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Non-Inclusive 1-Good-Neighbor Diagnosability of Augmented Cubes

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Abstract

The diagnosability of interconnection networks serves as a critical metric for evaluating the reliability of multiprocessor systems, as it quantifies the system's capability to identify faulty processors. Among various diagnosability models, the non-inclusive g-good-neighbor diagnosability offers a more precise characterization of fault tolerance by considering both the non-inclusive nature of fault sets and the requirement of maintaining a certain number of good neighbors for non-faulty vertices. This study focuses on the non-inclusive 1-good-neighbor diagnosability of augmented cubes (AQ_n), a class of interconnection networks with excellent topological properties. Through systematic analysis using structural construction, proof by contradiction, and induction, we derive the exact values of the non-inclusive 1-good-neighbor diagnosability of AQ_n under two classic diagnosis models: under the PMC model, $t_{N_1}(AQ_n) = 8n - 27$ for $n \geq 23$; under the MM* model, $t_{N_1}(AQ_n) = 6n - 17$ for $n \geq 13$. These results provide valuable insights for the design and optimization of reliable multiprocessor systems, and lay a foundation for extending the research to $g \geq 2$ or other network topologies.

Keywords

Non-Inclusive Diagnosability, g-Good-Neighbor, Augmented Cubes

1. Introduction

The exponential growth of data volume and the increasing demand for high-performance computing have driven the rapid development of large-scale parallel processing systems. The interconnection network, as the communication backbone of multiprocessor systems, directly determines the efficiency and reliability of the entire system. Its topology, which describes the connection pattern between processors, plays a pivotal role in influencing system performance such as communication latency and fault tolerance [1] [2]. In practical operations, processor faults are inevitable due to hardware aging, environmental interference, or other factors. Therefore, accurately identifying faulty processors (i.e., fault diagnosis) is essential to ensure the continuous and stable operation of the system [3].

To address the fault diagnosis problem, two classic diagnosis models have been proposed: the PMC model [1] and the MM* model [2]. The PMC model diagnoses faults through mutual testing between adjacent processors, while the MM* model relies on comparison-based testing. For these models, two key propositions for distinguishing fault sets have been widely recognized [3] [4]:

Proposition 1. [3] For any two distinct sets $F_1, F_2 \subseteq V(G)$, F_1 and F_2 are distinguishable under the PMC model if and only if there exists a vertex $u \in \overline{F_1 \cup F_2}$ and there exists a vertex $v \in F_1 \Delta F_2$ such that $uv \in E(G)$ (See **Figure 1(a)**).

Proposition 2. [4] For any two distinct sets $F_1, F_2 \subseteq V(G)$, F_1 and F_2 are distinguishable under the MM* model if and only if at least one of the following conditions is satisfied (See **Figure 1(b)**).

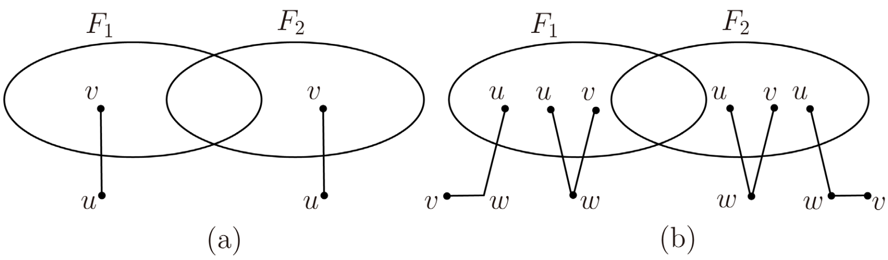

Figure 1. Illustration of Propositions 1.1 and 1.2.

(1) There exist three vertices $u \in F_1 \Delta F_2$ and $v, w \in \overline{F_1 \cup F_2}$ such that $uw, vw \in E(G)$.

