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Physical Properties of the Types of Dark Energy Introduced in Light of the "DESI-Pressure" upon the ΛCDM Universe Model

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Abstract

This paper presents an introduction to the main point in the changes of the ΛCDM-universe model that the cosmologists have proposed in light of the DESI-observational data and other types of observations: New types of dark energy characterized by different equations of state than that of Lorentz invariant vacuum energy, LIVE, with constant density in the ΛCDM-universe model. Gravitational properties and time evolution of these new types of dark energy are discussed and illustrated graphically by employing analytical expressions for dimensionless density factors, resulting from integrating the equation of continuity of the different models. The density factors are employed to calculate the time evolution of the deceleration parameters of universe models with different types of dark energy. It is also shown that the introduction of the $w_0 w_a$CDM-universe models changes the value of Hubble's constant relative to that of the ΛCDM-universe model, but with a too small magnitude to solve the Hubble tension. It is argued that this is a general feature of all dynamical dark energy models constrained by low redshift data.

Keywords

Dark Energy, Cosmological Model, Equation of State, Hubble Tension, Deceleration Parameter

1. Introduction

DESI is an acronym for Dark Energy Spectroscopic Instrument. It is a large and advanced spectrograph installed on a 4 m-telescope at Kitt Peak National Observatory 2100 m above sea level which has been operative since 2022. Its first main data release was published 19 March 2025 [1] and contained spectra from galaxies at distances out to 11 billion light years.

In the reports [1] and [2], the DESI Collaboration analysed cosmic parameters based upon baryon acoustic oscillation (BAO) measurements from more than 14 million galaxies and quasars drawn from the DESI Data Release 2 (DR2), obtained during three years of operation, together with data sets obtained in other projects with different types of observations. In this report, the authors wrote that "probing the behaviour and nature of dark energy is the main goal of DESI. The question of perhaps greatest interest, and the one that BAO measurements can best illuminate, is the value of the equation-of-state parameter $w = p/\rho c^2$ and its possible evolution with time".

In [1], they wrote: "To examine this, we will primarily use the so-called Chevallier-Polarski-Linder (CPL) parametrization [3] [4] of Equation (8). While this form of $w(a)$ does not arise directly from an underlying physical model, it is a flexible parametrization that is capable of matching the predictions for observable quantities obtained in a wide range of models that are physically motivated [5]. The accompanying paper [2] explores various other parametrizations of $w(z)$".

By means of DESI, cosmic distance measurements are measured in a setting where baryonic acoustic oscillations (BAO) are utilized. These oscillations are pressure waves in the cosmic plasma which existed in the early universe before the recombination time around 380,000 years after the Big Bang.

The maximal distance sound waves created at the time of the Big Bang could move before the plasma disappeared, and is called the *sound horizon*. In a given universe model, for example, the ΛCDM universe model, the radius of the sound horizon can be calculated, and can therefore function as a cosmic standard ruler. It is $r_d = 490000$ light years long. Radiation emitted at this so called *drag epoch* in the universe has a redshift $z_d = 1050$ when it is observed at the present time.

These quantities have been utilized by DESI to give a picture of the evolution of the cosmic matter distribution during the last 4 billion years. From a thorough analysis of such data in combination with data from other types of observations, it was concluded that the cosmic dark energy did not have a constant density in this period. Hence, the observations favour that the dark energy is not of the LIVE-type as in the ΛCDM-universe model, but is of a dynamical type with time-dependent density.

The main results were summarized in the following way: "The DR2 BAO results are consistent with DESI DR1 and SDSS, and their distance-redshift relationship matches those from recent compilations of supernovae over the same redshift range. The results are well described by a flat ΛCDM model, but the parameters preferred by BAO are in mild, 2.3σ tension, with those determined from the cosmic microwave background (CMB), although the DESI results are consistent with the acoustic angular scale that is well-measured by Planck. This tension is alleviated by dark energy with a time-evolving equation of state parametrized by w_0 and w_a, which provides a better fit to the data, with a favoured solution in the quadrant with $w_0 > -1$ and $w_a < 0$. This solution is preferred over ΛCDM at 3.1σ for the combination of DESI BAO and CMB data. When also including SNe, the preference for a dynamical dark energy model over ΛCDM ranges from 2.8 - 4.2σ depending on which SNe sample is used".

The authors concluded that "unless there is an unknown systematic error associated with one or more datasets, it is clear that ΛCDM is being challenged by the combination of DESI BAO with other measurements, and that dynamical dark energy offers a possible solution".

The DESI-results have resulted in an enormous research activity the last year. Searching on "DESI dark energy" and "abstract" in the ArXiv gave around 450 preprints from March 2025 to December 2025. Hence, I refer the reader to the ArXiv and mention only a few of the articles here. Most of the authors have analysed observational data in different ways and confirmed the DESI-results.

In nearly all of the universe models proposed to obtain agreement with the DESI-results, the modification of the ΛCDM-universe model has been to introduce a type of dynamical dark energy in the proposed universe model. Hence in this paper, I shall focus upon the gravitational properties of the proposed models of dark energy.

Recently (11 December 2025), S. Capozziello and co-workers have published an interesting review article [6] with title: "Is Dark Energy Dynamical in the DESI Era? A Critical Review". Here they wrote: "DESI DR2 consistently favours the quadrant $\omega_0 > -1$ and $\omega_a < 0$, indicating a preference for dynamical dark energy of the Quintom-B type at the $\lesssim 3\sigma$ level for most dataset combinations, rising to $\sim 3.8\sigma$ only when the DES-SN5Y supernova sample is included". But they also found that the preference for dynamical dark energy is biased by the low-z ($z <$ 0.1) DES-SN5Y SNe Ia sample from the CfA/CSP sample. When these low-z SNe Ia are excluded, their analysis no longer requires a dynamical dark energy and fully restores the ΛCDM model.

I will here focus primarily upon the types of dark energy considered in [1] and [2] where the equation of state parameter of the dark energy is assumed to depend upon the scale factor, or equivalently upon the cosmic redshift, in a given way. However, a more realistic scalar field dark energy has been considered by Z. Zhang and co-workers [7] in light of the DESI-result. They also investigated restrictions from observations upon the evolution of the dark energy equation of state parameter, w, and found that high-z CMB data prefer phantom behaviour ($w < -1$), while low-z BAO and SNIa data favour quintessence ($w > -1$).

In [8], it was concluded that our universe may presently be in a state of decelerated expansion. The state of acceleration or retardation of the expansion is represented by the deceleration parameter q. In this paper, I will therefore have particular focus upon the time evolution of the deceleration parameter of different universe models that have been introduced as response to the DESI-results.

It would be a great strength for a universe model if it turns out both to be in agreement with the combined data behind the conclusion that the LIVE of the

ΛCDM-universe model should be replaced by a dynamical dark energy, and is also able to solve the conflict between the early time- and late time determination of the Hubble constant, often called the *Hubble tension*. Hence I shall calculate the change of the Hubble constant when LIVE is replaced by dynamical dark energy.

This paper is written for students and teachers who are not performing research on these matters, but are familiar with relativistic cosmology, and want to obtain an updated knowledge of what is going on in cosmology right now. It seems that we are at the beginning of an era where new observations lead to a new understanding which requires that the present standard model of the universe—the ΛCDM-universe model-should be replaced by a new model with another type of dark energy than LIVE.

We have lived with the ΛCDM-universe model as our standard model of the universe in 25 years, since it was discovered that the universe is in a state of accelerated expansion. Hence, in order to appreciate why we may consider the present time as the beginning of a new revolution in cosmology, it is necessary to have a good understanding of the ΛCDM-universe model. Therefore, Section 2 is devoted to give a conceptual description of this model. In the following sections I then compare the proposed new universe models with the ΛCDM-universe model—focusing upon how different types of dark energy that dominate the different universe models, determine the expansion history of the universe. Also, the age of the different universe models is calculated. Finally, this type of modified ΛCDM-universe models is applied to the Hubble tension, in order to judge whether such modifications can solve this tension.

2. The Standard Model of the Universe

In the general theory of relativity the gravitational energy density, ρ_G, is given by (using units so that $c=1$)

$$\rho_G = \rho + 3p, \tag{1}$$

where ρ is the density of the cosmic fluid as measured by an observer co-moving with the fluid, and p is the pressure ($p > 0$) or strain ($p < 0$). *If* $\rho_G > 0$ *the gravitational energy density of matter or energy causes attractive gravity with decelerated expansion of the universe*, *and if* $\rho_G < 0$ *the energy causes repulsive gravity and accelerated expansion of the universe.* Thinking upon the expanding space as a river, it is often called the *Hubble flow*.

In cosmology, and considering dark energy, it is usual to call the relationship

$$\rho = w\, p \tag{2}$$

the *equation of state* of the dark energy, and w *the equation of state parameter*. Hence

$$\rho_G = (1+3w)\,\rho. \tag{3}$$

This shows that *dark energy with* $w < -1/3$ *causes accelerated expansion of the universe*.

In the ΛCDM-universe model the universe is filled by a vacuum energy which exists according to the quantum theory, having a density which we have not been able to calculate from the quantum theory. It is possible that a quantum gravity theory will be needed to calculate it. However, Georges Lemaître [9] pointed out already in 1934 that in order not to be in conflict with the principle of relativity the vacuum energy must have Lorentz invariant properties, it must be a Lorentz Invariant Vacuum Energy, LIVE. He also showed that this type of vacuum energy has an equation of state parameter $w_0 = -1$. Hence LIVE causes accelerated expansion of the universe.

We shall here consider a flat, isotropic and homogeneous universe. Einstein's field equations applied to such a universe give the energy conservation equation,

$$\mathrm{d}\ln\rho = -3\left(1+w\right)\mathrm{d}\ln a\,, \tag{4}$$

where *a* is the scale factor representing the ratio of the distance at an arbitrary point of time between to objects following the Hubble flow, and their present distance. It follows that *LIVE has constant energy density* in spite of the expansion of the universe. This motivated Lemaître to give a new interpretation of the cosmological constant (different from Einstein's interpretation as representing a tendency of space to expand). He said that it represents the density of LIVE,

$$\Lambda = \kappa\rho_{LIVE}\,,\ \text{where}\quad \kappa = \frac{8\pi G}{c^4} \tag{5}$$

is Einstein's gravitational constant. In the present standard model of the universe LIVE makes out around 70% of the contents of the standard model of the universe, while 26% of the contents is cold, dark matter, which is why it is called the ΛCDM universe. Only around 4% is ordinary matter.

3. Dynamical Dark Energy of the CPL-Type

Different types of dark energy have different values of the equation of state parameter, and the density changes with time. We cannot observe directly how the distance between clusters of galaxies, change with time. Our life is too short. Fortunately there is an indirect way: When we observe outwards in space, we observe backwards in time, because we observe how an object was when it emitted the observed radiation. So, after all we *can* observe the state of the universe at different cosmic times, billions of years ago, by observing objects billion of light years away from us.

The mathematical description of this is in terms of the scale factor *a*. This describes how the distance between to objects following the Hubble flow changes with time. But again this cannot be directly observed. What *is* observed is the red shift, *z*, of the spectral lines from the objects. That is what DESI observes.

The general relativistic interpretation of the cosmic redshift is that it is due to the lengthening of the electromagnetic waves due to the expansion of the universe, when they travel from an object to an observer. This implies that there is a simple relationship between the redshift and the scale factor.

$$1+z=\frac{1}{a}. \tag{6}$$

In an expanding universe the scale factor *a* increases with time. Thus, looking to the past, as we do when we observe distant objects, the scale factor has a smaller value the more remote in the past the emission time is, and the larger is the redshift. Hence cosmologists often use the scale factor or the cosmic redshift as a clock. Therefore, it is usual that they write the equation of state parameter as a function of the scale factor or the cosmic redshift to describe how the equation of state of some sort of dark energy varies during the history of the universe.

Integrating Equation (4) gives the result that if the equation of state parameter is constant, the density of the dark energy varies with the scale factor in the following way

$$\rho=\rho_0 a^{-3(1+w)}=\rho_0\left(1+z\right)^{3(1+w)}. \tag{7}$$

This equation shows that if $w>-1$ the density of the dark energy decreases with time, and if $w<-1$ the density of the dark energy increases with time. The density is constant if $w=-1$ which is the value of the equation of state parameter of LIVE.

The DESI-measurements showed that during the last four billion years the density of the dark energy has decreased by about 10%. This indicates that at least in most parts of this period the equation of state parameter of the dark energy had a value $w>-1$, not necessarily constant. Such types of dark energy are called *dynamical dark energy*.

One may divide the types of dynamical dark energy into two classes: A. Dynamical energy with constant value of the equation of state parameter, $w>-1$. These models have decreasing energy density given by Equation (7). B. Dynamical energy with varying value of the equation of state parameter. These models have energy density with a change of rate which varies and even may change sign during the evolution of the universe. Since type A is a special case of type B we shall mainly consider dark energy of type B in the present article.

A large number of dark energy models have been introduced in cosmology. Hence we need to make some limitations. The main criteria used in order to decide which types of dark energy that shall be considered here are: 1) The types of dark energy used by the DESI-team, and later in follow up articles by others, to analyse the observational data are given high priority. As stated in the heading, also the present paper is a follow up article to the DESI-reports. 2) Simplicity. Since we know so little about the type of the dark energy in the universe, the types of dark energy that can be described mathematically in the simplest way are chosen.

The DESI-reports, and most of the follow up articles, considered universe models with dark energy of the $w_0 w_a$-type having an evolving equation of state parameter of the form

$$w(a)=w_0+w_a\left(1-a\right),\quad w_z\left(z\right)=w_0+w_a\frac{z}{1+z}, \tag{8}$$

where w_0 and w_a are constants with values determined either by observations or by theoretical constraints. This scale-factor dependence of the equation of state parameter w was introduced by Chevallier, Polarski [3] and Linder [4] and is therefore usually called the CPL-model of dark energy. It has also been called the $w_0 w_a$-model, but since there are several models that use the constants w_0 and w_a in different expressions for w, we shall use the name CPL-model here for the models with equation of state parameter given by Equation (8).

The requirement of Lorentz-invariance of all the components of the energy-momentum tensor of the LIVE-energy of the ΛCDM-universe model leads to the values $w_0 = -1$, $w_a = 0$ [10]. In a sense all the $w_0 w_a$-models are 'weaker' universe models than the ΛCDM-model since they have more arbitrary parameters associated with the dark energy than the ΛCDM-model.

In the CPL-models the equation of state parameter changes from an initial value $w_0 + w_a$ to the present value w_0. We shall see later that observations favour negative sign for both w_0 and w_a, so we assume this is the case. The evolution of this type of dark energy is rather strange. It will first pass a stage where the gravity of the dark energy vanishes. At this moment the gravity changes from being repulsive to being attractive. It follows from Equation (3) that this happens for a scale factor a_{1CPL} given by $w(a_{1CPL}) = -1/3$, which leads to

$$a_{1CPL} = 1 + \frac{w_0 + 1/3}{w_a}. \tag{9}$$

This will happen at the present time if $w_0 = -1/3$ and in the future if $a_{1CPL} > 1$, *i.e.* if $w_0 < -1/3$. Somewhat later the dark energy passes through a state where it is like dust or cold matter with vanishing pressure. This happens for $w(a_{2CPL}) = 0$ giving

$$a_{2CPL} = 1 + \frac{w_0}{w_a}. \tag{10}$$

At this point of time the pressure of the dark energy changes sign. If both w_0 and w_a are negative the dark energy will then change from being in a state with negative pressure, *i.e.* strain, to a state with positive pressure.

Heat is defined as transport of energy due to temperature differences. In a homogeneous universe there are no large scale temperature differences. Hence *the universe expands adiabatically*. The adiabatic energy conservation equation, Equation (4), for such a universe is a consequence of Einstein's field equations.

We now apply this equation to flat universe with non-interacting dust and dark energy modelled as a perfect fluid with equation of state (2) where the equation of state parameter is in general not constant but may vary with time during the evolution of the universe. Since the scale factor is monotonously increasing with time in an expanding universe, the time-dependence of w may be represented by considering w to be a function of the scale factor, a. Due to the relationship (6) between the cosmic redshift and the scale factor, the differentials are related by

$$\frac{\mathrm{d}z}{1+z} = -\frac{\mathrm{d}a}{a}. \tag{11}$$

Because of the fact that z can be directly measured, it is also usual to consider the equation of state parameter as a function of the redshift for example as in Equation (8). Note that *z increases when we look backwards in time*. We shall here use both these representations of the time dependence of the equation of state parameter.

