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Mapping Properties for Fractional Integral Related to Schrödinger Operator on Generalized Fractional Mixed Morrey Spaces

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Abstract

This paper establishes the boundedness of fractional integral operators associated with Schrödinger operators and their commutators with $BMO_\theta(\rho)$ functions on generalized fractional mixed Morrey spaces. This is achieved by employing function decomposition and stratification techniques in these spaces, while drawing on known boundedness results in the context of mixed Lebesgue spaces.

Keywords

Schrödinger Operator, Fractional Integral, Generalized Fractional Mixed Morrey Space, Commutator, Boundedness

1. Introduction

The Schrödinger operator holds significant importance in harmonic analysis and partial differential equations. In 1983, Fefferman [1] studied the eigenvalues of the Schrödinger operator L, highlighted the limitations of the volume counting method, and provided more precise eigenvalue estimates via the Sobolev, Aronszajn, and K.T. Smith (SAK) principle. In 1995, Shen [2] investigated the L^p boundedness of the Schrödinger operator L with specific potentials. Later, in 1999, Zhong [3] studied estimates for solutions of the Schrödinger operator L. In 2009, Yang *et al.* [4], while investigating the endpoint estimates for Riesz transforms, maximal operators, and fractional integrals related to the Schrödinger operator L, established a characterization of local Hardy spaces. In recent years, the mapping properties of integrals associated with the Schrödinger operator on various function spaces have attracted widespread attention. In 2024, Guliyev *et al.* [5] studied the boundedness of fractional integrals related to the Schrödinger operator on generalized mixed Morrey spaces. In 2025, Xue *et al.* [6] studied the boundedness of Marcinkiewicz integrals related to the Schrödinger operator on Campanato-type spaces.

From a broader perspective, establishing the boundedness of operators on various function spaces within the framework of Schrödinger operator with perturbed potentials is a central issue in harmonic analysis and PDE theory. Generalized fractional mixed Morrey spaces, as an extension of mixed-norm spaces and classical Morrey spaces, provide a suitable functional framework for studying the regularity of solutions to partial differential equations with anisotropic or non-standard growth conditions. This paper aims to extend the boundedness results of fractional integral operators and their commutators associated with Schrödinger operator from classical Lebesgue spaces, Morrey spaces, and their mixed-norm variants to generalized fractional mixed Morrey spaces. Such results are of significant importance for studying the regularity, uniqueness of solutions to Schrödinger equations, and the well-posedness of related variational problems, particularly when dealing with potential functions V exhibiting critical growth or solutions with localized anisotropic behaviors.

This paper studies second-order Schrödinger differential operators defined on $\mathbb{R}^n\left(n\geq 3\right)$:

$$L=-\Delta+V,$$

where $-\Delta$ is the Laplace operator, and V is a non-negative function belonging to the reverse Hölder class RH_q with $q\geq\frac{n}{2}$. Specifically, let V be a non-negative, locally L_q-integrable function on $\mathbb{R}^n$. If there exists a constant $C>0$ such that the reverse Hölder inequality

$$\left(\frac{1}{|B|}\int_B V^q\left(x\right)\mathrm{d}x\right)^{\frac{1}{q}}\leq\frac{C}{|B|}\int_B V\left(x\right)\mathrm{d}x$$

holds for every ball B, then V is said to belong to $V\in RH_q\left(1<p\leq\infty\right)$. pical examples include non-negative polynomials $V\in RH_\infty$, particularly $|x|\in RH_\infty$.

For a potential function $V\in RH_q$ with $q\geq\frac{n}{2}$, the critical radius function is defined as

$$\rho\left(x\right)=\frac{1}{m_V\left(x\right)}=\sup_{r>0}\left\{r:\frac{1}{r^{n-2}}\int_{B(x,r)}V\left(y\right)\mathrm{d}y\leq 1\right\},\ x\in\mathbb{R}^n.$$

It can be observed that: when $V\neq 0$, $0<m_V\left(x\right)<\infty$; when $V=1$, $m_V\left(x\right)=1$; and when $V\left(x\right)=|x|^2$, $m_V\left(x\right)\approx 1+|x|$.

Based on the heat diffusion semigroup e^{-tL}, for a sufficiently regular function f, the negative power integral operator $L^{\frac{-\beta}{2}}\left(\beta>0\right)$ associated with the Schrödinger operator L can be expressed as

$$I_\beta^L f(x) = L^{\frac{-\beta}{2}} f(x) = \int_0^\infty e^{-tL}(f)(x) t^{\frac{\beta}{2}-1} dt,\ 0<\beta<n.$$

According to existing research results [7], for sufficiently regular functions f

$$I_\beta^L f(x) = \int_{\mathbb{R}^n} K_\beta(x,y) f(y) dy,\ 0<\beta<n, \tag{1.1}$$

where the integral kernel $K_\beta(x,y)$ satisfies the estimate:

$$\left|K_\beta(x,y)\right| \le \frac{C}{\left(1+\frac{|x-y|}{\rho(x)}\right)^N} \frac{1}{|x-y|^{n-\beta}}.$$

In particular, for $0<\beta<n$, $\left|K_\beta(x,y)\right| \le \frac{C}{|x-y|^{n-\beta}}$.

The commutator operator [5] is defined as

$$\left[b, I_\beta^L\right] f(x) = b(x) I_\beta^L f(x) - I_\beta^L(bf)(x).$$

It is noteworthy that when $L=-\Delta$, I_β^L and $\left[b, I_\beta^L\right]$ reduce to the classical Riesz potential and its commutator, respectively:

$$I_\beta f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^n} dy,\ \left[b, I_\beta\right] f(x) = \int_{\mathbb{R}^n} \frac{b(x)-b(y)}{|x-y|^n} f(y) dy.$$

In 1961, Benedek and Panzone [8] first introduced the definition of mixed-norm Lebesgue spaces. Mixed Lebesgue spaces play a crucial role in mathematics and have garnered widespread attention from researchers; see [9]-[11]. In 1938, to study the regularity of second-order elliptic partial differential equations, Morrey introduced the definition of Morrey spaces [7]. In 2019, Nogayama [12] introduced the definition of mixed Morrey spaces and proved the boundedness of some operators on these spaces, In 2021, Nogayama *et al.* [13] established new norm estimates for mixed Morrey spaces. In 2022, Wei [14] proposed the definition of generalized mixed Morrey spaces. In 2023, the concept of generalized fractional mixed Morrey spaces was defined in [15] and the properties of these spaces, as well as the boundedness of various operators on them, have been thoroughly studied; see [15].

Although the techniques used in the proofs—such as ball decompositions, truncations, and the known boundedness in Lebesgue spaces—are standard in harmonic analysis, extending the fractional integral operators for Schrödinger operators to the generalized fractional mixed Morrey spaces $L_{\alpha,V}^{\vec{p},\eta,\varphi}\left(\mathbb{R}^n\right)$ presents several nontrivial challenges. First, the mixed norms cause the norm of the characteristic function χ_B of a ball to no longer be simply $|B|^{\frac{1}{p}}$ (Lemma 2.3), which complicates the scaling analysis. Second, the introduction of the generalized function φ makes the geometric structure of the space depend on the growth of the scale r, and its doubling condition (1.3) must be precisely controlled in the summation estimates throughout the proof to ensure the convergence of the series. Third, the Schrödinger potential V introduces a new "metric" through the critical radius function $\rho(x)$, which interacts with the Euclidean metric. This requires simultaneously handling the decay factor $\left(1+\frac{2^k r}{\rho(x)}\right)^{-\alpha}$ and the growth factor of $BMO_\theta(\rho)$ functions in pointwise estimates and commutator estimates. Managing the interplay among these three elements and deriving the precise constant condition C_φ to ensure the convergence of the series constitute the core difficulties of this work.