(2) There exist three vertices $u, v \in F_1 - F_2$ and $w \in \overline{F_1 \cup F_2}$ such that $uw, vw \in E(G)$.

(3) There exist three vertices $u, v \in F_2 - F_1$ and $w \in \overline{F_1 \cup F_2}$ such that $uw, vw \in E(G)$.

In recent years, extensive research has been conducted on the diagnosability of various interconnection networks, including regular networks, matching composition networks, and lexicographic product networks [5]-[9]. As a generalized form of conditional diagnosability, the g-good-neighbor conditional diagnosability requires that each non-faulty vertex retains at least g neighbors within the non-faulty set, which more closely reflects the actual operating conditions of the system [10]. This metric has been applied to analyze the diagnosability of networks such as k-ary n-cubes, star graphs, and locally exchanged twisted cubes [11]-[13].

However, existing studies primarily focus on inclusive fault sets (i.e., one fault set is a subset of the other), while the probability of non-inclusive fault sets (i.e., neither $F_1 \subseteq F_2$ nor $F_2 \subseteq F_1$) cannot be ignored in practical scenarios [14]. To fill this gap, the concept of non-inclusive diagnosability has been proposed, and relevant research has been carried out for networks such as alternating group graphs, hypercubes, and (n,k)-star networks [15]-[18]. The non-inclusive g-good-neighbor diagnosability, which combines the advantages of non-inclusive fault sets and g-good-neighbor constraints, provides a more comprehensive measure of the system's diagnosis capability [19]. For instance, in large-scale data centers where processors are geographically distributed across multiple clusters, failures often occur independently due to localized issues such as power outages, cooling system malfunctions, or hardware aging. These failures typically affect separate clusters rather than concentrating around a single vertex's neighborhood, resulting in fault sets that do not include all neighbors of any vertex. Similarly, in distributed computing networks spanning multiple regions, communication delays or regional network disruptions can lead to simultaneous but isolated failures, forming non-inclusive fault patterns. The non-inclusive g-good-neighbor diagnosability has been studied in interconnection networks, hypercubes [20] [14].

The augmented cube AQ_n is a promising interconnection network topology with superior properties such as high connectivity, symmetry, and fault tolerance [21]. Previous studies have investigated the diagnosability of AQ_n under different models, and derived results such as its conditional diagnosability and g-good-neighbor conditional diagnosability [22]-[25]. However, the non-inclusive 1-good-neighbor diagnosability of AQ_n remains unaddressed. This study aims to fill this research gap by systematically analyzing the structural characteristics of AQ_n and deriving the exact values of its non-inclusive 1-good-neighbor diagnosability under the PMC and MM* models.

2. Preliminaries

For a simple undirected graph $G=(V(G),E(G))$, let $V(G)$ and $E(G)$ denote the vertex set and edge set of G, respectively. For a vertex $u \in V(G)$, $N_G(u)$ represents the neighborhood of u (i.e., the set of vertices adjacent to u), and $d_G(u)=|N_G(u)|$ denotes the degree of u. A graph G is called k-regular if $d_G(u)=k$ for all $u \in V(G)$. A vertex subset $S \subseteq V(G)$ is referred to as a vertex cut if $G-S$ is disconnected.

The symmetric difference of two sets F_1 and F_2 is defined as $F_1 \Delta F_2 = (F_1 \cup F_2) \setminus (F_1 \cap F_2)$, which consists of elements that belong to exactly one of the two sets. Two sets F_1 and F_2 are defined as non-inclusive if and only if neither F_1 is a subset of F_2 nor F_2 is contained in F_1, mathematically expressed as $F_1 \subsetneq F_2$ and $F_2 \subsetneq F_1$.

Definition 1. [19] A system $G=(V,E)$ is non-inclusive g-good-neighbor t-diagnosable if and only if F_1 and F_2 are distinguishable for any two distinct non-inclusive g-good-neighbor faulty subsets F_1 and F_2 of $V(G)$ with

$|F_1| \le t$ and $|F_2| \le t$. The non-inclusive g-good-neighbor diagnosability of G, denoted by $t_{Ng}(G)$, is the maximum value of the integer t such that G is non-inclusive g-good-neighbor t-diagnosable.