Let us now consider dark energy. Inserting Equation (2) into Equation (4) and using a dot for differentiating with respect to time, gives

$$\frac{\dot{\rho}_{de}}{\rho_{de}} = -3(1+w)\frac{\dot{a}}{a}, \tag{12}$$

or

$$(\ln \rho_{de})^{\cdot} = -3(1+w)(\ln a)^{\cdot}. \tag{13}$$

Integration with the boundary condition $a(t_0)=1$, where t_0 is the present time, gives

$$\rho_{de} = \rho_{de0} f_{de}(a), \quad f_{de}(a) = \exp\left[-3\int_1^a \frac{1+w(a')}{a'}\mathrm{d}a'\right] = a^{-3}\exp\left[-3\int_1^a \frac{w(a')}{a'}\mathrm{d}a'\right]. \tag{14}$$

Expressed in terms of the redshift, using Equation (11) and that light emitted at the present time has vanishing redshift, $z(t_0)=0$, this equation takes the form

$$\rho_{dw} = \rho_{de0} g_{de}(z),$$

$$g_{de}(z) = \exp\left[3\int_0^z \frac{1+w_z(z')}{1+z'}\mathrm{d}z'\right] = (1+z)^3 \exp\left[3\int_0^z \frac{w_z(z')}{1+z'}\mathrm{d}z'\right]. \tag{15}$$

The functions $f_{de}(a)$ and $g_{de}(z)$ represent the density of the dark energy at an arbitrary point of time relative to its present density. This is analogous to the scale factor, a, which represents the ratio of cosmic distances at an arbitrary point of time and the present distances. Therefore it is natural to call $f_{de}(a)$ and $g_{de}(z)$ the *density factors*. The value of the density factors for the ΛCDM-universe model is obtained by inserting $w=-1$, and is equal to the present value of the density factors $f_{de}(1) = g_{de}(0) = 1$.

Inserting Equation (8) into Equations (14) and (15) gives

$$f_{deCPL} = a^{-3(1+w_0+w_a)} \mathrm{e}^{-3w_a(1-a)} \tag{16}$$

and

$$g_{deCPL} = (1+z)^{3(1+w_0+w_a)} \mathrm{e}^{-3w_a \frac{z}{1+z}}. \tag{17}$$

A combination of DESI-measurements and other types of measurement have given several favoured values for w_0 and w_a depending upon which datasets are taken into account. In table V in DESI 2 the values of w_0 vary between the CMB-value −1.23 and the BAO + CMB-value −0.42. The BAO + CMB + SN (supernova)-value is $w_0 = -1.09^{+0.31}_{-027}$. Similarly the values of w_a vary between −0.17

and −1.75 with the BAO + CMB + SN-value $w_a = -0.67 \pm 0.09$. Hence, in the graphs below I will use the values $w_0 = -0.7$ and $w_a = -1$ as representative for the DESI-results [1]. According to Equation (9) and (10) the dark energy then changes from causing repulsive to attractive gravitation at $a_1 = 1.37$, and changes from having negative pressure to positive pressure at $a_2 = 1.7$.

With these values the density factor, $f_{dew_0w_a}$, of the dark energy, as given in Equation (16), is shown in **Figure 1**.

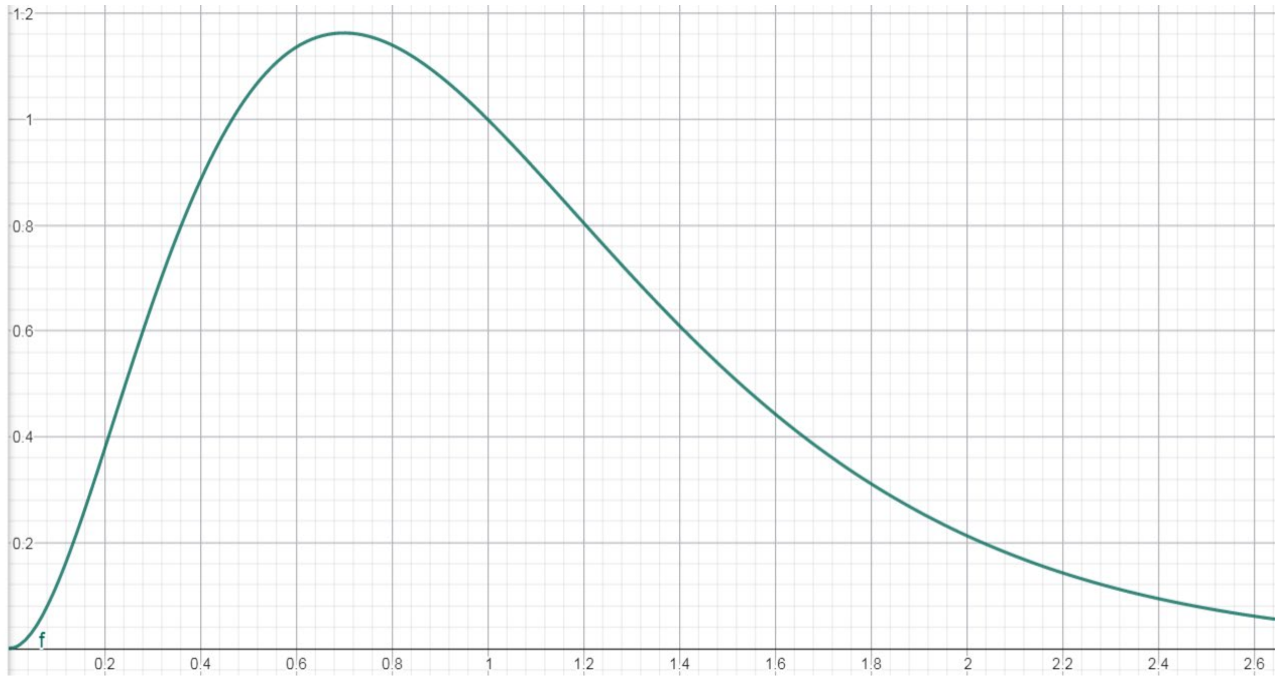


Figure 1. The density factor of the dark energy as a function of the scale factor in the w_0w_aCDM universe model with $w_0 = -0.7$ and $w_a = -1$.

It is seen that the density had a maximum $\rho_{max} = 1.16\rho_0$ for $a = 2/3$ corresponding to a redshift $z = 1/2$. This result has been commented by M. L. Abreu and M. S. Turner in an interesting article [11] with title "DESI Dark Secrets". They write: "The first year results of DESI (DR1) provide evidence that dark energy may not be quantum vacuum energy (Λ). If true, this would be an extraordinary development in the 25-year quest to understand cosmic acceleration. The best-fit DESI w_0 w_a models for dark energy, which underpin the claim, have strange behaviour. They achieve a maximum energy density around $z \simeq 0.5$ and rapidly decrease before and after". Hence if the dark energy is of this type, we live at a special time in the evolution of the universe, just after the time when the dark energy rather suddenly became dominant and then became sub-dominant again.

Increasing redshift means looking backwards in time. The density of the dark energy relative to its present density is given as a function of redshift by the density factor $g_{dew_0w_a}$ in Equation (17). It is shown in **Figure 2** with $w_0 = -0.7$ and $w_a = -1$.

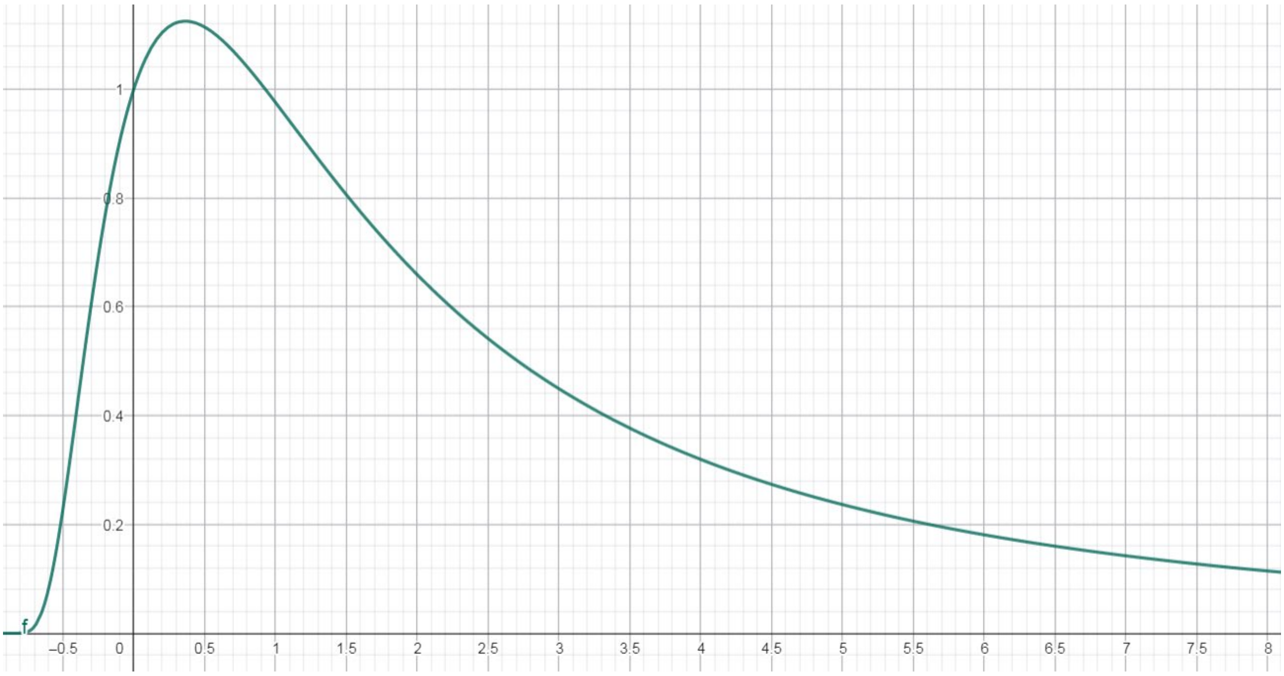


Figure 2. The density of the dark energy relative to its present density as a function of the redshift in the $w_0 w_a$CDM universe model with $w_0 = -0.7$ and $w_a = -1$.

If we look at universe models with dark energy having an equation of state parameter obeying Equation (8), there are several possibilities for the future evolution depending upon the values of the constants w_0 and w_a. In this connection several scenarios have been considered. They are organized in four classes [12].

- *Quintom A*: $w_0 < -1$, $w_a > -w_0 - 1$.
- *Phantom energy*: $w_0 < -1$, $w_a < -w_0 - 1$.
- *Quintom B*: $-1 < w_0 < 0$, $w_a < -w_0 - 1$.
- *Quintessence*: $-1 < w_0 < 0$, $w_a > -w_0 - 1$.

This is illustrated in **Figure 3**.

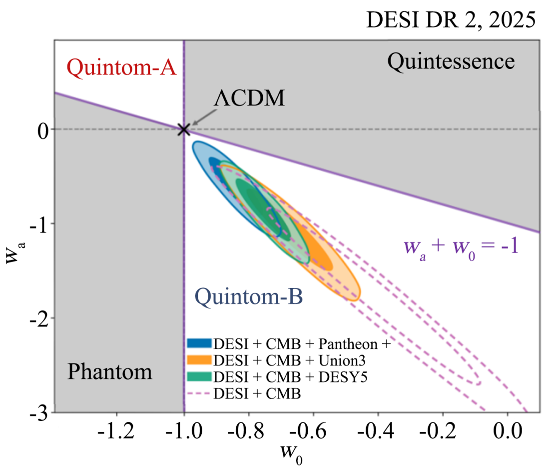


Figure 3. The four classes of CPL-universe models [13]. This classification is based upon Equation (8) for the equation of state parameter.

4. Deceleration Parameter of Flat Universe Models with Dust and Dark Energy

Our point of departure for calculating the deceleration parameter is the standard formula for the Hubble parameter, Equations (11.123), (11.124)] in [14]. The ratio of the Hubble parameter at an arbitrary point of time and its present value, Hubble's constant, is usually denoted by *h*. It may be called the *Hubble factor*. Introducing the density factors $f_{de}(a)$ and $g_{de}(z)$ it takes the forms

$$h_a(a) = \left[\Omega_{m0}a^{-3} + \Omega_{de0}f_{de}(a)\right]^{1/2}, \tag{18}$$

and

$$h_z(z) = \left[\Omega_{m0}(1+z)^3 + \Omega_{de0}g_{de}(z)\right]^{1/2}. \tag{19}$$

The deceleration parameter can be calculated from the formula

$$q = -1 - \frac{\dot{h}}{h^2}. \tag{20}$$

We shall first express the deceleration parameter as a function of the scale factor, *a*. Using the chain rule of differentiation gives

$$\dot{h}_a = \frac{\mathrm{d}h_a}{\mathrm{d}a}\dot{a} = ah_a\frac{\mathrm{d}h_a}{\mathrm{d}a}. \tag{21}$$

Hence, the expression for the deceleration parameter can be written as

$$q_a = -1 - \frac{a}{h_a}\frac{\mathrm{d}h_a}{\mathrm{d}a}. \tag{22}$$

Inserting Equation (19) into Equation (22) we get after a small rearrangement

$$q_a = \frac{1}{2}\frac{A - a^3\left[2f_w(a) + a\mathrm{d}f_{de}/\mathrm{d}a\right]}{A + a^3 f_{de}(a)}, \quad A = \frac{\Omega_{m0}}{\Omega_{de0}}. \tag{23}$$

Differentiating the function $f_{de}(a)$ we get

$$\frac{\mathrm{d}f_{de}}{\mathrm{d}a} = -3(1+w)\frac{f_{de}(a)}{a}. \tag{24}$$

Inserting this into Equation (23) gives

$$q_a = \frac{1}{2}\frac{A + \left[1 + 3w(a)\right]a^3 f_{de}(a)}{A + a^3 f_{de}(a)}. \tag{25}$$

We shall now express the deceleration parameter as a function of the redshift, *z*. Using that

$$\dot{h}_z = \frac{\mathrm{d}h_z}{\mathrm{d}z}\dot{z} = -\frac{\mathrm{d}h_z}{\mathrm{d}z}\frac{\dot{a}}{a^2} = -\frac{\mathrm{d}h_z}{\mathrm{d}z}\frac{h_z}{a} = -(1+z)h_z\frac{\mathrm{d}h_z}{\mathrm{d}z}, \tag{26}$$

the deceleration parameter can be written

$$q_z = -1 + (1+z)\frac{1}{h_z}\frac{\mathrm{d}h_z}{\mathrm{d}z}. \tag{27}$$

Inserting Equation (19) into Equation (27) and performing the differentiation, we get [15]

$$q_z = -1 + \frac{3A(1+z)^3 + (1+z)g'_{de}(z)}{2\left[A(1+z)^3 + g_{de}(z)\right]}. \tag{28}$$

Differentiating the function $g_{de}(z)$ gives

$$g'_{de}(z) = 3\frac{1+w_z(z)}{1+z}g_{de}(z). \tag{29}$$

Inserting this into Equation (28) leads to

$$q_z = \frac{1}{2}\frac{A(1+z)^3 + \left[1+3w_z(z)\right]g_{de}(z)}{A(1+z)^3 + g_{de}(z)}. \tag{30}$$

Equations (25) and (30) may be written as

$$q_a = \frac{1}{2}\left[1 + \frac{3w(a)}{1 + A\left[a^3 f_{de}(a)\right]^{-1}}\right] \tag{31}$$

and

$$q_z = \frac{1}{2}\left[1 + \frac{3w_z(z)}{1 + A(1+z)^3\left[g_{de}(z)\right]^{-1}}\right]. \tag{32}$$

It should be noted that a universe with only cold dark matter has $\Omega_{de0} = 0$, and hence for such a universe $A \to \infty$. This means that such a matter dominated universe has a deceleration parameter $q_m = 0.5$.

