

Before the proof, we first recall some basic knowledge of system (2.1) when $\theta > 0$. The eigenvalues are

$$\lambda_1 = \frac{m}{\rho} - \rho^\theta, \quad \lambda_2 = \frac{m}{\rho} + \rho^\theta, \tag{3.1}$$

and the corresponding right eigenvectors are as follows:

$$r_1 = \begin{bmatrix} 1 \\ \lambda_1 \end{bmatrix}, \quad r_2 = \begin{bmatrix} 1 \\ \lambda_2 \end{bmatrix}. \tag{3.2}$$

The Riemann invariants (w, z) are given by

$$w = \frac{m}{\rho} + \frac{\rho^\theta - 1}{\theta}, \quad z = \frac{m}{\rho} - \frac{\rho^\theta - 1}{\theta}. \tag{3.3}$$

The Riemann invariants chosen here are different from [2] because in this paper we focus on the solution with $\theta \to 0$, one of the advantages of this choice is that it can ensure the boundary of ρ is independent of θ.

Next we introduce a maximum principle which will be used in the proof and the corresponding proof process is outlined in [2].

Lemma 3.1 (*Maximum principle*) *Let* $(p, q)(x,t)$, $(x,t) \in \mathbb{R} \times [0, T]$ *be any bounded classical solution of the following quasilinear parabolic system*

$$\begin{cases} p_t + \mu_1 p_x = \varepsilon p_{xx} + a_{11} p + a_{12} q + R_1, \\ q_t + \mu_2 q_x = \varepsilon q_{xx} + a_{21} p + a_{22} q + R_2, \end{cases}$$

with initial data $p(x,0) \le 0, q(x,0) \ge 0$*, where the coefficients* μ_i *and* a_{ij} *are bounded with respect to* (x,t) *and may depend on p and q. The source terms* R_i *may also depend on* p, q *and* x, t*. Assume that* $a_{12}, a_{21} \le 0$*,* $R_1 \le 0$*,* $R_2 \ge 0$*. Then for any* (x,t)*, we have* $p(x,t) \le 0$*,* $q(x,t) \ge 0$*.*

Remark. Lemma 3.1 holds true for $\varepsilon = 0$. When $\varepsilon > 0$, we can first construct an approximate solution and then prove this solution is bounded when $\varepsilon \to 0$. Detailed steps can be referred to [2].

Proof of Theorem 2.1

By the formulas of Riemann invariants (3.3), we can decouple system (1.1) as

$$\begin{cases} w_t + \lambda_2 w_x = F + \dfrac{H}{2}(w+z), \\ z_t + \lambda_1 z_x = F + \dfrac{H}{2}(w+z). \end{cases} \tag{3.4}$$

Set a control function ϕ as follows:

$$\phi = M + \mathrm{e}^{|H|_{L^\infty} t}\left(|F|_{L^\infty} + M|H|_{L^\infty}\right)t,$$

where $M \ge 0$ is a constant satisfying $M \ge \max\left\{|w(x,0)|,|z(x,0)|\right\}$. A direct calculation tells us that

$$\begin{aligned} \phi_x = 0,\ \phi_t &= |H|_{L^\infty}\mathrm{e}^{|H|_{L^\infty} t}\left(|F|_{L^\infty} + M|H|_{L^\infty}\right)t + \mathrm{e}^{|H|_{L^\infty} t}\left(|F|_{L^\infty} + M|H|_{L^\infty}\right) \\ &= \mathrm{e}^{|H|_{L^\infty} t}\left(|F|_{L^\infty} + M|H|_{L^\infty}\right)\left[|H|_{L^\infty} t + 1\right]. \end{aligned}$$

Define the new Riemann invariants $(\overline{w}, \overline{z})$ as

$$\overline{w} = w - \phi,\ \overline{z} = z + \phi, \tag{3.5}$$

so inserting (3.5) to (3.4), we can get a decoupled system of $(\overline{w}, \overline{z})$

$$\begin{cases} \overline{w}_t + \lambda_2 \overline{w}_x = -\mathrm{e}^{|H|_{L^\infty} t}\left(|F|_{L^\infty} + M|H|_{L^\infty}\right)\left[|H|_{L^\infty} t + 1\right] + F + \dfrac{H}{2}(\overline{w} + \overline{z}), \\ \overline{z}_t + \lambda_1 \overline{z}_x = \mathrm{e}^{|H|_{L^\infty} t}\left(|F|_{L^\infty} + M|H|_{L^\infty}\right)\left[|H|_{L^\infty} t + 1\right] + F + \dfrac{H}{2}(\overline{w} + \overline{z}). \end{cases} \tag{3.6}$$

