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Solution of a Two-Degree-of-Freedom Nonlinear Coupled Vibration System Based on the Multiple Scales Method

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Abstract

This paper takes the two-degree-of-freedom nonlinear coupled vibration system, which is widely present in engineering scenarios such as mechanical coupled structures and electromechanical systems, as the research object. Focusing on its cubic coupling nonlinear terms and the complex form of coexisting external excitation and parametric excitation, the multiple scales method is adopted to carry out dynamic solution and analysis. First, the governing equations of the system are defined, and the perturbation expansion of displacement responses ($y_1 = y_{10} + \varepsilon y_{11}$, $y_2 = y_{20} + \varepsilon y_{21}$) is performed based on the small parameter ε. Then, the expansion is substituted into the original equations, and the coefficients of the same order of ε are matched to derive the solvability conditions under the combined action of 2nd-order superharmonic resonance and 1/2-order subharmonic resonance. Finally, the amplitude-frequency response equations of the system are solved, and the influence laws of key parameters such as nonlinear stiffness coefficients, excitation amplitudes, and damping ratios on the dynamic behaviors of the system are further discussed. The research results show that the multiple scales method can effectively capture the nonlinear dynamic characteristics of the two-degree-of-freedom coupled system, providing a theoretical reference for the design and stability control of related engineering structures.

Keywords

Multiple Scales Method, Nonlinear Dynamical System

1. Introduction

Nonlinear coupled vibration phenomena are widely prevalent in engineering fields such as mechanical coupled structures, mechatronic systems, and aerospace equipment. From the coupled oscillators in micro-electro-mechanical systems (MEMS) to the shafting coupled vibration in large rotating machinery, their dynamic behaviors are directly related to the operational stability and service life of equipment [1]. As a fundamental model of multi-degree-of-freedom coupled systems, the two-degree-of-freedom nonlinear coupled vibration system exhibits more complex dynamic characteristics compared with single-degree-of-freedom systems due to the energy transfer between oscillators and nonlinear coupling effects, making it a research focus in the field of nonlinear dynamics [2]. Among the analytical methods for two-degree-of-freedom nonlinear coupled vibration systems [3]-[5], numerical methods can obtain accurate numerical solutions but fail to intuitively reveal the analytical relationship between system parameters and dynamic behaviors. In contrast, perturbation methods, as classical analytical approaches, can derive analytical solutions of the system through asymptotic expansion. Among them, the multiple scales method is widely applied to the analysis of vibration systems with nonlinear terms and composite excitations due to its ability to handle wide-band vibration problems and effectively separate motions on different time scales. At present, existing studies mostly focus on two-degree-of-freedom coupled systems under a single excitation form, or only analyze quadratic nonlinear terms and single resonance types. For the more common complex scenarios in engineering, such as cubic coupling nonlinear terms and the coexistence of external excitation and parametric excitation, the research on dynamic solution and parameter influence laws remains to be further explored [6]. In view of this, this paper takes the two-degree-of-freedom nonlinear coupled vibration system with cubic coupling nonlinear terms and the combined action of external excitation and parametric excitation as the research object, and adopts the multiple scales method to carry out dynamic solution and analysis of the system. First, the governing equations of the system are constructed, and the perturbation expansion of displacement responses is performed based on the small parameter. Then, by substituting the expansion into the original equations and matching the coefficients of the same order of the small parameter [7], the solvability conditions of the system under the combined action of superharmonic resonance and subharmonic resonance are derived. Finally, the amplitude-frequency response equations are solved, and the influence laws of key parameters on the dynamic behaviors of the system are discussed. The research results of this paper aim to provide a reference for the analytical analysis of multi-degree-of-freedom nonlinear coupled vibration systems, and also offer a theoretical basis for the design and stability control of related engineering structures.

2. Modeling of a Two-Degree-of-Freedom Nonlinear Coupled Vibration System

The dynamic equations of such a two-degree-of-freedom nonlinear coupled vibration system are derived based on the Lagrange's equation, which is more suitable for multi-degree-of-freedom systems with complex nonlinearities and coupling terms compared with Newton's second law. The core form of Lagrange's equation is [7]:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{y}_i}\right)-\frac{\partial L}{\partial y_i}+\frac{\partial D}{\partial \dot{y}_i}=Q_i \quad (i=1,2)$$

where, $L=T-V$ denotes the Lagrangian function, with T being the kinetic energy and V the potential energy of the system; D represents the dissipation function, which describes the energy dissipation caused by damping only containing quadratic terms of velocity; Q_i is the generalized external force, including harmonic excitation, parametric excitation and other non-conservative forces; y_1, y_2 are the generalized displacements of the two degrees of freedom after dimensionless processing.

The core idea of equation establishment is to first define the kinetic energy, potential energy, dissipation function and generalized force of the system, then substitute them into Lagrange's equation for expansion and arrangement, and finally obtain the dynamic control equations. The general structure of the equation is inertial term + damping term + linear stiffness term + parametric excitation term + nonlinear coupling term = external excitation term. The source and physical meaning of each term are analyzed as follows:

2.1. Inertial Terms

1) Basic inertial terms: $\ddot{y}_1, \ddot{y}_2$

Source: Derived by taking the time derivative of the partial derivative of the system's kinetic energy T with respect to velocity. If the system is mass-normalized letting the equivalent mass of the two degrees of freedom be 1, the kinetic energy can be simplified as $T=\frac{1}{2}\dot{y}_1^2+\frac{1}{2}\dot{y}_2^2+\varepsilon b_{21}\dot{y}_1\dot{y}_2$ the last term is the coupled inertial term. Taking the partial derivative of T with respect to $\dot{y}_2$ and then the time derivative yields $\ddot{y}_2+\varepsilon b_{21}\ddot{y}_1$.

