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Cohomology of Homogeneous Spaces H_1/H_2

Yingling Liu, Yu Wang*

College of Mathematics and Statistics, Sichuan University of Science and Engineering, Zigong, China
Email: 18982859249@163.com

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Abstract

Let G be a compact connected Lie group with Lie algebra $\mathfrak{g}$, H_1 a normal subgroup of G with Lie algebra $\mathfrak{h}_1$, and $H_2 \subseteq H_1$ a normal subgroup with Lie algebra $\mathfrak{h}_2$. This paper establishes an isomorphism between the de Rham cohomology of the quotient manifold H_1/H_2 and the Lie algebra cohomology of the quotient Lie algebra $\mathfrak{h}_1/\mathfrak{h}_2$. By constructing explicit isomorphisms of cochain complexes, we prove $H_{dR}^{\bullet}(H_1/H_2) \cong H^{\bullet}(\mathfrak{h}_1/\mathfrak{h}_2, \mathbb{R}) \cong \left(\left(\Lambda^{\bullet}(\mathfrak{h}_1/\mathfrak{h}_2)\right)^*\right)^{\mathfrak{g}}$, where $\left(\left(\Lambda^{\bullet}(\mathfrak{h}_1/\mathfrak{h}_2)\right)^*\right)^{\mathfrak{g}}$ denotes the space of $\mathfrak{g}$-invariant elements. This result transforms the geometric problem of computing the de Rham cohomology of the quotient H_1/H_2 into an algebraic computation on the Lie algebra $\mathfrak{h}_1/\mathfrak{h}_2$.

Keywords

Homogeneous Space, Cochain Complex, de Rham Cohomology, Lie Algebra Cohomology, Invariant Forms

1. Introduction

Let G be a compact connected Lie group with Lie algebra $\mathfrak{g}$, H_1 a normal subgroup of G with Lie algebra $\mathfrak{h}_1$, and $H_2 \subseteq H_1$ a normal subgroup of H_1 with Lie algebra $\mathfrak{h}_2$. Chevalley and Eilenberg [1] established a fundamental bridge between the geometry of Lie groups and the algebra of their Lie algebras by proving that the de Rham cohomology of G is isomorphic to the cohomology of its Lie algebra:

$$H^{\bullet}(\mathfrak{g}, \mathbb{R}) \cong H_{dR}^{\bullet}(G) \cong \left(\left(\Lambda^{\bullet}\mathfrak{g}\right)^*\right)^{\mathfrak{g}}.$$

This isomorphism allows one to translate geometric problems into tractable algebraic ones.

This paper investigates the cohomology of the quotient manifold H_1/H_2.

Specifically, we study the relationship between the de Rham cohomology $H^{\bullet}_{dR}(H_1/H_2)$ of the quotient and the Lie algebra cohomology $H^{\bullet}(\mathfrak{h}_1/\mathfrak{h}_2,\mathbb{R})$ of the quotient Lie algebra $\mathfrak{h}_1/\mathfrak{h}_2$, which provides a powerful tool for analyzing the topology and geometry of such spaces. We establish the following isomorphism, generalizing the Chevalley-Eilenberg result:

$$H^{\bullet}(\mathfrak{h}_1/\mathfrak{h}_2,\mathbb{R}) \cong H^{\bullet}_{dR}(H_1/H_2) \cong \left(\left(\Lambda^{\bullet}(\mathfrak{h}_1/\mathfrak{h}_2)\right)^{*}\right)^{\mathfrak{g}}.$$

Here, the group G acts on H_1/H_2 via left multiplication, and $\mathfrak{g}$ acts on the Lie algebra $\mathfrak{h}_1/\mathfrak{h}_2$ via the adjoint representation. Our approach constructs explicit isomorphisms of cochain complexes linking invariant differential forms on H_1/H_2 to cochains on $\mathfrak{h}_1/\mathfrak{h}_2$. These isomorphisms collectively yield the main theorem (Theorem 1). This result extends the foundational work of Chevalley and Eilenberg to a broader class of quotient manifolds and provides a concrete algebraic framework for computing their cohomology.

This paper is organized as follows: In Sec. 2, we recall basic concepts of de Rham cohomology and Lie algebra cohomology. In Sec. 3, we give a detailed proof of the isomorphism between de Rham cohomology of the quotient manifold and its Lie algebra cohomology.

2. Preliminary Knowledge

We recall the definitions and basic properties of Lie group cohomology and Lie algebras cohomology.

Definition 1. [2] *Let K be a field. A Lie algebra $\mathfrak{g}$ is a vector space over K with a bilinear bracket $[-,-]$:*

$$\mathfrak{g}\times\mathfrak{g}\to\mathfrak{g},$$

satisfying the following axioms for all $X,Y,Z\in\mathfrak{g}$ and $\lambda_1,\lambda_2\in K$:

1) *Bilinearity*: $[\lambda_1 X+\lambda_2 Y,Z]=\lambda_1[X,Z]+\lambda_2[Y,Z]$;

2) *Antisymmetry*: $[X,X]=0$;

3) *Jacobi identity*: $[X,[Y,Z]]+[Y,[Z,X]]+[Z,[X,Y]]=0$.

Definition 2. [2] *Let K be a field and A be a unital ring. If the additive group of A forms a K-vector space, and*

$$\lambda(ab)=(\lambda a)b=a(\lambda b)$$

for all $\lambda\in K$, and $a,b\in A$, then A is called an associative algebra over K, or simply a K-algebra.

Let A be an associative algebra over a field K. The commutator bracket $[a,b]=ab-ba$ defines a Lie algebra structure on the underlying K-vector space of A.

Definition 3. [2] *A subspace $\mathfrak{h}$ of a Lie algebra $\mathfrak{g}$ is called an ideal of $\mathfrak{g}$ if it satisfies $[\mathfrak{g},\mathfrak{h}]\subset\mathfrak{h}$.*

Definition 4. [2] *Let $\mathfrak{h}$ be an ideal of a Lie algebra $\mathfrak{g}$. Define a Lie bracket operation in the quotient space $\mathfrak{g}/\mathfrak{h}$ as follows:*

$$[X+\mathfrak{h},Y+\mathfrak{h}]=[X,Y]+\mathfrak{h},$$

then the quotient space $\mathfrak{g}/\mathfrak{h}$ *is a Lie algebra, called the quotient algebra of* $\mathfrak{g}$ *by* $\mathfrak{h}$.

Definition 5. [3] *Let* $\mathfrak{g}_1$ *and* $\mathfrak{g}_2$ *be Lie algebras over a field* K. *A* K*-linear map* $\mathscr{A}:\mathfrak{g}_1\to\mathfrak{g}_2$ *is a Lie algebra homomorphism if for any* $X,Y\in\mathfrak{g}_1$

$$\left[\mathscr{A}(X),\mathscr{A}(Y)\right]=\mathscr{A}\left([X,Y]\right).$$

A canonical example is a homomorphism from a Lie algebra $\mathfrak{g}$ to $\mathrm{gl}(V)$, the general linear Lie algebra on a vector space V.

Definition 6. [3] *A representation of a Lie algebra* $\mathfrak{g}$ *over a field* K *is a pair* (V,ρ), *where* V *is a* K*-vector space and* $\rho:\mathfrak{g}\to\mathrm{gl}(V)$ *is a Lie algebra homomorphism.*

Definition 7. [3] *For any Lie algebra* $\mathfrak{g}$, *the adjoint representation is the homomorphism* $ad:\mathfrak{g}\to\mathrm{gl}(\mathfrak{g})$, *defined by* $ad(X)(Y)=[X,Y]$.

We now recall some fundamental concepts of smooth manifolds and establish the correspondence between Lie groups and Lie algebras.

Definition 8. [4] *Let* M *be an* n*-dimensional topological manifold. If a smooth structure* Σ *is specified on* M, *then* (M,Σ) *is called an* n*-dimensional smooth manifold.*

Definition 9. [4] *Let* M *be a smooth manifold. A function* $f:M\to\mathbb{R}$ *is called smooth if it is smooth with respect to the smooth structure of* M. *The set of all smooth functions on* M *is denoted by* $C^\infty(M)$.

Definition 10. [5] *Let* M *be a smooth manifold and* $p\in M$. *Denote by* C_p^∞ *the algebra of germs of smooth functions at* p. *A tangent vector at* p *is a linear map* $v:C_p^\infty\to\mathbb{R}$ *satisfying the following axioms: for all* $f,g\in C_p^\infty$ *and* $\lambda\in\mathbb{R}$,

1) $v(f+\lambda g)=v(f)+\lambda v(g)$;

2) $v(f\cdot g)=v(f)g(p)+f(p)v(g)$.

The tangent space at p, denoted T_pM, is the vector space of all tangent vectors.

Definition 11. [5] *A cotangent vector at* $p\in M$ *is a linear functional* $\alpha:T_pM\to\mathbb{R}$. *The cotangent space* T_p^*M *is the dual space of* T_pM.

Definition 12. [5] *Let* M *be a smooth manifold. The tangent bundle of* M *is defined as* $TM=\bigcup_{p\in M}T_pM$, *equipped with a natural smooth structure that makes it a smooth manifold.*

Definition 13. [6] *A smooth vector field on a smooth manifold* M *is a smooth map* $X:M\to TM$ *such that* $\pi\circ X=id_M$, *where* $\pi:TM\to M$ *is the canonical projection. In other words,* X *is a smooth section of the tangent bundle.*

The set of all smooth vector fields on M is denoted by $\mathfrak{X}(M)$.

Definition 14. [7] *Let* M *be a smooth manifold. A differential* k*-form on* M *is a smooth section of the* k*-th exterior power of the cotangent bundle, i.e., a smooth map*

$$\omega:M\to\Lambda^kT^*M=\bigcup_{p\in M}\Lambda^k\left(T_p^*M\right),$$

such that $\omega(p)\in\Lambda^k\left(T_p^*M\right)$ *for each* $p\in M$. *The set of all differential* k*-*

forms on M *is denoted by* $\Omega^k(M)$.

Property 1. [6] *Let* M *be a smooth manifold. There exists a unique operator* $d:\Omega^r(M)\to\Omega^{r+1}(M)(r\geq 0)$, *called the exterior derivative, satisfying the following properties*:

1) $d:\Omega^r(M)\to\Omega^{r+1}(M)$ *is a linear map.*

2) $d(\varphi\wedge\psi)=d\varphi\wedge\psi+(-1)^r\varphi\wedge d\psi$, $\varphi\in\Omega^r(M)$, $\psi\in\Omega^s(M)$.

3) *For* $f\in C^\infty(M)=\Omega^0(M)$, df *is the ordinary differential of* f.

4) $d\circ d=0$.