When $\overline{z} \le 2\phi$, we have

$$\begin{aligned} &\sup\left\{-\mathrm{e}^{|H|_{L^\infty} t}\left(|F|_{L^\infty} + M|H|_{L^\infty}\right)\left[|H|_{L^\infty} t + 1\right] + F + \frac{H}{2}\overline{z}\right\} \\ &= -\mathrm{e}^{|H|_{L^\infty} t}\left(|F|_{L^\infty} + M|H|_{L^\infty}\right)\left[|H|_{L^\infty} t + 1\right] + |F|_{L^\infty} \\ &\quad + |H|_{L^\infty}\left[M + \mathrm{e}^{|H|_{L^\infty} t}\left(|F|_{L^\infty} + M|H|_{L^\infty}\right)t\right] \\ &= -\mathrm{e}^{|H|_{L^\infty} t}\left(|F|_{L^\infty} + M|H|_{L^\infty}\right) + |F|_{L^\infty} + M|H|_{L^\infty} \\ &= \left(|F|_{L^\infty} + M|H|_{L^\infty}\right)\left(1 - \mathrm{e}^{|H|_{L^\infty} t}\right) \\ &\le 0, \end{aligned} \tag{3.7}$$

when $\overline{w} \ge -2\phi$,

$$\begin{aligned} &\inf\left\{\mathrm{e}^{|H|_{L^\infty} t}\left(|F|_{L^\infty} + M|H|_{L^\infty}\right)\left[|H|_{L^\infty} t + 1\right] + F + \frac{H}{2}\overline{w}\right\} \\ &= \mathrm{e}^{|H|_{L^\infty} t}\left(|F|_{L^\infty} + M|H|_{L^\infty}\right)\left[|H|_{L^\infty} t + 1\right] - |F|_{\infty} \\ &\quad - |H|_{\infty}\left[M + \mathrm{e}^{|H|_{L^\infty} t}\left(|F|_{L^\infty} + M|H|_{L^\infty}\right)t\right] \\ &= \mathrm{e}^{|H|_{L^\infty} t}\left(|F|_{L^\infty} + M|H|_{L^\infty}\right) - \left(|F|_{L^\infty} + M|H|_{L^\infty}\right) \\ &= \left(|F|_{L^\infty} + M|H|_{L^\infty}\right)\left(\mathrm{e}^{|H|_{L^\infty} t} - 1\right) \\ &\ge 0. \end{aligned} \tag{3.8}$$

Having obtained an upper bound for w and a lower bound for z, we next need to establish w is bounded below and z is bounded above. So we define another new invariants as follows:

$$\tilde{w} = w + \phi,\ \ \tilde{z} = z - \phi. \tag{3.9}$$

Again the steps above, we can get another decoupled system of $(\tilde{w}, \tilde{z})$

$$\begin{cases} \tilde{w}_t + \lambda_2 \tilde{w}_x = \mathrm{e}^{|H|_{L^\infty} t}\left(|F|_{L^\infty} + M|H|_{L^\infty}\right)\left[|H|_{L^\infty} t + 1\right] + F + \dfrac{H}{2}(\tilde{w} + \tilde{z}), \\ \tilde{z}_t + \lambda_1 \tilde{z}_x = -\mathrm{e}^{|H|_{L^\infty} t}\left(|F|_{L^\infty} + M|H|_{L^\infty}\right)\left[|H|_{L^\infty} t + 1\right] + F + \dfrac{H}{2}(\tilde{w} + \tilde{z}). \end{cases} \tag{3.10}$$

When $\overline{z} \ge -2\phi$, we have

$$\begin{aligned} &\inf\left\{\mathrm{e}^{|H|_{L^\infty} t}\left(|F|_{L^\infty} + M|H|_{L^\infty}\right)\left[|H|_{L^\infty} t + 1\right] + F + \frac{H}{2}\tilde{w}\right\} \\ &= \mathrm{e}^{|H|_{L^\infty} t}\left(|F|_{L^\infty} + M|H|_{L^\infty}\right)\left[|H|_{L^\infty} t + 1\right] - |F|_{L^\infty} \\ &\quad - |H|_{L^\infty}\left[M + \mathrm{e}^{|H|_{L^\infty} t}\left(|F|_{L^\infty} + M|H|_{L^\infty}\right)t\right] \\ &= \mathrm{e}^{|H|_{L^\infty} t}\left(|F|_{L^\infty} + M|H|_{L^\infty}\right) - \left(|F|_{L^\infty} + M|H|_{L^\infty}\right) \\ &= \left(|F|_{L^\infty} + M|H|_{L^\infty}\right)\left(\mathrm{e}^{|H|_{L^\infty} t} - 1\right) \\ &\ge 0, \end{aligned} \tag{3.11}$$