Inspired by the above results, this paper will study the boundedness of fractional integrals associated with the Schrödinger operator on generalized fractional mixed Morrey spaces. To this end, it is necessary to recall some concepts and notations.

Definition 1.1 [8] The Lebesgue space with mixed norm $L^{\vec{p}}(\mathbb{R}^n)$ was introduced by Benedek and Panzone, as follows

$$L^{\vec{p}}(\mathbb{R}^n)=\left\{f\in L^0(\mathbb{R}^n):\|f\|_{L^{\vec{p}}(\mathbb{R}^n)}<\infty\right\},$$

where $L^0(\mathbb{R}^n)$ denotes the set of all Lebesgue measurable functions on $\mathbb{R}^n$, $\vec{p}=(p_1,\cdots,p_n)\in(0,\infty]^n$ and the definition of the mixed Lebesgue norm $\|\cdot\|_{L^{\vec{p}}(\mathbb{R}^n)}$ is

$$\|f\|_{\vec{p}}=\|f\|_{(p_1,p_2,\cdots,p_n)}$$
$$=\left(\int_R\cdots\left(\int_R\left(\int_R|f(x_1,x_2,\cdots,x_n)|^{p_1}\mathrm{d}x_1\right)^{\frac{p_2}{p_1}}\mathrm{d}x_2\right)^{\frac{p_3}{p_2}}\cdots\mathrm{d}x_n\right)^{\frac{1}{p_n}}. \quad (1.2)$$

Definition 1.2 Let φ be a positive, increasing function defined on $(0,\infty)$ and there exists a positive constant C_φ such that

$$\varphi(2t)\le C_\varphi\varphi(t),\ \text{for any } t>0. \quad (1.3)$$

The best possible constant C_φ in (1.3) is called the **doubling constant** for the φ.

Let $\eta\in[0,n)$ and $\vec{p}=(p_1,\cdots,p_n)\in(0,\infty]^n$ satisfy

$$\sum_{j=1}^n\frac{1}{p_j}>\eta,$$

and $\alpha\ge0$, $V\in RH_q$, $q\ge1$, then the generalized fractional mixed Morrey space norm $\|\cdot\|_{L^{\vec{p},\eta,\varphi}_{\alpha,V}(\mathbb{R}^n)}$ is defined by

$$\|f\|_{L^{\vec{p},\eta,\varphi}_{\alpha,V}(\mathbb{R}^n)}=\sup_{x\in\mathbb{R}^n,r>0}\left[\varphi(r)\right]^{\frac{\eta}{n}-\frac{1}{n}\sum_{j=1}^n\frac{1}{p_j}}\left\|f\chi_{B(x,r)}\right\|_{\vec{p}} \quad (1.4)$$

for $f\in L^0(\mathbb{R}^n)$. We define the generalized fractional mixed Morrey space $L^{\vec{p},\eta,\varphi}_{\alpha,V}(\mathbb{R}^n)$ to be the set of all $f\in L^0(\mathbb{R}^n)$ with $\|f\|_{L^{\vec{p},\eta,\varphi}_{\alpha,V}(\mathbb{R}^n)}<\infty$.

Remark 1.3 If we set $p_j=p\,(j=1,\cdots,n)$, in (1.4), then the generalized fractional mixed Morrey space $L^{\vec{p},\eta,\varphi}_{\alpha,V}(\mathbb{R}^n)$ coincides with the generalized fractional Morrey space $L^{p,\eta,\varphi}_{\alpha,V}(\mathbb{R}^n)$.

Definition 1.4 [16] $\left(BMO_\theta(\rho)(\mathbb{R}^n)\right)$ For $\theta > 0$, this space is defined as the set of all locally integrable functions b satisfying

$$\frac{1}{|B(x,r)|}\int_{B(x,r)}|b(y)-b_B|\,\mathrm{d}y \le C\left(1+\frac{r}{\rho(x)}\right)^{\theta}$$

for all $x \in \mathbb{R}^n$ and $r > 0$, where b_B denotes the average of b over the ball B. The norm of $b \in BMO_\theta(\rho)$ is denoted by $[b]_\theta$. Clearly, $BMO(\mathbb{R}^n) \subset BMO_\theta(\rho)(\mathbb{R}^n)$.

Throughout this paper, the letter $\vec{p}$ denotes n-tuples of the number in $(0,\infty]$, $n \ge 1$, $\vec{p} = (p_1, p_2, \cdots, p_n)$, $1 \le \vec{p} < \infty$ means $1 \le p_i < \infty$ for each i, For $1 \le \vec{p} \le \infty$, we denote $\vec{p}' = (p_1', p_2', \cdots, p_n')$, where p_i' satisfies $1/p_i + 1/p_i' = 1$.

2. Preliminaries

Before proving the main results, we need to recall the following lemmas.

Lemma 2.1 [8] (Hölder's inequality on mixed Lebesgue spaces) Let $1 \le \vec{p} \le \infty$ and $\frac{1}{\vec{p}} + \frac{1}{\vec{p}'} = 1$.Then for any $f \in L_{\vec{p}}$ and $g \in L_{\vec{p}'}$, we have

$$\int_{\mathbb{R}^n} f(x)g(x)\,\mathrm{d}x \le \|f\|_{L_{\vec{p}}}\|g\|_{L_{\vec{p}'}}.$$

Lemma 2.2 [8] (Minkowski's inequality on mixed Lebesgue spaces) Let $1 \le \vec{p} \le \infty$, If $f, g \in L_{\vec{p}}$, then

$$\|f+g\|_{L_{\vec{p}}} \le \|f\|_{L_{\vec{p}}} + \|g\|_{L_{\vec{p}}}.$$

Lemma 2.3 [12] Let $0 < \vec{p} < \infty$, and B be a ball in $\mathbb{R}^n$. Then

$$\|\chi_B\|_{L_{\vec{p}}} = \|\chi_B\|_{WL_{\vec{p}}} = |B|^{\frac{1}{n}\sum_{i=1}^{n}\frac{1}{p_i}}.$$

Lemma 2.4 [2] Let $V \in RH_{\frac{n}{2}}$. For the associated function ρ, there exist constants C and $k_0 \ge 1$, such that for all $x, y \in \mathbb{R}^n$,

$$C^{-1}\rho(x)\left(1+\frac{|x-y|}{\rho(x)}\right)^{-k_0} \le \rho(y) \le C\rho(x)\left(1+\frac{|x-y|}{\rho(x)}\right)^{\frac{k_0}{1+k_0}}.$$

Lemma 2.5 [17] Let $x \in B(x_0, r)$. Then for $k \in \mathbb{Z}$,

$$\frac{1}{\left(1+\frac{2^k r}{\rho(x)}\right)^{N}} \le C\frac{1}{\left(1+\frac{2^k r}{\rho(x_0)}\right)^{\frac{N}{k_0+1}}}.$$

Lemma 2.6 [16] Let $1 \le s < \infty$. If $b \in BMO_\theta(\rho)(\mathbb{R}^n)$, then for any ball $B = B(x,r)$ with center $x \in \mathbb{R}^n$ and radius $r > 0$,

$$\left(\frac{1}{|B|}\int_B |b(y)-b_B|^s\,\mathrm{d}x\right)^{\frac{1}{s}} \le \|b\|_{BMO_\theta(\rho)(\mathbb{R}^n)}\left(1+\frac{r}{\rho(x)}\right)^{\theta'},$$

where $\theta' = (k_0+1)\theta$, and k_0 is the constant from Lemma 2.4.