2) Coupled inertial term: $\varepsilon b_{21}\ddot{y}_1$

ε is a small parameter characterizing the "weak effect" of coupling, nonlinearity and excitation, which is the application premise of the multiple scales method, and b_{21} is the inertial coupling coefficient, reflecting the inertial interaction between the two degrees of freedom.

2.2. Damping Terms

1) Self-damping terms: $\varepsilon\mu_1\dot{y}_1, \varepsilon\mu_2\dot{y}_2$

Source: Derived from the dissipation function

$D=\frac{1}{2}\varepsilon\mu_1\dot{y}_1^2+\frac{1}{2}\varepsilon\mu_2\dot{y}_2^2+\frac{1}{2}\varepsilon b_{22}\dot{y}_1\dot{y}_2$ the form of dissipation function for viscous damping by taking the partial derivative with respect to velocity; μ_1, μ_2 are self-damping coefficients, and ε indicates weak damping.

2) Coupled damping term: $\varepsilon b_{22}\dot{y}_1$

b_{22} is the damping coupling coefficient, reflecting the damping effect of the velocity of one degree of freedom on the other.

2.3. Linear Stiffness Terms

1) Self-stiffness terms: $\omega_1^2 y_1, \omega_2^2 y_2$

Source: Derived from the system's linear potential energy $V_{\text{linear}} = \frac{1}{2}\omega_1^2 y_1^2 + \frac{1}{2}\omega_2^2 y_2^2 \frac{1}{2}\varepsilon b_{23} y_1 y_2$ by taking the partial derivative with respect to displacement; ω_1, ω_2 are the linear natural angular frequencies of the two degrees of freedom $\omega^2 = k/m$, and $k = \omega^2$ after mass normalization.

2) Linear coupling term: $-\varepsilon b_{23} y_1$

b_{23} is the linear coupling stiffness coefficient, and the negative sign indicates the directional relationship between the coupling force and displacement, reflecting the linear elastic coupling between the two degrees of freedom.

2.4. Parametric Excitation Term

1) Term: $\varepsilon f_2 \cos\Omega_2 t \cdot y_1$

Source: Derived from the parametric excitation potential energy $V_{\text{parametric}} = \frac{1}{2}\varepsilon f_2 \cos\Omega_2 t \cdot y_1^2$ parametric excitation is essentially "stiffness varying periodically with time" rather than a direct acting force by taking the partial derivative with respect to y_1; Ω_2 is the parametric excitation frequency, f_2 is the excitation amplitude, and ε indicates weak parametric excitation.

2.5. Nonlinear Coupling Terms

All cubic terms containing εa_{ij} are cubic nonlinear terms derived from the system's nonlinear potential energy:

$$V_{\text{nonlinear}} = \frac{1}{4}\varepsilon\left(a_{14}y_1^4 + a_{21}y_2^4 + a_{11}y_1^2 y_2^2 + a_{13}y_1^3 y_2 + a_{22}y_1 y_2^3 + a_{23}y_1^2 y_2^2 + a_{24}y_1^4\right)$$

Taking the partial derivative with respect to displacement converts the quartic terms into cubic terms e.g., $\partial V_{\text{nonlinear}}/\partial y_1 = \varepsilon a_{14} y_1^3 + \varepsilon a_{11} y_1 y_2^2 + \varepsilon a_{13} y_1^2 y_2$, and the negative sign indicates the directional relationship between the nonlinear restoring force and displacement. Self-nonlinear terms: y_1^3, y_2^3, reflecting the nonlinear stiffness of a single degree of freedom; Coupled nonlinear terms: $y_1 y_2^2, y_1^2 y_2$, reflecting the nonlinear coupling effect between the two degrees of freedom; a_{11}, a_{13}, a_{21}, etc.: coefficients of different nonlinear forms, determined by the physical characteristics of the system (e.g., geometric nonlinearity, material nonlinearity).

2.6. External Excitation Terms

1) $\varepsilon f_{11}\cos\Omega_1 t, \varepsilon f_{12}\cos\Omega_1 t$

Source: Harmonic external excitation forces directly acting on the two degrees of freedom generalized force Q_i; Ω_1 is the external excitation frequency, f_{11}, f_{12} are the excitation amplitudes, and ε indicates weak external excitation. The resonance conditions given in the hint:

$$\omega_1^2 = 4\Omega_1^2 + \varepsilon\sigma_1;\ \omega_2^2 = \frac{1}{4}\Omega_1^2 + \varepsilon\sigma_1;\ \Omega_1 = \frac{1}{2}\Omega_2$$

are not "derived" but artificially set resonance scenarios during modeling adapted to the solution by the multiple scales method, with the core purposes: $\omega_1^2 \approx 4\Omega_1^2$: the 1st degree of freedom of the system is in superharmonic resonance natural frequency 2 times the external excitation frequency; $\omega_2^2 \approx \frac{1}{4}\Omega_1^2$: the 2nd degree of freedom of the system is in subharmonic resonance natural frequency 1/2 times the external excitation frequency; σ_1 is a standard frequency detuning parameter. $\varepsilon\sigma_1$: detuning parameter, characterizing the small deviation between the natural frequency and the resonance frequency since ε is a small parameter, the deviation is also a small quantity, which is the key to handling resonance problems with the multiple scales method; $\frac{1}{2}\Omega_2$: correlating the external excitation frequency Ω_1 and the parametric excitation frequency Ω_2 to form combined resonance of the two excitations a complex resonance scenario closer to engineering practice.

3. Dynamic Solution of the System Based on the Multiple Scales Method

3.1. Basic Principles of the Multiple Scales Method

In 1957, the American physicist Peter A. Sturrock discovered that the nonlinear effects of plasmas have time scales of different speeds, so multiple time scales can be introduced for research, and this idea is universal [8]-[10]. If time t is used to describe the vibration of a single-degree-of-freedom autonomous system, the time scale for describing the changes in its amplitude and initial phase is εt, and $0 < \varepsilon \ll 1$. Therefore, two different time scales can be used to study the vibration of the autonomous system this is the multiple scales method.