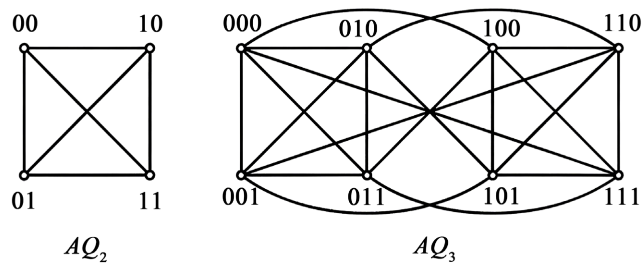

Figure 2. AQ_2 and AQ_3.

Definition 2. [26] AQ_1 is a complete graph K_2 with the vertex set $\{0,1\}$. For $n \ge 2$, AQ_n is obtained by taking two copies of the augmented cube AQ_{n-1}, denoted by AQ_{n-1}^0 and AQ_{n-1}^1, and adding $2 \times 2^{n-1}$ edges between the two as follows:

Let $V(AQ_{n-1}^0) = \{0u_{n-1} \cdots u_2u_1 : u_i = 0 \text{ or } 1\}$ and $V(AQ_{n-1}^1) = \{1u_{n-1} \cdots u_2u_1 : u_i = 0 \text{ or } 1\}$. A vertex $u = 0a_{n-1} \cdots a_2a_1$ of $V(AQ_n^0)$ is adjacent to $v = 1b_{n-1} \cdots b_2b_1 \in V(AQ_n^1)$ of AQ_{n-1}^1 if and only if either

(1) $a_i = b_i$ for $1 \le i \le n-1$; or

(2) $a_i = \overline{b_i}$ for $1 \le i \le n-1$, where $\overline{b_i} = 1 - b_i$.

According to Definition 1 of augmented cubes, we write this recursive construction of AQ_n symbolically as $AQ_n = L \oplus R$, where $L \cong AQ_{n-1}^0$ and $R \cong AQ_{n-1}^1$. We call the edges between L and R crossed edges. Clearly, every vertex of AQ_n is incident with two crossed edges. For an n-bit binary string $u = a_na_{n-1} \cdots a_1$, we use u_i (respectively, $\overline{u}_i$) to denote the binary string $a_n \cdots \overline{a}_i \cdots a_1$ (respectively, $a_n \cdots \overline{a}_i \cdots \overline{a}_1$) which differs with u in the i th bit position (respectively, from the first to the i th bit positions). It is clear that $u_1 = \overline{u}_1$. We use u_1 rather than $\overline{u}_1$ [26].

'The augmented cubes AQ_2 and AQ_3 are shown in Figure 2.

An alternative definition of AQ_n is given in the following.

Definition 3. [26] The augmented cube AQ_n of dimension n has 2^n vertices. Each vertex is labeled by a unique n-bit binary string as its address. Two vertices $u = a_na_{n-1} \cdots a_1$ and $v = b_nb_{n-1} \cdots b_1$ are joined iff either (1) There exists an integer i, $1 \le i \le n$, such that $v = u_i$; in this case, the edge is called a hypercube edge of dimension i, denoted by (u, u_i), or (2) There exists an integer i, $2 \le i \le n$, such that $v = \overline{u}_i$; in this case, the edge is called a complement edge of dimension i, denoted by $(u, \overline{u}_i)$.

Clearly, AQ_n is $(2n-1)$-regular. From the definition of AQ_n, we have the useful properties as follows.

Lemma 3. [27] Let a cycle of four nodes $\{u, v, x, y\}$ in AQ_n, where $v = \overline{u}_2$, $x = \overline{u}_4$, and $y = \overline{x}_2$. $|N(\{u, v, x, y\})| = 8n - 28$ for $n \ge 5$.