Proposition 1. [6] *The space* $\Omega^k(M)$ *of differential* k*-forms on a smooth manifold* M *is isomorphic as a* $C^\infty(M)$*-module to the space of alternating* $C^\infty(M)$*-multilinear maps* $\mathfrak{X}(M)^k\to C^\infty(M)$.

Proposition 2. [8] (*Invariant formula*) *For any* $\omega\in\Omega^k(M)$ *and* $X_0,\cdots,X_k\in\mathfrak{X}(M)$,

$$d\omega(X_0,\cdots,X_k)=\sum_{i=0}^{k}(-1)^i X_i\left(\omega\left(X_0,\cdots,\hat{X}_i,\cdots,X_k\right)\right)$$
$$+\sum_{0\leq i<j\leq k}(-1)^{i+j}\,\omega\left(\left[X_i,X_j\right],X_0,\cdots,\hat{X}_i,\cdots,\hat{X}_j,\cdots,X_k\right),$$

where $\hat{X}_i$ *denotes that the element* X_i *is omitted.*

Definition 15. [6] *Let* M *and* N *be smooth manifolds and* $f:M\to N$ *a smooth map. For each* $p\in M$, *the pushforward of* f *at* p *is the linear map* $f_{*p}:T_pM\to T_{f(p)}N$ *defined by*

$$\left(f_{*p}(X)\right)(g)=X(g\circ f)$$

for all $X\in T_pM$ *and* $g\in C^\infty_{f(p)}$.

Definition 16. [6] *Let* $f:M\to N$ *be a smooth map. The pullback induced by* f *is the map* $f^*:\Omega^k(N)\to\Omega^k(M)$ *defined by*

$$f^*(\omega)(X_1,\cdots,X_k)=\omega\left(f_*(X_1),\cdots,f_*(X_k)\right)$$

for all $\omega\in\Omega^k(N)$ *and* $X_1,\cdots,X_k\in\mathfrak{X}(M)$.

Property 2. [6] *Let* $f:M\to N$ *be a smooth map. The pullback* $f^*:\Omega^k(N)\to\Omega^k(M)$ *satisfies the following properties*:

1) f^* *is a linear map.*

2) *For all* $\omega\in\Omega^r(N)$ *and* $\eta\in\Omega^s(N)$, $f^*(\omega\wedge\eta)=f^*\omega\wedge f^*\eta$.

3) $f^*(d\omega)=d\left(f^*\omega\right)$.

Definition 17. [9] *A Lie group is a group* G *that is also a smooth manifold such that the group operations* $\varphi:G\times G\to G,(g,h)\mapsto gh$ *and inversion* $\tau:G\to G,g\mapsto g^{-1}$ *are smooth maps.*

For any fixed $g\in G$, the maps

$$L_g:G\to G,\ L_g(h)=gh$$

and

$$R_g:G\to G,\ R_g(h)=hg^{-1}$$

are smooth diffeomorphisms, called left multiplication and right multiplication, respectively.

The group $G \times G$ acts smoothly on G via

$$(g,h)\cdot x = R_h L_g(x) = gxh^{-1},\ \ g,h,x \in G.$$

If $g = h$, this action gives the conjugation by g, denoted $c_g = R_g L_g$.

Definition 18. [7] *Let a Lie group* G *acts smoothly on a smooth manifold* M *via a map* $G \times M \to M$. *For each* $g \in G$, *denote by* g *the smooth map* $M \to M$ *given by the action. A differential form* $\omega \in \Omega^k(M)$ *is called* G*-invariant if* $g^*\omega = \omega$, for all $g \in G$.

The space of all G-invariant k-forms on M is denoted by $\Omega^k(M)^G$.

Definition 19. [9] *Let* G *be a Lie group. A vector field* $X \in \mathfrak{X}(G)$ *is left-invariant if for all* $g,h \in G$,

$$(L_g)_{*_h}(X(h)) = X(gh).$$

The set of all left-invariant vector fields on G forms a Lie algebra under the Lie bracket of vector fields. This Lie algebra is denoted by $\mathfrak{g}$ and is isomorphic to the tangent space T_eG at the identity element $e \in G$. It is called the Lie algebra of G. Next, we introduce the basic concepts of homology.

Definition 20. [10] *A chain complex* $\mathbf{C}$ *of* R*-modules is a family* $\{C_n\}_{n\in Z}$ *of* R*-modules together with* R*-modules map* $d_n : C_n \to C_{n-1}$ *such that the sequence*

$$\cdots \to C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \to \cdots$$

satisfies $d_n \circ d_{n+1} = 0$ for all $n \in Z$.

Definition 21. [10] *Let* $(\mathbf{C}_\bullet, d_\bullet)$ *and* $(\mathbf{C}'_\bullet, d'_\bullet)$ *be chain complexes. A chain map*

$$f = f_\bullet : (\mathbf{C}_\bullet, d_\bullet) \to (\mathbf{C}'_\bullet, d'_\bullet)$$

is a family of morphisms $\{f_n : C_n \to C'_n\}_{n\in\mathbb{Z}}$ such that diagram

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & C_{n+1} & \xrightarrow{d_{n+1}} & C_n & \xrightarrow{d_n} & C_{n-1} & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle f_{n+1}} & & \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f_{n-1}} & & \\
\cdots & \longrightarrow & C'_{n+1} & \xrightarrow[d'_{n+1}]{} & C'_n & \xrightarrow[d'_n]{} & C'_{n-1} & \longrightarrow & \cdots
\end{array}$$

commutes, i.e., $f_{n-1} \circ d_n = d'_n \circ f_n$ *for all* $n \in Z$.

Definition 22. [10] *Let* $\mathbf{C} : \cdots \to C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \to \cdots$ *be a chain complex. Its* n*-th homology is defined as the quotient module*

$$H_n(\mathbf{C}) = \frac{\ker d_n}{\operatorname{Im} d_{n+1}}.$$

Definition 23. [10] *A chain map* $f : (A,d) \to (C,\delta)$ *is called a quasi-isomorphism if for every integer* n *the induced map* $f_* : H_n(A) \to H_n(C)$ *are an isomorphism.*

Definition 24. [11] *Let* K *be a feld,* $\mathfrak{g}$ *a Lie algebra over* K, *and* Γ *a* $\mathfrak{g}$

-module. Define

$$C^n(\mathfrak{g},\Gamma) := \mathrm{Hom}_K(\Lambda^n\mathfrak{g},\Gamma),\ n>0,\ C^0(\mathfrak{g},\Gamma) := \Gamma.$$

The space $C^n(\mathfrak{g},\Gamma)$ *can be identified with the space of alternating* n*-linear maps* $\mathfrak{g}^n \to \Gamma$. *For* $c \in C^n(\mathfrak{g},\Gamma)$, *define* $dc \in C^{n+1}(\mathfrak{g},\Gamma)$ *by*

$$\begin{aligned} dc(X_1,\cdots,X_{n+1}) &= \sum_{i=1}^{n+1}(-1)^{i+1} X_i\left(c\left(X_1,\cdots,\hat{X}_i,\cdots,X_{n+1}\right)\right) \\ &+ \sum_{1\le i<j\le n+1}(-1)^{i+j} c\left(\left[X_i,X_j\right],X_1,\cdots,\hat{X}_i,\cdots,\hat{X}_j,\cdots,X_{n+1}\right) \end{aligned}$$

for all $X_1,\cdots,X_{n+1} \in \mathfrak{g}$.

One can verify that $d \circ d = 0$, so we obtain a cochain complex

$$\cdots \to C^{n-1}(\mathfrak{g},\Gamma) \xrightarrow{d^{n-1}} C^n(\mathfrak{g},\Gamma) \xrightarrow{d^n} C^{n+1}(\mathfrak{g},\Gamma) \to \cdots.$$

The Lie algebra cohomology of $\mathfrak{g}$ with coefficients in Γ is

$$H^n(\mathfrak{g},\Gamma) := H^n\left(\left(C^*(\mathfrak{g},\Gamma),d\right)\right) = \frac{\ker d^n}{\operatorname{Im} d^{n-1}}.$$

3. Isomorphisms between de Rham Cohomology and Lie Algebra Cohomology for H_1/H_2

Let G be a compact connected Lie group with Lie algebra $\mathfrak{g}$, H_1 a normal subgroup of G with Lie algebra $\mathfrak{h}_1$, and H_2 a normal subgroup of G such that $H_2 \subseteq H_1$ with Lie algebra $\mathfrak{h}_2$. G acts by multiplication on the quotient H_1/H_2, and $\mathfrak{g}$ acts on $\mathfrak{h}_1/\mathfrak{h}_2$, the Lie algebra of H_1/H_2, by the adjoint action. The section establishes an isomorphism between the de Rham cohomology of the quotient space H_1/H_2 and the cohomology of the Lie algebra $\mathfrak{h}_1/\mathfrak{h}_2$. This is achieved by constructing chain complex isomorphisms via evaluation at the identity $\bar{e} \in H_1/H_2$. This construction links G-invariant and $G \times G$-invariant differential forms to $\mathfrak{g}$-invariant Lie algebra cochains.

Proposition 3. [11] *Suppose* G *acts on a manifold* H_1/H_2 *via an action* $\alpha: G \times H_1/H_2 \to H_1/H_2$. *Then the inclusion* $\varphi: \Omega^*(H_1/H_2)^G \hookrightarrow \Omega^*(H_1/H_2)$ *is a quasi-isomorphism.*

Let V be a vector space and $\pi: G \to \mathrm{Aut}(V)$ a representation of G, with derivative $\rho = D_e\pi: \mathfrak{g} \to End(V)$. Recall that for all $X \in \mathfrak{g}$,

$$X = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \exp(tX)$$

where $\exp: \mathfrak{g} = T_eG \to G$ is the exponential map, defined by $X \mapsto \theta_X(1)$. Here θ_X is the maximal integral curve of the left-invariant vector field defined by X, satisfying $\theta_X(0) = 1$. Then, by the chain rule,

$$\rho(X) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \pi(\exp(tX)).$$

Definition 25. [12] *A vector* $v \in V$ *is called* G*-invariant if* $\pi(g)(v) = v$ *for all* $g \in G$. *The subspace of all* G*-invariant elements is denoted by* V^G. *A vector*

$v \in V$ *is called* $\mathfrak{g}$ *-invariant if* $\rho(X)(v) = 0$ *for all* $X \in \mathfrak{g}$. *The subspace of all* $\mathfrak{g}$ *-invariant elements is denoted by* $V^{\mathfrak{g}}$.

We shall now prove that these two subspaces are equal.

Proposition 4. $V^G = V^{\mathfrak{g}}$.

Proof. We establish the equality by proving two inclusions.

1) $V^G \subseteq V^{\mathfrak{g}}$.

Let $v \in V^G$. Then $\pi(g)(v) = v$ for all $g \in G$. For any $X \in \mathfrak{g}$,

$$\rho(X)(v) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \pi(\exp(tX))(v) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} v = 0,$$

so $v \in V^{\mathfrak{g}}$.