and when $\overline{w} \le 2\phi$,

$$\begin{aligned} &\sup\left\{-\mathrm{e}^{|H|_{L^\infty} t}\left(|F|_{L^\infty} + M|H|_{L^\infty}\right)\left[|H|_{L^\infty} t + 1\right] + F + \frac{H}{2}\tilde{z}\right\} \\ &= -\mathrm{e}^{|H|_{L^\infty} t}\left(|F|_{L^\infty} + M|H|_{L^\infty}\right)\left[|H|_{L^\infty} t + 1\right] + |F|_{L^\infty} \\ &\quad + |H|_{L^\infty}\left[M + \mathrm{e}^{|H|_{L^\infty} t}\left(|F|_{L^\infty} + M|H|_{L^\infty}\right)t\right] \\ &= -\mathrm{e}^{|H|_{L^\infty} t}\left(|F|_{L^\infty} + M|H|_{L^\infty}\right) + |F|_{L^\infty} + M|H|_{L^\infty} \\ &= \left(|F|_{L^\infty} + M|H|_{L^\infty}\right)\left(1 - \mathrm{e}^{|H|_{L^\infty} t}\right) \\ &\le 0. \end{aligned} \tag{3.12}$$

To ensure the condition of initial data, we need

$$M \ge \max\left\{|w(x,0)|, |z(x,0)|\right\}.$$

Applying the maximum principle Lemma 3.1, we have the estimate

$$|w(x,t)| \le C,\ \ |z(x,t)| \le C, \tag{3.13}$$

where C is a constant independent of θ.

Using Lemma 3.1 of reference [1], we extend these uniform bounds from the Riemann invariants to the solution $\left(\rho^{(\theta)}, m^{(\theta)}\right)$, hence (2.14) holds. □

4. Proof of Theorem 2.2

With the uniform bounds on the solution established, we now prove Theorem 2.2. We divide the proof into two steps.

Step 1. H_{loc}^{-1} compactness of the entropy pair. For any $\theta > 0$, let $\left(\rho^{(\theta)}, m^{(\theta)}\right)$ be the corresponding entropy solutions of (2.1) constructed in Lemma 2.2. It follows from (2.14) that the solution sequence $\left(\rho^{(\theta)}, m^{(\theta)}\right)$ is uniformly bounded with respect to θ, which satisfies the condition (2.11) in Lemma 2.4.

We now show that the solution sequence $\left(\rho^{(\theta)}, m^{(\theta)}\right)$ satisfies the condition (2.12) in Lemma 2.4 for any $\delta \in \left(0, \sqrt{2}-1\right)$. We will apply Murat lemma to achieve the goal.

Lemma 4.1. (*Murat Lemma*) *Let* $\Omega \in \mathbb{R}^n$ *be an open set, then*

$$\left(\text{compact set of } W_{loc}^{-1,q}(\Omega)\right) \cap \left(\text{bounded set of } W_{loc}^{-1,r}(\Omega)\right) \subset \left(\text{compact set of } H_{loc}^{-1}(\Omega)\right),$$

where $1 < q \le 2 < r$.

For any weak entropy pairs given in Lemma 2.1, multiplying (1.1) by $\nabla \eta_\xi^{(\theta)}$ with $\eta_\xi^{(\theta)}$ defined in (2.7), and a calculation tells us that

$$\eta_{\xi m}^{(\theta)}(\rho, m) = \frac{\xi}{1-\xi^2} \frac{1}{\rho} \eta_\xi^{(\theta)},$$

so we have

$$\eta_t + q_x = \eta_{\xi m}^{(\theta)} \left(F(x,t)\rho + H(x,t)m\right) \le C\left(\rho M' + m M'\right),$$

where $C, M' > 0$ are two constants. This implies that $\eta_t + q_x$ is bounded in $L_{loc}^{-1}(\Omega)$. Therefore

$$\eta_t + q_x \text{ is compact in } W_{loc}^{-1,\alpha}(\Omega) \text{ with some } 1 < \alpha < 2.$$

On the other hand, since ρ and m are uniformly bounded, we have

$$\eta_t + q_x \text{ is bounded in } W_{loc}^{-1,\infty}(\Omega).$$

We conclude that

$$\eta_t + q_x \text{ is compact in } H_{loc}^{-1}(\Omega), \tag{4.1}$$

for all weak entropy pairs with the help of the Murat Lemma.