Lemma 4. [26] Any two vertices in AQ_n have at most four common neighbors for $n \ge 3$.

Lemma 5 [28] Let X be a vertex set of AQ_n ($n \ge 2$).

1) If $|X| = 2$, then $|N(X)| \ge 4n-8$,

2) If $|X| = 3$, then $|N(X)| \ge 6n-17$,

3) If $|X| = 4$, then $|N(X)| \ge 8n-28$,

4) If $|X| = 5$, then $|N(X)| \ge 10n-31$.

Lemma 6. [27] Let F be a vertices set of AQ_n, $|F| \le 6n-18$ for $n \ge 4$, $AQ_n - F$ is either connected, or it consists of a large component, and all the small components contain at most 2 vertices.

Lemma 7. [28] Let F be a vertices set of AQ_n, $|F| \le 8n-42$ for $n \ge 8$, $AQ_n - F$ is either connected, or it consists of a large component, and all the small components contain at most 3 vertices.

Lemma 8. [28] Let F be a vertices set of AQ_n, $|F| \le 10n-74$ for $n \ge 8$, $AQ_n - F$ is either connected, or it consists of a large component, and all the small components contain at most 4 vertices.

Lemma 9. [29] Let $P = (Y, X, Z)$ be a path of length two in AQ_n between Y and Z for $n \ge 5$. Then $|N_{AQ_n}(P)| \ge 6n-17$ and
$|N_{AQ_n}(X) \cap N_{AQ_n}(Y) \cap N_{AQ_n}(Z)| \le 1$. Furthermore, if $Z = \bar{X}_n$, we have
$|N_{AQ_n}(P)| \ge 6n-15$.

Lemma 10. [30] A graph G has a perfect matching if and only if
$o(G-X) \le |X|$ for all $X \subseteq V$, where $o(G-X)$ is the number of connected components of $G-X$ with odd order.

3. Non-Inclusive 1-Good-Neighbor Diagnosability of AQ_n

3.1. Result under the PMC Model

In this section, we determine the non-inclusive g-good-neighbor diagnosability of AQ_n under the PMC model for $g = 1$.

Lemma 11. Let AQ_n be an n-dimensional augmented cube with $n \ge 9$. Then $t_{N_1}(AQ_n) \le 8n-27$.

Proof. Consider a 4-cycle $A = \{u, v, x, y\}$ in AQ_n, where $v = \bar{u}_2$, $x = \bar{u}_4$, and $y = \bar{x}_2$ (the notation $\bar{u}_i$ denotes the vertex obtained by flipping the i-th bit of u). Let $D = N_{AQ_n}(A)$; define two fault sets $F_1 = \{u, v\} \cup D$ and $F_2 = \{x, y\} \cup D$. By Lemma 5, $|D| = |N_{AQ_n}(\{u, v, x, y\})| = 8n-28$ for $n \ge 5$. It is easy to verify that $F_1 \Delta F_2 = \{u, v, x, y\}$, and neither $F_1 \subseteq F_2$ nor $F_2 \subseteq F_1$, indicating that F_1 and F_2 are non-inclusive sets.

Since $uv \in E(AQ_n)$ and $v \notin F_2$, we have $d_{AQ_n - F_2}(u) \ge 1$ and $d_{AQ_n - F_2}(v) \ge 1$; similarly, $xy \in E(AQ_n)$ implies $d_{AQ_n - F_1}(x) \ge 1$ and $d_{AQ_n - F_1}(y) \ge 1$. Suppose there exists an isolated vertex $w \in V(AQ_n) \setminus (F_1 \cup F_2)$, then $N_{AQ_n}(w) \subseteq F_1 \cap F_2$. By Lemma 4, $|N_{AQ_n}(w) \cap N_{AQ_n}(z)| \le 4$ for any $z \in \{u, v, x, y\}$. Since AQ_n is $(2n-1)$-regular, $|N_{AQ_n}(w)| = 2n-1 \ge 17$ when $n \ge 9$, which implies that at least one neighbor of w is not in $F_1 \cap F_2$. This contradiction shows that

$\delta(AQ_n - F_1) \geq 1$ and $\delta(AQ_n - F_2) \geq 1$, that is, each vertex in $AQ_n - F_1$ and $AQ_n - F_2$ has at least one good neighbors, so F_1 and F_2 are non-inclusive 1-good-neighbor fault sets.