The frequency of the periodic vibration of the autonomous system is expanded into a power series of ε. Then, the fast-varying phase of the periodic vibration can be expressed as8:

$$\omega t = \omega_0 t + \varepsilon\omega_1 t + \varepsilon^2\omega_2 t + \cdots = \omega_0 t + \omega_1(\varepsilon t) + \omega_2(\varepsilon^2 t) + \cdots \tag{3.1.1}$$

where εt and $\varepsilon^2 t$ are increasingly slow time scales. After this, introduce successively slow multiple time scales:

$$T_k = \varepsilon^k t,\ k = 0,1,2,\cdots \tag{3.1.2}$$

and treat these time scales as independent variables. Express the solution of the equation as (where $k = 0$):

$$u(t,\varepsilon) = u_0(T_0,T_1,\cdots) + \varepsilon u_1(T_0,T_1,\cdots) + \varepsilon^2 u_2(T_0,T_1,\cdots) + \cdots \tag{3.1.3}$$

For simplicity, define the following partial derivative operators to represent the derivative operations with respect to time:

$$D_k = \frac{\partial}{\partial T_k},\ \frac{\mathrm{d}u}{\mathrm{d}t} = \sum_{k=0}^{\infty} \frac{\mathrm{d}T_k}{\mathrm{d}t}\frac{\partial}{\partial T_k} = \sum_{k=0}^{\infty} \varepsilon^k D_k \tag{3.1.4a}$$

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_{k=0}^{\infty} \varepsilon^k D_k\right) = D_0^2 + 2\varepsilon D_0 D_1 + \varepsilon^2\left(D_1^2 + 2D_0 D_2\right) + \cdots \tag{3.1.4b}$$

Substitute Eqs. (3.1.3) and (3.1.4) into

$$\begin{cases} \ddot{u} + \omega_0^2 u + \varepsilon f\left(u, \dot{u}\right) = 0 \\ u\left(0\right) = a_0,\ \dot{u}\left(0\right) = 0 \end{cases},$$

and compare the coefficients of the same powers of ε. A series of linear partial differential equations are obtained:

$$\varepsilon^0:\ \ D_0^2 u_0 + \omega_0^2 u_0 = 0 \tag{3.1.5a}$$

$$\varepsilon^1:\ \ D_0^2 u_1 + \omega_0^2 u_1 = -\left(D_0^2 + 2D_0 D_2\right)u_0 - 2D_0 D_1 u_0 - f\left(u_0, D_0 u_0\right) \tag{3.1.5b}$$

$$\varepsilon^2:\ \ D_0^2 u_2 + \omega_0^2 u_2 = -\left(D_1^2 + 2D_0 D_2\right)u_0 - 2D_0 D_1 u_1 + \cdots \tag{3.1.5c}$$

It is not difficult to see that this set of equations can be solved successively. Now, we introduce how to solve the above linear partial differential equations. First, it is easy to see that the solution of Eq. (3.1.5a) has the form:

$$u_0 = A\left(T_1, T_2, \cdots\right)\cos\left(\omega_0 T_0 + \varphi\left(T_1, T_2, \cdots\right)\right) \tag{3.1.6}$$

To facilitate solving u_1, according to Euler's formula, introduce the complex amplitude corresponding to the amplitude term in Eq. (3.1.6):

$$A\left(T_1, T_2, \cdots\right) \equiv \frac{a\left(T_1, T_2, \cdots\right)}{2}\exp\left(\mathrm{i}\varphi\left(T_1, T_2, \cdots\right)\right) \tag{3.1.7}$$

Rewrite Eq. (3.1.6) in the form of a complex function:

$$u_0 = A\left(T_1, T_2, \cdots\right)\exp\left(\mathrm{i}\omega_0 T_0\right) + \mathrm{cc} \tag{3.1.8}$$

where "cc" represents the complex conjugate of the preceding terms (not repeated later). Substitute Eq. (3.1.8) into Eq. (3.1.5b) to get:

$$\begin{aligned} D_0^2 u_1 + \omega_0^2 u_1 = &-2\mathrm{i}\omega_0 D_1 A \exp\left(\mathrm{i}\omega_0 T_0\right) + \mathrm{cc} \\ &- f\left(A\exp\left(\mathrm{i}\omega_0 T_0\right) + \mathrm{cc}, \mathrm{i}\omega_0 A\exp\left(\mathrm{i}\omega_0 T_0\right) + \mathrm{cc}\right) \end{aligned} \tag{3.1.9}$$

This can be understood as an undamped system under periodic excitation, whose natural frequency is ω_0. To avoid secular terms, the right-hand side of the above equation cannot contain $\exp\left(\mathrm{i}\omega_0 T_0\right)$ or $\exp\left(-\mathrm{i}\omega_0 T_0\right)$. This requires that the Fourier coefficient corresponding to the right-hand side of the above equation is zero, i.e.,

$$-2\mathrm{i}\omega_0 D_1 A - \frac{\omega_0}{2\pi}\int_0^{2\pi/\omega_0} f\left(A\exp\left(\mathrm{i}\omega_0 T_0\right) + \mathrm{cc}, \mathrm{i}\omega_0 A\exp\left(\mathrm{i}\omega_0 T_0\right) + \mathrm{cc}\right)\exp\left(-\mathrm{i}\omega_0 T_0\right)\mathrm{d}T_0 = 0$$

Substitute Eq. (3.1.7) into Eq. (3.1.10) to obtain the trigonometric function form of the condition:

$$\left(\mathrm{i}D_1 a + \mathrm{i}\omega_1 a\right) + \frac{1}{2\pi\omega_0}\int_0^{2\pi} f\left(a\cos\psi - \omega_0 a\sin\psi\right)\left(\cos\psi - \mathrm{i}\sin\psi\right)\mathrm{d}\psi = 0 \tag{3.1.10}$$