Lemma 2.7 [16] Let $1 \le s < \infty$. If $b \in BMO_{\theta}(\rho)(\mathbb{R}^n)$, and $B = B(x,r)$, then for all $k \in \mathbb{N}$

$$\left(\frac{1}{\left|2^k B\right|}\int_{2^k B}\left|b(y)-b_B\right|^s \mathrm{d}x\right)^{\frac{1}{s}} \le \|b\|_{BMO_{\theta}(\rho)(\mathbb{R}^n)}\, k\left(1+\frac{2^k r}{\rho(x)}\right)^{\theta'},$$

where $\theta' = (k_0+1)\theta$.

3. Main Results

Theorem 3.1. Let $V \in RH_{\frac{n}{2}}$, $\alpha \ge 0$, $0 < \beta < \eta < n$, $0 < C_{\varphi} < 2^{n+\alpha\Big/\left(\frac{1}{n}\sum_{j=1}^{n}\frac{1}{p_j}-\frac{\eta}{n}\right)}$, $1 < \vec{p} < \vec{q} < \infty$, and satisfy $\sum_{j=1}^{n}\frac{1}{q_j} = \sum_{j=1}^{n}\frac{1}{p_j} - \beta$. Assume that I_{β}^{L} is defined as in (1.1), and $\sum_{j=1}^{n}\frac{1}{p_j} > \eta$. Then there exists a positive constant C such that for all $f \in L_{\alpha,V}^{\vec{p},\eta,\varphi}(\mathbb{R}^n)$,

$$\left\|I_{\beta}^{L}(f)\right\|_{L_{\alpha,V}^{\vec{q},\eta-\beta,\varphi}(\mathbb{R}^n)} \le C\|f\|_{L_{\alpha,V}^{\vec{p},\eta,\varphi}(\mathbb{R}^n)}.$$

Proof of Theorem 3.1 Let $B = B(x_0,r)$ be an open ball centered at $x_0 \in \mathbb{R}^n$, with radius $r > 0$. We can decompose the function f as

$$f = f_1 + f_2 = f\chi_{2B} + f\chi_{\mathbb{R}^n\setminus(2B)}.$$

By Lemma 2.2, we have

$$\left\|I_{\beta}^{L}(f)\right\|_{L_{\alpha,V}^{\vec{q},\eta-\beta,\varphi}(\mathbb{R}^n)} \le \left\|I_{\beta}^{L}(f_1)\right\|_{L_{\alpha,V}^{\vec{q},\eta-\beta,\varphi}(\mathbb{R}^n)} + \left\|I_{\beta}^{L}(f_2)\right\|_{L_{\alpha,V}^{\vec{q},\eta-\beta,\varphi}(\mathbb{R}^n)} = D_1 + D_2.$$

By (1.3), (1.4) and the boundedness of I_{β}^{L} from $L_{\vec{p}}(\mathbb{R}^n)$ to $L_{\vec{q}}(\mathbb{R}^n)$ [8], we have

$$\begin{aligned}
D_1 &= \sup_{x_0\in\mathbb{R}^n, r>0}\left(1+\frac{r}{\rho(x_0)}\right)^{\alpha}\left[\varphi(r)\right]^{\frac{\eta-\beta}{n}-\frac{1}{n}\sum_{j=1}^{n}\frac{1}{q_j}}\left\|I_{\beta}^{L}(f_1)\right\|_{L^{\vec{q}}(\mathbb{R}^n)}\\
&\le C\sup_{x_0\in\mathbb{R}^n, r>0}\left(1+\frac{r}{\rho(x_0)}\right)^{\alpha}\left[\varphi(r)\right]^{\frac{\eta-\beta}{n}-\frac{1}{n}\sum_{j=1}^{n}\frac{1}{q_j}}\left\|f\chi_{2B}\right\|_{L^{\vec{p}}(\mathbb{R}^n)}\\
&\le C\sup_{x_0\in\mathbb{R}^n, r>0}\left(1+\frac{2r}{\rho(x_0)}\right)^{\alpha}\left[\varphi(r)\right]^{\frac{\eta-\beta}{n}-\frac{1}{n}\sum_{j=1}^{n}\frac{1}{q_j}}\left[\varphi(2r)\right]^{\frac{1}{n}\sum_{j=1}^{n}\frac{1}{p_j}-\frac{\eta}{n}}\\
&\quad\times\left[\varphi(2r)\right]^{\frac{\eta}{n}-\frac{1}{n}\sum_{j=1}^{n}\frac{1}{p_j}}\left\|f\chi_{2B}\right\|_{L^{\vec{p}}(\mathbb{R}^n)}\\
&\le C\|f\|_{L_{\alpha,V}^{\vec{p},\eta,\varphi}(\mathbb{R}^n)}\sup_{x_0\in\mathbb{R}^n, r0}\left[\varphi(r)\right]^{\frac{\eta-\beta}{n}-\frac{1}{n}\sum_{j=1}^{n}\frac{1}{p_j}+\frac{\beta}{n}}\left[\varphi(2r)\right]^{\frac{1}{n}\sum_{j=1}^{n}\frac{1}{p_j}-\frac{\eta}{n}}\\
&\le C\|f\|_{L_{\alpha,V}^{\vec{p},\eta,\varphi}(\mathbb{R}^n)}\sup_{x_0\in\mathbb{R}^n, r0}\left[\frac{C_{\varphi}\varphi(r)}{\varphi(r)}\right]^{\frac{1}{n}\sum_{j=1}^{n}\frac{1}{p_j}-\frac{\eta}{n}}\\
&\le C\|f\|_{L_{\alpha,V}^{\vec{p},\eta,\varphi}(\mathbb{R}^n)}.
\end{aligned}$$

To estimate D_2, we first consider $\left|I_\beta^L\left(f_2\right)(y)\right|$ for $y\in B\left(x_0,r\right)$. By (1.1), (1.4), and Lemma 2.1, we have