According to the PMC model's diagnosis rules, F_1 and F_2 are indistinguishable because there is no vertex $v \notin F_1 \cup F_2$ that can distinguish them. Thus, AQ_n is not non-inclusive 1-good-neighbor $(8n-26)$-diagnosable, leading to $t_{N_1}(AQ_n) \leq 8n-27$.

Lemma 12. Let AQ_n be an n-dimensional augmented cube with $n \geq 23$. Then $t_{N_1}(AQ_n) \geq 8n-27$.

Proof. Assume for contradiction that there exist two distinct non-inclusive 1-good-neighbor fault sets F_1 and F_2 of AQ_n with $|F_1| \leq 8n-27$ and $|F_2| \leq 8n-27$, such that F_1 and F_2 are indistinguishable under the PMC model. First, we have $|F_1 \cup F_2| \leq |F_1| + |F_2| \leq 16n-54 < 2^n$ for $n \geq 23$, which implies that $V(AQ_n) \setminus (F_1 \cup F_2)$ is non-empty.

Since F_1 and F_2 are 1-good-neighbor fault sets, $\delta(AQ_n - F_1) \geq 1$ and $\delta(AQ_n - F_2) \geq 1$. For $n \geq 23$, $|F_1 \Delta F_2| \leq 8n-29 \leq 10n-74$, and $|V(AQ_n) \setminus (F_1 \cup F_2)| \geq 2^n - (16n-54) > 4$ for $n \geq 7$. This indicates that $AQ_n - (F_1 \cup F_2)$ contains a large connected component with at least 4 vertices.

Since F_1 and F_2 are indistinguishable, according to Proposition 1, all the vertices in $F_1 \Delta F_2$ are not adjacent to each other in $AQ_n - (F_1 \cup F_2)$, which is $N_{AQ_n}(F_1 \Delta F_2) \subseteq F_1 \cap F_2$. By Lemma 5, $|N_{AQ_n}(F_1 \Delta F_2)| \geq 8n-28$ when $|F_1 \Delta F_2| = 4$, which implies $|F_1 \cap F_2| \geq 8n-28$. However,

$$|F_1 \cap F_2| = |F_1| + |F_2| - |F_1 \cup F_2| \leq (8n-27) + (8n-27) - (|F_1 \Delta F_2|) \leq 16n-54-4 = 16n-58$$, which

contradicts $|F_1 \cap F_2| \geq 8n-28$ for $n \geq 23$. Thus, the lemma holds.

Theorem 13. For $n \geq 23$, we have $t_{N_1}(AQ_n) = 8n-27$.

Proof. The conclusion follows directly from Lemmas 11 and 12.

3.2. Result under the MM* Model

Lemma 14. For $n \geq 13$, we have $t_{N_1}(AQ_n) \leq 6n-17$.

Proof. Consider a path $A = \{u, v, w\}$ of order 3 (with two edges) in AQ_n, where $v = \bar{u}_i$ and $w = \bar{u}_j$ (here, $\bar{u}_k$ denotes the vertex formed by flipping the k-th bit of u). Let $D = N_{AQ_n}(A)$, and define two fault sets $F_1 = \{u\} \cup D$ and $F_2 = \{w\} \cup D$. By Lemma 5, the size of D (i.e., the neighborhood of path A) is $|D| = 6n-17$.