2) $V^{\mathfrak{g}} \subseteq V^G$.

Let $v \in V^{\mathfrak{g}}$. Then $\rho(X)(v) = 0$ for all $X \in \mathfrak{g}$. Define the evaluation map $\mathrm{ev}_v : \mathrm{Aut}(V) \to V$ by $\mathrm{ev}_v(A) = A(v)$. Then

$$D_e(\mathrm{ev}_v \circ \pi)(X) = \mathrm{ev}_v \circ D_e\pi(X) = \mathrm{ev}_v(\rho(X)) = \rho(X)(v) = 0.$$

Since G is connected, $\mathrm{ev}_v \circ \pi$ is constant. As $\mathrm{ev}_v \circ \pi(e) = \pi(e)(v) = v$, where e is the identity of G. We obtain $\mathrm{ev}_v \circ \pi(g) = \pi(g)(v) = v$. Thus, $v \in V^G$. Combing (1) and (2), we conclude that $V^G = V^{\mathfrak{g}}$.

We now construct an isomorphism relating the invariant differential forms to the Lie algebra chain complex, thereby linking de Rham cohomology with Lie algebra cohomology.

Proposition 5. *The evaluation map at the identity* $\bar{e} = e + H_2 \in H_1/H_2$, $\varepsilon : \Omega^m(H_1/H_2)^G \to C^m(\mathfrak{h}_1/\mathfrak{h}_2, \mathbb{R})$, $\omega \mapsto \omega_{\bar{e}}$, *defines an isomorphism of chain complexes.*

Proof. First, we verify that ε is well-defined. Identifying $\mathfrak{h}_1/\mathfrak{h}_2$ with the tangent space $T_{\bar{e}}(H_1/H_2)$, we have

$$\begin{aligned}\varepsilon(\omega) = \omega_{\bar{e}} &\in \mathrm{Hom}_{\mathbb{R}}\left(\Lambda^m\left(T_{\bar{e}}(H_1/H_2)\right), \mathbb{R}\right) = \mathrm{Hom}_{\mathbb{R}}\left(\Lambda^m(\mathfrak{h}_1/\mathfrak{h}_2), \mathbb{R}\right) \\ &= C^m(\mathfrak{h}_1/\mathfrak{h}_2, \mathbb{R}).\end{aligned}$$

Hence ε is well-defined.

We consider the sequence

$$\cdots \to \Omega^{m-1}(H_1/H_2)^G \xrightarrow{d^{m-1}} \Omega^m(H_1/H_2)^G \xrightarrow{d^m} \Omega^{m+1}(H_1/H_2)^G \to \cdots.$$

For any $\omega \in \Omega^{m-1}(H_1/H_2)^G$ and $g \in G$, we have

$$g^*\left(d^{m-1}\omega\right) = d^{m-1}\left(g^*\omega\right) = d^{m-1}\omega,$$

which shows that $d^{m-1}\omega \in \Omega^m(H_1/H_2)^G$. Let $\bar{X}_1, \cdots, \bar{X}_{m+1}$ be left-invariant vector fields on H_1/H_2, then

$$\begin{aligned}&d^m\left(d^{m-1}\omega\right)\left(\bar{X}_1, \cdots, \bar{X}_{m+1}\right) \\ &= \sum_{i=1}^{m+1}(-1)^{i+1}\bar{X}_i\left(d^{m-1}\omega\left(\bar{X}_1, \cdots, \hat{\bar{X}}_i, \cdots, \bar{X}_{m+1}\right)\right) \\ &+ \sum_{i<j}(-1)^{i+j} d^{m-1}\omega\left(\left[\bar{X}_i, \bar{X}_j\right], \bar{X}_1, \cdots, \hat{\bar{X}}_i, \cdots, \hat{\bar{X}}_j, \cdots, \bar{X}_{m+1}\right)\end{aligned}$$

$$= \sum_{i<j} (-1)^{i+j} \left[\bar{X}_i, \bar{X}_j\right]\left(\omega\left(\bar{X}_1, \cdots, \hat{\bar{X}}_i, \cdots, \hat{\bar{X}}_j, \cdots, \bar{X}_{m+1}\right)\right)$$
$$+ \sum_{r<i<j} (-1)^{i+j+r} \bar{X}_r \left(\omega\left(\left[\bar{X}_i, \bar{X}_j\right], \bar{X}_1, \cdots, \hat{\bar{X}}_r, \cdots, \hat{\bar{X}}_i, \cdots, \hat{\bar{X}}_j, \cdots, \bar{X}_{m+1}\right)\right)$$
$$+ \sum_{i<r<j} (-1)^{i+j+r-1} \bar{X}_r \left(\omega\left(\left[\bar{X}_i, \bar{X}_j\right], \bar{X}_1, \cdots, \hat{\bar{X}}_i, \cdots, \hat{\bar{X}}_r, \cdots, \hat{\bar{X}}_j, \cdots, \bar{X}_{m+1}\right)\right)$$
$$+ \sum_{i<j<r} (-1)^{i+j+r-2} \bar{X}_r \left(\omega\left(\left[\bar{X}_i, \bar{X}_j\right], \bar{X}_1, \cdots, \hat{\bar{X}}_i, \cdots, \hat{\bar{X}}_j, \cdots, \hat{\bar{X}}_r, \cdots, \bar{X}_{m+1}\right)\right)$$
$$+ \sum_{s<i<j} (-1)^{i+j+s} \omega\left(\left[\left[\bar{X}_i, \bar{X}_j\right], \bar{X}_s\right], \bar{X}_1, \cdots, \hat{\bar{X}}_s, \cdots, \hat{\bar{X}}_i, \cdots, \hat{\bar{X}}_j, \cdots, \bar{X}_{m+1}\right)$$
$$+ \sum_{i<s<j} (-1)^{i+j+s+1} \omega\left(\left[\left[\bar{X}_i, \bar{X}_j\right], \bar{X}_s\right], \bar{X}_1, \cdots, \hat{\bar{X}}_i, \cdots, \hat{\bar{X}}_s, \cdots, \hat{\bar{X}}_j, \cdots, \bar{X}_{m+1}\right)$$
$$+ \sum_{i<j<s} (-1)^{i+j+s} \omega\left(\left[\left[\bar{X}_i, \bar{X}_j\right], \bar{X}_s\right], \bar{X}_1, \cdots, \hat{\bar{X}}_i, \cdots, \hat{\bar{X}}_j, \cdots, \hat{\bar{X}}_s, \cdots, \bar{X}_{m+1}\right)$$
$$+ \sum_{k<l<i<j} (-1)^{i+j+k+l} \omega\left(\left[\bar{X}_k, \bar{X}_l\right], \left[\bar{X}_i, \bar{X}_j\right], \bar{X}_1, \cdots, \hat{\bar{X}}_k, \cdots, \hat{\bar{X}}_l, \cdots, \hat{\bar{X}}_i, \cdots, \hat{\bar{X}}_j, \cdots, \bar{X}_{m+1}\right)$$
$$+ \sum_{k<i<l<j} (-1)^{i+j+k+l+1} \omega\left(\left[\bar{X}_k, \bar{X}_l\right], \bar{X}_i, \bar{X}_j, \bar{X}_1, \cdots, \hat{\bar{X}}_k, \cdots, \hat{\bar{X}}_i, \cdots, \hat{\bar{X}}_l, \cdots, \hat{\bar{X}}_j, \cdots, \bar{X}_{m+1}\right)$$
$$+ \sum_{k<i<j<l} (-1)^{i+j+k+l} \omega\left(\left[\bar{X}_k, \bar{X}_l\right], \left[\bar{X}_i, \bar{X}_j\right], \bar{X}_1, \cdots, \hat{\bar{X}}_k, \cdots, \hat{\bar{X}}_i, \cdots, \hat{\bar{X}}_j, \cdots, \hat{\bar{X}}_l, \cdots, \bar{X}_{m+1}\right)$$
$$+ \sum_{i<k<l<j} (-1)^{i+j+k+l} \omega\left(\left[\bar{X}_k, \bar{X}_l\right], \left[\bar{X}_i, \bar{X}_j\right], \bar{X}_1, \cdots, \hat{\bar{X}}_i, \cdots, \hat{\bar{X}}_k, \cdots, \hat{\bar{X}}_l, \cdots, \hat{\bar{X}}_j, \cdots, \bar{X}_{m+1}\right)$$
$$+ \sum_{i<k<j<l} (-1)^{i+j+k+l-1} \omega\left(\left[\bar{X}_k, \bar{X}_l\right], \left[\bar{X}_i, \bar{X}_j\right], \bar{X}_1, \cdots, \hat{\bar{X}}_i, \cdots, \hat{\bar{X}}_k, \cdots, \hat{\bar{X}}_j, \cdots, \hat{\bar{X}}_l, \cdots, \bar{X}_{m+1}\right)$$
$$+ \sum_{i<j<k<l} (-1)^{i+j+k+l} \omega\left(\left[\bar{X}_k, \bar{X}_l\right], \left[\bar{X}_i, \bar{X}_j\right], \bar{X}_1, \cdots, \hat{\bar{X}}_i, \cdots, \hat{\bar{X}}_j, \cdots, \hat{\bar{X}}_k, \cdots, \hat{\bar{X}}_l, \cdots, \bar{X}_{m+1}\right)$$
$$= 0.$$

We obtain $d^m \circ d^{m-1} = 0$. Thus, $\left(\Omega^*\left(H_1/H_2\right)^G, d^\bullet\right)$ is a chain complex.