Inserting Equations (8), (16) and (17) in Equations (31) and (32) gives the deceleration parameter of a flat universe model with pressure-free matter and dark energy of the CPL-type as function of the scale factor and as function of the red shift,

$$q_{aCPLm} = \frac{1}{2}\left[1 + 3\frac{w_0 + w_a(1-a)}{1 + Aa^{3(w_0+w_a)}e^{3w_a(1-a)}}\right], \tag{33}$$

$$q_{zCPLm} = \frac{1}{2}\left[1 + 3\frac{w_0 + w_a\frac{z}{1+z}}{1 + A(1+z)^{-3(w_0+w_a)}e^{3w_a\frac{z}{1+z}}}\right]. \tag{34}$$

It is interesting to note that all of the models have the same expression for the present value of the deceleration meter. Utilizing that the present value of the density factors is per definition 1 independently of the form of the equation of state of the dark energy, Equation (31) shows that the present value of the deceleration parameter is

$$q_0 = \frac{1}{2}\left(1 + \frac{3w_0}{1+A}\right), \tag{35}$$

where w_0 is the present value of the equation of state parameter. Noting that in a flat universe

$$1 + A = 1 + \frac{\Omega_{m0}}{\Omega_{de0}} = \frac{\Omega_{de0} + \Omega_{m0}}{\Omega_{de0}} = \frac{1}{\Omega_{de0}} = \frac{1}{1-\Omega_{m0}}, \tag{36}$$

and solving Equation (35) with respect to w_0 gives [16]

$$w_0 = \frac{1}{3}\frac{2q_0 - 1}{1 - \Omega_{m0}}. \tag{37}$$

Hence measurements of the present values of the deceleration parameter of the universe, and the mass parameter of the cold matter, determines the present value of the equation of state parameter of the dark energy. The measurements indicate that the universe is at the present time in a state of accelerated expansion, i. e. that $q_0 < 0$. Hence $w_0 < -\left[3\left(1-\Omega_{m0}\right)\right]^{-1}$. With $\Omega_{m0} = 0.3$ this gives $w_0 < -0.48$.

The deceleration parameter of the ΛCDM-universe model is found by inserting $w_0 = -1$, $w_a = 0$, in Equation (33) and (34) which gives

$$q_{a\Lambda CDM} = \frac{1}{2}\frac{A - 2a^3}{A + a^3}, \quad q_{z\Lambda CDM} = \frac{1}{2}\frac{A(1+z)^3 - 2}{A(1+z)^3 + 1} \tag{38}$$

which are shown graphically in Figure 4 and Figure 5.

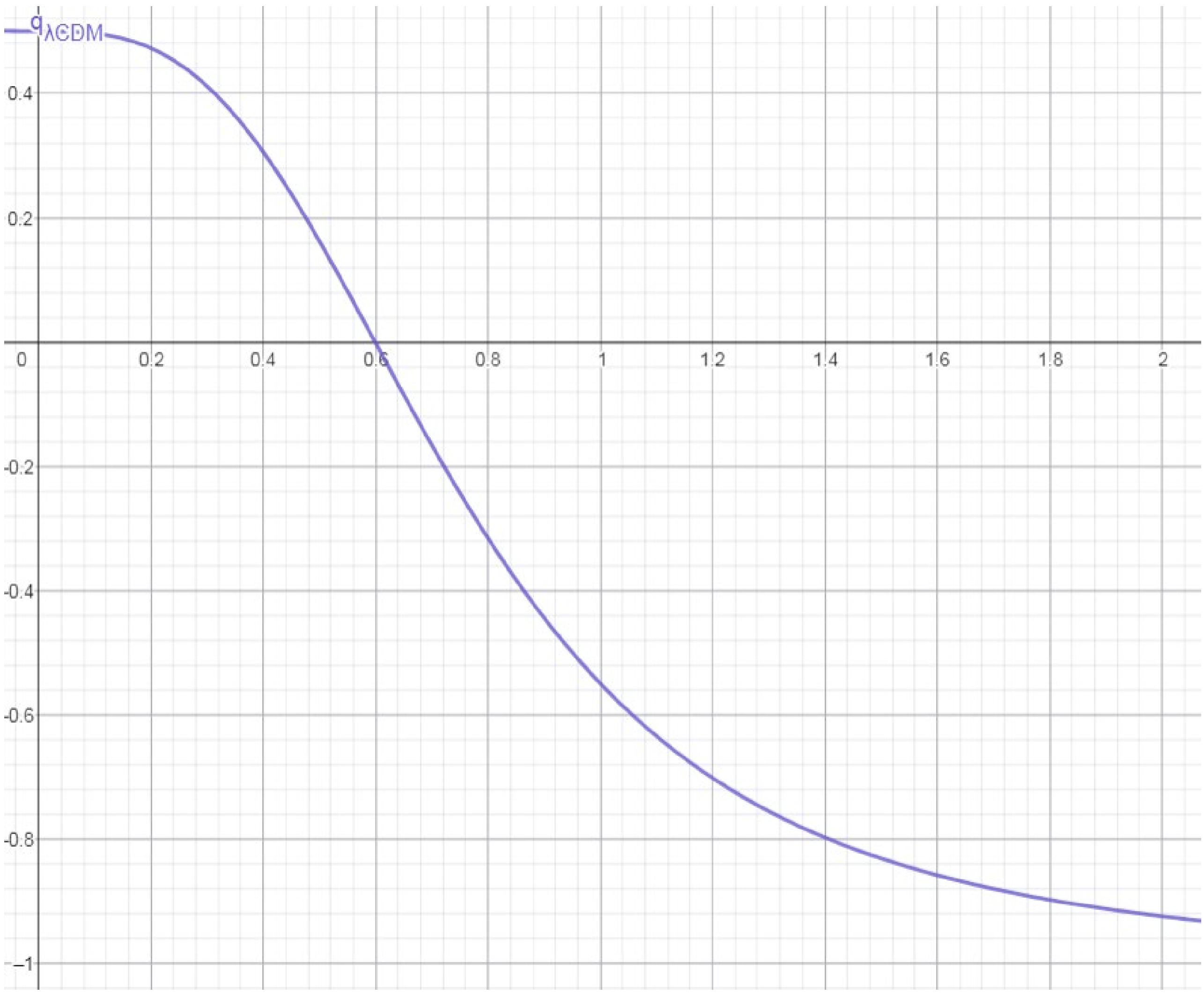


Figure 4. The deceleration parameter of the ΛCDM-universe model as a function of the scale factor.

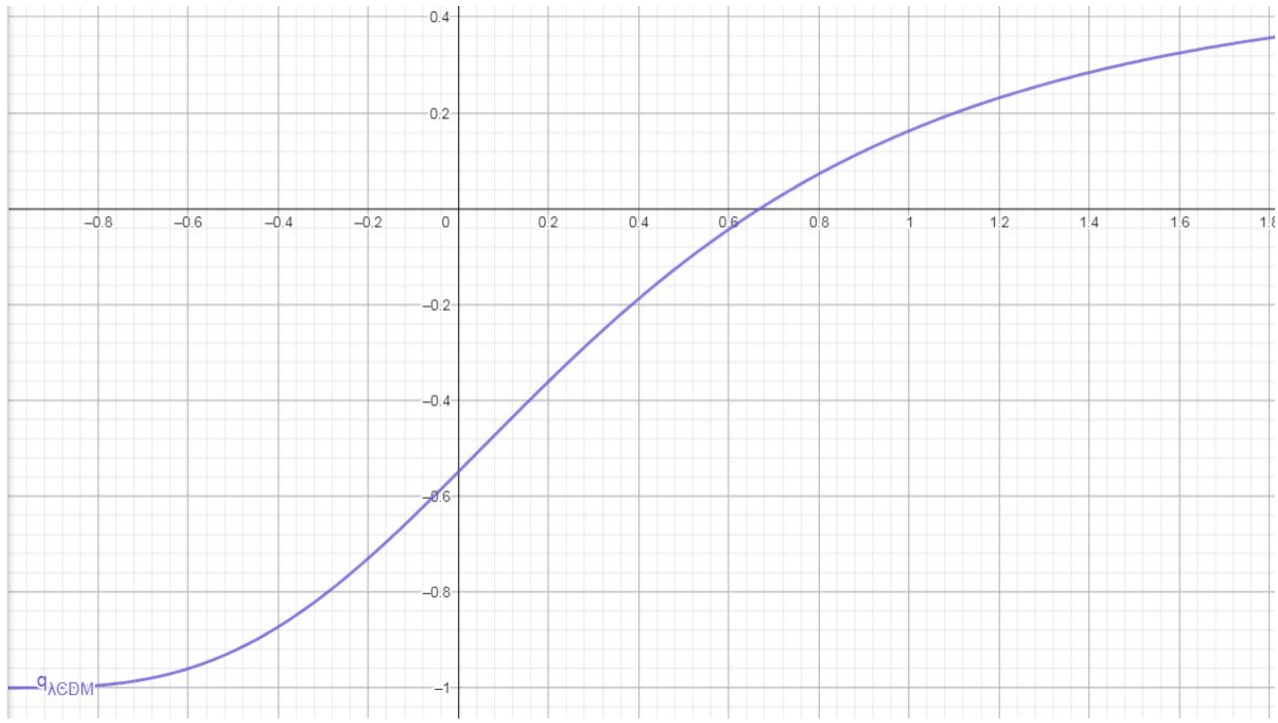


Figure 5. The deceleration parameter of the ΛCDM-universe model as a function of the cosmic redshift.

In the standard model of the universe the rate of change of the expansion was initially dominated by the attractive gravity of the dust. The deceleration parameter was positive and the expansion was decelerated. But the density of the dust decreased, and the density of LIVE remained constant, and about seven billion years ago the repulsive gravity of LIVE began dominating. The expansion turned from being decelerated to accelerated for a scale factor $a_t \approx 0.6$. In the future there will be eternal accelerated expansion according to this universe model.

The present value of $q_{a\Lambda CDM}$ is

$$q_{a\Lambda CDM\,0} = \frac{1}{2}\frac{A-2}{A+1}. \tag{39}$$

Inserting $A = 0.43$ gives $q_{a\Lambda CDM\,0} = -0.55$; accelerating expansion of the universe corresponding to $w_0 = -1$.

5. Universe with Only Dark Energy

In order to obtain some intuition about the gravitational properties of the new types of dark energy, we shall now consider an era where the dark energy dominates. Hence, in this section we shall neglect dust. Then $A = 0$ and the expression (33) for the deceleration parameter reduces to

$$q_{aCPL} = \frac{1}{2}\left[1 + 3w_0 + 3w_a\left(1-a\right)\right]. \tag{40}$$

The same expression is obtained from the Friedmann equation which says that the acceleration of the scale factor, $\ddot{a}$, is proportional to the gravitational mass density as given in Equation (3). It is a linear function of the scale factor.

Note that $a<1$ in the past, and $a>1$ in the future, and that negative deceleration parameter means acceleration, and positive deceleration parameter means deceleration.

There is a transition with vanishing acceleration when the scale factor has the value a_1 given in Equation (9). If $w_0=-1/3$, $w_a=0$ there will be uniform expansion with no acceleration. The reason is that the gravitational mass, Equation (3), of this type of dark energy vanishes. A universe filled by this type of dark energy behaves like the empty Milne universe.

If $w_0=-1/3$ and $w_a>0$ there will be a transition from deceleration to acceleration at the present time, and if $w_0=-1/3$ and $w_a<0$ there will be a transition from acceleration to deceleration at the present time. If $w_0=-1/3$ and $w_a>0$ there will be a transition from deceleration to acceleration at the present time.

In general there are four cases for a universe dominated by dark energy with an equation of state parameter (8):

1) $w_0<-1/3$, $w_a<0$: Transition from acceleration to deceleration in the future.

2) $w_0<-1/3$, $w_a>0$: Transition from deceleration to acceleration in the past.

3) $w_0>-1/3$, $w_a<0$: Transition from acceleration to deceleration in the past.

4) $w_0>-1/3$, $w_a>0$: Transition from deceleration to acceleration in the future.

The DESI-results favour the values $w_0=-0.7$, $w_a=-1$. With these values we are presently in a quintom-B era. This corresponds to the first case, transition from accelerated expansion to decelerated expansion in the future. Hence, even in a universe containing only dark energy, *i.e.* without help from the attractive gravitation of cold matter, there will be a transition from accelerated to decelerated expansion. With $w_0=-0.7$, $w_a=-1$ Equation (40) gives a transition from accelerated to decelerated expansion at $a_t=1.4$.

This transition happens a little earlier in a flat universe model with dust and this type of dark energy, namely at $a_t=1.27$. This is due to the attractive gravitation of the dust. The reason that the difference in time, with and without dust, for the transition from acceleration to deceleration is so small, is that the transition happens while the dark energy still makes up around 35% of the contents of the universe.

We are used to think of the repulsive gravity of dark energy as an explanation of the accelerated expansion of the universe. However these new types of dark energy have different gravitational properties. They can change from causing repulsive gravity to attractive gravity.

It should be noted that it is highly doubtful whether these models of dark energy are realistic. They are introduced in an ad hoc way to explain observations in the most simple way mathematically.

Also the existence of LIVE is a consequence of quantum physics in combination with the requirement that the quantum mechanical vacuum energy shall not act as a sort of ether making it possible to introduce absolute velocity in conflict with the principle of relativity. Hence if the DESI-results turn up to be correct, we must expect that there are at least to types of dark energy in the universe. LIVE and another type.

In order to obtain a more realistic model of this other type of dark energy one should possibly introduce one of the types of energy that have been introduced in connection with the early inflationary era of the universe, for example a Starobinsky type of dark energy [17].

6. Other Types of Dark Energy

In the report [2], the DESI-team presented the following figure (**Figure 6** here):

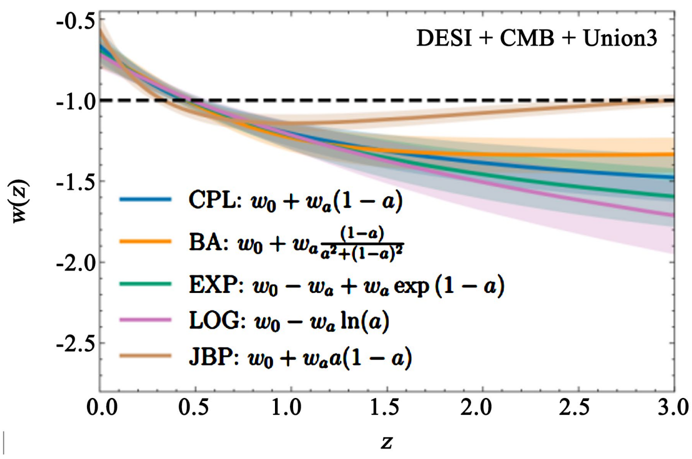


Figure 6. Different expressions for the equation of state parameter w as functions of the scale factor, and graphs showing how w depends upon z for these expressions with $w_0 = -0.7$, $w_a = -1$.

In this figure the equation of state parameter as function of the scale factor was presented for five different types of dark energy, all containing the constants w_0 and w_a. We have considered the CPL-dark energy above. Here we shall take a closer look at the other types.

18 December 2025 W. Hossain published a preprint [18] with title "Current observations favour phantom-enhanced nature of dark energy". In addition to that with equation of state parameter (8) he investigated observational constraints upon dark energy with other equations of state parameter. Three of them shall be considered here. First dark energy of the BA-type,

$$w_{BA}(a) = w_0 + w_a \frac{1-a}{a^2 + (1-a)^2} = w_0 + w_a \frac{z(1+z)}{1+z^2}. \tag{41}$$

The index BA is used because this model was introduced by E. M. Barboza Jr. and J. S. Alcaniz [16]. Next Hossain considered two types that were not included in **Figure 4** (we shall come back to the last tree models of this figure below),

$$w_{PL}(a) = w_0 a^{-\alpha}, \tag{42}$$

$$w_{MPL}(a) = \frac{2w_0}{1+a^{\alpha}}. \tag{43}$$

They are shown graphically together with the linear equation of state parameter from Equation (8) in **Figure 7** with the values $w_0 = -0.9$, $w_a = -0.4$ in all the plots, $\alpha = 0.6$ in w_{PL} and $\alpha = 2$ in w_{MPL}, that Hossain found were favoured by the observational data. There were some differences in values. But Hossain chose to use the same values for all the models in the graphs for the sake of comparing the models. We shall follow him here.