It is straightforward to verify that $F_1 \Delta F_2 = \{u, w\}$, and neither F_1 is a subset of F_2 nor F_2 is contained in F_1, which confirms that F_1 and F_2 form a pair of non-inclusive sets. Since $uv \in E(AQ_n)$ and $v \notin F_1 \cup F_2$, we have $d_{AQ_n - F_2}(u) \geq 1$ and $d_{AQ_n - F_2}(v) \geq 1$; similarly, $vw \in E(AQ_n)$ implies $d_{AQ_n - F_1}(v) \geq 1$ and $d_{AQ_n - F_1}(w) \geq 1$.

By the construction of F_1 and F_2, the neighborhood of the vertex set $V(AQ_n) \setminus (F_1 \cup F_2 \cup \{v\})$ is entirely contained in $F_1 \cap F_2$. For $n \geq 13$, $|D| = 6n-17 < 8n-42$, so Lemma 7 guarantees that $AQ_n - (F_1 \cap F_2)$ remains connected. Additionally, the size of $V(AQ_n)\setminus(F_1\cup F_2\cup\{v\})$ satisfies:

$$|V(AQ_n)|-|F_1\cup F_2|-1\geq 2^n-(6n-15)-1=2^n-6n+14>3$$

for $n\geq 13$. This implies that $AQ_n-(F_1\cup F_2\cup\{v\})$ contains a large connected component, and thus $\delta(AQ_n-(F_1\cup F_2\cup\{v\}))\geq 1$.

Under the MM* model, F_1 and F_2 are indistinguishable because there is no vertex outside $F_1\cup F_2$ that can distinguish the two fault sets. Therefore, AQ_n is not non-inclusive 1-good-neighbor $(6n-16)$ -diagnosable, leading to the conclusion that $t_{N_1}(AQ_n)\leq 6n-17$.

Lemma 15. For $n\geq 6$, we have $t_{N_1}(AQ_n)\geq 6n-17$.

Proof. Assume for contradiction that there exist two distinct non-inclusive 1-good-neighbor fault sets F_1 and F_2 of AQ_n such that $|F_1|<6n-17$, $|F_2|<6n-17$, and F_1 and F_2 are indistinguishable under the MM* model. Without loss of generality, suppose $F_1\subsetneq F_2$ and $F_2\subsetneq F_1$.

According to Proposition 2, we can obtain the definition of the MM* model's indistinguishability, that is, for any vertex $w\in V(AQ_n)\setminus(F_1\cup F_2)$, the neighborhood $N_{AQ_n}(w)$ cannot contain vertices from both $F_1\setminus F_2$ and $F_2\setminus F_1$ simultaneously. Let W denote the set of vertices in $V(AQ_n)\setminus(F_1\cup F_2)$ whose neighborhoods intersect with $F_1\Delta F_2$. For $AQ_n=AQ_{n-1}^0\oplus AQ_{n-1}^1$ (the recursive decomposition of augmented cubes), we have $N_{AQ_n}(w)\setminus F_1\subseteq F_2\setminus F_1$ or $N_{AQ_n}(w)\setminus F_2\subseteq F_1\setminus F_2$ for any $w\in W$.

By the structural properties of AQ_n, $|N_{AQ_n}(w)\cap(F_2\setminus F_1)|\leq 6n-19$ and $|N_{AQ_n}(w)\cap(F_1\setminus F_2)|\leq 6n-19$. Since AQ_n is $(2n-1)$ -regular, the size of $V(AQ_n)$ satisfies:

$$2^n=|F_1\cup F_2|+|W|\leq 2(6n-15)=12n-30$$

However, for $n>6$, $2^n>12n-30$, which contradicts the above inequality. Furthermore, $|V(AQ_n)\setminus(F_1\cup F_2\cup W)|>2^n-12n+30>2$ for $n>6$, implying the existence of a connected component G_1 in $AQ_n\setminus(F_1\cup F_2)$ with $|V(G_1)|\geq 3$.