Next, we consider the sequence

$$\cdots \to C^{m-1}\left(\mathfrak{h}_1/\mathfrak{h}_2, \mathbb{R}\right) \xrightarrow{d'^{m-1}} C^m\left(\mathfrak{h}_1/\mathfrak{h}_2, \mathbb{R}\right) \xrightarrow{d'^m} C^{m+1}\left(\mathfrak{h}_1/\mathfrak{h}_2, \mathbb{R}\right) \to \cdots.$$

For $c \in C^{m-1}\left(\mathfrak{h}_1/\mathfrak{h}_2, \mathbb{R}\right)$, the differential $d'^{m-1}c \in C^m\left(\mathfrak{h}_1/\mathfrak{h}_2, \mathbb{R}\right)$ is defined by

$$d'^{m-1}c\left(\bar{X}_1, \cdots, \bar{X}_m\right) = \sum_{i=1}^{m} (-1)^{i+1} \bar{X}_i \left(c\left(\bar{X}_1, \cdots, \hat{\bar{X}}_i, \cdots, \bar{X}_m\right)\right)$$
$$+ \sum_{i<j} (-1)^{i+j} c\left(\left[\bar{X}_i, \bar{X}_j\right], \bar{X}_1, \cdots, \hat{\bar{X}}_i, \cdots, \hat{\bar{X}}_j, \cdots, \bar{X}_m\right),$$

where $\bar{X}_1, \cdots, \bar{X}_m \in \mathfrak{h}_1/\mathfrak{h}_2$. Then

$$d'^m\left(d'^{m-1}c\right)\left(\bar{X}_1, \cdots, \bar{X}_{m+1}\right)$$
$$= \sum_{i=1}^{m+1} (-1)^{i+1} \bar{X}_i \left(d'^{m-1}c\left(\bar{X}_1, \cdots, \hat{\bar{X}}_i, \cdots, \bar{X}_{m+1}\right)\right)$$
$$+ \sum_{i<j} (-1)^{i+j} d'^{m-1}c\left(\left[\bar{X}_i, \bar{X}_j\right], \bar{X}_1, \cdots, \hat{\bar{X}}_i, \cdots, \hat{\bar{X}}_j, \cdots, \bar{X}_{m+1}\right)$$
$$= \sum_{r<i} (-1)^{i+r} \bar{X}_i \bar{X}_r \left(c\left(\bar{X}_1, \cdots, \hat{\bar{X}}_r, \cdots, \hat{\bar{X}}_i, \cdots, \bar{X}_{m+1}\right)\right)$$

$$
\begin{aligned}
&+\sum_{i<r}(-1)^{i+r+1}\,\overline{X}_i\overline{X}_r\left(c\left(\overline{X}_1,\cdots,\hat{\overline{X}}_i,\cdots,\hat{\overline{X}}_r,\cdots,\overline{X}_{m+1}\right)\right)\\
&+\sum_{r<s<i}(-1)^{i+r+s+1}\,\overline{X}_i\left(c\left(\left[\overline{X}_r,\overline{X}_s\right],\overline{X}_1,\cdots,\hat{\overline{X}}_r,\cdots,\hat{\overline{X}}_s,\cdots,\hat{\overline{X}}_i,\cdots,\overline{X}_{m+1}\right)\right)\\
&+\sum_{r<i<s}(-1)^{i+r+s}\,\overline{X}_i\left(c\left(\left[\overline{X}_r,\overline{X}_s\right],\overline{X}_1,\cdots,\hat{\overline{X}}_r,\cdots,\hat{\overline{X}}_i,\cdots,\hat{\overline{X}}_s,\cdots,\overline{X}_{m+1}\right)\right)\\
&+\sum_{i<r<s}(-1)^{i+r+s+1}\,\overline{X}_i\left(c\left(\left[\overline{X}_r,\overline{X}_s\right],\overline{X}_1,\cdots,\hat{\overline{X}}_i,\cdots,\hat{\overline{X}}_r,\cdots,\hat{\overline{X}}_s,\cdots,\overline{X}_{m+1}\right)\right)\\
&+\sum_{i<j}(-1)^{i+j}\left[\overline{X}_i,\overline{X}_j\right]\left(c\left(\overline{X}_1,\cdots,\hat{\overline{X}}_i,\cdots,\hat{\overline{X}}_j,\cdots,\overline{X}_{m+1}\right)\right)\\
&+\sum_{r<i<j}(-1)^{i+j+r}\,\overline{X}_r\left(c\left(\left[\overline{X}_i,\overline{X}_j\right],\overline{X}_1,\cdots,\hat{\overline{X}}_r,\cdots,\hat{\overline{X}}_i,\cdots,\hat{\overline{X}}_j,\cdots,\overline{X}_{m+1}\right)\right)\\
&+\sum_{i<r<j}(-1)^{i+j+r-1}\,\overline{X}_r\left(c\left(\left[\overline{X}_i,\overline{X}_j\right],\overline{X}_1,\cdots,\hat{\overline{X}}_i,\cdots,\hat{\overline{X}}_r,\cdots,\hat{\overline{X}}_j,\cdots,\overline{X}_{m+1}\right)\right)\\
&+\sum_{i<j<r}(-1)^{i+j+r-2}\,\overline{X}_r\left(c\left(\left[\overline{X}_i,\overline{X}_j\right],\overline{X}_1,\cdots,\hat{\overline{X}}_i,\cdots,\hat{\overline{X}}_j,\cdots,\hat{\overline{X}}_r,\cdots,\overline{X}_{m+1}\right)\right)\\
&+\sum_{s<i<j}(-1)^{i+j+s}\,c\left(\left[\left[\overline{X}_i,\overline{X}_j\right],\overline{X}_s\right],\overline{X}_1,\cdots,\hat{\overline{X}}_s,\cdots,\hat{\overline{X}}_i,\cdots,\hat{\overline{X}}_j,\cdots,\overline{X}_{m+1}\right)\\
&+\sum_{i<s<j}(-1)^{i+j+s+1}\,c\left(\left[\left[\overline{X}_i,\overline{X}_j\right],\overline{X}_s\right],\overline{X}_1,\cdots,\hat{\overline{X}}_i,\cdots,\hat{\overline{X}}_s,\cdots,\hat{\overline{X}}_j,\cdots,\overline{X}_{m+1}\right)\\
&+\sum_{i<j<s}(-1)^{i+j+s}\,c\left(\left[\left[\overline{X}_i,\overline{X}_j\right],\overline{X}_s\right],\overline{X}_1,\cdots,\hat{\overline{X}}_i,\cdots,\hat{\overline{X}}_j,\cdots,\hat{\overline{X}}_s,\cdots,\overline{X}_{m+1}\right)\\
&+\sum_{k<l<i<j}(-1)^{i+j+k+l}\,c\left(\left[\overline{X}_k,\overline{X}_l\right],\left[\overline{X}_i,\overline{X}_j\right],\overline{X}_1,\cdots,\hat{\overline{X}}_k,\cdots,\hat{\overline{X}}_l,\cdots,\hat{\overline{X}}_i,\cdots,\hat{\overline{X}}_j,\cdots,\overline{X}_{m+1}\right)\\
&+\sum_{k<i<l<j}(-1)^{i+j+k+l+1}\,c\left(\left[\overline{X}_k,\overline{X}_l\right],\left[\overline{X}_i,\overline{X}_j\right],\overline{X}_1,\cdots,\hat{\overline{X}}_k,\cdots,\hat{\overline{X}}_i,\cdots,\hat{\overline{X}}_l,\cdots,\hat{\overline{X}}_j,\cdots,\overline{X}_{m+1}\right)\\
&+\sum_{k<i<j<l}(-1)^{i+j+k+l}\,c\left(\left[\overline{X}_k,\overline{X}_l\right],\left[\overline{X}_i,\overline{X}_j\right],\overline{X}_1,\cdots,\hat{\overline{X}}_k,\cdots,\hat{\overline{X}}_i,\cdots,\hat{\overline{X}}_j,\cdots,\hat{\overline{X}}_l,\cdots,\overline{X}_{m+1}\right)\\
&+\sum_{i<k<l<j}(-1)^{i+j+k+l}\,c\left(\left[\overline{X}_k,\overline{X}_l\right],\left[\overline{X}_i,\overline{X}_j\right],\overline{X}_1,\cdots,\hat{\overline{X}}_i,\cdots,\hat{\overline{X}}_k,\cdots,\hat{\overline{X}}_l,\cdots,\hat{\overline{X}}_j,\cdots,\overline{X}_{m+1}\right)\\
&+\sum_{i<k<j<l}(-1)^{i+j+k+l-1}\,c\left(\left[\overline{X}_k,\overline{X}_l\right],\left[\overline{X}_i,\overline{X}_j\right],\overline{X}_1,\cdots,\hat{\overline{X}}_i,\cdots,\hat{\overline{X}}_k,\cdots,\hat{\overline{X}}_j,\cdots,\hat{\overline{X}}_l,\cdots,\overline{X}_{m+1}\right)\\
&+\sum_{i<j<k<l}(-1)^{i+j+k+l}\,c\left(\left[\overline{X}_k,\overline{X}_l\right],\left[\overline{X}_i,\overline{X}_j\right],\overline{X}_1,\cdots,\hat{\overline{X}}_i,\cdots,\hat{\overline{X}}_j,\cdots,\hat{\overline{X}}_k,\cdots,\hat{\overline{X}}_l,\cdots,\overline{X}_{m+1}\right)\\
&=0.
\end{aligned}
$$

We obtain $d'^m \circ d'^{m-1} = 0$. Thus, $C^*\left(\mathfrak{h}_1/\mathfrak{h}_2,\mathbb{R},d'^{\bullet}\right)$ is a chain complex.

Consider the following diagram:

$$
\begin{array}{ccccccccc}
\cdots \longrightarrow & \Omega^{m-1}(H_1/H_2)^G & \xrightarrow{d^{m-1}} & \Omega^{m}(H_1/H_2)^G & \xrightarrow{d^{m}} & \Omega^{m+1}(H_1/H_2)^G & \longrightarrow \cdots \\
 & \downarrow{\scriptstyle \varepsilon^{m-1}} & & \downarrow{\scriptstyle \varepsilon^{m}} & & \downarrow{\scriptstyle \varepsilon^{m+1}} & \\
\cdots \longrightarrow & C^{m-1}(\mathfrak{h}_1/\mathfrak{h}_2,\mathbb{R}) & \xrightarrow[d'^{m-1}]{} & C^{m}(\mathfrak{h}_1/\mathfrak{h}_2,\mathbb{R}) & \xrightarrow[d'^{m}]{} & C^{m+1}(\mathfrak{h}_1/\mathfrak{h}_2,\mathbb{R}) & \longrightarrow \cdots .
\end{array}
$$

Next, we show that ε is a chain map, i.e., $\varepsilon^{m+1}\circ d^m = d'^m \circ \varepsilon^m$. Let $\omega \in \Omega^m\left(H_1/H_2\right)^G$ and $\overline{v}_1,\cdots,\overline{v}_{m+1} \in T_{\overline{e}}\left(H_1/H_2\right)$. Let $\overline{X}_i$ be the left-invariant vector field on H_1/H_2 with $\overline{v}_i = v_i + \mathfrak{h}_2 = \overline{X_i(e)}$. Since ω and all the $\overline{X}_i$ are left-invariant, we have $\omega\left(\bar{X}_1,\cdots,\bar{X}_m\right)$ is left-invariant and therefore is constant.