Separate the real and imaginary parts of the above equation to get:

$$\begin{cases} D_1 a = \dfrac{1}{2\pi\omega_0}\displaystyle\int_0^{2\pi} f\left(a\cos\psi - \omega_0 a\sin\psi\right)\sin\psi \mathrm{d}\psi \\ aD_1\varphi = \dfrac{1}{2\pi\omega_0}\displaystyle\int_0^{2\pi} f\left(a\cos\psi - \omega_0 a\sin\psi\right)\cos\psi \mathrm{d}\psi \end{cases} \tag{3.1.11}$$

By solving Eq. (3.1.9) under this condition, a first-order corrected solution $u_1\left(T_0, T_1, \cdots\right)$ can be obtained. Substitute it together with $u_0\left(T_0, T_1, \cdots\right)$ into Eq. (4.3.5c), and the solvability condition for eliminating secular terms can be obtained, thereby obtaining the second-order corrected solution $u_2\left(T_0, T_1, \cdots\right)$.

3.2. Solution of the Two-Degree-of-Freedom Nonlinear Coupled Vibration System

$$\begin{aligned} &\ddot{y}_1 + \varepsilon\mu_1\dot{y}_1 + \left(\omega_1^2 + \varepsilon f_2\cos\left(\Omega_2 t\right)\right)y_1 - \varepsilon a_{11}y_1y_2^2 - \varepsilon a_{13}y_1^2y_2 - \varepsilon a_{14}y_1^3 \\ &= \varepsilon f_{11}\cos\left(\Omega_1 t\right) \end{aligned}$$

$$\begin{aligned} &\ddot{y}_2 + \varepsilon b_{21}\ddot{y}_1 + \varepsilon\mu_2\dot{y}_2 + \varepsilon b_{22}\dot{y}_1 + \omega_2^2 y_2 - \varepsilon b_{23}y_1 - \varepsilon a_{21}y_2^3 - \varepsilon a_{22}y_1y_2^2 \\ &- \varepsilon a_{23}y_1^2y_2 - \varepsilon a_{24}y_1^3 = \varepsilon f_{12}\cos\left(\Omega_1 t\right) \end{aligned}$$

Special instance Let $\begin{aligned} y_1 &= y_{10} + \varepsilon y_{11} \\ y_2 &= y_{20} + \varepsilon y_{21} \end{aligned}$

Consider resonance of the form $\begin{aligned} \omega_1^2 &= 4\Omega_1^2 + \varepsilon\sigma_1 \\ \omega_2^2 &= \frac{1}{4}\Omega_1^2 + \varepsilon\sigma_1 \\ \Omega_1 &= \frac{1}{2}\Omega_2 \end{aligned}$

Solution: Introduce the multiple time scales and the corresponding differential operators:

$$T_0 = t,\ T_1 = \varepsilon t,$$

$$D_0 = \frac{\partial}{\partial T_0},\ D_1 = \frac{\partial}{\partial T_1}.$$

Expand the unknown functions y_1 and y_2 as asymptotic series in the small parameter ε:

$$y_1 = y_{10}\left(T_0, T_1\right) + \varepsilon y_{11}\left(T_0, T_1\right) + O\left(\varepsilon^2\right),$$

$$y_2 = y_{20}\left(T_0, T_1\right) + \varepsilon y_{21}\left(T_0, T_1\right) + O\left(\varepsilon^2\right).$$

Accordingly, the differential operators become:

$$\frac{\mathrm{d}}{\mathrm{d}t} = D_0 + \varepsilon D_1 + O\left(\varepsilon^2\right),$$

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} = D_0^2 + 2\varepsilon D_0 D_1 + O\left(\varepsilon^2\right).$$

Assume the following resonance relations:

$$\omega_1^2 = 4\Omega_1^2 + \varepsilon\sigma_1, \tag{1}$$

$$\omega_2^2 = \frac{1}{4}\Omega_1^2 + \varepsilon\sigma_1, \tag{2}$$

$$\Omega_2 = 2\Omega_1. \tag{3}$$

Substitute the resonance conditions into the original system equations to obtain the following form, which is explicit in ε:

$$\begin{aligned}\ddot{y}_1 + 4\Omega_1^2 y_1 = \varepsilon\big[&f_{11}\cos(\Omega_1 t) - \mu_1\dot{y}_1 - \sigma_1 y_1 - f_2\cos(\Omega_2 t)\, y_1 \\ &+ a_{11}y_1y_2^2 + a_{13}y_1^2y_2 + a_{14}y_1^3\big],\end{aligned} \tag{4}$$

$$\begin{aligned}\ddot{y}_2 + \frac{1}{4}\Omega_1^2 y_2 = \varepsilon\big[&-b_{21}\ddot{y}_1 - \mu_2\dot{y}_2 - b_{22}\dot{y}_1 - \sigma_1 y_2 + b_{23}y_1 + a_{21}y_2^3 \\ &+ a_{22}y_1y_2^2 + a_{23}y_1^2y_2 + a_{24}y_1^3 + f_{12}\cos(\Omega_1 t)\big].\end{aligned} \tag{5}$$

$O(1)$-Order Equations Insert the asymptotic expansions for y_1, y_2, and the derivative operators into equations (4) and (5), and collect terms of order ε^0 (i.e., $O(1)$). This yields the leading-order linear homogeneous equations:

$$D_0^2 y_{10} + 4\Omega_1^2 y_{10} = 0, \tag{6}$$

$$D_0^2 y_{20} + \frac{1}{4}\Omega_1^2 y_{20} = 0. \tag{7}$$

The general solutions to equations (6) and (7) are harmonic oscillations. It is convenient to introduce phase functions that incorporate both the fast oscillation and the slow phase modulation:

$$\theta_1 = 2\Omega_1 T_0 + \beta_1(T_1), \tag{8}$$

$$\theta_2 = \frac{1}{2}\Omega_1 T_0 + \beta_2(T_1). \tag{9}$$

Here, $\beta_1(T_1)$ and $\beta_2(T_1)$ are slowly varying phase shifts to be determined at the next order of approximation.