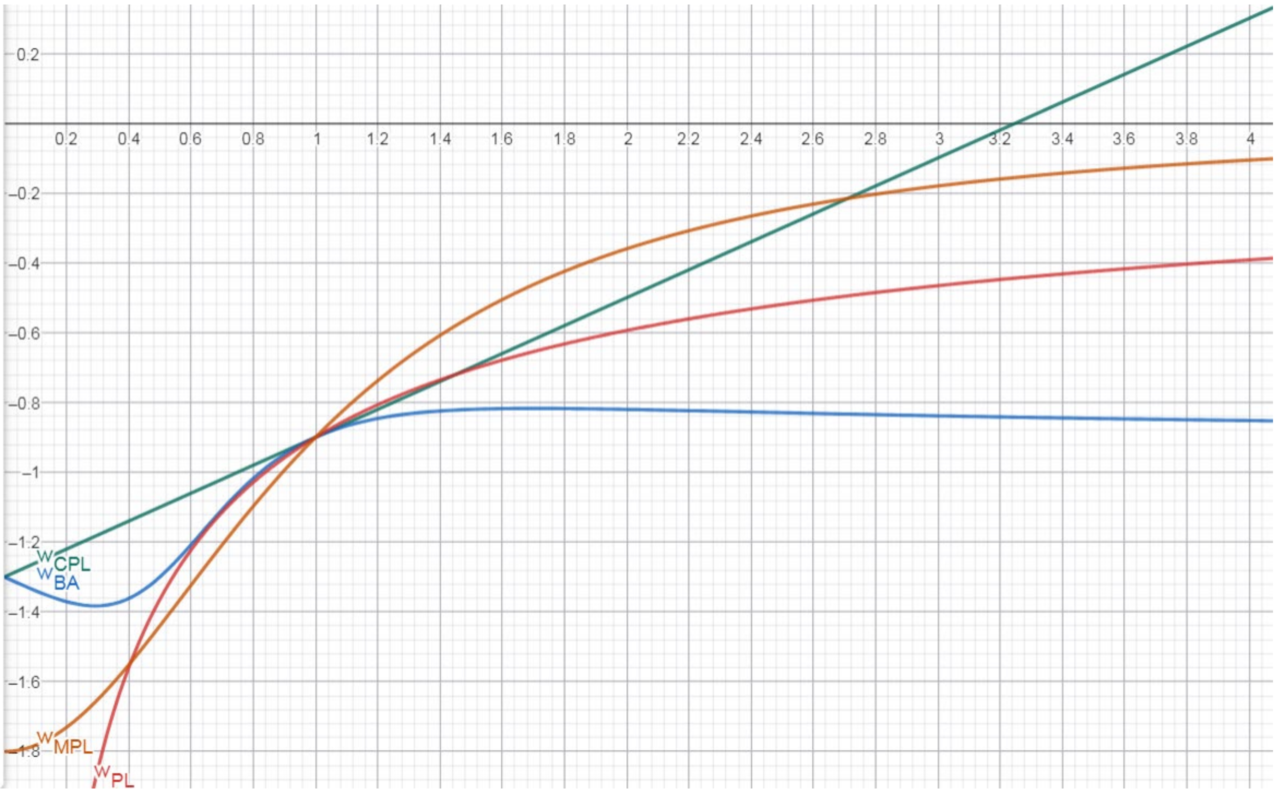


Figure 7. The equation of state parameters of different types of dark energy given in Equations (8) and (41) - (43) as functions of the scale factor with the values $w_0 = -0.9$, $w_a = -0.4$ in all the plots, $\alpha = 0.6$ in w_{PL} and $\alpha = 2$ in w_{MPL}.

In order to talk about the different types of dark energy with these equations of state we need some definitions, generalizing the classification above based upon **Figure 3**.

Phantom energy: $w < -1$. This type of dark energy causes repulsive gravity and accelerated cosmic expansion. It is seen from Equation (7) that the density of phantom energy increases as the universe expands.

Quintessence energy: $-1 < w < 0$. The density of quintessence energy decreases with expansion. This type of energy causes repulsive gravity if $w < -1/3$ and attractive gravity, *i.e.* decelerated expansion, if $w > -1/3$. For the equation of state parameter given in Equation (8) the scale factor at this transition is given in Equation (9).

Quintom-A energy: Dark energy which evolves from a quintessence stage across the "phantom divide", $w=-1$, and into a phantom stage.

Quintom-B energy: Dark energy which evolves from a phantom stage across the "phantom divide", $w=-1$, and into a quintessence stage.

The scale factor at the "phantom divide" or "phantom crossing" is given by $w(a_{PG})=-1$. With the equation of state (8) this gives

$$a_{PD}=1+\frac{1+w_0}{w_a} \tag{44}$$

with $w_0<-1$ and $w_a<w_0-1$. Hence, $a_{PD}>1$; the phantom crossing happens in the future and is a crossing from a phantom state into a quintessence state. It may be noted that if the dark energy comes from a scalar field ϕ with potential $V(\phi)$, the equation of state parameter is

$$w=\frac{(1/2)\dot{\phi}^2-V}{(1/2)\dot{\phi}^2+V}. \tag{45}$$

where the dot denotes differentiation with respect to time. For this type of dark energy $w\geq-1$, and there exists no phantom state. A static scalar field has $\dot{\phi}=0$ and $w=-1$, which characterizes LIVE, the dark energy of the ΛCDM-universe model.

With an evolving equation of state parameter, w, a dark energy may exist in different stages at different points of time. It may even appear in a dust like stage if $w=0$.

Figure 7 shows that with the parameter values $w_0=-0.9$, $w_a=-0.4$, a universe with dark energy obeying the equation of state (8), was in the past, up to $a=0.75$, in the quintom-B stage but then entered a quintessence state, where it is now and will evolve into a dust-like stage at $a=3.25$. Relativistically stiff matter has $w=1$, and this is the upper limit of the values of w which is allowed by the theory of relativity. Since Equation (8) gives $w(5.75)=1$ the equation of state (8) is without meaning for $a>5.75$ with these parameter values. As noted by Hossein [18] this equation of state is only expected to be realistic for values of a close to 1, *i.e.* with small redshifts.

The dark energy of the types w_{BA} and w_{PL} will begin in a phantom state and enter a quintessence state a little before the present time, but never enter into a state with $w>-1/3$ where the dark energy causes attractive gravity. Also the dark energy of the type w_{MPL} begins in a phantom state, but it enters an era with $w_{MPL}>-1/3$ and attractive gravity at $a_1=2.1$.

Inserting the expressions (41) - (43) into Equation (14) the corresponding density factors are

$$f_{BA}(a)=\left[a^2+(1-a)^2\right]^{3w_a/2}a^{-3(1+w_0+w_a)}, \tag{46}$$

$$f_{PL}=a^{-3}\mathrm{e}^{\frac{3w_0}{\alpha}\left(a^{-\alpha}-1\right)}, \tag{47}$$

$$f_{MPL}(a)=a^{-3}\left(\frac{1+a^{-\alpha}}{2}\right)^{6w_0/\alpha}. \tag{48}$$

These density factors are plotted in Figure 8.

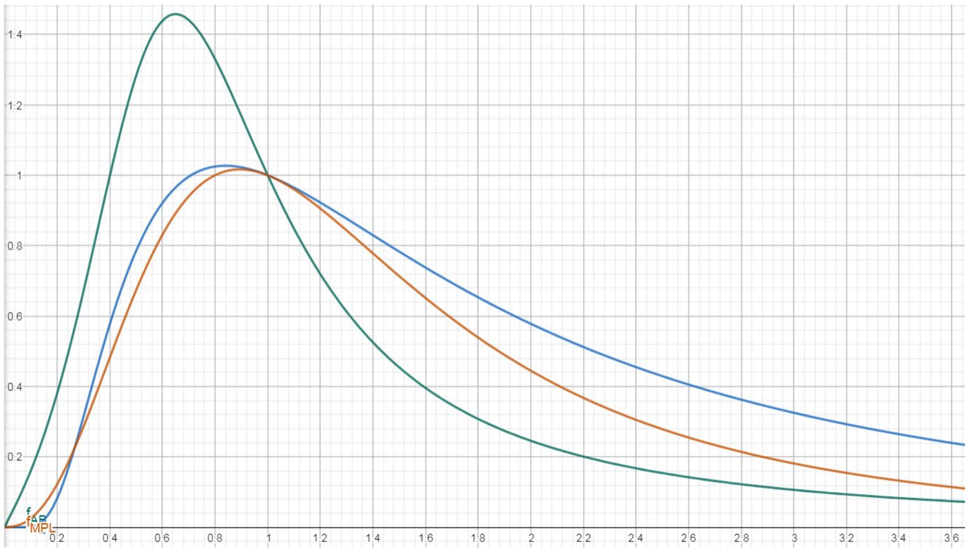


Figure 8. The density factors of dark energy as functions of the scale factor with equations of state (41) - (43). Here f_{BA} is green, f_{PL} blue and f_{MPL} brown, with the values $w_0 = -0.9$, $w_a = -0.4$ in all the plots, $\alpha = 0.6$ in f_{PL} and $\alpha = 2$ in f_{MPL}.

The corresponding expressions of the density factors (46) - (48) expressed as functions of the redshift are

$$g_{BA}(z) = (1+z)^{3(1+w_0)} \left(1+z^2\right)^{3w_a/2}, \tag{49}$$

$$g_{PL} = (1+z)^3 \, \mathrm{e}^{\frac{3w_0}{\alpha}\left[(1+z)^\alpha - 1\right]}, \tag{50}$$

$$g_{MPL}(a) = (1+z)^3 \left(\frac{1+(1+z)^\alpha}{2}\right)^{6w_0/\alpha}. \tag{51}$$

These density factors are plotted in Figure 9.

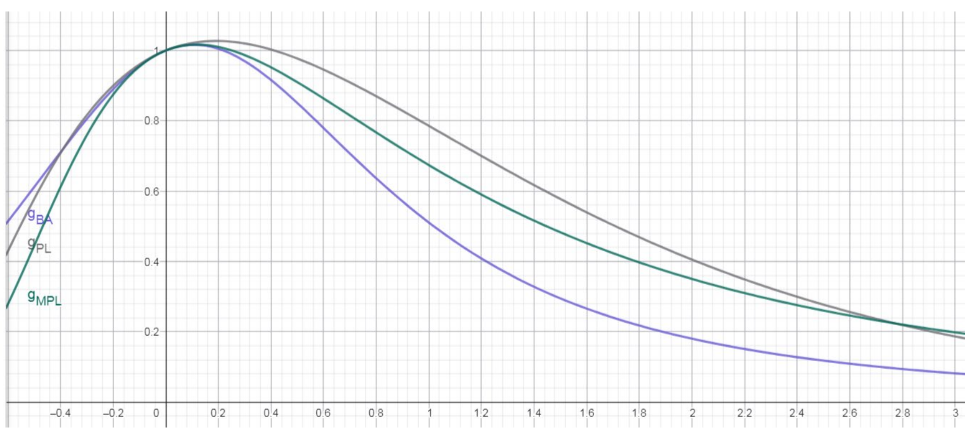


Figure 9. The density factors of dark energy as functions of the redshift with equations of state (52) - (54). Here g_{BA} is blue, g_{PL} green and g_{MPL} grey, with the values $w_0 = -0.9$, $w_a = -0.4$ in all the plots, $\alpha = 0.6$ in f_{PL} and $\alpha = 2$ in f_{MPL}.

Inserting the equation of state parameters (41) - (43) and density factors (46) - (48) into Equation (30) gives the deceleration parameters of flat universe models with dust and these types of dark energy,

$$q_{BA} = \frac{1}{2} + \frac{3}{2} \frac{w_0 + w_a \dfrac{1-a}{a^2 + (1-a)^2}}{1 + A\left[a^2 + (1-a)^2\right]^{-3w_a/2} a^{-3(w_0 + w_a)}}, \tag{52}$$

$$q_{PL} = \frac{1}{2} + \frac{3}{2} w_0 a^{-\alpha} \left[1 + A\mathrm{e}^{-(3w_0/\alpha)(a^{-\alpha}-1)}\right]^{-1}, \tag{53}$$

$$q_{MPL} = \frac{1}{2} + \frac{3w_0}{1+a^{\alpha}} \left[1 + A\left(\frac{1+a^{-\alpha}}{2}\right)^{-6w_0/\alpha}\right]^{-1}, \tag{54}$$

These deceleration parameters are plotted in **Figure 10**.

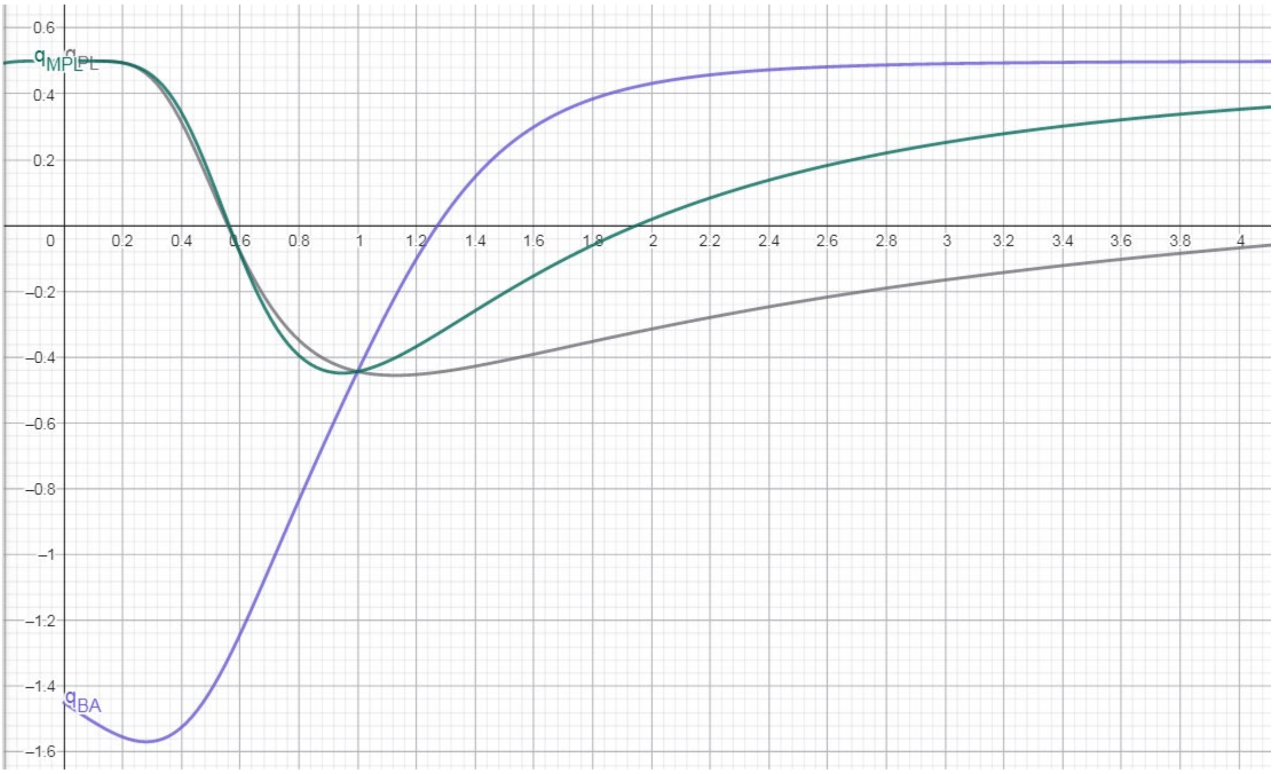


Figure 10. The deceleration parameters (52) - (54) plotted as functions of the scale parameter with the values $w_0 = -0.9$, $w_a = -0.4$ in all the plots, $\alpha = 0.6$ in f_{PL} and $\alpha = 2$ in f_{MPL}.

It is seen that although the evolution of the density factors of these the universe models are rather similar (**Figure 9**), the PL and MPL-models have very different expansion histories from that of the BA-model. The reason is that in the past PL-models had rather different values of the equation of state parameters from that of the BA-model (**Figure 7**). But all of the models have the same value, Equation (35), of the deceleration parameter at the present time. With $w_0 = -0.9$, $A = 0.43$ this gives a negative present value of the deceleration parameter, $q(1) = -0.42$, *i.e.* accelerated expansion at the present time. The BA-model started with accelerated expansion having $q_{BA}(0) = -1.45$. There is a transition from accelerated to decelerated expansion in a relatively near future, at $a_{BA} = 1.28$.

The PL-model and MPL-model both started with decelerated expansion having $q_{PL}(0) = 0.5$. This may seem a little surprising since **Figure 7** shows that the PL-dark energy has a very large negative value of the equation of state parameter at the beginning causing repulsive gravitation and accelerated expansion. However, looking at the expression f_{PL} for the density parameter, we find that for small values of a, say $a = 1/p$ with $p \gg 1$ we have $f_{PL} \approx p^3 / e^{4.5p} \ll 1$. Hence the dust dominates initially, and this is the reason for the initial decelerated expansion of this universe model. Later on the density of the dark energy rises, and the repulsive gravity dominates over the attractive gravity of the dust, and the expansion becomes accelerated.

We shall now briefly consider the last three models of **Figure 6** and start with the LOG-model having an equation of state parameter

$$w = w_0 - w_a \ln a. \tag{55}$$

This model was considered by G. Efstathiou [19] already in 1999. It is shown graphically in **Figure 11**.