$$
\begin{aligned}
&\left|I_\beta^L\left(f_2\right)(y)\right|\\
&\le C\int_{\mathbb{R}^n\setminus(2B)}\frac{\left|f(z)\right|}{\left|y-z\right|^{n-\beta}}\mathrm{d}z\le C\sum_{k=1}^{\infty}\int_{2^{k+1}B\setminus\left(2^kB\right)}\frac{\left|f(z)\right|}{\left|y-z\right|^{n-\beta}}\mathrm{d}z\\
&\le C\sum_{k=1}^{\infty}\frac{1}{\left|2^kB\right|^{1-\frac{\beta}{n}}}\int_{2^{k+1}B}\left|f(z)\right|\mathrm{d}z\\
&\le C\sum_{k=1}^{\infty}\frac{1}{\left|2^kB\right|^{1-\frac{\beta}{n}}}\left\|f\chi_{2^{k+1}B}\right\|_{L^{\vec{p}}\left(\mathbb{R}^n\right)}\left\|\chi_{2^{k+1}B}\right\|_{L^{\vec{p}'}\left(\mathbb{R}^n\right)}\\
&\le C\sum_{k=1}^{\infty}\frac{\left\|\chi_{2^{k+1}B}\right\|_{L^{\vec{p}'}\left(\mathbb{R}^n\right)}}{\left|2^kB\right|^{1-\frac{\beta}{n}}}\left[\varphi\left(2^{k+1}r\right)\right]^{\frac{1}{n}\sum_{j=1}^{n}\frac{1}{p_j}-\frac{\eta}{n}}\left[\varphi\left(2^{k+1}r\right)\right]^{\frac{\eta}{n}-\frac{1}{n}\sum_{j=1}^{n}\frac{1}{p_j}}\left\|f\chi_{2^{k+1}B}\right\|_{L^{\vec{p}}\left(\mathbb{R}^n\right)}\\
&\le C\left\|f\right\|_{L_{\alpha,V}^{\vec{p},\eta,\varphi}\left(\mathbb{R}^n\right)}\sum_{k=1}^{\infty}\left(1+\frac{2^{k+1}r}{\rho\left(x_0\right)}\right)^{-\alpha}\frac{\left\|\chi_{2^{k+1}B}\right\|_{L^{\vec{p}'}\left(\mathbb{R}^n\right)}}{\left|2^kB\right|^{1-\frac{\beta}{n}}}\left[\varphi\left(2^{k+1}r\right)\right]^{\frac{1}{n}\sum_{j=1}^{n}\frac{1}{p_j}-\frac{\eta}{n}}
\end{aligned}
$$

Furthermore, by (1.3), (1.4) and Lemma 2.3, we have

$$
\begin{aligned}
D_2&=\sup_{x_0\in\mathbb{R}^n,r>0}\left(1+\frac{r}{\rho\left(x_0\right)}\right)^{\alpha}\left[\varphi(r)\right]^{\frac{\eta-\beta}{n}-\frac{1}{n}\sum_{j=1}^{n}\frac{1}{q_j}}\left\|I_\beta^L\left(f_2\right)\right\|_{L^{\vec{q}}\left(B\left(x_0,r\right)\right)}\\
&\le C\left\|f\right\|_{L_{\alpha,V}^{\vec{p},\eta,\varphi}\left(\mathbb{R}^n\right)}\sup_{x_0\in\mathbb{R}^n,r>0}\left(1+\frac{r}{\rho\left(x_0\right)}\right)^{\alpha}\left[\varphi(r)\right]^{\frac{\eta-\beta}{n}-\frac{1}{n}\sum_{j=1}^{n}\frac{1}{q_j}}\left\|\chi_{B\left(x_0,r\right)}\right\|_{L^{\vec{q}}\left(\mathbb{R}^n\right)}\\
&\quad\times\sum_{k=1}^{\infty}\left(1+\frac{2^{k+1}r}{\rho\left(x_0\right)}\right)^{-\alpha}\frac{\left\|\chi_{2^{k+1}B}\right\|_{L^{\vec{p}'}\left(\mathbb{R}^n\right)}}{\left|2^kB\right|^{1-\frac{\beta}{n}}}\left[\varphi\left(2^{k+1}r\right)\right]^{\frac{1}{n}\sum_{j=1}^{n}\frac{1}{p_j}-\frac{\eta}{n}}\\
&\le C\left\|f\right\|_{L_{\alpha,V}^{\vec{p},\eta,\varphi}\left(\mathbb{R}^n\right)}\sup_{x_0\in\mathbb{R}^n,r>0}\left[\varphi(r)\right]^{\frac{\eta-\beta}{n}-\frac{1}{n}\sum_{j=1}^{n}\frac{1}{q_j}}\left\|\chi_{B\left(x_0,r\right)}\right\|_{L^{\vec{q}}\left(\mathbb{R}^n\right)}\\
&\quad\times\sum_{k=1}^{\infty}\left(1+\frac{2^{k+2}r}{\rho\left(x_0\right)}\right)^{-\alpha}\frac{\left\|\chi_{2^{k+1}B}\right\|_{L^{\vec{p}'}\left(\mathbb{R}^n\right)}}{\left|2^kB\right|^{1-\frac{\beta}{n}}}\left[\varphi\left(2^{k+1}r\right)\right]^{\frac{1}{n}\sum_{j=1}^{n}\frac{1}{p_j}-\frac{\eta}{n}}\\
&\le C\left\|f\right\|_{L_{\alpha,V}^{\vec{p},\eta,\varphi}\left(\mathbb{R}^n\right)}\sup_{x_0\in\mathbb{R}^n,r>0}\left[\varphi(r)\right]^{\frac{\eta}{n}-\frac{1}{n}\sum_{j=1}^{n}\frac{1}{p_j}}\left|B\left(x_0,r\right)\right|^{\frac{1}{n}\sum_{j=1}^{n}\frac{1}{q_j}}\\
&\quad\times\sum_{k=1}^{\infty}\left(1+\frac{2^{k+2}r}{\rho\left(x_0\right)}\right)^{-\alpha}\frac{\left|B\left(x_0,2^{k+1}r\right)\right|^{\frac{1}{n}\sum_{j=1}^{n}\frac{1}{p_j'}}}{\left|2^kB\right|^{1-\frac{\beta}{n}}}\left[\varphi\left(2^{k+1}r\right)\right]^{\frac{1}{n}\sum_{j=1}^{n}\frac{1}{p_j}-\frac{\eta}{n}}\\
&\le C\left\|f\right\|_{L_{\alpha,V}^{\vec{p},\eta,\varphi}\left(\mathbb{R}^n\right)}\sup_{x_0\in\mathbb{R}^n,r>0}\sum_{k=1}^{\infty}\frac{\left|B\left(x_0,r\right)\right|^{\frac{1}{n}\sum_{j=1}^{n}\frac{1}{p_j}-\frac{\beta}{n}}\left(1+\frac{2^{k+2}r}{\rho\left(x_0\right)}\right)^{-\alpha}}{\left|B\left(x_0,2^{k+1}r\right)\right|^{\frac{1}{n}\sum_{j=1}^{n}\frac{1}{p_j}-\frac{\beta}{n}}}\left[\frac{\varphi\left(2^{k+1}r\right)}{\varphi(r)}\right]^{\frac{1}{n}\sum_{j=1}^{n}\frac{1}{p_j}-\frac{\eta}{n}}
\end{aligned}
$$