$$y_{10} = A_1(T_1)\cos\theta_1, \qquad y_{20} = A_2(T_1)\cos\theta_2,$$

$$D_0 y_{10} = -2\Omega_1 A_1\sin\theta_1, \qquad D_0 y_{20} = -\frac{1}{2}\Omega_1 A_2\sin\theta_2,$$

$$D_1 y_{10} = A_1'\cos\theta_1 - A_1\beta_1'\sin\theta_1, \qquad D_1 y_{20} = A_2'\cos\theta_2 - A_2\beta_2'\sin\theta_2,$$

$$D_0D_1 y_{10} = -2\Omega_1 A_1'\sin\theta_1 - 2\Omega_1 A_1\beta_1'\cos\theta_1,$$
$$D_0D_1 y_{20} = -\frac{1}{2}\Omega_1 A_2'\sin\theta_2 - \frac{1}{2}\Omega_1 A_2\beta_2'\cos\theta_2.$$

$O(\varepsilon)$-Order Equation for y_1:

$$D_0^2 y_{11} + 4\Omega_1^2 y_{11} = R_1(T_0, T_1)$$

Among them,

$$R_1 = -2D_0D_1y_{10} - \mu_1 D_0 y_{10} - \sigma_1 y_{10} - f_2 \cos\Omega_2 T_0 y_{10} + a_{11}y_{10}y_{20}^2 + a_{13}y_{10}^2 y_{20} + a_{14}y_{10}^3 + f_{11}\cos\Omega_1 T_0$$

Only retain the resonant terms with the same frequency as $\cos\theta_1, \sin\theta_1$: Linear Slowly Varying Part:

$$-2D_0D_1y_{10} = 4\Omega_1 A_1' \sin\theta_1 + 4\Omega_1 A_1 \beta_1' \cos\theta_1$$

$$-\mu_1 D_0 y_{10} = 2\Omega_1 \mu_1 A_1 \sin\theta_1$$

$$-\sigma_1 y_{10} = -\sigma_1 A_1 \cos\theta_1$$

Nonlinear Terms:

$$y_{20}^2 = A_2^2 \cos^2\theta_2 = \frac{1}{2}A_2^2\left(1+\cos 2\theta_2\right)$$

$$a_{11}y_{10}y_{20}^2 = \frac{1}{2}a_{11}A_1A_2^2\cos\theta_1 + \left(\text{non-resonant high-frequency terms}\right)$$

From

$$\cos^3\theta_1 = \frac{1}{4}\left(3\cos\theta_1 + \cos 3\theta_1\right)$$

we obtain

$$a_{14}y_{10}^3 = \frac{3}{4}a_{14}A_1^3\cos\theta_1 + \left(\text{non-resonant high-frequency terms}\right)$$

Other terms $f_2\cos\Omega_2 t y_{10}$, $a_{13}y_{10}^2 y_{20}$, $f_{11}\cos\Omega_1 t$ only contain frequencies such as $0, \Omega_1, 3\Omega_1, \cdots$, which do not resonate with $2\Omega_1$ they do not appear in the amplitude equation. Thus, the resonant part is:

$$R_1^{(\text{res})} = \left(4\Omega_1 A_1' + 2\Omega_1\mu_1 A_1\right)\sin\theta_1 + \left(4\Omega_1 A_1\beta_1' - \sigma_1 A_1 + \frac{1}{2}a_{11}A_1A_2^2 + \frac{3}{4}a_{14}A_1^3\right)\cos\theta_1$$

Eliminate the secular terms set the coefficients to zero:

$$4\Omega_1 A_1' + 2\Omega_1\mu_1 A_1 = 0$$

$$4\Omega_1 A_1\beta_1' - \sigma_1 A_1 + \frac{1}{2}a_{11}A_1A_2^2 + \frac{3}{4}a_{14}A_1^3 = 0$$

$$\Rightarrow\ A_1' = -\frac{1}{2}\mu_1 A_1$$

$$\Rightarrow\ 4\Omega_1\beta_1' = \sigma_1 - \frac{1}{2}a_{11}A_2^2 - \frac{3}{4}a_{14}A_1^2$$

$$\Rightarrow\ \beta_1' = \frac{1}{4\Omega_1}\left(\sigma_1 - \frac{1}{2}a_{11}A_2^2 - \frac{3}{4}a_{14}A_1^2\right)$$

Equation of order $O(\varepsilon)$ y_2:

$$D_0^2 y_{21} + \frac{1}{4}\Omega_1^2 y_{21} = R_2\left(T_0, T_1\right) \tag{10}$$

where the inhomogeneous term $R_2\left(T_0, T_1\right)$ is given by:

$$R_2 = -2D_0D_1y_{20} - b_{21}D_0^2y_{10} - \mu_2D_0y_{20} - b_{22}D_0y_{10} - \sigma_1y_{20} + b_{23}y_{10} + a_{21}y_{20}^3 + a_{22}y_{10}y_{20}^2 + a_{23}y_{10}^2y_{20} + a_{24}y_{10}^3 + f_{12}\cos\left(\Omega_1T_0\right). \tag{11}$$