Since $\mathfrak{g}$ acts trivially on $\mathbb{R}$, we have

$$\begin{aligned}
&\varepsilon^{m+1}\left(d^m\omega\right)\left(\bar{v}_1,\cdots,\bar{v}_{m+1}\right)\\
&=\left(d^m\omega\right)_{\bar{e}}\left(\bar{v}_1,\cdots,\bar{v}_{m+1}\right)=d^m\omega\left(\bar{X}_1,\cdots,\bar{X}_{m+1}\right)\left(\bar{e}\right)\\
&=\sum_{i=1}^{m+1}(-1)^{i+1}\bar{X}_i\left(\omega\left(\bar{X}_1,\cdots,\hat{\bar{X}}_i,\cdots,\bar{X}_{m+1}\right)\right)\left(\bar{e}\right)\\
&\quad+\sum_{1\le i<j\le m+1}(-1)^{i+j}\omega\left(\left[\bar{X}_i,\bar{X}_j\right],\bar{X}_1,\cdots,\hat{\bar{X}}_i,\cdots,\hat{\bar{X}}_j,\cdots,\bar{X}_{m+1}\right)\left(\bar{e}\right)\\
&=\sum_{1\le i<j\le m+1}(-1)^{i+j}\omega\left(\left[\bar{X}_i,\bar{X}_j\right],\bar{X}_1,\cdots,\hat{\bar{X}}_i,\cdots,\hat{\bar{X}}_j,\cdots,\bar{X}_{m+1}\right)\left(\bar{e}\right)\\
&=\sum_{1\le i<j\le m+1}(-1)^{i+j}\omega_{\bar{e}}\left(\left[\bar{v}_i,\bar{v}_j\right],\bar{v}_1,\cdots,\hat{\bar{v}}_i,\cdots,\hat{\bar{v}}_j,\cdots,\bar{v}_{m+1}\right).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
d'^m\left(\varepsilon^m\omega\right)\left(\bar{v}_1,\cdots,\bar{v}_{m+1}\right)&=d'^m\left(\omega_{\bar{e}}\right)\left(\bar{v}_1,\cdots,\bar{v}_{m+1}\right)\\
&=\sum_{i=1}^{m+1}(-1)^{i+1}\bar{v}_i\left(\omega_{\bar{e}}\left(\bar{v}_1,\cdots,\hat{\bar{v}}_i,\cdots,\bar{v}_{m+1}\right)\right)\\
&\quad+\sum_{1\le i<j\le m+1}(-1)^{i+j}\omega_{\bar{e}}\left(\left[\bar{v}_i,\bar{v}_j\right],\bar{v}_1,\cdots,\hat{\bar{v}}_i,\cdots,\hat{\bar{v}}_j,\cdots,\bar{v}_{m+1}\right)\\
&=\sum_{1\le i<j\le m+1}(-1)^{i+j}\omega_{\bar{e}}\left(\left[\bar{v}_i,\bar{v}_j\right],\bar{v}_1,\cdots,\hat{\bar{v}}_i,\cdots,\hat{\bar{v}}_j,\cdots,\bar{v}_{m+1}\right).
\end{aligned}$$

This implies that $\varepsilon^{m+1}\circ d^m=d'^m\circ\varepsilon^m$, so ε is a chain map.

Next, we prove that ε is an isomorphism. Let $\omega\in\Omega^m\left(H_1/H_2\right)^G$, $\bar{g}\in H_1/H_2$ and $\bar{u}_1,\cdots,\bar{u}_m\in T_{\bar{g}}\left(H_1/H_2\right)$. We have

$$\omega_{\bar{g}}\left(\bar{u}_1,\cdots,\bar{u}_m\right)=\left(L_{g^{-1}}^*\omega\right)_{\bar{g}}\left(\bar{u}_1,\cdots,\bar{u}_m\right)=\omega_{\bar{e}}\left(D_{\bar{g}}L_{g^{-1}}\left(\bar{u}_1\right),\cdots,D_{\bar{g}}L_{g^{-1}}\left(\bar{u}_m\right)\right).$$

Hence $\ker\varepsilon=\left\{\omega\in\Omega^m\left(H_1/H_2\right)^G\mid\varepsilon(\omega)=\omega_{\bar{e}}=0\right\}=0$. It follows that ε is injective.

Let $c\in C^m\left(\mathfrak{h}_1/\mathfrak{h}_2,\mathbb{R}\right)$, there exists $\omega\in\Omega^m\left(H_1/H_2\right)$ such that

$$\omega_{\bar{g}}\left(\bar{u}_1,\cdots,\bar{u}_m\right)=c\left(D_{\bar{g}}L_{g^{-1}}\left(\bar{u}_1\right),\cdots,D_{\bar{g}}L_{g^{-1}}\left(\bar{u}_m\right)\right).$$

For any $\bar{g}=g+H_2\in H_1/H_2, h\in G$,

$$\begin{aligned}
\left(\left(L_h\right)^*\omega\right)_{\bar{g}}\left(\bar{u}_1,\cdots,\bar{u}_m\right)&=\omega_{\overline{hg}}\left(\left(L_h\right)_{*\bar{g}}\left(\bar{u}_1\right),\cdots,\left(L_h\right)_{*\bar{g}}\left(\bar{u}_m\right)\right)\\
&=\omega_{\overline{hg}}\left(D_{\bar{g}}L_h\left(\bar{u}_1\right),\cdots,D_{\bar{g}}L_h\left(\bar{u}_m\right)\right)\\
&=c\left(D_{\overline{hg}}L_{(hg)^{-1}}\left(D_{\bar{g}}L_h\left(\bar{u}_1\right)\right),\cdots,D_{\overline{hg}}L_{(hg)^{-1}}\left(D_{\bar{g}}L_h\left(\bar{u}_m\right)\right)\right)\\
&=c\left(D_{\bar{g}}L_{g^{-1}}\left(\bar{u}_1\right),\cdots,D_{\bar{g}}L_{g^{-1}}\left(\bar{u}_m\right)\right)\\
&=\omega_{\bar{g}}\left(\bar{u}_1,\cdots,\bar{u}_m\right).
\end{aligned}$$

Thus, $\omega \in \Omega^m(H_1/H_2)^G$. Additionally, since

$$\begin{aligned}\varepsilon(\omega)(\bar{v}_1,\cdots,\bar{v}_m) &= \omega_{\bar{e}}(\bar{v}_1,\cdots,\bar{v}_m)\\ &= c\left(D_{\bar{e}}L_{e^{-1}}(\bar{v}_1),\cdots,D_{\bar{e}}L_{e^{-1}}(\bar{v}_m)\right)\\ &= c\left(\left(L_{e^{-1}}\right)_{*\bar{e}}\left(\overline{X_1(e)}\right),\cdots,\left(L_{e^{-1}}\right)_{*\bar{e}}\left(\overline{X_m(e)}\right)\right)\\ &= c\left(\overline{X_1(e^{-1}\cdot e)},\cdots,\overline{X_m(e^{-1}\cdot e)}\right)\\ &= c\left(\overline{X_1(e)},\cdots,\overline{X_m(e)}\right)\\ &= c(\bar{v}_1,\cdots,\bar{v}_m),\end{aligned}$$

it follows that $\varepsilon(\omega)=c$. Thus, ε is surjective. We conclude that ε is an isomorphism.

Next, we construct a representation of the Lie group G with Lie algebra $\mathfrak{g}$ on the cochain complex $C^*(\mathfrak{h}_1/\mathfrak{h}_2,\mathbb{R})$, and show that the subspace of G-invariant cochains coincides with the subspace of $\mathfrak{g}$-invariant cochains.

Proposition 6. $\left(C^*(\mathfrak{h}_1/\mathfrak{h}_2,\mathbb{R})^G,d\right)=\left(C^*(\mathfrak{h}_1/\mathfrak{h}_2,\mathbb{R})^{\mathfrak{g}},d\right)$.

Proof. First, we construct the action $\pi: G\to \mathrm{Aut}\left(C^m(\mathfrak{h}_1/\mathfrak{h}_2,\mathbb{R})\right)$.

For $g,h\in G$ and $\bar{X},\bar{X}_1,\cdots,\bar{X}_m\in \mathfrak{h}_1/\mathfrak{h}_2$, recall that G acts on $\mathfrak{h}_1/\mathfrak{h}_2$ via the adjoint action

$$\mathrm{Ad}: G\to \mathrm{Aut}(\mathfrak{h}_1/\mathfrak{h}_2),\ \mathrm{Ad}(g)(\bar{X})=T_{\bar{e}}c_g(\bar{X}).$$

This action extends naturally to the exterior power $\Lambda^m(\mathfrak{h}_1/\mathfrak{h}_2)$, which we also denote by $\mathrm{Ad}: G\to \mathrm{Aut}\left(\Lambda^m(\mathfrak{h}_1/\mathfrak{h}_2)\right)$, satisfying

$$\mathrm{Ad}(g)(\bar{X}_1\wedge\cdots\wedge\bar{X}_m):=\mathrm{Ad}(g)(\bar{X}_1)\wedge\cdots\wedge\mathrm{Ad}(g)(\bar{X}_m).$$

Dualising gives an action π of G on $C^m(\mathfrak{h}_1/\mathfrak{h}_2,\mathbb{R})$,

$\pi: G\to \mathrm{Aut}\left(C^m(\mathfrak{h}_1/\mathfrak{h}_2,\mathbb{R})\right)$,

$$g\mapsto \mathrm{Ad}(g)^*,\ \left(\mathrm{Ad}(g)^*c\right)(\bar{X}_1,\cdots,\bar{X}_m):=c\left(\mathrm{Ad}(g^{-1})(\bar{X}_1),\cdots,\mathrm{Ad}(g^{-1})(\bar{X}_m)\right).$$

Next we verify that π is a homomorphism. For $g,h\in G$ and $c\in C^m(\mathfrak{h}_1/\mathfrak{h}_2,\mathbb{R})$,

$$\begin{aligned}\left(\pi(gh)c\right)(\bar{X}_1,\cdots,\bar{X}_m) &= \mathrm{Ad}(gh)^*c(\bar{X}_1,\cdots,\bar{X}_m)\\ &= c\left(\mathrm{Ad}\left((gh)^{-1}\right)(\bar{X}_1),\cdots,\mathrm{Ad}\left((gh)^{-1}\right)(\bar{X}_m)\right)\\ &= c\left(T_{\bar{e}}c_{(gh)^{-1}}(\bar{X}_1),\cdots,T_{\bar{e}}c_{(gh)^{-1}}(\bar{X}_m)\right),\end{aligned}$$

$$\begin{aligned}\left(\pi(g)\circ\pi(h)c\right)(\bar{X}_1,\cdots,\bar{X}_m) &= \left((\mathrm{Ad}g)^*\circ(\mathrm{Ad}h)^*\right)c(\bar{X}_1,\cdots,\bar{X}_m)\\ &= (\mathrm{Ad}g)^*\left((\mathrm{Ad}h)^*c\right)(\bar{X}_1,\cdots,\bar{X}_m)\\ &= (\mathrm{Ad}h)^*c\left(T_{\bar{e}}c_{g^{-1}}(\bar{X}_1),\cdots,T_{\bar{e}}c_{g^{-1}}(\bar{X}_m)\right)\\ &= c\left(T_{\bar{e}}c_{h^{-1}}\left(T_{\bar{e}}c_{g^{-1}}(\bar{X}_1)\right),\cdots,T_{\bar{e}}c_{h^{-1}}\left(T_{\bar{e}}c_{g^{-1}}(\bar{X}_m)\right)\right)\\ &= c\left(T_{\bar{e}}c_{(gh)^{-1}}(\bar{X}_1),\cdots,T_{\bar{e}}c_{(gh)^{-1}}(\bar{X}_m)\right),\end{aligned}$$

it follows that $\pi(gh)=\pi(g)\circ\pi(h)$. Thus, π is a representation of G.