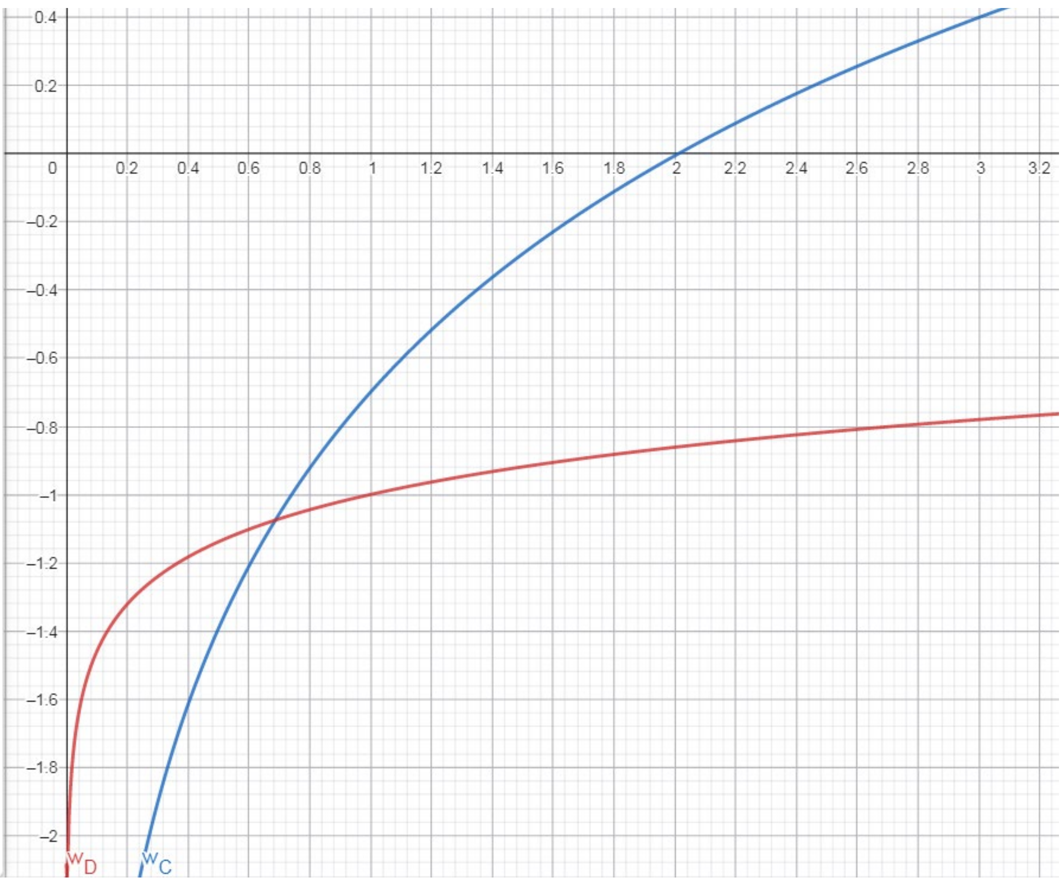


Figure 11. The equation of state parameter with the DESI-values $w_{0D} = -0.7$ and $w_{aD} = -1$ (red) and the CMB-values $w_{0C} = -1$, $w_{aC} = -0.2$ of [19] (blue).

For both of these parameter sets the value of the equation of state parameter increases. The dark energy is initially in a phantom state with $w_C < w_D$, and $w_C = w_D$ at

$$a_E = \exp\frac{w_{0C} - w_{0D}}{w_{aC} - w_{aD}}. \quad (56)$$

Inserting the values from the text of **Figure 9** gives $a_E = 0.69$. The dark energy passes the phantom divide, $w = -1$, at

$$a_{PDE} = \exp\frac{1 + w_0}{w_a}. \quad (57)$$

Inserting $w_{0D} = -0.7$ and $w_{aD} = -1$ gives $a_{PDEC} = 0.74$, and $w_{0C} = -1$, $w_{aC} = -0.2$ gives $a_{PDED} = 1$. The transition from repulsive to attractive gravity happens at

$$a_{1LOG} = \exp\frac{3w_0 + 1}{3w_a}, \quad (58)$$

giving $a_{1LOGC} = 1.4$ and $a_{1LOGD} \approx 28$.

Efstathiou [19] showed that the density factor of this dark energy is

$$f_{LOG}(a) = a^{-3(1+w_0)} e^{-(3/2)w_a(\ln a)^2}. \quad (59)$$

This is shown graphically in **Figure 12** with $w_0 = -0.7$ and $w_a = -1$.

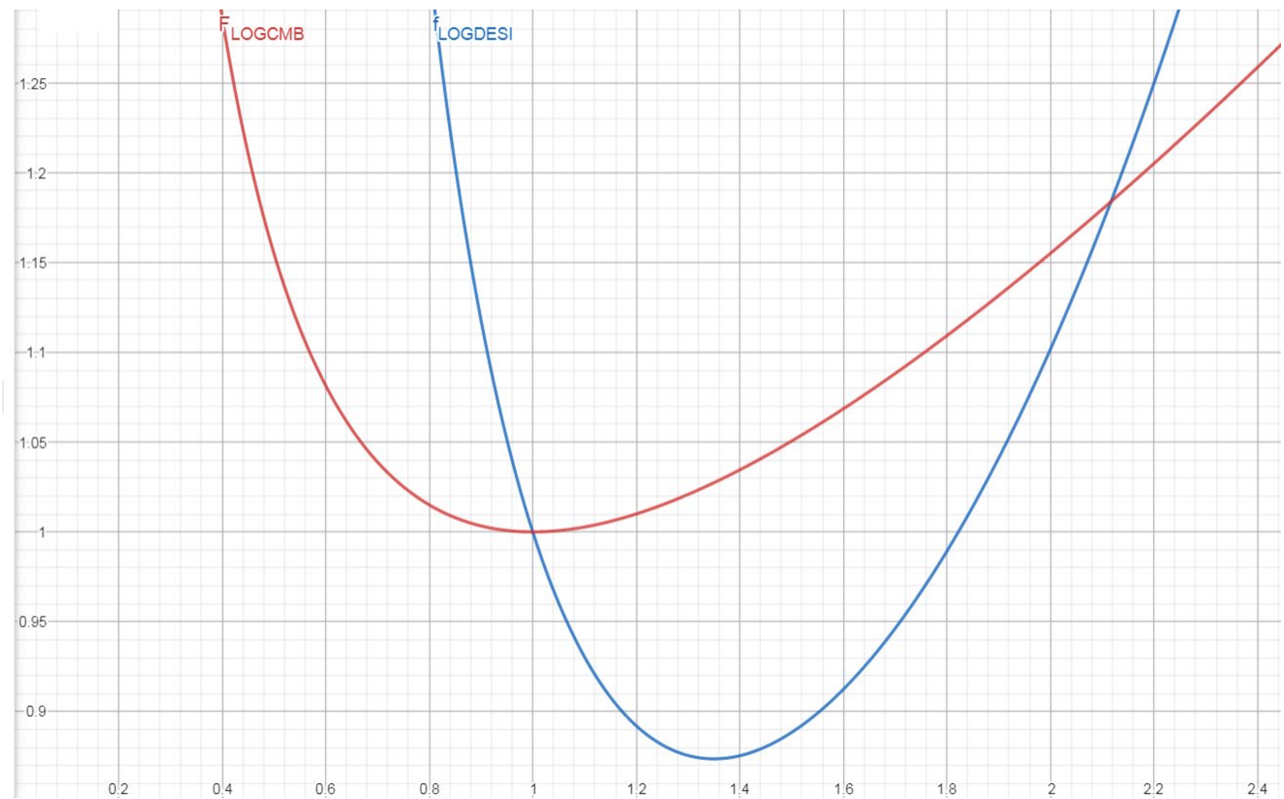


Figure 12. The density factor for the LOG-model of dark energy with the DESI-values $w_0 = -0.7$ and $w_a = -1$ (blue) and the values $w_0 = -1$, $w_a = -0.2$ of [19] (red).

The deceleration parameter is

$$q=\frac{1}{2}\left[1+\frac{3\left(w_0-w_a\ln a\right)}{1+Aa^{3w_0}e^{(3/2)w_a(\ln a)^2}}\right] \tag{60}$$

This is shown graphically in **Figure 13**.

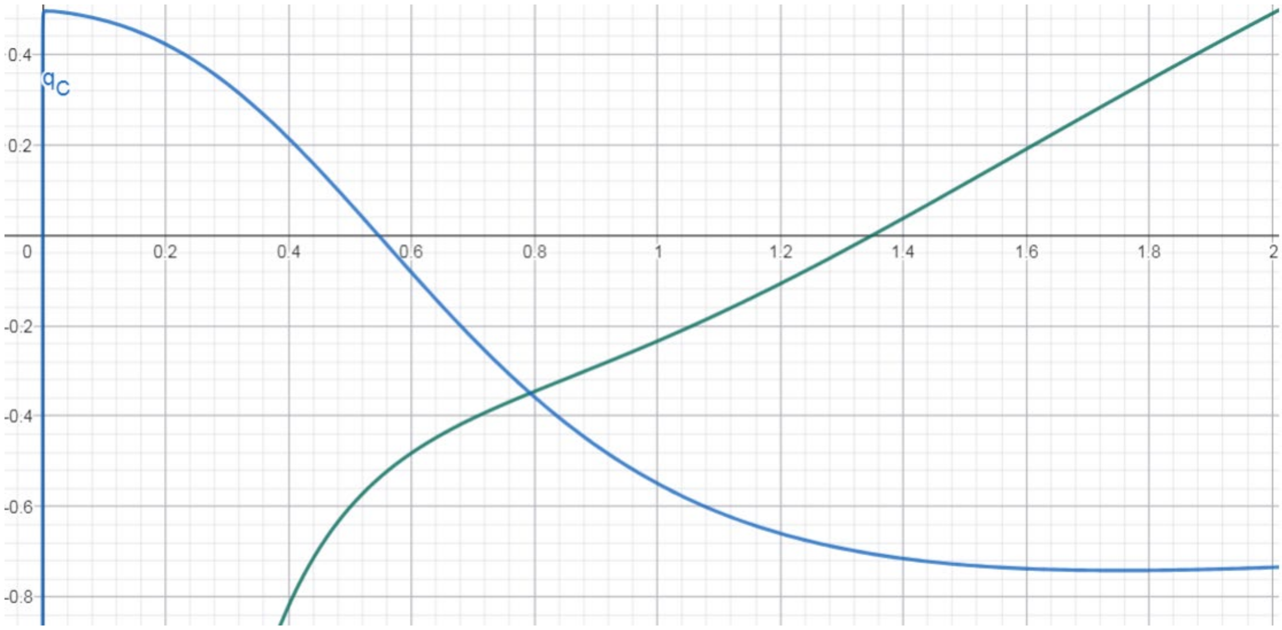


Figure 13. The deceleration factor as function of the scale factor for the LOG-model with the DESI-values $w_0=-0.7$ and $w_a=-1$ (green), and the values $w_0=-1$, $w_a=-0.2$ of Efstathiou [19] (blue).

The evolution of the deceleration parameter for this universe model depends dramatically upon the values of the constants w_0 and w_a. With DESI-values it is increasing and passes in the future from acceleration of the expansion to deceleration at $a=1.35$. With the CMP-values the evolution is similar to that of the ΛCDM-universe model, and the deceleration parameter has a decreasing value with transition in the past from decelerated to accelerated expansion for $a=0.55$. However with the CMB-values this universe was in a state of extremely accelerated expansion in its very early history. The reason is seen in **Figure 10** and **Figure 11**. The dark energy had a great density and caused repulsive gravitation.

One might then wonder: Why was this universe model in an early state of decelerated expansion? Also with the DESI-values of the constants the dark energy had large initial density and caused repulsive gravity. The answer follows by looking at the blue curve of **Figure 13**. It shows that the initial value of the deceleration parameter with the CMB-values of the constants was $q(0)=0.5$. As noted after Equation (32) this is the value of the deceleration parameter of a universe without dark energy, only with cold matter. Hence initially the cold matter had a much larger density than the dark energy and had a dominating effect upon the expansion motion of the universe, in spite of the increased density of the dark energy with deceasing small value of the scale factor. Furthermore it is seen from the blue curve in Figure 13 that the value of the deceleration parameter decreases. Hence the dark energy gives an increasing repulsive contribution to the expansion of the universe, and the expansion becomes accelerated at $a \approx 0.55$.

Next we consider the JBP-model with equation of state parameter

$$w(a) = w_0 + w_a a(1-a), \quad w_z(z) = w_0 + w_a \frac{z}{(1+z)^2}. \tag{61}$$

This is shown graphically in Figure 14.

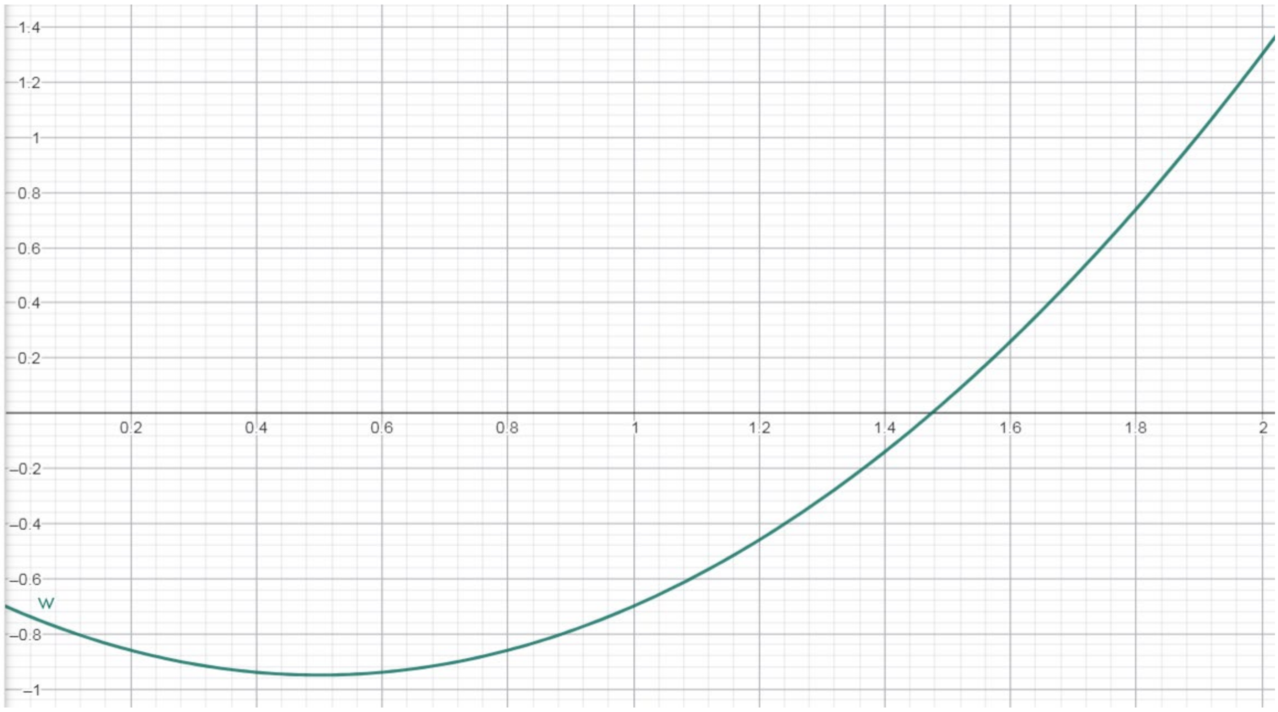


Figure 14. The equation of state parameter of the JBP-model as function of the scale-factor with the DESI-values of the constants.

The minimum of the equation of state parameter is $w_{\min} = w(1/2) = w_0 + (1/4)w_a$, which is $w_{\min} = -0.95$ with $w_0 = -0.7$, $w_a = -1$. Then this dark energy is never in the phantom state. At $a = 1.3$ it passes from a state with repulsive gravitation to attractive gravitation.

The model (61) was presented by H. J. Jassal, J. S. Bagla and T. Badmanabhan [20] in 2005. They showed that the density factor of this model is

$$f_{deJBP} = a^{-3(1+w_0)} e^{\frac{3}{2}w_a(1-a)^2}, \quad g_{deJBP} = (1+z)^{3(1+w_0)} e^{\frac{3}{2}w_a\left(\frac{z}{1+z}\right)^2}. \tag{62}$$

The density factor as a function of the scale factor is shown in Figure 15.

According to this universe model the dark energy have recently had maximal density. We saw in Figure 14 that the gravity of this dark energy became attractive at $a = 1.3$. At the present time the dark energy makes up 70% of the contents of the universe. We see from Figure 15 that the dark energy still makes up 80% of this, *i.e.* 56% of the contents of the universe at this point of time.