To eliminate secular terms and obtain the solvability conditions, we must identify and retain only those terms in R_2 that are resonant with the homogeneous solution of (10). The homogeneous solution has frequency $\Omega_1/2$, corresponding to the phase function $\theta_2 = \frac{1}{2}\Omega_1T_0 + \beta_2\left(T_1\right)$. Therefore, we retain only terms proportional to $\cos\theta_2$ and $\sin\theta_2$. Resonant Terms from the Linear/ Slowly-Varying Part

$$-2D_0D_1y_{20} = \Omega_1A_2'\sin\theta_2 + \Omega_1A_2\beta_2'\cos\theta_2, \tag{12}$$

$$-\mu_2D_0y_{20} = \frac{1}{2}\Omega_1\mu_2A_2\sin\theta_2, \tag{13}$$

$$-\sigma_1y_{20} = -\sigma_1A_2\cos\theta_2. \tag{14}$$

Here, the prime symbol ($'$) denotes the derivative with respect to the slow time T_1, e.g., $A_2' = \mathrm{d}A_2/\mathrm{d}T_1$. Resonant Terms from the Nonlinear Part Applying trigonometric identities, we expand the cubic nonlinearities and identify their resonant components:

$$a_{21}y_{20}^3 = a_{21}A_2^3\cos^3\theta_2 = \frac{3}{4}a_{21}A_2^3\cos\theta_2 + \left(\text{higher harmonics}\right), \tag{15}$$

$$y_{10}^2 = A_1^2\cos^2\theta_1 = \frac{1}{2}A_1^2\left(1+\cos2\theta_1\right), \tag{16}$$

$$\begin{aligned} a_{23}y_{10}^2y_{20} &= a_{23}\left(\frac{1}{2}A_1^2\left(1+\cos2\theta_1\right)\right)A_2\cos\theta_2 \\ &= \frac{1}{2}a_{23}A_1^2A_2\cos\theta_2 + \left(\text{non-resonant high-frequency terms}\right). \end{aligned} \tag{17}$$

Note that the term $\cos2\theta_1\cdot\cos\theta_2$ produces sum and difference frequencies, none of which are resonant with $\cos\theta_2$ or $\sin\theta_2$ given the assumed frequency relations ($\theta_1 = 2\Omega_1T_0 + \beta_1$, $\theta_2 = \frac{1}{2}\Omega_1T_0 + \beta_2$). Terms Requiring Further Analysis The following terms from (11) do not directly yield resonant components proportional to $\cos\theta_2$ or $\sin\theta_2$ in their basic form. They may, however, contribute to resonant terms through product expansions with other leading-order solutions or require specific phase relationships (e.g., internal resonance) to become resonant. Their explicit resonant contributions must be calculated by substituting the expressions for y_{10} and y_{20} and applying trigonometric identities:

$$b_{21}D_0^2y_{10},\ b_{22}D_0y_{10},\ b_{23}y_{10},\ a_{22}y_{10}y_{20}^2,\ a_{24}y_{10}^3,\ f_{12}\cos\left(\Omega_1T_0\right). \tag{18}$$

The evaluation of these terms typically involves checking if products like $\cos\theta_1\cdot\cos^2\theta_2$ or $\cos^3\theta_1$ contain terms with frequency $\Omega_1/2$. Purpose of This Step The process of extracting only the $\cos\theta_2$ and $\sin\theta_2$ components from the full expression for R_2 is crucial. Setting the coefficients of these resonant terms to zero constitutes the solvability condition for equation (10). This condition yields the slow-flow equations or amplitude/phase modulation equations that govern the long-term evolution of the amplitude $A_2(T_1)$ and phase $\beta_2(T_1)$. Extraction of Resonant Terms and Secularity Condition The resonant component $R_2^{(\text{res})}(T_0,T_1)$ of the first-order forcing term is obtained by collecting all terms proportional to $\sin\theta_2$ and $\cos\theta_2$, where $\theta_2=\frac{1}{2}\Omega_1 T_0+\beta_2(T_1)$. This component is responsible for generating secular (i.e., unbounded) terms in the solution y_{21} and must be eliminated to ensure a uniform expansion. Its expression is:

$$R_2^{(\text{res})}=\left(\Omega_1 A_2'+\frac{1}{2}\Omega_1\mu_2 A_2\right)\sin\theta_2 +\left(\Omega_1 A_2\beta_2'-\sigma_1 A_2+\frac{3}{4}a_{21}A_2^3+\frac{1}{2}a_{23}A_1^2A_2\right)\cos\theta_2 \tag{19}$$

Here, the prime (′) denotes differentiation with respect to the slow time variable T_1, i.e., $A_2'=\mathrm{d}A_2/\mathrm{d}T_1$, $\beta_2'=\mathrm{d}\beta_2/\mathrm{d}T_1$. Elimination of Secular Terms To prevent the appearance of secular terms which grow linearly in T_0 in the solution for y_{21}, the coefficients of the resonant terms $\sin\theta_2$ and $\cos\theta_2$ in Eq. (19) must be set to zero. This yields the solvability conditions:

$$\Omega_1 A_2'+\frac{1}{2}\Omega_1\mu_2 A_2=0, \tag{20}$$

$$\Omega_1 A_2\beta_2'-\sigma_1 A_2+\frac{3}{4}a_{21}A_2^3+\frac{1}{2}a_{23}A_1^2A_2=0. \tag{21}$$

Derivation of the Slow-Flow Equations Solving the solvability conditions (20) and (21) provides the differential equations governing the slow evolution of the amplitude $A_2(T_1)$ and the phase correction $\beta_2(T_1)$. From Eq. (20), we obtain the amplitude evolution equation:

$$A_2'=-\frac{1}{2}\mu_2 A_2.$$

This is a simple linear decay equation, indicating that the amplitude A_2 decays exponentially on the slow time scale if $\mu_2>0$. From Eq. (21), assuming $A_2\neq 0$, we can solve for the phase evolution. First, rearrange:

$$\Omega_1\beta_2'=\sigma_1-\frac{3}{4}a_{21}A_2^2-\frac{1}{2}a_{23}A_1^2.$$

Finally, the equation for the phase derivative β_2' is:

$$\beta_2'=\frac{1}{\Omega_1}\left(\sigma_1-\frac{3}{4}a_{21}A_2^2-\frac{1}{2}a_{23}A_1^2\right).$$