Now we construct the corresponding Lie algebra action $\rho:\mathfrak{g}\to\mathrm{End}\left(C^m\left(\mathfrak{h}_1/\mathfrak{h}_2,\mathbb{R}\right)\right)$. The adjoint action of $\mathfrak{g}$ on $\mathfrak{h}_1/\mathfrak{h}_2$ is

$$\mathrm{ad}:\mathfrak{g}\to\mathrm{End}\left(\mathfrak{h}_1/\mathfrak{h}_2\right),\ \mathrm{ad}(X)\left(\overline{Y}\right)=\overline{[X,Y]},$$

which we can extend to an action of $\mathfrak{g}$ on $\Lambda^m\left(\mathfrak{h}_1/\mathfrak{h}_2\right)$ by

$$\mathrm{ad}(X)\left(\overline{X}_1\wedge\cdots\wedge\overline{X}_m\right)=\sum_{i=1}^{m}\overline{X}_1\wedge\cdots\wedge\overline{[X,X_i]}\wedge\cdots\wedge\overline{X}_m.$$

Again this dualises to an action on $C^m\left(\mathfrak{h}_1/\mathfrak{h}_2,\mathbb{R}\right)$,

$$\rho:\mathfrak{g}\to\mathrm{End}\left(C^m\left(\mathfrak{h}_1/\mathfrak{h}_2,\mathbb{R}\right)\right),\ X\mapsto\mathrm{ad}(X)^*,$$

satisfying

$$\left(\mathrm{ad}(X)^*c\right)\left(\overline{X}_1,\cdots,\overline{X}_m\right)=\sum_{i=1}^{m}c\left(\overline{X}_1,\cdots,\overline{[X_i,X]},\cdots,\overline{X}_m\right).$$

Next we show that ρ is a Lie algebra homomorphism, i.e., $\rho([X,Y])=[\rho(X),\rho(Y)]$ for any $X,Y\in\mathfrak{g}$. Recall that $[\rho(X),\rho(Y)]=\rho(X)\circ\rho(Y)-\rho(Y)\circ\rho(X)$,

$$\begin{aligned}
&\left(\rho(X)\circ\rho(Y)c\right)\left(\overline{X}_1,\cdots,\overline{X}_m\right)\\
&=\sum_{i=1}^{m}\rho(Y)c\left(\overline{X}_1,\cdots,\overline{[X_i,X]},\cdots,\overline{X}_m\right)\\
&=\sum_{1\le j<i\le m}c\left(\overline{X}_1,\cdots,\overline{[X_j,Y]},\cdots,\overline{[X_i,X]},\cdots,\overline{X}_m\right)\\
&\quad+\sum_{1\le j=i\le m}c\left(\overline{X}_1,\cdots,\overline{[[X_i,X],Y]},\cdots,\overline{X}_m\right)\\
&\quad+\sum_{1\le i<j\le m}c\left(\overline{X}_1,\cdots,\overline{[X_i,X]},\cdots,\overline{[X_j,Y]},\cdots,\overline{X}_m\right).
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\left(\rho(Y)\circ\rho(X)c\right)\left(\overline{X}_1,\cdots,\overline{X}_m\right)\\
&=\sum_{j=1}^{m}\rho(X)c\left(\overline{X}_1,\cdots,\overline{[X_j,Y]},\cdots,\overline{X}_m\right)\\
&=\sum_{1\le j<i\le m}c\left(\overline{X}_1,\cdots,\overline{[X_j,Y]},\cdots,\overline{[X_i,X]},\cdots,\overline{X}_m\right)\\
&\quad+\sum_{1\le j=i\le m}c\left(\overline{X}_1,\cdots,\overline{[[X_j,Y],X]},\cdots,\overline{X}_m\right)\\
&\quad+\sum_{1\le i<j\le m}c\left(\overline{X}_1,\cdots,\overline{[X_i,X]},\cdots,\overline{[X_j,Y]},\cdots,\overline{X}_m\right).
\end{aligned}$$

Hence

$$
\begin{aligned}
&\left(\left[\rho(X),\rho(Y)\right]c\right)\left(\bar{X}_1,\cdots,\bar{X}_m\right)\\
&=\left(\rho(X)\circ\rho(Y)-\rho(Y)\circ\rho(X)\right)c\left(\bar{X}_1,\cdots,\bar{X}_m\right)\\
&=\sum_{1\le j=i\le m} c\left(\bar{X}_1,\cdots,\overline{\left[\left[X_i,X\right],Y\right]-\left[\left[X_j,Y\right],X\right]},\cdots,\bar{X}_m\right)\\
&=\sum_{i=1}^{m} c\left(\bar{X}_1,\cdots,\overline{\left[\left[X_i,X\right],Y\right]-\left[\left[X_i,Y\right],X\right]},\cdots,\bar{X}_m\right)\\
&=\sum_{i=1}^{m} c\left(\bar{X}_1,\cdots,\overline{\left[X_i,\left[X,Y\right]\right]},\cdots,\bar{X}_m\right)\\
&=\left(\rho\left(\left[X,Y\right]\right)c\right)\left(\bar{X}_1,\cdots,\bar{X}_m\right),
\end{aligned}
$$

it follows that $\rho\left(\left[X,Y\right]\right)=\left[\rho(X),\rho(Y)\right]$. Thus, ρ is a representation of $\mathfrak{g}$.

Next we prove that $\rho(X)=D_e\pi(X)$ for every $X\in\mathfrak{g}$. Since

$$
\begin{aligned}
&\left(D_e\pi(X)c\right)\left(\bar{X}_1,\cdots,\bar{X}_m\right)\\
&=\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}\pi\left(\exp(tX)\right)c\left(\bar{X}_1,\cdots,\bar{X}_m\right)\\
&=\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}\mathrm{Ad}\left(\exp(tX)\right)^{*}c\left(\bar{X}_1,\cdots,\bar{X}_m\right)\\
&=\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}c\left(\mathrm{Ad}\left(\exp(tX)\right)^{-1}\left(\bar{X}_1\right),\cdots,\mathrm{Ad}\left(\exp(tX)\right)^{-1}\left(\bar{X}_m\right)\right)\\
&=\sum_{i=1}^{m} c\left(\bar{X}_1,\cdots,ad(-X)\left(\bar{X}_i\right),\cdots,\bar{X}_m\right)\\
&=\sum_{i=1}^{m} c\left(\bar{X}_1,\cdots,\overline{\left[X_i,X\right]},\cdots,\bar{X}_m\right)\\
&=\left((adX)^{*}c\right)\left(\bar{X}_1,\cdots,\bar{X}_m\right)\\
&=\left(\rho(X)c\right)\left(\bar{X}_1,\cdots,\bar{X}_m\right),
\end{aligned}
$$

it follows that $\rho=D_e\pi$. According to Proposition 4, we have $C^m\left(\mathfrak{h}_1/\mathfrak{h}_2,\mathbb{R}\right)^G=C^m\left(\mathfrak{h}_1/\mathfrak{h}_2,\mathbb{R}\right)^{\mathfrak{g}}$.

Finally, we prove that for every $c\in C^m\left(\mathfrak{h}_1/\mathfrak{h}_2,\mathbb{R}\right)^G$, $dc\in C^{m+1}\left(\mathfrak{h}_1/\mathfrak{h}_2,\mathbb{R}\right)^G$. Let $g\in G$ and $\bar{X}_1,\cdots,\bar{X}_{m+1}\in\mathfrak{h}_1/\mathfrak{h}_2$, Because c is G-invariant,

$$\pi(g)c\left(\bar{X}_1,\cdots,\bar{X}_m\right)=(\mathrm{Adg})^{*}c\left(\bar{X}_1,\cdots,\bar{X}_m\right)=c\left(\bar{X}_1,\cdots,\bar{X}_m\right).$$

So

$$
\begin{aligned}
&\left((\mathrm{Adg})^{*}dc\right)\left(\bar{X}_1,\cdots,\bar{X}_{m+1}\right)\\
&=dc\left(\mathrm{Ad}\left(g^{-1}\right)\left(\bar{X}_1\right),\cdots,\mathrm{Ad}\left(g^{-1}\right)\left(\bar{X}_{m+1}\right)\right)\\
&=\sum_{i=1}^{m+1}(-1)^{i+1}\mathrm{Ad}\left(g^{-1}\right)\left(\bar{X}_i\right)\left(c\left(\mathrm{Ad}\left(g^{-1}\right)\left(\bar{X}_1\right),\cdots,\mathrm{Ad}\left(\widehat{g^{-1}}\right)\left(\bar{X}_i\right),\cdots,\mathrm{Ad}\left(g^{-1}\right)\left(\bar{X}_{m+1}\right)\right)\right)\\
&\quad+\sum_{i<j}(-1)^{i+j}c\left(\left[\mathrm{Ad}\left(g^{-1}\right)\left(\bar{X}_i\right),\mathrm{Ad}\left(g^{-1}\right)\left(\bar{X}_j\right)\right],\mathrm{Ad}\left(g^{-1}\right)\left(\bar{X}_1\right),\cdots,\right.\\
&\quad\left.\mathrm{Ad}\left(\widehat{g^{-1}}\right)\left(\bar{X}_i\right),\cdots,\mathrm{Ad}\left(\widehat{g^{-1}}\right)\left(\bar{X}_j\right),\cdots,\mathrm{Ad}\left(g^{-1}\right)\left(\bar{X}_{m+1}\right)\right)
\end{aligned}
$$

$$=\sum_{i<j}(-1)^{i+j}c\left(\mathrm{Ad}\left(g^{-1}\right)\left(\overline{\left[X_i,X_j\right]}\right),\mathrm{Ad}\left(g^{-1}\right)\left(\bar{X}_1\right),\cdots,\mathrm{Ad}\left(\widehat{g^{-1}}\right)\left(\bar{X}_i\right),\cdots,\right.$$
$$\left.\mathrm{Ad}\left(\widehat{g^{-1}}\right)\left(\bar{X}_j\right),\cdots,\mathrm{Ad}\left(g^{-1}\right)\left(\bar{X}_{m+1}\right)\right)$$
$$=\sum_{i<j}(-1)^{i+j}\left((\mathrm{Adg})^*c\right)\left(\overline{\left[X_i,X_j\right]},\bar{X}_1,\cdots,\hat{\bar{X}}_i,\cdots,\hat{\bar{X}}_j,\cdots,\bar{X}_{m+1}\right)$$
$$=\sum_{i<j}(-1)^{i+j}c\left(\overline{\left[X_i,X_j\right]},\bar{X}_1,\cdots,\hat{\bar{X}}_i,\cdots,\hat{\bar{X}}_j,\cdots,\bar{X}_{m+1}\right)$$
$$=dc\left(\bar{X}_1,\cdots,\bar{X}_{m+1}\right).$$

It follows that $dc\in C^{m+1}(\mathfrak{h}_1/\mathfrak{h}_2,\mathbb{R})^G$. We finally obtain $\left(C^*(\mathfrak{h}_1/\mathfrak{h}_2,\mathbb{R})^G,d\right)=\left(C^*(\mathfrak{h}_1/\mathfrak{h}_2,\mathbb{R})^{\mathfrak{g}},d\right)$.