$$\begin{aligned}
&\le C\|f\|_{L_{\alpha,V}^{\vec{p},\eta,\varphi}(\mathbb{R}^n)} \sup_{x_0\in\mathbb{R}^n, r>0} \sum_{k=1}^{\infty} \frac{\left(1+\frac{2^{k+2}r}{\rho(x_0)}\right)^{-\alpha}}{\left[2^{(k+1)n}\right]^{\frac{1}{n}\sum_{j=1}^{n}\frac{1}{p_j}-\frac{\beta}{n}}} \left[\frac{C_\varphi^{k+1}\varphi(r)}{\varphi(r)}\right]^{\frac{1}{n}\sum_{j=1}^{n}\frac{1}{p_j}-\frac{\eta}{n}} \\
&\le C\|f\|_{L_{\alpha,V}^{\vec{p},\eta,\varphi}(\mathbb{R}^n)} \sum_{k=1}^{\infty}\left(1+\frac{2^{k+2}r}{\rho(x_0)}\right)^{-\alpha} \frac{\left[C_\varphi^{k+1}\right]^{\frac{1}{n}\sum_{j=1}^{n}\frac{1}{p_j}-\frac{\eta}{n}}}{\left[2^{(k+1)n}\right]^{\frac{1}{n}\sum_{j=1}^{n}\frac{1}{p_j}-\frac{\beta}{n}}} \\
&\le C\|f\|_{L_{\alpha,V}^{\vec{p},\eta,\varphi}(\mathbb{R}^n)} \sum_{k=1}^{\infty}\left(1+\frac{2^{k+2}r}{\rho(x_0)}\right)^{-\alpha} \left[\frac{C_\varphi}{2^n}\right]^{(k+1)\left(\frac{1}{n}\sum_{j=1}^{n}\frac{1}{p_j}-\frac{\eta}{n}\right)} \\
&\le C\|f\|_{L_{\alpha,V}^{\vec{p},\eta,\varphi}(\mathbb{R}^n)},
\end{aligned}$$

where the series

$$\begin{aligned}
&\sum_{k=1}^{\infty}\left(1+\frac{2^{k+2}r}{\rho(x_0)}\right)^{-\alpha} \left[\frac{C_\varphi}{2^n}\right]^{(k+1)\left(\frac{1}{n}\sum_{j=1}^{n}\frac{1}{p_j}-\frac{\eta}{n}\right)} \\
&\approx \sum_{k=1}^{\infty} C\cdot 2^{-\alpha k} a^{(k+1)b} = \sum_{k=1}^{\infty} C\left(a^b\cdot 2^{-\alpha}\right)^k = \frac{C\left(a^b\cdot 2^{-\alpha}\right)}{1-\left(a^b\cdot 2^{-\alpha}\right)},
\end{aligned}$$

when $k\to\infty$, $\left(1+\frac{2^{k+2}r}{\rho(x_0)}\right)^{-\alpha} \approx \left(\frac{2^{k+2}r}{\rho(x_0)}\right)^{-\alpha} = C\cdot 2^{-\alpha k}$.

Explanation on the convergence condition: The series convergence requires $a^b\cdot 2^{-\alpha}<1$, where $a=\frac{C_\varphi}{2^n}$, and $b=\frac{1}{n}\sum_{j=1}^{n}\frac{1}{p_j}-\frac{\eta}{n}>0$. Taking logarithms yields:

$$b\left(\ln C_\varphi - n\ln 2\right)-\alpha\ln 2<0,$$

Rearranging gives:

$$\ln C_\varphi < n\ln 2+\frac{\alpha}{b}\ln 2,$$

which is equivalent to:

$$C_\varphi < 2^{n+\frac{\alpha}{b}} = 2^{n+\alpha\Big/\left(\frac{1}{n}\sum_{j=1}^{n}\frac{1}{p_j}-\frac{\eta}{n}\right)}.$$

Combining the estimates for D_1, Theorem 3.1 is proved.

Theorem 3.2. Let $V\in RH_{\frac{n}{2}}$, $\alpha\ge 0$, $b\in BMO_\theta(\rho)(\mathbb{R}^n)$, $0<\beta<\eta<n$, $0<C_\varphi<2^{n+\alpha\Big/\left(\frac{1}{n}\sum_{j=1}^{n}\frac{1}{p_j}-\frac{\eta}{n}\right)}$, $1<\vec{p}<\vec{q}<\infty$, and satisfy $\sum_{j=1}^{n}\frac{1}{q_j}=\sum_{j=1}^{n}\frac{1}{p_j}-\beta$. Assume that I_β^L is defined as in (1.1), and $\sum_{j=1}^{n}\frac{1}{p_j}>\eta$. Then there exists a positive constant C, such that for all $f\in L_{\alpha,V}^{\vec{p},\eta,\varphi}(\mathbb{R}^n)$,

$$\left\|\left[b,I_\beta^L\right](f)\right\|_{L_{\alpha,V}^{\vec{q},\eta-\beta,\varphi}(\mathbb{R}^n)} \le C\|b\|_{BMO_\theta(\rho)(\mathbb{R}^n)}\|f\|_{L_{\alpha,V}^{\vec{p},\eta,\varphi}(\mathbb{R}^n)}.$$

Proof of Theorem 3.2 Let $B=B(x_0,r)$ be an open ball centered at $x_0\in\mathbb{R}^n$

with radius $r>0$. We can decompose the function f as

$$f=f_1+f_2=f\chi_{2B}+f\chi_{\mathbb{R}^n\setminus(2B)}.$$

Using Lemma 2.2, we have

$$\begin{aligned}\left\|\left[b,I_\beta^L\right](f)\right\|_{L_{\alpha,V}^{\vec{q},\eta-\beta,\varphi}\left(\mathbb{R}^n\right)} &\le \left\|\left[b,I_\beta^L\right](f_1)\right\|_{L_{\alpha,V}^{\vec{q},\eta-\beta,\varphi}\left(\mathbb{R}^n\right)}+\left\|\left[b,I_\beta^L\right](f_2)\right\|_{L_{\alpha,V}^{\vec{q},\eta-\beta,\varphi}\left(\mathbb{R}^n\right)}\\ &=E_1+E_2.\end{aligned}$$

By (1.3), (1.4) and the boundedness of $\left[b,I_\beta^L\right]$ form $L_{\vec{p}}\left(\mathbb{R}^n\right)$ to $L_{\vec{q}}\left(\mathbb{R}^n\right)$ [18], we have

$$\begin{aligned}E_1 &= \sup_{x_0\in\mathbb{R}^n,r>0}\left(1+\frac{r}{\rho(x_0)}\right)^{\alpha}\left[\varphi(r)\right]^{\frac{\eta-\beta}{n}-\frac{1}{n}\sum_{j=1}^{n}\frac{1}{q_j}}\left\|\left[b,I_\beta^L\right](f_1)\right\|_{L^{\vec{q}}\left(\mathbb{R}^n\right)}\\
&\le C\|b\|_{BMO_\theta(\rho)\left(\mathbb{R}^n\right)}\sup_{x_0\in\mathbb{R}^n,r>0}\left(1+\frac{r}{\rho(x_0)}\right)^{\alpha}\left[\varphi(r)\right]^{\frac{\eta-\beta}{n}-\frac{1}{n}\sum_{j=1}^{n}\frac{1}{q_j}}\left\|f\chi_{2B}\right\|_{L^{\vec{p}}\left(\mathbb{R}^n\right)}\\
&\le C\|b\|_{BMO_\theta(\rho)\left(\mathbb{R}^n\right)}\sup_{x_0\in\mathbb{R}^n,r>0}\left(1+\frac{2r}{\rho(x_0)}\right)^{\alpha}\left[\varphi(r)\right]^{\frac{\eta-\beta}{n}-\frac{1}{n}\sum_{j=1}^{n}\frac{1}{q_j}}\left[\varphi(2r)\right]^{\frac{1}{n}\sum_{j=1}^{n}\frac{1}{p_j}-\frac{\eta}{n}}\\
&\quad\times\left[\varphi(2r)\right]^{\frac{\eta}{n}-\frac{1}{n}\sum_{j=1}^{n}\frac{1}{p_j}}\left\|f\chi_{2B}\right\|_{L^{\vec{p}}\left(\mathbb{R}^n\right)}\\
&\le C\|b\|_{BMO_\theta(\rho)\left(\mathbb{R}^n\right)}\|f\|_{L_{\alpha,V}^{\vec{p},\eta,\varphi}\left(\mathbb{R}^n\right)}\sup_{x_0\in\mathbb{R}^n,r>0}\left[\varphi(r)\right]^{\frac{\eta-\beta}{n}-\frac{1}{n}\sum_{j=1}^{n}\frac{1}{p_j}+\frac{\beta}{n}}\left[\varphi(2r)\right]^{\frac{1}{n}\sum_{j=1}^{n}\frac{1}{p_j}-\frac{\eta}{n}}\\
&\le C\|b\|_{BMO_\theta(\rho)\left(\mathbb{R}^n\right)}\|f\|_{L_{\alpha,V}^{\vec{p},\eta,\varphi}\left(\mathbb{R}^n\right)}\sup_{x_0\in\mathbb{R}^n,r>0}\left[\frac{C_\varphi\varphi(r)}{\varphi(r)}\right]^{\frac{1}{n}\sum_{j=1}^{n}\frac{1}{p_j}-\frac{\eta}{n}}\\
&\le C\|b\|_{BMO_\theta(\rho)\left(\mathbb{R}^n\right)}\|f\|_{L_{\alpha,V}^{\vec{p},\eta,\varphi}\left(\mathbb{R}^n\right)}.\end{aligned}$$