This equation describes how the phase β_2 (and thus the oscillation frequency) is modified by the detuning parameter σ_1 and the nonlinear interactions quantified by a_{21} and a_{23}. Physical Interpretation Equation for A_2': Governs the amplitude modulation. The term $-\frac{1}{2}\mu_2 A_2$ represents linear damping. Equation for β_2': Governs the frequency modulation or phase drift. The right-hand side includes: σ_1/Ω_1 : Linear frequency shift due to detuning. $-\frac{3}{4\Omega_1}a_{21}A_2^2$: Nonlinear frequency shift due to self-interaction (cubic nonlinearity in y_2). $-\frac{1}{2\Omega_1}a_{23}A_1^2$: Nonlinear frequency shift due to cross-mode interaction with y_1. These two equations together form the slow-flow subsystem for the (A_2, β_2) variables. Amplitude-Phase Slow-Flow Equations and First-Order Approximate Solution. The evolution of the slowly-varying amplitudes and phases on the time scale $T_1 = \varepsilon t$ is governed by the following slow-flow equations:

$$\text{Amplitude Equations}: \begin{cases} A_1' = -\frac{1}{2}\mu_1 A_1, \\ A_2' = -\frac{1}{2}\mu_2 A_2, \end{cases} \tag{22}$$

$$\text{Phase Equations}: \begin{cases} \beta_1' = \frac{1}{4\Omega_1}\left(\sigma_1 - \frac{1}{2}a_{11}A_2^2 - \frac{3}{4}a_{14}A_1^2\right), \\ \beta_2' = \frac{1}{\Omega_1}\left(\sigma_1 - \frac{3}{4}a_{21}A_2^2 - \frac{1}{2}a_{23}A_1^2\right), \end{cases} \tag{23}$$

where the prime ($'$) denotes differentiation with respect to the slow time variable $T_1 = \varepsilon t$. First-Order Multiple Scale Approximate Solution. Therefore, the first-order multiple scale approximate solutions for the system variables are:

$$y_1(t) \approx A_1(T_1)\cos\left(2\Omega_1 t + \beta_1(T_1)\right), \tag{24}$$

$$y_2(t) \approx A_2(T_1)\cos\left(\frac{1}{2}\Omega_1 t + \beta_2(T_1)\right), \tag{25}$$

where $T_1 = \varepsilon t$ is explicitly substituted to show the slow modulation. The functions $A_1(T_1)$, $A_2(T_1)$, $\beta_1(T_1)$, and $\beta_2(T_1)$ appearing in the solutions are determined by solving the slow-flow equations (22) and (23). The solutions (24) and (25) exhibit clear slow-fast separation: The arguments of the cosine functions ($2\Omega_1 t$ and $\Omega_1 t/2$) represent the fast oscillations at the linear system's natural frequencies (or their combinations). The amplitudes A_i and phases β_i are not constants but evolve on a much slower time scale $T_1 = \varepsilon t$, governed by the simpler, first-order differential equations (22) and (23). The amplitude equations (22) show simple exponential decay due to linear damping coefficients μ_1 and μ_2. The phase equations (23) reveal how the oscillation frequencies are modified (a phenomenon known as frequency pulling or detuning by: The linear detuning parameter σ_1. Nonlinear self-interaction terms (e.g., $a_{14}A_1^2$, $a_{21}A_2^2$). Nonlinear cross-mode interaction terms (e.g., $a_{11}A_2^2$, $a_{23}A_1^2$).

This result is the standard output of the method of multiple scales for weakly nonlinear oscillators: the original complex, nonlinear differential equations are reduced to a set of simpler equations describing the slow evolution of the amplitudes and phases, from which the physical motion can be reconstructed via (24) and (25).

4. Analysis of System Dynamic Behavior

Numerical simulation of coupled nonlinear oscillators: Dynamic Behaviors of the

System from the Graphs:

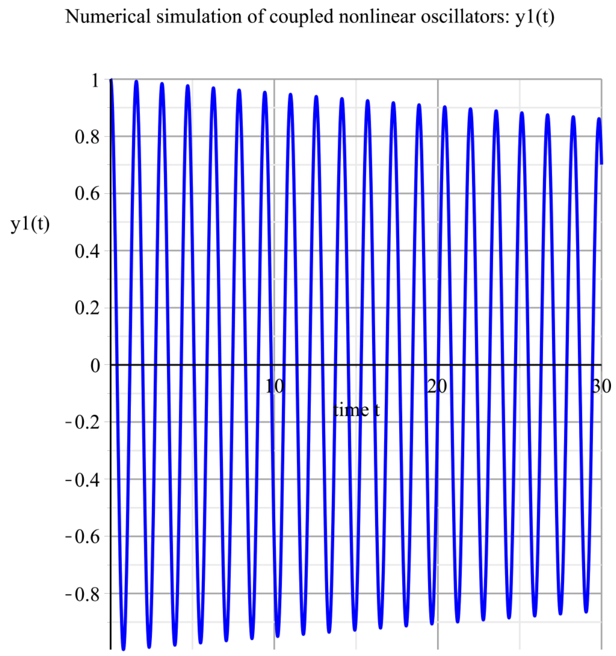

Figure 1. Numerical simulation of coupled nonlinear oscillators: $y_1(t)$.