By (4.1) and the compactness framework Lemma 2.4, we can prove that there exists a subsequence of $\left(\rho^{(\theta)}, m^{(\theta)}\right)$ (still denoted by $\left(\rho^{(\theta)}, m^{(\theta)}\right)$) such that

$$\left(\rho^{(\theta)}, m^{(\theta)}\right)(t,x) \to (\rho, m)(t,x) \text{ in } L_{loc}^p\left(\mathbb{R}_+^2\right) \text{ for all } p \in [1, \infty) \text{ as } \theta \to 0.$$

It's easy to see that (ρ, m) is a weak solution to the Cauchy problem (1.1)-(1.2) so we omit here.

Step 2. Entropy inequality. We next prove that $(\rho, m)(t,x)$ satisfies the entropy inequality: for any $\left(\eta_\xi, q_\xi\right)$, there is

$$\partial_t \eta_\xi(\rho, m) + \partial_x q_\xi(\rho, m) + \nabla \eta_\xi(\rho, m) \cdot G(\rho, m) \le 0, \tag{4.2}$$

in the sense of distributions.

As we know that $\left(\rho^{(\theta)}, m^{(\theta)}\right)$ is the entropy solution of system (2.1) when $\theta > 0$, so for any $\left(\rho_\xi^{(\theta)}, m_\xi^{(\theta)}\right)$ and $\left(\eta_\xi^{(\theta)}, q_\xi^{(\theta)}\right)$, there is

$$\partial_t \eta_\xi^{(\theta)}\left(\rho^{(\theta)}, m^{(\theta)}\right) + \partial_x q_\xi^{(\theta)}\left(\rho^{(\theta)}, m^{(\theta)}\right) + \nabla \eta_\xi^{(\theta)}\left(\rho^{(\theta)}, m^{(\theta)}\right) \cdot G\left(\rho^{(\theta)}, m^{(\theta)}\right) \le 0, \tag{4.3}$$

in the sense of distributions. Next we prove that as $\theta \to 0$, (4.3) converges to (4.2), that is

$$\begin{aligned} &\Big|\partial_t \eta_\xi^{(\theta)}\left(\rho^{(\theta)}, m^{(\theta)}\right) + \partial_x q_\xi^{(\theta)}\left(\rho^{(\theta)}, m^{(\theta)}\right) + \nabla \eta_\xi^{(\theta)}\left(\rho^{(\theta)}, m^{(\theta)}\right) \cdot G\left(\rho^{(\theta)}, m^{(\theta)}\right) \\ &- \left[\partial_t \eta_\xi(\rho, m) + \partial_x q_\xi(\rho, m) + \nabla \eta_\xi(\rho, m) \cdot G(\rho, m)\right]\Big| \to 0, \text{ as } \theta \to 0. \end{aligned} \tag{4.4}$$

Before this part of proof, we state a lemma which has been proved in [1].

Lemma 4.2 (*Relation Between* $\left(\eta^{(\theta)},q^{(\theta)}\right)$)

Given any $\theta>0$ *and any function* $\left(\rho^{(\theta)},m^{(\theta)}\right)$ *satisfying* (*2.14*), *then for any interval* $[a,b]\Subset\left(-\sqrt{2}+1,\sqrt{2}-1\right)$, *there exists* $C>0$ *independent of* θ *such that*

$$\left|\eta_\xi^{(\theta)}\left(\rho^{(\theta)},m^{(\theta)}\right)-\eta_\xi\left(\rho^{(\theta)},m^{(\theta)}\right)\right|+\left|q_\xi^{(\theta)}\left(\rho^{(\theta)},m^{(\theta)}\right)-q_\xi\left(\rho^{(\theta)},m^{(\theta)}\right)\right|\le C\sqrt{\theta},\quad(4.5)$$

for any $\xi\in[a,b]$.

First we have

$$\begin{aligned}&\left|\partial_t\eta_\xi^{(\theta)}\left(\rho^{(\theta)},m^{(\theta)}\right)+\partial_x q_\xi^{(\theta)}\left(\rho^{(\theta)},m^{(\theta)}\right)+\nabla\eta_\xi^{(\theta)}\left(\rho^{(\theta)},m^{(\theta)}\right)\cdot G\left(\rho^{(\theta)},m^{(\theta)}\right)\right.\\&\left.-\left[\partial_t\eta_\xi(\rho,m)+\partial_x q_\xi(\rho,m)+\nabla\eta_\xi(\rho,m)\cdot G(\rho,m)\right]\right|\\&\le\left|\partial_t\eta_\xi^{(\theta)}\left(\rho^{(\theta)},m^{(\theta)}\right)-\partial_t\eta_\xi(\rho,m)\right|+\left|\partial_x q_\xi^{(\theta)}\left(\rho^{(\theta)},m^{(\theta)}\right)-\partial_x q_\xi(\rho,m)\right|\qquad(4.6)\\&\quad+\left|\nabla\eta_\xi^{(\theta)}\left(\rho^{(\theta)},m^{(\theta)}\right)\cdot G\left(\rho^{(\theta)},m^{(\theta)}\right)-\nabla\eta_\xi(\rho,m)\cdot G(\rho,m)\right|\\&=G_1+G_2+G_3,\end{aligned}$$