We now consider the action of $G\times G$ on the quotient space H_1/H_2 and establish an isomorphism of chain complexes between the $G\times G$-invariant differential forms and the G-invariant space of Lie algebra chain complex.

Proposition 7. *Evaluation at* $\bar{e}=e+H_2\in H_1/H_2$, $\tau:\Omega^m(H_1/H_2)^{G\times G}\to C^m(\mathfrak{h}_1/\mathfrak{h}_2,\mathbb{R})^G$, $\omega\mapsto\omega_{\bar{e}}$ *defines an isomorphism of chain complexes.*

Proof. Let $\omega\in\Omega^m(H_1/H_2)^{G\times G}$, we show that $\omega_{\bar{e}}\in C^m(\mathfrak{h}_1/\mathfrak{h}_2,\mathbb{R})^G$, that is, $\omega_{\bar{e}}$ is G-invariant. For any $g\in G$ and $\bar{X}_1,\cdots,\bar{X}_m\in\mathfrak{h}_1/\mathfrak{h}_2$, the invariance of ω gives $c_g^*\omega=\omega$. Hence

$$\begin{aligned}\omega_{\bar{e}}\left(\bar{X}_1,\cdots,\bar{X}_m\right)&=\left(c_{g^{-1}}^*\omega\right)_{\bar{e}}\left(\bar{X}_1,\cdots,\bar{X}_m\right)\\&=\omega_{\bar{e}}\left(D_{\bar{e}}c_{g^{-1}}\left(\bar{X}_1\right),\cdots,D_{\bar{e}}c_{g^{-1}}\left(\bar{X}_m\right)\right)\\&=\omega_{\bar{e}}\left(\mathrm{Ad}\left(g^{-1}\right)\left(\bar{X}_1\right),\cdots,\mathrm{Ad}\left(g^{-1}\right)\left(\bar{X}_m\right)\right).\end{aligned}$$

Therefore

$$\begin{aligned}\left(\pi(g)\omega_{\bar{e}}\right)\left(\bar{X}_1,\cdots,\bar{X}_m\right)&=\left((\mathrm{Adg})^*\omega_{\bar{e}}\right)\left(\bar{X}_1,\cdots,\bar{X}_m\right)\\&=\omega_{\bar{e}}\left(\mathrm{Ad}\left(g^{-1}\right)\left(\bar{X}_1\right),\cdots,\mathrm{Ad}\left(g^{-1}\right)\left(\bar{X}_m\right)\right)\\&=\omega_{\bar{e}}\left(\bar{X}_1,\cdots,\bar{X}_m\right).\end{aligned}$$

It follows that $\pi(g)\omega_{\bar{e}}=\omega_{\bar{e}}$, which shows that $\omega_{\bar{e}}$ is G-invariant. Thus, τ is well-defined.

Consider the following diagram:

$$\begin{array}{ccccccc}\longrightarrow\Omega^{m-1}(H_1/H_2)^{G\times G}&\xrightarrow{d'^{m-1}}&\Omega^m(H_1/H_2)^{G\times G}&\xrightarrow{d'^m}&\Omega^{m+1}(H_1/H_2)^{G\times G}\longrightarrow\\ \downarrow\tau^{m-1}&&\downarrow\tau^m&&\downarrow\tau^{m+1}\\ \longrightarrow C^{m-1}(\mathfrak{h}_1/\mathfrak{h}_2,\mathbb{R})^G&\xrightarrow[d^{m-1}]{}&C^m(\mathfrak{h}_1/\mathfrak{h}_2,\mathbb{R})^G&\xrightarrow[d^m]{}&C^{m+1}(\mathfrak{h}_1/\mathfrak{h}_2,\mathbb{R})^G\longrightarrow.\end{array}$$

Let $\omega\in\Omega^m(H_1/H_2)^{G\times G}$ and $g,h\in G$. Since ω is $G\times G$-invariant, we have

$$\left(R_gL_h\right)^*\left(d'^m\omega\right)=L_h^*\left(R_g^*\left(d'^m\omega\right)\right)=L_h^*d'^m\left(R_g^*\omega\right)=d'^m\left(L_h^*R_g^*\omega\right)=d'^m\omega,$$

so $d'^m\omega\in\Omega^{m+1}(H_1/H_2)^{G\times G}$.

Next we show that τ is a chain map, i.e., $\tau^{m+1}\circ d'^m = d^m\circ\tau^m$. Let $\omega\in\Omega^m\left(H_1/H_2\right)^{G\times G}$ and $\overline{v}_1,\cdots,\overline{v}_{m+1}\in T_{\overline{e}}\left(H_1/H_2\right)$. Since ω and all the $\overline{X}_i$ are left-invariant, the function $\omega\left(\overline{X}_1,\cdots,\overline{X}_m\right)$ is left-invariant and therefore is constant. Moreover, $\mathfrak{g}$ acts trivially on $\mathbb{R}$. Consequently,

$$
\begin{aligned}
&\left(\tau^{m+1}\left(d'^m\omega\right)\right)\left(\overline{v}_1,\cdots,\overline{v}_{m+1}\right)\\
&=\left(d'^m\omega\right)_{\overline{e}}\left(\overline{v}_1,\cdots,\overline{v}_{m+1}\right)=d'^m\omega\left(\overline{X}_1,\cdots,\overline{X}_m\right)\left(\overline{e}\right)\\
&=\sum_{i=1}^{m+1}(-1)^{i+1}\,\overline{X}_i\left(\omega\left(\overline{X}_1,\cdots,\hat{\overline{X}}_i,\cdots,\overline{X}_{m+1}\right)\right)\left(\overline{e}\right)\\
&\quad+\sum_{i<j}(-1)^{i+j}\,\omega\left(\overline{\left[X_i,X_j\right]},\overline{X}_1,\cdots,\hat{\overline{X}}_i,\cdots,\hat{\overline{X}}_j,\cdots,\overline{X}_{m+1}\right)\left(\overline{e}\right)\\
&=\sum_{i<j}(-1)^{i+j}\,\omega\left(\overline{\left[X_i,X_j\right]},\overline{X}_1,\cdots,\hat{\overline{X}}_i,\cdots,\hat{\overline{X}}_j,\cdots,\overline{X}_{m+1}\right)\left(\overline{e}\right)\\
&=\sum_{i<j}(-1)^{i+j}\,\omega_{\overline{e}}\left(\overline{\left[v_i,v_j\right]},\overline{v}_1,\cdots,\hat{\overline{v}}_i,\cdots,\hat{\overline{v}}_j,\cdots,\overline{v}_{m+1}\right).
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
\left(d^m\left(\tau^m\omega\right)\right)\left(\overline{v}_1,\cdots,\overline{v}_{m+1}\right)&=d^m\left(\omega_{\overline{e}}\right)\left(\overline{v}_1,\cdots,\overline{v}_{m+1}\right)\\
&=\sum_{i=1}^{m+1}(-1)^{i+1}\,\overline{v}_i\left(\omega_{\overline{e}}\left(\overline{v}_1,\cdots,\hat{\overline{v}}_i,\cdots,\overline{v}_{m+1}\right)\right)\\
&\quad+\sum_{i<j}(-1)^{i+j}\,\omega_{\overline{e}}\left(\overline{\left[v_i,v_j\right]},\overline{v}_1,\cdots,\hat{\overline{v}}_i,\cdots,\hat{\overline{v}}_j,\cdots,\overline{v}_{m+1}\right)\\
&=\sum_{i<j}(-1)^{i+j}\,\omega_{\overline{e}}\left(\overline{\left[v_i,v_j\right]},\overline{v}_1,\cdots,\hat{\overline{v}}_i,\cdots,\hat{\overline{v}}_j,\cdots,\overline{v}_{m+1}\right).
\end{aligned}
$$

This implies that $\tau^{m+1}\circ d'^m = d^m\circ\tau^m$. Thus, τ is chain map.

Next we prove that τ is an isomorphism. Let $\omega\in\Omega^m\left(H_1/H_2\right)^{G\times G}$, $\overline{g}\in H_1/H_2$ and $\overline{u}_1,\cdots,\overline{u}_m\in T_{\overline{g}}\left(H_1/H_2\right)$. Since a $G\times G$-invariant differential form is in particular left-invariant, it follows that

$$
\omega_{\overline{g}}\left(\overline{u}_1,\cdots,\overline{u}_m\right)=\left(L^*_{g^{-1}}\omega\right)_{\overline{g}}\left(\overline{u}_1,\cdots,\overline{u}_m\right)=\omega_{\overline{e}}\left(D_{\overline{g}}L_{g^{-1}}\left(\overline{u}_1\right),\cdots,D_{\overline{g}}L_{g^{-1}}\left(\overline{u}_m\right)\right).
$$

This gives $\ker\tau=\left\{\omega\in\Omega^m\left(H_1/H_2\right)^{G\times G}\mid\tau(\omega)=\omega_{\overline{e}}=0\right\}=0$. Thus, τ is injective.