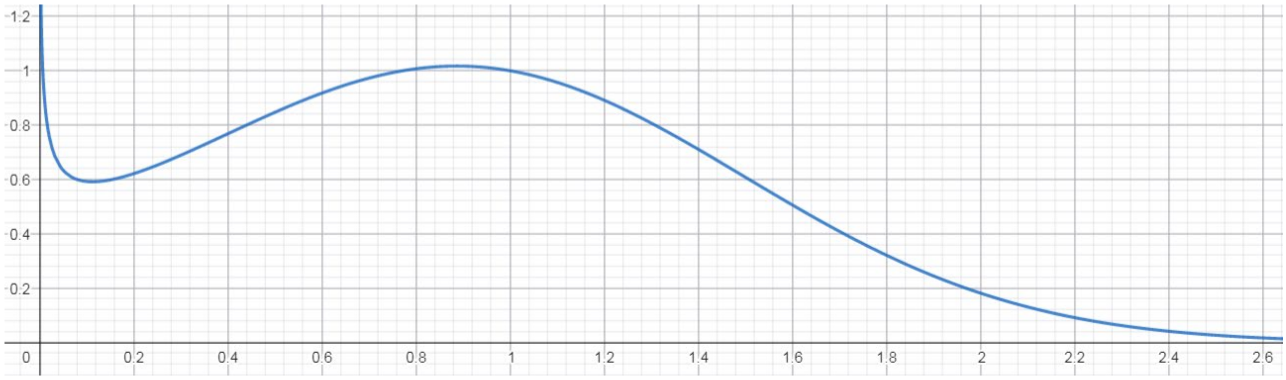


Figure 15. The density factor of the JBP-model as function of the scale factor with $w_0 = -0.7$, $w_a = -1$.

The deceleration parameter as a function of the scale factor is now found by inserting the expressions (62) into Equation (31), which gives

$$q_{JBP}(a) = \frac{1}{2}\left[1 + 3\frac{w_0 + w_a a(1-a)}{1 + 0.43 a^{3w_0} e^{-(3/2)w_a(1-a)^2}}\right]. \quad (63)$$

This is shown graphically in **Figure 16**.

Figure 16. The deceleration parameter of the JBP-universe model as a function of the scale factor with $w_0 = -0.7$, $w_a = -1$.

The deceleration parameter of this universe model varies in an interesting way. Initially it is positive, *i.e.* there is decelerated expansion. It is seen from **Figure 12** that at this time the dark energy is of the quintessence type and contributes with repulsive gravity. But the density is now only about 60% of its present density. This means that the dark energy only made up about 42% of the contents of the universe at this time. Hence the cold matter, causing attractive gravity, dominated at this time. Therefore the expansion was decelerated at this early time. At $a \approx 0.56$ this universe entered a period with accelerated expansion. At this time the equation of state parameter had near its minimum value, *i.e.* and the density of the dark energy had increased to about 90% of its present value. It now made up about 63% of the contents of the universe. Hence the universe entered a period where the repulsive gravity of the dark energy dominated over the attractive gravity of the cold matter. At the present time the acceleration is still accelerated, but in a relatively near future, at $a \approx 1.2$ the universe entered a period with decelerated expansion. The equation of state parameter of the dark energy had now increased to around $w \approx -0.45$, the repulsive gravity of the dark energy became weaker and the attractive gravity of the cold matter dominated the rate of change of the cosmic expansion. **Figure 12** shows that soon after this, at about $a = 1.3$, the value of w became larger than −1.3, and from now on also the dark energy caused attractive gravity and deceleration of the expansion.

Finally, N. Dimakis and co-workers [21] considered dark energy related to scalar fields with an equation of state parameter depending exponentially upon the scale factor. It was called the EXP-type dark energy and given the form

$$w = w_0 - w_a \left(1 - \mathrm{e}^{1-a}\right) \tag{64}$$

in Figure 4 by the DESI-team [2]. It is shown graphically with the DESI-values $w_0 = 0.7$, $w_a = -1$ in **Figure 17**.

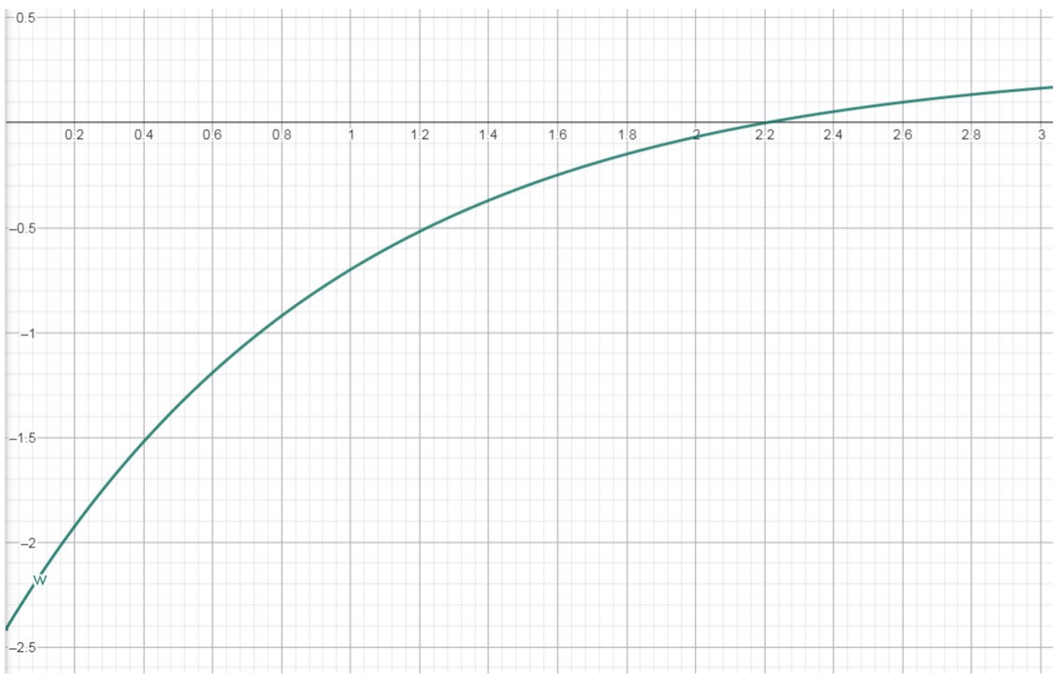


Figure 17. The equation of state parameter as a function of the scale factor for the EXP-type dark energy with the DESI values $w_0 = 0.7$, $w_a = -1$ of the constants.

Solving Equation (64) with respect to a gives

$$a = 1 - \ln\left(1 + \frac{w - w_0}{w_a}\right). \quad (65)$$

For this model, and with the DESI-values of the constants, the phantom crossing for $w = -1$ happens at $a_{PDEXP} = 0.74$. The transition for $w = -1/3$ from accelerated to decelerated expansion happens at $a_{1EXP} = 1.46$.

7. The Padé Universe Models

Recently Y. Carloni and co-workers [15] published a preprint where they considered three universe models with the following redshift dependence of the equation of state parameter.

$$w_{P^w(01)} = \frac{w_0 a}{(1 - b_1)a + b_1} = \frac{w_0}{1 + b_1 z}, \quad (66)$$

called the Padé^w(01) universe model, and

$$w_{P^w(11)} = \frac{(w_0 - a_1)a + a_1}{(1 - b_1)a + b_1} = \frac{w_0 + a_1 z}{1 + b_1 z}, \quad (67)$$

called the Padé^w(11) universe model. The third model is called the Padé^q(01) universe model and has an equation of state parameter of the dark energy:

$$w_{P^q(01)} = -\frac{1}{3}\frac{z - 62}{z - 20}, \quad w_{P^q(01)} = -\frac{1}{3}\frac{63a - 1}{21a - 1}. \quad (68)$$

Here the value $q_{de} = -1.05$ preferred by observations has been inserted for a parameter q_{de} which appears in the corresponding equation of Carloni *et al.* [15]. As usual w_0 is the present value of the equation of state parameter. Note that $w_0 = -1$, $a_1 = b_1 = 0$ gives LIVE. Comparing with observations, Carloni and co-workers found that for the Padé^w(01) model the best fit values are $w_0 = -1.05$, $b_1 = 3.7 \times 10^{-4}$. Hence this fluid is very similar to LIVE. For the Padé^w(11) model they found $w_0 = -0.98$, $a_1 = -0.51$, $b_1 = 0.31$.

Inserting Equation (66) into Equation (13) gives the density factor of the Padé^w(01)-dark energy

$$f_{P^w(01)de} = a^{-3}\left[(1 - b_1)a + b_1\right]^{-\frac{3w_0}{1-b_1}}, \quad (69)$$

having the constant LIVE-value $f = 1$ for $w_0 = -1$, $b_1 = 0$. The density factor (69) is shown graphically with the values $w_0 = -1.05$, $b_1 = 3.7 \times 10^{-4}$ in **Figure 18**.

It is seen that for this model the density of the dark energy is monotonously increasing with the scale factor, *i.e.* with time, which is different from the density of the CPL-model of the dark matter shown in **Figure 1**. The density changes rather slowly. Hence this universe model behaves not very different from the ΛCDM-universe model.

Inserting Equations (67) and (69) into Equation (31) leads to the expression for the deceleration parameter of the Padé^w(01)-universe model as a function of the scale factor

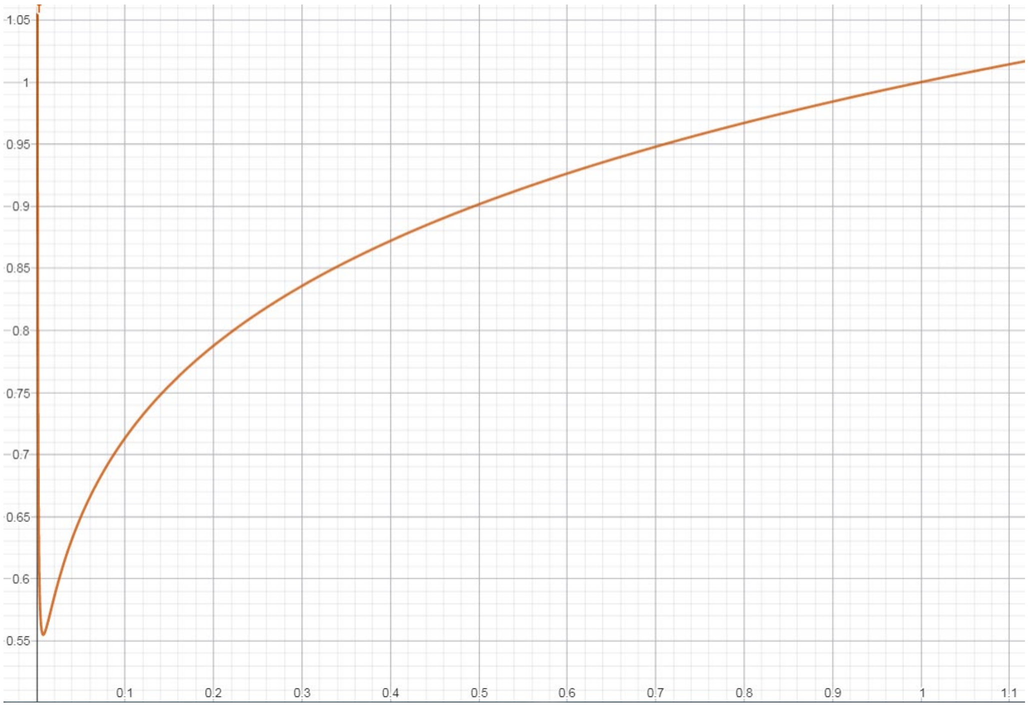


Figure 18. The graph shows the density factor of the dark energy of the Pw(01)-universe model as a function of the scale factor with $w_0 = -1.05$, $b_1 = 3.7 \times 10^{-4}$.

$$q_{aP^w(01)} = \frac{1}{2}\left\{1 + \frac{3w_0 a}{1 + A\left[\left(1-b_1\right)a + b_1\right]^{1-\frac{3w_0}{1-b_1}}}\right\}. \tag{70}$$

This is shown graphically with the values $A = 0.43$, $w_0 = -1.05$, $b_1 = 3.7 \times 10^{-4}$ in **Figure 19**. As a function of redshift it takes the form

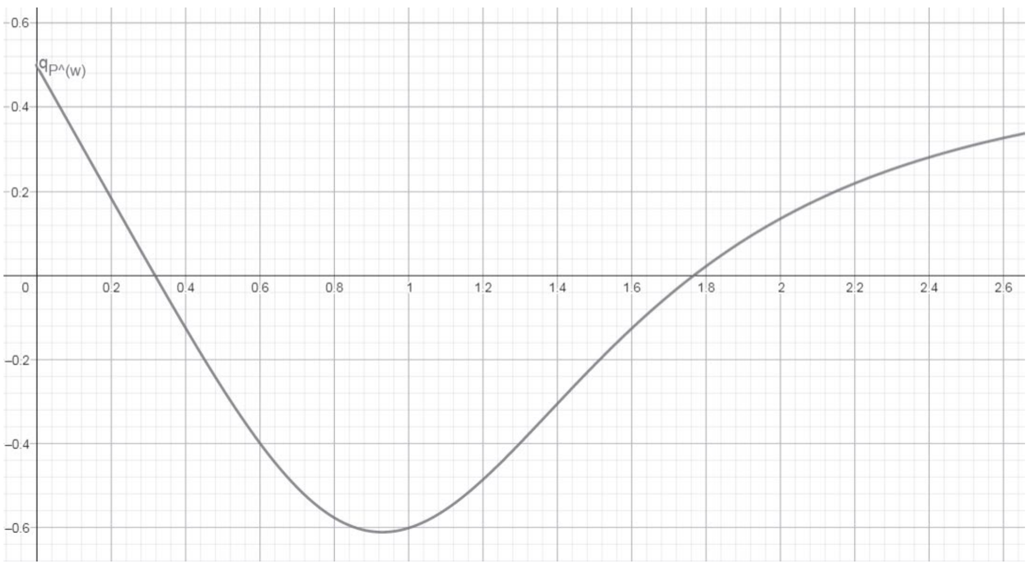


Figure 19. The graph shows the deceleration parameter of the Pw(01)-universe model as a function of the scale factor with $w_0 = -1.05$, $b_1 = 3.7 \times 10^{-4}$.

$$q_{zP^w(01)} = \frac{1}{2}\left\{1 + \frac{3w_0}{1 + A(1+z)\left[(1-b_1)(1+z)^{-1} + b_1\right]^{1-\frac{3w_0}{1-b_1}}}\right\}, \quad (71)$$

which is shown graphically in **Figure 20**. The present value of the deceleration parameter for this model is given by the same expression, (35), as for the $w_0 w_a$-models. With the Padé(01)-values of the constant this gives $q_{P(01)0} = -0.60$, which means accelerated expansion with a magnitude close to that of the ΛCDM-universe model.

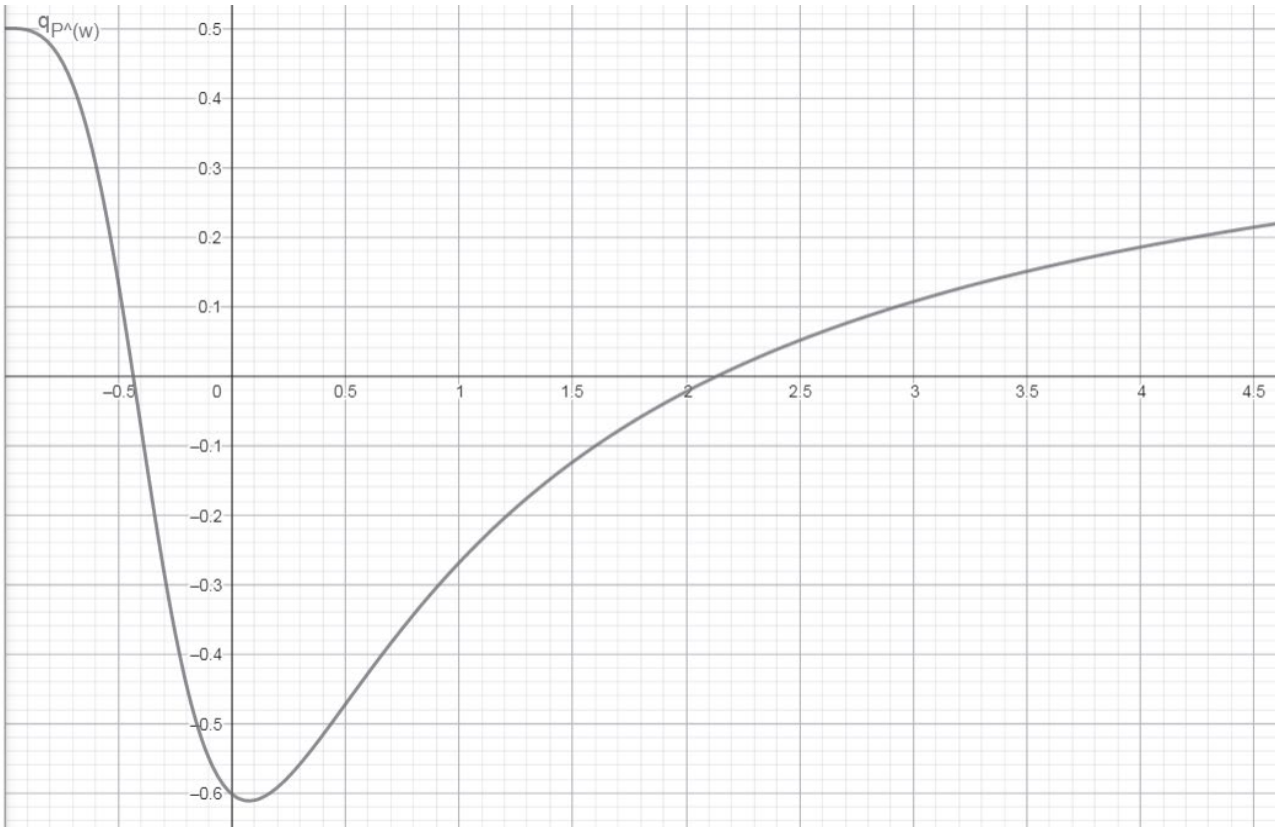


Figure 20. The deceleration parameter of the Pw(01)-universe model as a function of the cosmic redshift with $w_0 = -1.05$, $b_1 = 3.7 \times 10^{-4}$. It has the same shape as that in Fig. 3 of Carloni *et al.* [15].