For any $y\in B(x_0,r)$, since

$$\left|\left[b,I_\beta^L\right](f_2)(y)\right|\le\left|\left(b(y)-b_{2B}\right)\right|\left|I_\beta^L(f_2)(y)\right|+\left|I_\beta^L\left((b-b_{2B})f_2\right)(y)\right|,$$

we have

$$\begin{aligned}E_2 &= \sup_{x_0\in\mathbb{R}^n,r>0}\left(1+\frac{r}{\rho(x_0)}\right)^{\alpha}\left[\varphi(r)\right]^{\frac{\eta-\beta}{n}-\frac{1}{n}\sum_{j=1}^{n}\frac{1}{q_j}}\left\|\left[b,I_\beta^L\right](f_2)\right\|_{L^{\vec{q}}(B(x_0,r))}\\
&\le \sup_{x_0\in\mathbb{R}^n,r>0}\left(1+\frac{r}{\rho(x_0)}\right)^{\alpha}\left[\varphi(r)\right]^{\frac{\eta-\beta}{n}-\frac{1}{n}\sum_{j=1}^{n}\frac{1}{q_j}}\left\|\left(b(\cdot)-b_{2B}\right)I_\beta^L(f_2)\right\|_{L^{\vec{q}}(B(x_0,r))}\\
&\quad+\sup_{x_0\in\mathbb{R}^n,r>0}\left(1+\frac{r}{\rho(x_0)}\right)^{\alpha}\left[\varphi(r)\right]^{\frac{\eta-\beta}{n}-\frac{1}{n}\sum_{j=1}^{n}\frac{1}{q_j}}\left\|I_\beta^L\left(b(\cdot)-b_{2B}\right)(f_2)\right\|_{L^{\vec{q}}(B(x_0,r))}\\
&=E_2^1+E_2^2.\end{aligned}$$

By (1.4), Lemma 2.3, Lemma 2.6, the estimate for $\left|I_\beta^L(f_2)(y)\right|$ in D_2, and taking $N \geq (k_0+1)\theta$, we have

$$
\begin{aligned}
E_2^1 &= \sup_{x_0\in\mathbb{R}^n, r>0}\left(1+\frac{r}{\rho(x_0)}\right)^{\alpha}\left[\varphi(r)\right]^{\frac{\eta-\beta}{n}-\frac{1}{n}\sum_{j=1}^{n}\frac{1}{q_j}}\left\|\left(b(\cdot)-b_{2B}\right)I_\beta^L(f_2)\right\|_{L^{\vec{q}}(B(x_0,r))} \\
&\leq C\|f\|_{L_{\alpha,V}^{\vec{p},\eta,\varphi}(\mathbb{R}^n)}\sup_{x_0\in\mathbb{R}^n, r>0}\left(1+\frac{r}{\rho(x_0)}\right)^{\alpha}\left[\varphi(r)\right]^{\frac{\eta-\beta}{n}-\frac{1}{n}\sum_{j=1}^{n}\frac{1}{q_j}}\left\|b(\cdot)-b_{2B}\right\|_{L^{\vec{q}}(B(x_0,r))} \\
&\quad\times\sum_{k=1}^{\infty}\left(1+\frac{2^{k+1}r}{\rho(x_0)}\right)^{-\alpha}\frac{\left\|\chi_{2^{k+1}B}\right\|_{L^{\vec{p}'}(\mathbb{R}^n)}}{\left|2^kB\right|^{1-\frac{\beta}{n}}}\left[\varphi\left(2^{k+1}r\right)\right]^{\frac{1}{n}\sum_{j=1}^{n}\frac{1}{p_j}-\frac{\eta}{n}} \\
&\leq C\|f\|_{L_{\alpha,V}^{\vec{p},\eta,\varphi}(\mathbb{R}^n)}\sup_{x_0\in\mathbb{R}^n, r>0}\left[\varphi(r)\right]^{\frac{\eta-\beta}{n}-\frac{1}{n}\sum_{j=1}^{n}\frac{1}{q_j}}\left\|b(\cdot)-b_{2B}\right\|_{L^{\vec{q}}(B(x_0,r))} \\
&\quad\times\sum_{k=1}^{\infty}\left(1+\frac{2^{k+2}r}{\rho(x_0)}\right)^{-\alpha}\frac{\left\|\chi_{2^{k+1}B}\right\|_{L^{\vec{p}'}(\mathbb{R}^n)}}{\left|2^kB\right|^{1-\frac{\beta}{n}}}\left[\varphi\left(2^{k+1}r\right)\right]^{\frac{1}{n}\sum_{j=1}^{n}\frac{1}{p_j}-\frac{\eta}{n}} \\
&\leq C\|b\|_{BMO_\theta(\rho)(\mathbb{R}^n)}\|f\|_{L_{\alpha,V}^{\vec{p},\eta,\varphi}(\mathbb{R}^n)}\sup_{x_0\in\mathbb{R}^n, r>0}\left(1+\frac{r}{\rho(x_0)}\right)^{\theta-\frac{N}{k_0+1}}\left[\varphi(r)\right]^{\frac{\eta-\beta}{n}-\frac{1}{n}\sum_{j=1}^{n}\frac{1}{q_j}}\left\|\chi_{B(x_0,r)}\right\|_{L^{\vec{q}}(\mathbb{R}^n)} \\
&\quad\times\sum_{k=1}^{\infty}\left(1+\frac{2^{k+2}r}{\rho(x_0)}\right)^{-\alpha}\frac{\left\|\chi_{2^{k+1}B}\right\|_{L^{\vec{p}'}(\mathbb{R}^n)}}{\left|2^kB\right|^{1-\frac{\beta}{n}}}\left[\varphi\left(2^{k+1}r\right)\right]^{\frac{1}{n}\sum_{j=1}^{n}\frac{1}{p_j}-\frac{\eta}{n}} \\
&\leq C\|b\|_{BMO_\theta(\rho)(\mathbb{R}^n)}\|f\|_{L_{\alpha,V}^{\vec{p},\eta,\varphi}(\mathbb{R}^n)} \\
&\quad\times\sup_{x_0\in\mathbb{R}^n, r>0}\sum_{k=1}^{\infty}\left(1+\frac{2^{k+2}r}{\rho(x_0)}\right)^{-\alpha}\frac{\left\|\chi_{2^{k+1}B}\right\|_{L^{\vec{p}'}(\mathbb{R}^n)}\left\|\chi_{B(x_0,r)}\right\|_{L^{\vec{q}}(\mathbb{R}^n)}}{\left|2^kB\right|^{1-\frac{\beta}{n}}}\left[\frac{\varphi\left(2^{k+1}r\right)}{\varphi(r)}\right]^{\frac{1}{n}\sum_{j=1}^{n}\frac{1}{p_j}-\frac{\eta}{n}} \\
&\leq C\|b\|_{BMO_\theta(\rho)(\mathbb{R}^n)}\|f\|_{L_{\alpha,V}^{\vec{p},\eta,\varphi}(\mathbb{R}^n)} \\
&\quad\times\sup_{x_0\in\mathbb{R}^n, r>0}\sum_{k=1}^{\infty}\frac{\left(1+\frac{2^{k+2}r}{\rho(x_0)}\right)^{-\alpha}\left|B\left(x_0,2^{k+1}r\right)\right|^{1-\frac{1}{n}\sum_{j=1}^{n}\frac{1}{p_j}}\left|B\left(x_0,r\right)\right|^{\frac{1}{n}\sum_{j=1}^{n}\frac{1}{q_j}}}{\left|2^kB\right|^{1-\frac{\beta}{n}}}\left[\frac{C_\varphi^{k+1}\varphi(r)}{\varphi(r)}\right]^{\frac{1}{n}\sum_{j=1}^{n}\frac{1}{p_j}-\frac{\eta}{n}} \\
&\leq C\|b\|_{BMO_\theta(\rho)(\mathbb{R}^n)}\|f\|_{L_{\alpha,V}^{\vec{p},\eta,\varphi}(\mathbb{R}^n)}\sum_{k=1}^{\infty}\left(1+\frac{2^{k+2}r}{\rho(x_0)}\right)^{-\alpha}\frac{\left[C_\varphi^{k+1}\right]^{\frac{1}{n}\sum_{j=1}^{n}\frac{1}{p_j}-\frac{\eta}{n}}}{\left[2^{(k+1)n}\right]^{\frac{1}{n}\sum_{j=1}^{n}\frac{1}{p_j}-\frac{\beta}{n}}} \\
&\leq C\|b\|_{BMO_\theta(\rho)(\mathbb{R}^n)}\|f\|_{L_{\alpha,V}^{\vec{p},\eta,\varphi}(\mathbb{R}^n)}\sum_{k=1}^{\infty}\left(1+\frac{2^{k+2}r}{\rho(x_0)}\right)^{-\alpha}\left[\frac{C_\varphi}{2^n}\right]^{(k+1)\left(\frac{1}{n}\sum_{j=1}^{n}\frac{1}{p_j}-\frac{\eta}{n}\right)} \\
&\leq C\|b\|_{BMO_\theta(\rho)(\mathbb{R}^n)}\|f\|_{L_{\alpha,V}^{\vec{p},\eta,\varphi}(\mathbb{R}^n)}.
\end{aligned}
$$