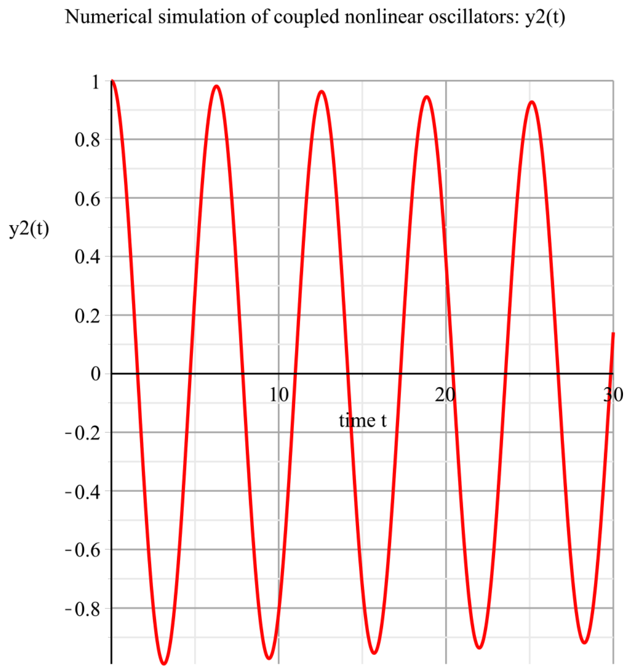

Figure 2. Numerical simulation of coupled nonlinear oscillators: $y_2(t)$.

1) Damped Vibration The vibration amplitudes of both oscillators $y_1(t)$ and $y_2(t)$ gradually decrease with time t, and there is no amplitude increase divergence or constant amplitude undamped behavior throughout the process. $y_1(t)$: The initial amplitude is about 0.1, which decays slowly over time, and the

overall vibration center has a slight offset gradually approaching the zero axis from the negative direction, as shown in **Figure 1**; $y_2(t)$: The initial amplitude is close to 0.15, with a more obvious decay rate than $y_1(t)$, and the vibration has good symmetry decaying around the zero axis. The system has viscous damping (or equivalent damping), and energy dissipates over time, which is a typical characteristic of damped nonlinear vibration systems, as shown in **Figure 2**.

2) Nonlinear Coupling Characteristics The vibration behaviors of the two oscillators are not independent but exhibit "coupling correlation": Their vibration frequencies show consistency in the time domain similar vibration periods, indicating that the coupling effect synchronizes the vibration frequencies of the two degrees of freedom; The amplitude decay rates are different (y_2 decays faster), reflecting the differential effects of nonlinear coupling terms on energy dissipation/transfer of different oscillators, which is the "coupling asymmetry" of nonlinear systems.

3) No Chaos/Bifurcation Behavior Within the entire time range (0 500), the vibrations of $y_1(t)$ and $y_2(t)$ maintain regular periodicity no random fluctuations, no amplitude mutations, no frequency jumps. Under this time scale and initial conditions, the system is in a periodic vibration state and does not enter complex nonlinear dynamic regions such as chaos or period-doubling bifurcation.

4) Amplitude Offset (Unique to $y_1(t)$) The vibration center of $y_1(t)$ moves from the initial negative direction approximately -0.05 to the zero axis gradually, while $y_2(t)$ vibrates around the zero axis all the time. This is the result of the combined action of nonlinear coupling and damping, reflecting the "asymmetric vibration" characteristic of nonlinear systems.

5. Conclusions Limitations and Future Work

This paper analyzes the dynamic behavior of a two-degree-of-freedom coupled nonlinear oscillator system through numerical simulation, yielding the following key conclusions. Under the time scale of $0 \sim 500$ and the given initial conditions, the system exhibits typical nonlinear periodic vibration characteristics with damping. Energy dissipates continuously over time, and the vibration amplitudes of both oscillators show a decaying trend. Numerical simulations show that this specific coupling form inherently suppresses chaotic behavior within the investigated range. No complex nonlinear dynamic phenomena such as chaos, bifurcation, or sudden changes in amplitude/frequency were observed. The nonlinear coupling effect causes the vibration frequencies of the two oscillators to tend to synchronize in the time domain. However, the coupling effect is significantly asymmetric: the amplitude decay rate of $y_2(t)$ is noticeably faster than that of $y_1(t)$. Moreover, $y_1(t)$ exhibits a shift of its vibration center from the negative side towards the zero axis, while $y_2(t)$ always decays symmetrically around zero. The unique amplitude offset of $y_1(t)$ results from the combined action of the nonlinear coupling terms and viscous damping. This characteristic further confirms the asymmetric nature of the nonlinear coupling in the system.

The analysis was confined to the system's behavior within the 0 ~ 500 time scale. The investigation did not explore whether complex dynamic phenomena such as bifurcation or chaos might occur over longer time scales. Furthermore, the simulation was based on only one set of initial conditions, without considering the impact of varying initial conditions on the system's behavior. The interpretation of the system's dynamic characteristics relied solely on time-domain waveform plots. Methods such as phase portraits, Poincar sections, and spectral analysis were not employed, limiting the comprehensive characterization of the system's nonlinear properties from frequency-domain and phase-space perspectives. The study did not investigate how variations in key parameters influence the system's characteristics, such as vibration frequency synchronization, amplitude decay rate, and asymmetric vibration features.

Extend the time scale of numerical simulations to investigate the system's dynamic behavior over longer periods. Modify initial conditions and coupling term types to analyze whether complex phenomena like bifurcation or chaos emerge under different conditions. Introduce phase portraits, Poincar sections, and Fourier spectral analysis to analyze the system's periodic characteristics, frequency components, and nonlinear coupling mechanisms from multiple dimensions. Combine theoretical derivations to quantify the influence weights of nonlinear coupling terms and damping on the system's behavior. Systematically study the effects of key parameters (e.g., coupling coefficients, damping coefficient, natural frequencies) on the oscillators' amplitude decay rates, frequency synchronization, and amplitude offset. Clarify the relationship between parameter thresholds and the system's dynamic behavior to provide a theoretical basis for the regulation and optimization of coupled nonlinear vibration systems.

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Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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