This universe model starts by being in a state with decelerated expansion. There is a transition to accelerated expansion at $a_{t1} = 0.32$ with a future transition to a new era with decelerated expansion at $a_{t2} = 1.76$.

This deceleration parameter is shown as a function of redshift in **Figure 20**.

Inserting Equation (67) into Equations (14) and (15) gives the density factor of the dark energy in the P^w(11) model as a function of the scale factor (**Figure 21**)

$$f_{P^w(11)} = a^{-3\left(1+\frac{a_1}{b_1}\right)}\left[b_1 + (1-b_1)a\right]^{3\frac{a_1 - w_0 b_1}{b_1(1-b_1)}}, \quad (72)$$

and as a function of redshift

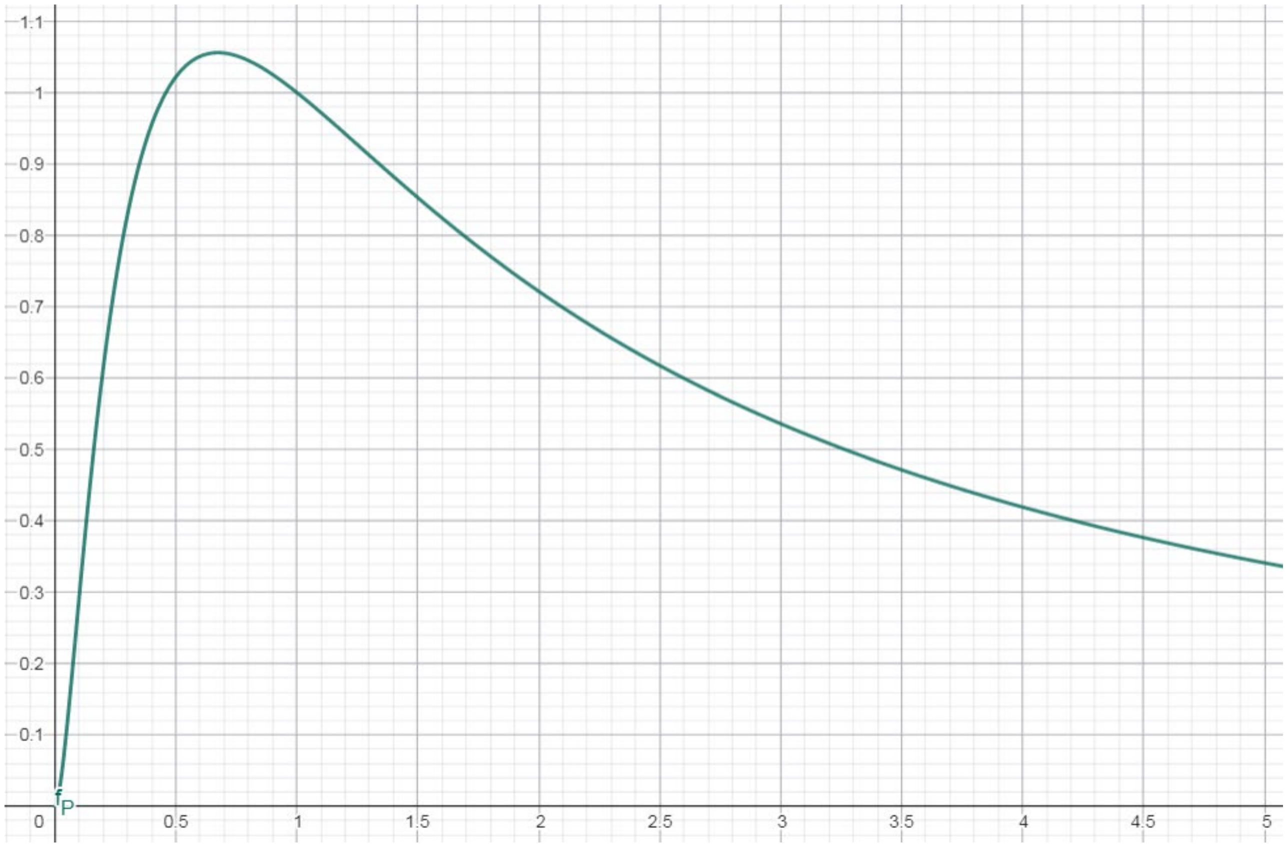


Figure 21. The graph shows the density factor of the dark energy of the Pw(11)-universe model as a function of the scale factor with $w_0 = -0.98$, $a_1 = -0.51$, $b_1 = 0.31$. Its evolution with time has a similar character as that of the dark energy of the CPL-type shown in **Figure 1**.

$$g_{P^w(11)} = (1+z)^{3\left(1+\frac{a_1}{b_1}\right)} \left(b_1 + \frac{1-b_1}{1+z}\right)^{3\frac{a_1 - w_0 b_1}{b_1(1-b_1)}}. \tag{73}$$

Inserting the expressions (67) and (72) into Equation (32) leads to the expression for the deceleration parameter of the Padé^w(11)-universe model as a function of the scale factor (**Figure 20**)

$$q_{P(11)} = \frac{1}{2}\left\{1 + 3\left[(w_0 - a_1)a + a_1\right]\left[(1-b_1)a + b_1\right]^{-1}\left[1 + A a^{3\frac{a_1}{b_1}}\left[(1-b_1)a + b_1\right]^{3\frac{w_0 b_1 - a_1}{b_1(1-b_1)}}\right]^{-1}\right\}. \tag{74}$$

This is shown graphically in **Figure 22** with the same values of the constants as in **Figure 21**.

Expressed as a function of redshift Equation (74) takes the form

$$q_{P^w(11)} = \frac{1}{2}\left\{1 + 3\frac{w_0 + a_1 z}{1 + b_1 z}\left[1 + A\frac{(1+z)^{3\frac{a_1 - w_0}{1-b_1}}}{(1+b_1 z)^{3\frac{a_1 - w_0 b_1}{b_1(1-b_1)}}}\right]^{-1}\right\}. \tag{75}$$

Again the present value of the deceleration parameter is given by Equation (35), giving $q_{P(11)0} = -0.53$, again close to the ΛCDM-universe model. The deceleration parameter as given in Equation (75) is shown in **Figure 23**.

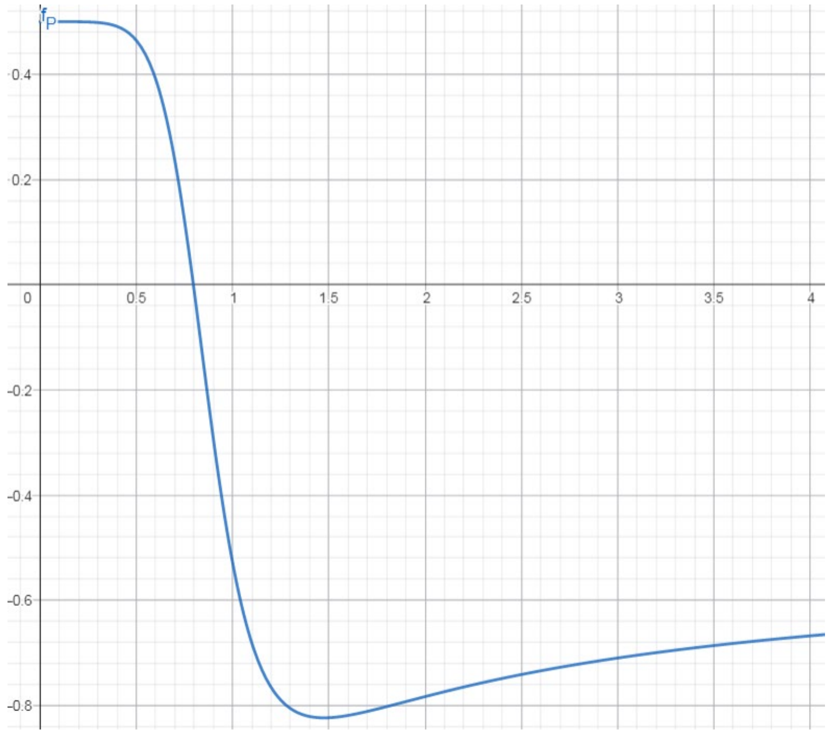


Figure 22. The graph shows the deceleration parameter of the Pw(11)-universe model as a function of the scale factor.

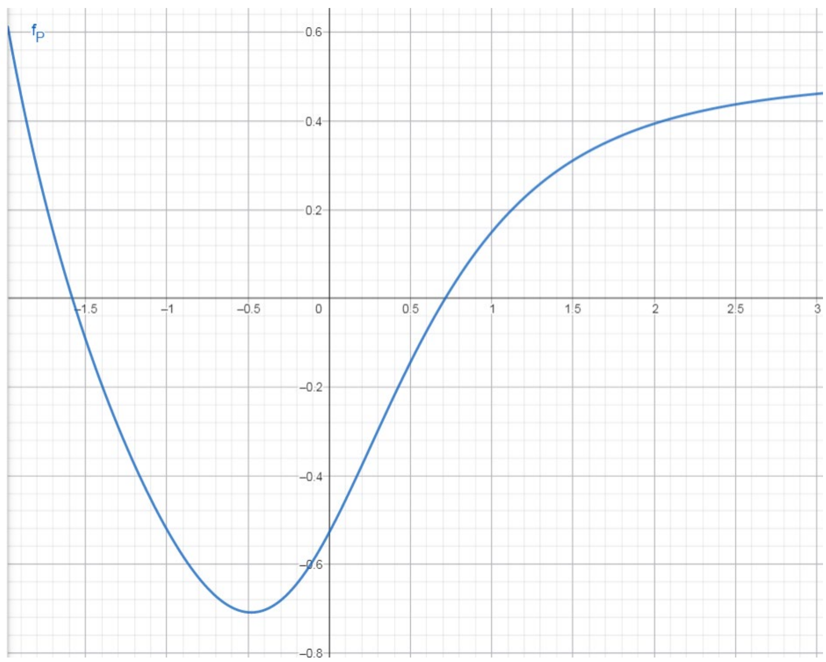


Figure 23. The graph shows the deceleration parameter of the Pw(11)-universe model as a function of the cosmic redshift, given in Equation (75).

Inserting Equation (68) into Equations (13) and (14) gives the density factors of the Padé^q(01)-model of the dark energy. **Figure 24** shows the density factor as a function of the scale factor.

Figure 24. The density of the dark energy of the Pq(01)-universe model relative to its present density as a function of the scale factor.

$$f_{P^q(01)} = a^{-2}\left(1.05a - 0.05\right)^{2.05}, \quad g_{P^q(01)} = \left(1+z\right)^2\left[1.05\left(1+z\right)^{-1} - 0.05\right]^{2.05}. \quad (76)$$

Inserting Equations (68) and Equation (76) into Equations (31) and (32) gives (**Figure 25**)

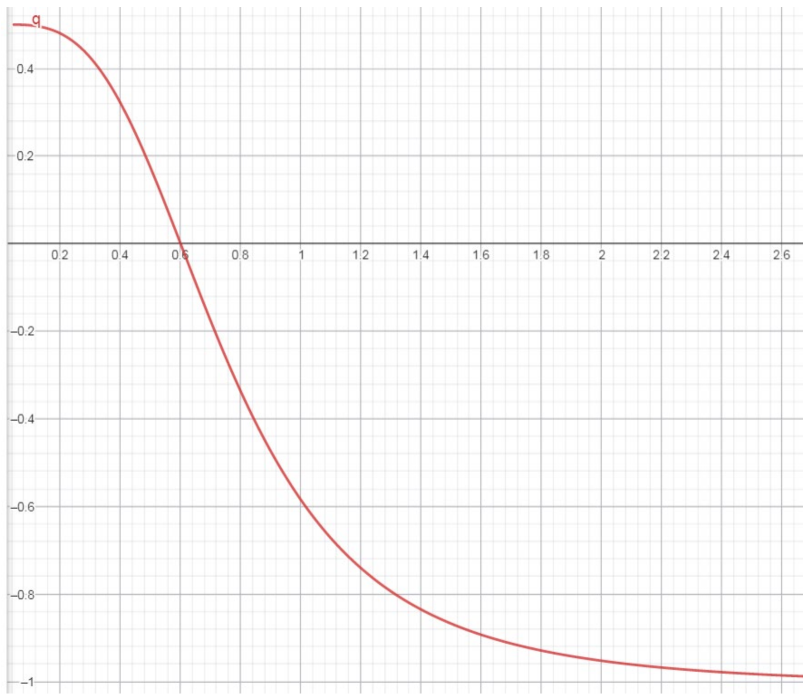


Figure 25. The deceleration parameter of the Pq(01)-universe model as a function of the scale factor.

$$q_{aP^q(01)} = \frac{1}{2}\left[1 - \frac{63a - 1}{(21a - 1)\left[1 + Aa^{-1}(1.05a - 0.05)^{-2.05}\right]}\right], \tag{77}$$

or (Figure 26)

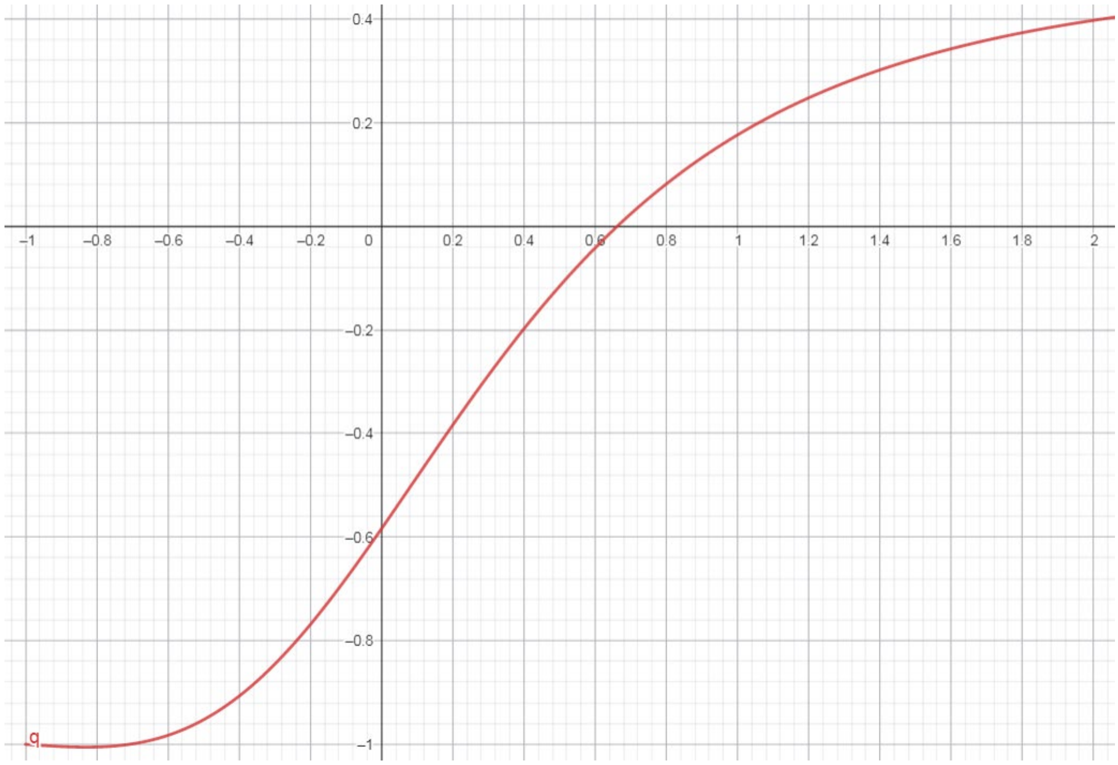


Figure 26. The deceleration parameter of the Pq(01)-universe model as a function of the cosmic redshift.

$$q_{zP^q(01)} = \frac{1}{2}\left[1 - \frac{z - 62}{(z - 20)\left[1 + A(1+z)\left[1.05(1+z)^{-1} - 0.05\right]^{-2.05}\right]}\right]. \tag{78}$$

The present value of the deceleration parameter with this universe model is $q_{P^q(01)0} = -0.58$.