To estimate E_2^2, we first consider $\left|I_\beta^L\left(\left(b(\cdot)-b_{2B}\right)f_2\right)(y)\right|$, for $y\in B(x_0,r)$. By (1.1), (1.4), Lemma 2.1, Lemma 2.5, and Lemma 2.7, we obtain

$$
\begin{aligned}
&\left|I_\beta^L\left(\left(b(\cdot)-b_{2B}\right)f_2\right)(y)\right| \\
&\le C\int_{\mathbb{R}^n\setminus(2B)} \frac{1}{\left(1+\frac{|y-z|}{\rho(y)}\right)^N}\frac{|b(z)-b_{2B}||f(z)|}{|y-z|^{n-\beta}}\mathrm{d}z \\
&\le C\sum_{k=1}^{\infty}\int_{2^{k+1}B\setminus(2^kB)} \frac{1}{\left(1+\frac{|y-z|}{\rho(y)}\right)^N}\frac{|b(z)-b_{2B}||f(z)|}{|x_0-z|^{n-\beta}}\mathrm{d}z \\
&\le C\sum_{k=1}^{\infty}\frac{1}{|2^kB|^{1-\frac{\beta}{n}}}\frac{1}{\left(1+\frac{2^kr}{\rho(y)}\right)^N}\int_{2^{k+1}B}|b(z)-b_{2B}||f(z)|\mathrm{d}z \\
&\le C\sum_{k=1}^{\infty}\frac{1}{|2^kB|^{1-\frac{\beta}{n}}}\frac{1}{\left(1+\frac{2^kr}{\rho(x_0)}\right)^{\frac{N}{k_0+1}}}\int_{2^{k+1}B}|b(z)-b_{2B}||f(z)|\mathrm{d}z \\
&\le C\sum_{k=1}^{\infty}\frac{1}{|2^kB|^{1-\frac{\beta}{n}}}\frac{1}{\left(1+\frac{2^kr}{\rho(x_0)}\right)^{\frac{N}{k_0+1}}}\left\|\left(b(\cdot)-b_{2B}\right)\chi_{2^{k+1}B}\right\|_{\vec{p}'}\left\|f\chi_{2^{k+1}B}\right\|_{\vec{p}} \\
&\le C\sum_{k=1}^{\infty}\frac{1}{|2^kB|^{1-\frac{\beta}{n}}}\frac{1}{\left(1+\frac{2^kr}{\rho(x_0)}\right)^{\frac{N}{k_0+1}}}\|b\|_{BMO_\theta(\rho)(\mathbb{R}^n)}(k+1)\left(1+\frac{2^{k+1}r}{\rho(x_0)}\right)^{\theta'}\left\|\chi_{2^{k+1}B}\right\|_{\vec{p}'}\left\|f\chi_{2^{k+1}B}\right\|_{\vec{p}} \\
&\le C\|b\|_{BMO_\theta(\rho)(\mathbb{R}^n)}\sum_{k=1}^{\infty}(k+1)\frac{\left\|\chi_{2^{k+1}B}\right\|_{\vec{p}'}}{|2^kB|^{1-\frac{\beta}{n}}}\left\|f\chi_{2^{k+1}B}\right\|_{\vec{p}} \\
&\le C\|b\|_{BMO_\theta(\rho)(\mathbb{R}^n)}\|f\|_{L_{\alpha,V}^{\vec{p},\eta,\varphi}(\mathbb{R}^n)}\sum_{k=1}^{\infty}(k+1)\left(1+\frac{2^{k+1}r}{\rho(x_0)}\right)^{-\alpha}\frac{\left\|\chi_{2^{k+1}B}\right\|_{\vec{p}'}}{|2^kB|^{1-\frac{\beta}{n}}}\left[\varphi\left(2^{k+1}r\right)\right]^{\frac{1}{n}\sum_{j=1}^{n}\frac{1}{p_j}-\frac{\eta}{n}},
\end{aligned}
$$