Let $c\in C^m\left(\mathfrak{h}_1/\mathfrak{h}_2,\mathbb{R}\right)^G$, there exists $\omega\in\Omega^m\left(H_1/H_2\right)^G$ such that

$$
\omega_{\overline{g}}\left(\overline{u}_1,\cdots,\overline{u}_m\right)=c\left(D_{\overline{g}}L_{g^{-1}}\left(\overline{u}_1\right),\cdots,D_{\overline{g}}L_{g^{-1}}\left(\overline{u}_m\right)\right).
$$

For any $x,y\in G$ and $\overline{g}\in H_1/H_2$,

$$
\begin{aligned}
&\left(\left(R_yL_x\right)^*\omega\right)_{\overline{g}}\left(\overline{u}_1,\cdots,\overline{u}_m\right)\\
&=\omega_{\overline{xgy^{-1}}}\left(D_{\overline{g}}\left(R_yL_x\right)\left(\overline{u}_1\right),\cdots,D_{\overline{g}}\left(R_yL_x\right)\left(\overline{u}_m\right)\right)\\
&=c\left(D_{\overline{xgy^{-1}}}L_{yg^{-1}x^{-1}}\left(D_{\overline{g}}\left(R_yL_x\right)\left(\overline{u}_1\right)\right),\cdots,D_{\overline{xgy^{-1}}}L_{yg^{-1}x^{-1}}\left(D_{\overline{g}}\left(R_yL_x\right)\left(\overline{u}_m\right)\right)\right)\\
&=c\left(D_{\overline{g}}\left(c_y\circ L_{g^{-1}}\right)\left(\overline{u}_1\right),\cdots,D_{\overline{g}}\left(c_y\circ L_{g^{-1}}\right)\left(\overline{u}_m\right)\right)\\
&=c\left(D_{\overline{e}}c_y\circ D_{\overline{g}}L_{g^{-1}}\left(\overline{u}_1\right),\cdots,D_{\overline{e}}c_y\circ D_{\overline{g}}L_{g^{-1}}\left(\overline{u}_m\right)\right)
\end{aligned}
$$

$$= c\left(\mathrm{Ad}y\left(D_{\bar{g}}L_{g^{-1}}\left(\bar{u}_1\right)\right),\cdots,\mathrm{Ad}y\left(D_{\bar{g}}L_{g^{-1}}\left(\bar{u}_m\right)\right)\right)$$
$$= c\left(D_{\bar{g}}L_{g^{-1}}\left(\bar{u}_1\right),\cdots,D_{\bar{g}}L_{g^{-1}}\left(\bar{u}_m\right)\right)$$
$$= \omega_{\bar{g}}\left(\bar{u}_1,\cdots,\bar{u}_m\right).$$

Thus, $\omega \in \Omega^m\left(H_1/H_2\right)^{G\times G}$. Additionally, since

$$\begin{aligned}\tau(\omega)\left(\bar{v}_1,\cdots,\bar{v}_m\right) &= \omega_{\bar{e}}\left(\bar{v}_1,\cdots,\bar{v}_m\right)\\ &= c\left(D_{\bar{e}}L_{e^{-1}}\left(\bar{v}_1\right),\cdots,D_{\bar{e}}L_{e^{-1}}\left(\bar{v}_m\right)\right)\\ &= c\left(\left(L_{e^{-1}}\right)_{*\bar{e}}\left(\overline{X_1(e)}\right),\cdots,\left(L_{e^{-1}}\right)_{*\bar{e}}\left(\overline{X_m(e)}\right)\right)\\ &= c\left(\overline{X_1\left(e^{-1}\cdot e\right)},\cdots,\overline{X_m\left(e^{-1}\cdot e\right)}\right)\\ &= c\left(\overline{X_1(e)},\cdots,\overline{X_m(e)}\right)\\ &= c\left(\bar{v}_1,\cdots,\bar{v}_m\right),\end{aligned}$$

it follows that $\tau(\omega)=c$. Thus, τ is surjective. We conclude that τ is an isomorphism.

The isomorphisms of chain complexes established in Propositions 5 and 7, together with the quasi-isomorphism in Proposition 3 and the equality in Proposition 6, induce isomorphisms in cohomology.

Theorem 1. *Let G be a compact connected Lie group with Lie algebra $\mathfrak{g}$. Let H_1 and H_2 be normal subgroup of G such that $H_2 \subseteq H_1$, with corresponding Lie algebra $\mathfrak{h}_1$ and $\mathfrak{h}_2$, respectively. The group G acts on the quotient H_1/H_2 by left multiplication, and its Lie algebra $\mathfrak{g}$ acts on the quotient Lie algebra $\mathfrak{h}_1/\mathfrak{h}_2$ via the adjoint action. Then*

$$H^\bullet\left(\mathfrak{h}_1/\mathfrak{h}_2,\mathbb{R}\right) \cong H^\bullet_{dR}\left(H_1/H_2\right) \cong \left(\left(\Lambda^\bullet\left(\mathfrak{h}_1/\mathfrak{h}_2\right)\right)^*\right)^{\mathfrak{g}},$$

where $\mathfrak{g}$ acts trivially on $\mathbb{R}$.

Proof. According to Proposition 3, we have $H^\bullet\left(\Omega^m\left(H_1/H_2\right)^G\right) \cong H^\bullet_{dR}\left(H_1/H_2\right)$. Moreover, Proposition 5 also gives an isomorphism of complex $\Omega^m\left(H_1/H_2\right)^G \cong C^m\left(\mathfrak{h}_1/\mathfrak{h}_2,R\right)^G$. Consequently,

$$\begin{aligned}H^m\left(\Omega^m\left(H_1/H_2\right)^G\right) &\cong H^m\left(\mathfrak{h}_1/\mathfrak{h}_2,R\right)\\ &\cong H^m\left(C^m\left(\mathfrak{h}_1/\mathfrak{h}_2,R\right)^G\right)\\ &= H^m\left(C^m\left(\mathfrak{h}_1/\mathfrak{h}_2,R\right)^{\mathfrak{g}}\right)\\ &= H^m\left(\left(\left(\wedge^m\left(\mathfrak{h}_1/\mathfrak{h}_2\right)\right)^*\right)^{\mathfrak{g}}\right).\end{aligned}$$

We obtain $H^\bullet\left(\mathfrak{h}_1/\mathfrak{h}_2,\mathbb{R}\right) \cong H^\bullet_{dR}\left(H_1/H_2\right) \cong H\left(\left(\left(\Lambda^\bullet\left(\mathfrak{h}_1/\mathfrak{h}_2\right)\right)^*\right)^{\mathfrak{g}}\right)$.

Now we prove that $d=0$ of the chain complex

$$\left(C^\bullet\left(\mathfrak{h}_1/\mathfrak{h}_2,\mathbb{R}\right)^G,d\right) = \left(C^\bullet\left(\mathfrak{h}_1/\mathfrak{h}_2,\mathbb{R}\right)^{\mathfrak{g}},d\right) = \left(\left(\left(\Lambda^\bullet\left(\mathfrak{h}_1/\mathfrak{h}_2\right)\right)^*\right)^{\mathfrak{g}},d\right).$$

For any $c \in C^m(\mathfrak{h}_1/\mathfrak{h}_2, \mathbb{R})^{\mathfrak{g}}$ and $\bar{X}_1, \cdots, \bar{X}_{m+1} \in \mathfrak{h}_1/\mathfrak{h}_2$. Because $\mathfrak{g}$ acts trivially on $\mathbb{R}$ and c is $\mathfrak{g}$-invariant, we have

$$\begin{aligned}
2dc\left(\bar{X}_1, \cdots, \bar{X}_{m+1}\right) = {} & 2\sum_{i=1}^{m+1}(-1)^{i+1}\bar{X}_i\left(c\left(\bar{X}_1, \cdots, \hat{\bar{X}}_i, \cdots, \bar{X}_{m+1}\right)\right) \\
& + 2\sum_{i<j}(-1)^{i+j} c\left(\overline{[X_i, X_j]}, \bar{X}_1, \cdots, \hat{\bar{X}}_i, \cdots, \hat{\bar{X}}_j, \cdots, \bar{X}_{m+1}\right) \\
= {} & \sum_{i<j}(-1)^{i+j} c\left(\overline{[X_i, X_j]}, \bar{X}_1, \cdots, \hat{\bar{X}}_i, \cdots, \hat{\bar{X}}_j, \cdots, \bar{X}_{m+1}\right) \\
& + \sum_{i<j}(-1)^{i+j} c\left(\overline{[X_i, X_j]}, \bar{X}_1, \cdots, \hat{\bar{X}}_i, \cdots, \hat{\bar{X}}_j, \cdots, \bar{X}_{m+1}\right) \\
= {} & \sum_{i<j}(-1)^{j} c\left(\bar{X}_1, \cdots, \overline{[X_i, X_j]}, \cdots, \hat{\bar{X}}_j, \cdots, \bar{X}_{m+1}\right) \\
& + \sum_{j<i}(-1)^{j} c\left(\bar{X}_1, \cdots, \hat{\bar{X}}_j, \cdots, \overline{[X_i, X_j]}, \cdots, \bar{X}_{m+1}\right) \\
= {} & \sum_{j=1}^{m+1}(-1)^{j}\left(\mathrm{ad}\bar{X}_j\right)^* c\left(\bar{X}_1, \cdots, \hat{\bar{X}}_j, \cdots, \bar{X}_{m+1}\right) \\
= {} & 0.
\end{aligned}$$

It follows that $d = 0$. Thus, $H\left(\left(\left(\wedge^m \mathfrak{h}_1/\mathfrak{h}_2\right)^*\right)^{\mathfrak{g}}\right) = \left(\left(\wedge^m\left(\mathfrak{h}_1/\mathfrak{h}_2\right)\right)^*\right)^{\mathfrak{g}}$.

Combining all isomorphisms we finally obtain

$$H^\bullet\left(\mathfrak{h}_1/\mathfrak{h}_2, \mathbb{R}\right) \cong H^\bullet_{dR}\left(H_1/H_2\right) \cong \left(\left(\Lambda^\bullet\left(\mathfrak{h}_1/\mathfrak{h}_2\right)\right)^*\right)^{\mathfrak{g}}.$$

Our work provides a concrete and effective algebraic framework for computing the de Rham cohomology of the homogeneous space H_1/H_2. It translates a topological problem into a purely algebraic one concerning the invariant multilinear forms on the Lie algebra $\mathfrak{h}_1/\mathfrak{h}_2$, which is often more tractable.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

References

[1] Chevalley, C. and Eilenberg, S. (1948) Cohomology Theory of Lie Groups and Lie Algebras. *Transactions of the American Mathematical Society*, **63**, 85-124. https://doi.org/10.1090/s0002-9947-1948-0024908-8

[2] Erdmann, K. and Wildon, M.J. (2006) Introduction to Lie Algebras. Springer.

[3] Humphreys, J.E. (2012) Introduction to Lie Algebras and Representation Theory. Springer Science & Business Media.

[4] Looijenga, E. (2010) Smooth Manifolds. Informal Notes.

[5] Christensen, J.D. and Wu, E. (2014) Tangent Spaces and Tangent Bundles for Diffeological Spaces.

[6] Khnel, W. (2015) Differential Geometry. American Mathematical Society.

[7] Morse, M. (2013) Differential Forms in Algebraic Topology.

[8] Tu, L.W. (2011) An Introduction to Manifolds. Springer, 47-83.

[9] Hall, B.C. (2015) Lie Groups, Lie Algebras, and Representations. Springer.

[10] Rotman, J.J. (2009) An Introduction to Homological Algebra. Springer.

[11] Larsen, M.J. and Lunts, V.A. (2021) A Note on Lie Algebra Cohomology. *Algebra & Number Theory*, **15**, 773-783. https://doi.org/10.2140/ant.2021.15.773

[12] Chevalley, C. (2018) Theory of Lie Groups. Courier Dover Publications.