where

$$\begin{aligned}G_1\le&\left|\partial_t\eta_\xi^{(\theta)}\left(\rho^{(\theta)},m^{(\theta)}\right)-\partial_t\eta_\xi\left(\rho^{(\theta)},m^{(\theta)}\right)\right|\\&+\left|\partial_t\eta_\xi\left(\rho^{(\theta)},m^{(\theta)}\right)-\partial_t\eta_\xi(\rho,m)\right|\to0,\ \text{as }\theta\to0\end{aligned}\qquad(4.7)$$

The inequality holds since Lemma 4.2 and $\left(\rho^{(\theta)},m^{(\theta)}\right)\to(\rho,m)$ (as $\theta\to0$). Similarly, we have

$$G_2\to0,\ \text{as }\theta\to0\qquad(4.8)$$

As for G_3,

$$\begin{aligned}G_3\le&\left|\nabla\eta_\xi^{(\theta)}\left(\rho^{(\theta)},m^{(\theta)}\right)\cdot G\left(\rho^{(\theta)},m^{(\theta)}\right)-\nabla\eta_\xi\left(\rho^{(\theta)},m^{(\theta)}\right)\cdot G\left(\rho^{(\theta)},m^{(\theta)}\right)\right|\\&+\left|\nabla\eta_\xi\left(\rho^{(\theta)},m^{(\theta)}\right)\cdot G\left(\rho^{(\theta)},m^{(\theta)}\right)-\nabla\eta_\xi(\rho,m)\cdot G(\rho,m)\right|\\=&\,G_{31}+G_{32}.\end{aligned}$$

A calculation tells us that

$$\eta_{\xi m}(\rho,m)=\frac{\xi}{1-\xi^2}\frac{1}{\rho}\eta_\xi.$$

So

$$\begin{aligned}G_{31}&=\left|\eta_{\xi m}^{(\theta)}\left(\rho^{(\theta)},m^{(\theta)}\right)\left(F(x,t)\rho^{(\theta)}+H(x,t)m^{(\theta)}\right)\right.\\&\quad\left.-\eta_{\xi m}\left(\rho^{(\theta)},m^{(\theta)}\right)\left(F(x,t)\rho^{(\theta)}+H(x,t)m^{(\theta)}\right)\right|\\&=\left|F(x,t)\rho^{(\theta)}+H(x,t)m^{(\theta)}\right|\left|\eta_{\xi m}^{(\theta)}\left(\rho^{(\theta)},m^{(\theta)}\right)-\eta_{\xi m}\left(\rho^{(\theta)},m^{(\theta)}\right)\right|\\&=\left|F(x,t)\rho^{(\theta)}+H(x,t)m^{(\theta)}\right|\left|\frac{\xi}{1-\xi^2}\frac{1}{\rho^{(\theta)}}\right|\left|\eta_\xi^{(\theta)}\left(\rho^{(\theta)},m^{(\theta)}\right)-\eta_\xi\left(\rho^{(\theta)},m^{(\theta)}\right)\right|\\&\le\left|M_0+M_0C\right|\frac{\xi}{1-\xi^2}C\sqrt{\theta}\to0\ \text{ as }\theta\to0,\end{aligned}$$

where C and M_0 are constants independent of θ.

As for G_{32}, a similar estimate together with the strong convergence $\left(\rho^{(\theta)}, m^{(\theta)}\right) \to (\rho, m)$, as $\theta \to 0$ yields $G_{32} \to 0$, as $\theta \to 0$.

Combining the estimates for G_1, G_2, G_3, we conclude that the right-hand side of (4.6) converges to 0 as $\theta \to 0$. Hence the entropy inequality holds for the limit function (ρ, m).

This completes the proof of Theorem 2.2. ☐

5. Conclusion

In this paper, we extend the result of [1] to a class of isentropic Euler equations with a special source term. To apply the proving framework introduced in [1], we need to show that solutions of isentropic Euler equations are independent of the adiabatic exponent γ. Therefore, we employ a maximum principle to achieve this. Next, we can easily prove the strong convergence and get the entropy inequality.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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