Since F_1 and F_2 are non-inclusive 1-good-neighbor fault sets, $\delta(AQ_n-F_1)\geq 1$ and $\delta(AQ_n-F_2)\geq 1$. The contradiction arises from the fact that the structural constraints of AQ_n force F_1 and F_2 to be distinguishable under the MM* model. Thus, $t_{N_1}(AQ_n)\geq 6n-17$.

Theorem 16. For $n\geq 13$, we have $t_{N_1}(AQ_n)=6n-17$ under the MM* model.

Proof. Combining the results of Lemmas 14 and 15, the non-inclusive 1-good-neighbor diagnosability of AQ_n under the MM* model is exactly $6n-17$ for $n\geq 13$.

4. Discussion

The main results of this paper impose constraints on n: $n\geq 23$ for the PMC model and $n\geq 13$ for the MM* model. These constraints stem from two key aspects of the proof process: 1) The induction step relies on the minimum size of subgraph components in AQ_n. For smaller n, the subgraphs may not retain the required connectivity and expansion properties, leading to the failure of key inequalities in the proof. 2) The verification of 1-good-neighbor condition requires sufficient neighbors for each fault-free vertex. For $n < 13$, the number of neighbors $(2n-1)$ is insufficient to guarantee at least one fault-free neighbor when facing the upper bound of fault sets.

Regarding the tightness of these constraints, we conjecture that they can be partially tightened. For example, preliminary analysis shows that the constraint for the MM* model may be reduced to $n \geq 10$ by optimizing the component size estimation in the proof. However, further verification requires refining the induction base and adjusting the fault set partitioning strategy, which will be the focus of future work.

Extending the current analysis to g-good-neighbor diagnosability with $g \geq 2$ faces three core challenges: 1) Fault set structure complexity: For $g = 2$, each fault-free vertex must have at least two fault-free neighbors, which requires stricter control over the distribution of fault sets. Unlike $g = 1$, the fault sets can no longer be concentrated in local regions, increasing the difficulty of partitioning and analysis. 2) Component connectivity requirements: The proof for $g = 1$ leverages the 1-extra connectivity of AQ_n, but for $g \geq 2$, higher-order extra connectivity parameters (e.g., 2-extra connectivity) are needed and integrating this into diagnosability analysis requires new technical tools. 3. Model-specific constraints: Under the MM* model, the distinguishability conditions for $g \geq 2$ involve more complex test outcome combinations. The mutual testing between fault-free and faulty vertices requires more detailed case analysis, especially for non-adjacent fault-free vertices.

5. Conclusions

This study focuses on the non-inclusive 1-good-neighbor diagnosability of augmented cubes AQ_n, a class of interconnection networks with excellent topological properties. Through systematic analysis of the structural characteristics of AQ_n and rigorous mathematical proof, we derive the exact values of the non-inclusive 1-good-neighbor diagnosability under two classic fault diagnosis models: 1) Under the PMC model, $t_{N_1}(AQ_n) = 8n - 27$ for $n \geq 23$; 2) Under the MM* model, $t_{N_1}(AQ_n) = 6n - 17$ for $n \geq 13$.

These findings quantify the fault tolerance capability of augmented cubes in the context of non-inclusive fault sets and 1-good-neighbor constraints, providing valuable theoretical support for the design and optimization of reliable multiprocessor systems. The proof methods adopted in this study, including structural construction, proof by contradiction, and induction, can be extended to the analysis of other interconnection network topologies.

Future research directions may include extending the results to $g \geq 2$ (i.e., non-inclusive g-good-neighbor diagnosability) and investigating the non-inclusive diagnosability of other network structures such as folded hypercubes, alternating group graphs, and (n,k)-star networks. Additionally, exploring the non-inclusive diagnosability under more realistic fault models (e.g., intermittent fault models) could further advance the research on network reliability.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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