Furthermore, by (1.3), (1.4), and Lemma 2.3, we have

$$
\begin{aligned}
E_2^2 &= \sup_{x_0\in\mathbb{R}^n,r>0}\left(1+\frac{r}{\rho(x_0)}\right)^{\alpha}\left[\varphi(r)\right]^{\frac{\eta-\beta}{n}-\frac{1}{n}\sum_{j=1}^{n}\frac{1}{q_j}}\left\|I_\beta^L\left(b(\cdot)-b_{2B}\right)(f_2)\right\|_{L^{\vec{q}}(B(x_0,r))} \\
&\le C\|b\|_{BMO_\theta(\rho)(\mathbb{R}^n)}\|f\|_{L_{\alpha,V}^{\vec{p},\eta,\varphi}(\mathbb{R}^n)}\sup_{x_0\in\mathbb{R}^n,r>0}\left(1+\frac{r}{\rho(x_0)}\right)^{\alpha}\left[\varphi(r)\right]^{\frac{\eta-\beta}{n}-\frac{1}{n}\sum_{j=1}^{n}\frac{1}{q_j}}\left\|\chi_{B(x_0,r)}\right\|_{L^{\vec{q}}(\mathbb{R}^n)} \\
&\quad\times\sum_{k=1}^{\infty}(k+1)\left(1+\frac{2^{k+1}r}{\rho(x_0)}\right)^{-\alpha}\frac{\left\|\chi_{2^{k+1}B}\right\|_{L^{\vec{p}'}}}{|2^kB|^{1-\frac{\beta}{n}}}\left[\varphi\left(2^{k+1}r\right)\right]^{\frac{1}{n}\sum_{j=1}^{n}\frac{1}{p_j}-\frac{\eta}{n}} \\
&\le C\|b\|_{BMO_\theta(\rho)(\mathbb{R}^n)}\|f\|_{L_{\alpha,V}^{\vec{p},\eta,\varphi}(\mathbb{R}^n)}\sup_{x_0\in\mathbb{R}^n,r>0}\left[\varphi(r)\right]^{\frac{\eta-\beta}{n}-\frac{1}{n}\sum_{j=1}^{n}\frac{1}{q_j}}\left\|\chi_{B(x_0,r)}\right\|_{L^{\vec{q}}(\mathbb{R}^n)} \\
&\quad\times\sum_{k=1}^{\infty}(k+1)\left(1+\frac{2^{k+2}r}{\rho(x_0)}\right)^{-\alpha}\frac{\left\|\chi_{2^{k+1}B}\right\|_{L^{\vec{p}'}}}{|2^kB|^{1-\frac{\beta}{n}}}\left[\varphi\left(2^{k+1}r\right)\right]^{\frac{1}{n}\sum_{j=1}^{n}\frac{1}{p_j}-\frac{\eta}{n}}
\end{aligned}
$$

$$
\begin{aligned}
&\le C\|b\|_{BMO_\theta(\rho)(\mathbb{R}^n)}\|f\|_{L^{\vec{p},\eta,\varphi}_{\alpha,V}(\mathbb{R}^n)}\\
&\quad\times\sup_{x_0\in\mathbb{R}^n,r>0}\sum_{k=1}^{\infty}(k+1)\frac{\left(1+\frac{2^{k+2}r}{\rho(x_0)}\right)^{-\alpha}\left\|\chi_{2^{k+1}B}\right\|_{L^{\vec{p}'}}\left\|\chi_{B(x_0,r)}\right\|_{L^{\vec{q}}(\mathbb{R}^n)}}{\left|2^kB\right|^{1-\frac{\beta}{n}}}\left[\frac{\varphi\left(2^{k+1}r\right)}{\varphi(r)}\right]^{\frac{1}{n}\sum_{j=1}^{n}\frac{1}{p_j}-\frac{\eta}{n}}\\
&\le C\|b\|_{BMO_\theta(\rho)(\mathbb{R}^n)}\|f\|_{L^{\vec{p},\eta,\varphi}_{\alpha,V}(\mathbb{R}^n)}\left[\frac{C_\varphi^{k+1}\varphi(r)}{\varphi(r)}\right]^{\frac{1}{n}\sum_{j=1}^{n}\frac{1}{p_j}-\frac{\eta}{n}}\\
&\quad\times\sup_{x_0\in\mathbb{R}^n,r>0}\sum_{k=1}^{\infty}(k+1)\left(1+\frac{2^{k+2}r}{\rho(x_0)}\right)^{-\alpha}\frac{\left|2^{k+1}B\right|^{1-\frac{1}{n}\sum_{j=1}^{n}\frac{1}{p_j}}\left|B(x_0,r)\right|^{\frac{1}{n}\sum_{j=1}^{n}\frac{1}{q_j}}}{\left|2^kB\right|^{1-\frac{\beta}{n}}}\\
&\le C\|b\|_{BMO_\theta(\rho)(\mathbb{R}^n)}\|f\|_{L^{\vec{p},\eta,\varphi}_{\alpha,V}(\mathbb{R}^n)}\sum_{k=1}^{\infty}(k+1)\left(1+\frac{2^{k+2}r}{\rho(x_0)}\right)^{-\alpha}\frac{\left[C_\varphi^{k+1}\right]^{\frac{1}{n}\sum_{j=1}^{n}\frac{1}{p_j}-\frac{\eta}{n}}}{\left[2^{(k+1)n}\right]^{\frac{1}{n}\sum_{j=1}^{n}\frac{1}{p_j}-\frac{\beta}{n}}}\\
&\le C\|b\|_{BMO_\theta(\rho)(\mathbb{R}^n)}\|f\|_{L^{\vec{p},\eta,\varphi}_{\alpha,V}(\mathbb{R}^n)}\sum_{k=1}^{\infty}(k+1)\left(1+\frac{2^{k+2}r}{\rho(x_0)}\right)^{-\alpha}\left[\frac{C_\varphi}{2^n}\right]^{(k+1)\left(\frac{1}{n}\sum_{j=1}^{n}\frac{1}{p_j}-\frac{\eta}{n}\right)}\\
&\le C\|b\|_{BMO_\theta(\rho)(\mathbb{R}^n)}\|f\|_{L^{\vec{p},\eta,\varphi}_{\alpha,V}(\mathbb{R}^n)},
\end{aligned}
$$

The estimate for the convergence of the series here is similar to Theorem 3.1.

Combining the estimates for E_1 and E_2^1, Theorem 3.2. is proved.

In this paper, we have established the boundedness of fractional integral associated with the Schrödinger operator and its commutator on the generalized fractional mixed Morrey spaces. The main results are stated in Theorems 3.1 and 3.2. Our results provide a unified framework that recovers several known results as special cases. For instance, by setting $p_j = p$, we obtain boundedness on generalized fractional Morrey spaces [15]. By further setting $\varphi(r) = r^\lambda$ and $\alpha = 0$, we recover results on classical mixed Morrey spaces [13]. When $L = -\Delta$, the results pertain to the classical Riesz potential and its commutator on these generalized spaces.

Future research directions naturally suggested by this work include: investigating the boundedness of other operators related to L on these and similar spaces, exploring the compactness properties of these operators on generalized fractional mixed Morrey spaces, considering more general potentials V or replacing the Laplacian $-\Delta$ with other elliptic operators, applying these boundedness results to study the regularity of solutions to non-linear partial differential equations, particularly of Schrödinger type, in the context of mixed-norm and Morrey-scale spaces, the estimates proven here could serve as crucial tools in establishing a priori estimates for such problems.

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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