It is seen that with the values of the equation of state constants determined by DESI- and other observations all of the models considered so far give a present state of accelerated expansion.

8. Is the Universe in a State of Decelerated Expansion?

6 November 2025 J. Son and co-workers [8] published an article where the heading ended with: "signs of a non-accelerating universe". I their abstract the authors wrote: "Supernova (SN) cosmology is based on the key assumption that the luminosity standardization process of Type Ia SNe remains invariant with progenitor age. However, direct and extensive age measurements of SN host galaxies reveal a significant (5.5σ) correlation between standardized SN magnitude and progenitor

age, which is expected to introduce a serious systematic bias with redshift in SN cosmology. This systematic bias is largely uncorrected by the commonly used mass-step correction, as progenitor age and host galaxy mass evolve very differently with redshift.

After correcting for this age bias as a function of redshift, the SN data set aligns more closely with the $w_0 w_a$ cold dark matter (CDM) model recently suggested by the Dark Energy Spectroscopic Instrument (DESI) baryon acoustic oscillations (BAO) project from a combined analysis using only BAO and cosmic microwave background (CMB) data.

This result is further supported by an evolution-free test that uses only SNe from young, coeval host galaxies across the full redshift range. When the three cosmological probes (SNe, BAO, and CMB) are combined, we find a significantly stronger (>9σ) tension with the CDM model than that reported in the DESI papers, suggesting a time-varying dark energy equation of state in a currently non-accelerating universe".

LIKE the DESI-team Son *et al.* [8] presented several values of w_0 and w_a depending upon which observations are taken into account. But the main point of their investigation was to show that it is necessary to correct the values of w_0 and w_a due to the observational bias they pointed out, namely that the earlier investigations did not take into account the correlation between standardized SN magnitude and progenitor age. Hence, I will here use representative values for their corrected magnitudes. With the last line of Table 2 in [8] as a point of departure I will use the values $w_0 = -0.3$, $w_a = -1.9$ when referring to [8].

Figure 27 shows the density factor of the dark energy of the CPL-type, as given in Equation (15) with the DESI-values and the corrected values of the equation of state parameters.

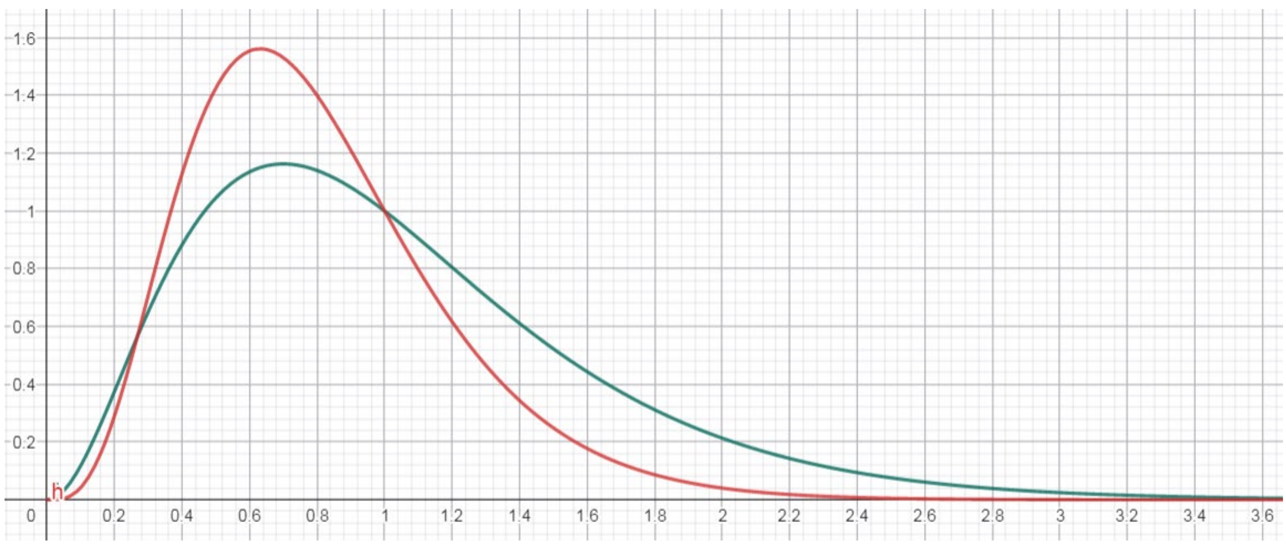


Figure 27. The density factor as function of the scale factor of the dynamical dark energy of the CPL-type, as given in Equation (14), with the DESI values $w_0 = -0.7$, $w_a = -1.0$ (green curve) and the corrected values, $w_0 = -0.3$, $w_a = -1.9$ (red curve), of the equation of state parameters. The density factor is equal to 1 for all values of the scale factor for the ΛCDM-universe model.

Figure 28 shows the results of inserting the corrected values taken from the last line of Table 2 in [8] with $\Omega_{m0} = 0.36$, $\Omega_{de0} = 0.64$ *i.e.* $A = 0.56$, and $w_0 = -0.3$, $w_a = -1.9$ (grey curve) in the analytical expression (33) for the deceleration parameter of a flat universe model with dust and dark energy of the CPL-type. Also graphs are shown with the uncorrected values with $\Omega_{m0} = 0.3$, $\Omega_{de0} = 0.7$ *i.e.* $A = 0.43$, and $w_0 = -0.7$, $w_a = -1.0$ for the dark energy of the CPL-type (red curve), and with $w_0 = -1$, $w_a = 0$ for the ΛCDM-universe model (green curve) in Figure 28. The results are similar to those in Figure 9 of Son *et al.* [8].

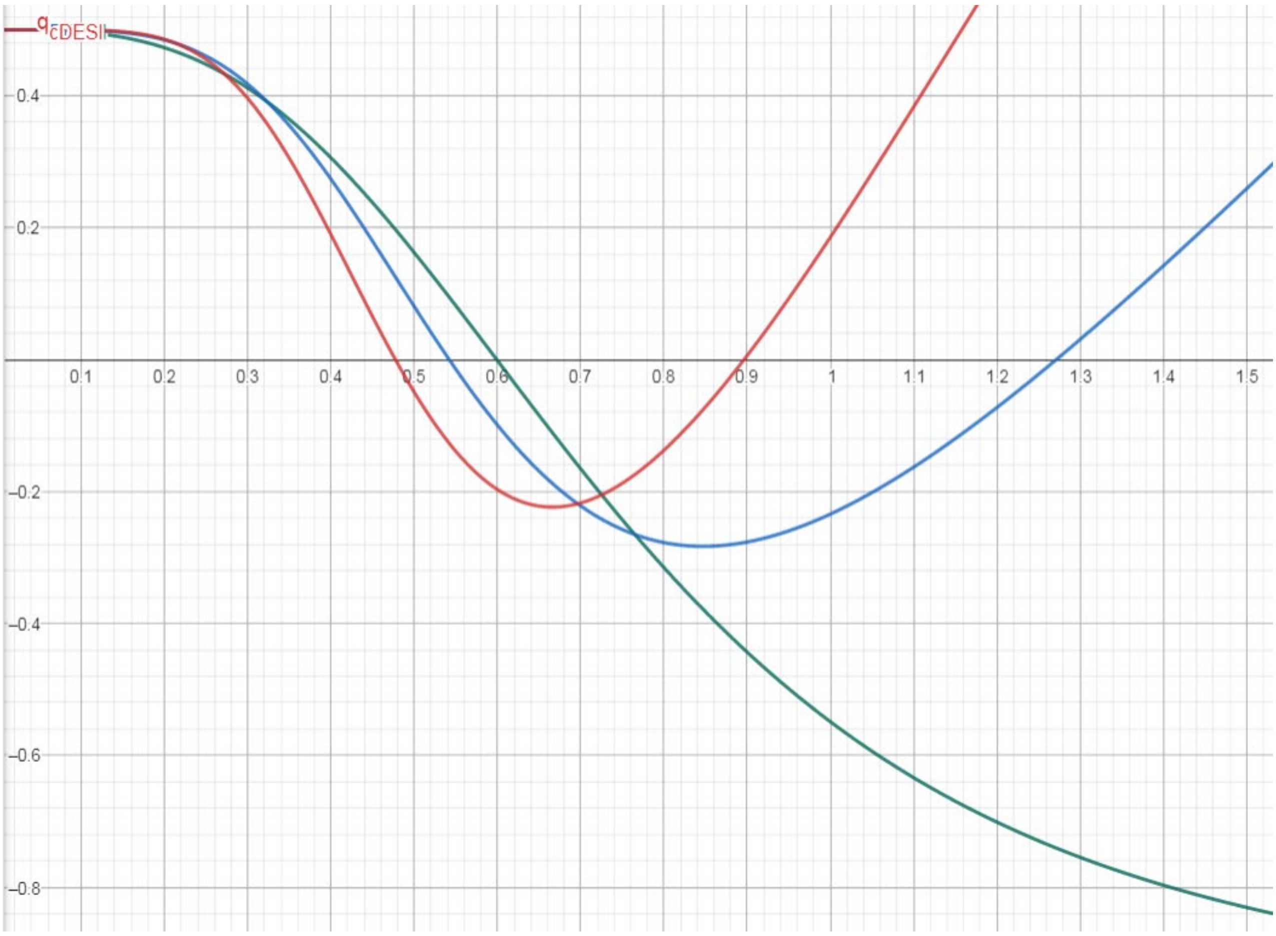


Figure 28. The deceleration parameter as function of the scale factor as given in Equation (33) with the values $w_0 = -1$, $w_a = 0$ of the ΛCDM-universe model (green curve), the DESI-values $w_0 = -0.7$, $w_a = -1.0$ (blue), and the corrected DESI-values $w_0 = -0.3$, $w_a = -1.9$ (red). The results shown here are similar to those shown in Figure 9 of Son *et al.* which were calculated numerically.

Again Equation (35) gives the present values of the deceleration parameter for these universe models. A flat universe model with dust and LIVE, *i.e.* the ΛCDM-universe model has $q(t_0) = -0.55$, while a flat universe with dynamical dark energy of the CPL-type and the DESI-values of the equation of state parameters, has $q(t_0) = -0.23$, still indicating accelerated cosmic expansion, but the CPL-universe model with corrected values of the equation of state parameters of the dark energy has $q(t_0)=0.21$, indicating a present state of decelerated expansion.

The figure shows that universe model with dark energy of the CPL-type have an initial period with decelerated expansion followed by a period with accelerated expansion and then re-enters a final period with decelerated expansion. For the uncorrected DESI-values of the constants the transition from the initial period with decelerated expansion to the intermediate period with accelerated expansion happened at $a=0.54$ and the re-entering to an era with decelerated expansion at $a=1.26$. With the corrected constants of Son *et al.* [4] the universe entered the intermediate era with accelerated expansion at $a=0.53$ and re-entered an era with decelerated expansion at $a=0.86$, *i.e.* before the present time. Hence according to this model the universe is presently in an era with decelerated cosmic expansion, as noted above.

The last of the examples in the $w_0 w_a$-class of models which we shall consider, is that of Dutra *et al.* [22]. Using the parameter values from the last line in their Table 3, $\Omega_{m0}=0.56$, $\Omega_{de0}=0.44$ *i.e.* $A=1.27$, and $w_0=-0.46$, $w_a=-16.4$, the equation of state parameter of the dark energy is $w(a)=-16.40(1.028-a)$. In a universe containing only this type of dark energy there is a transition from accelerated to decelerated expansion at a_T determined by $w(a_T)=-1/3$. This gives $a_T=1.008$ very close to the present time. In the universe model of Dutra *et al.* there is a little more dust than dark energy at the present time. Hence we expect a transition to decelerated expansion a little earlier. This is indeed the case as shown in **Figure 28**.

The graphs of the density factor of the dark energy and the deceleration parameter of a flat universe containing dust and this type of dark energy as a function of the scale factor, are shown in **Figure 29** and **Figure 30**.

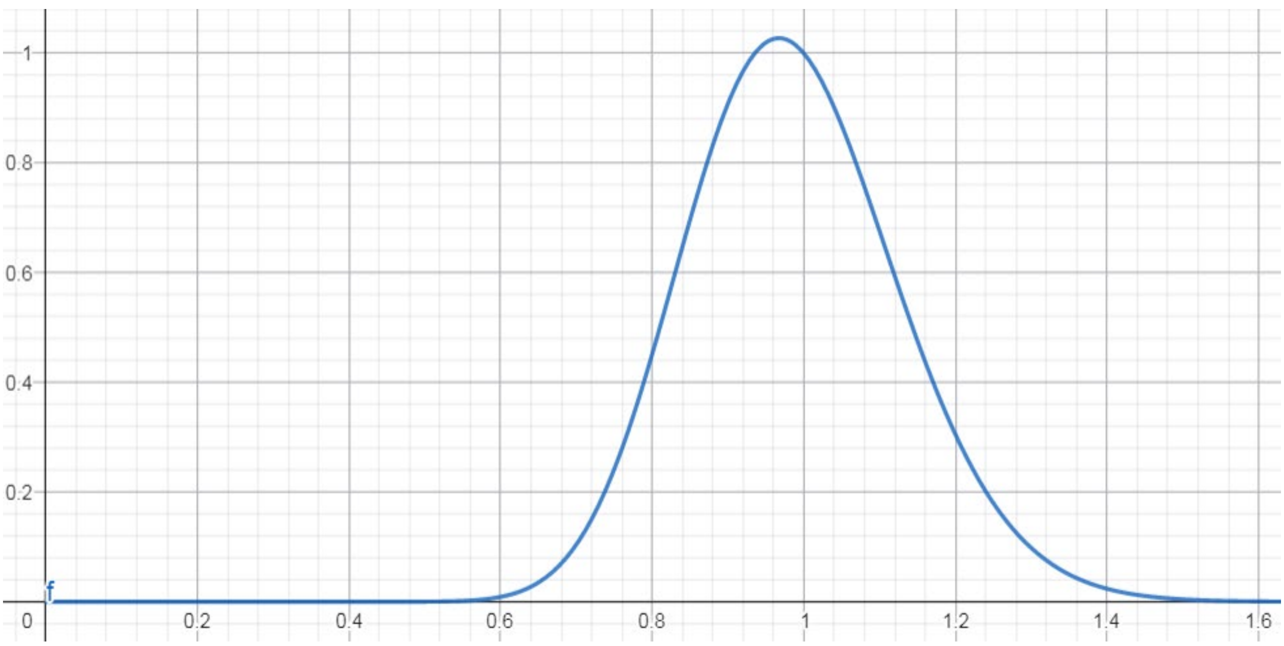


Figure 29. The density factor of the universe model favoured by Dutra *et al.*

Figure 30. The deceleration parameter of the universe model favoured by Dutra *et al.* [22]. It contains dust and dark energy with an equation of state parameter given in equation.

Looking at **Figure 29** and **Figure 30** we see an interesting phase difference. The dark energy has increasing density between $a = 0.6$ and $a = 1.4$ with maximum around $a = 0.95$. Thinking upon the dark energy as a source of accelerated expansion, one expects that the expansion should be accelerated during the period with over-density of the dark energy. But **Figure 30** shows that this is not the case. Here it is accelerated expansion only in a brief period between $a = 0.75$ and $a = 0.97$. Then there is a rapid change to decelerated expansion with maximal deceleration at $a = 1.2$ in a period with over-density of dark energy. This unexpected behaviour is due to the unusual properties of the dark energy.

In order to understand this we look at a universe with only the Dutra-type of dark energy [22], having $w_0 = -0.46$, $w_a = -16.4$. Then Equation (37) for the deceleration parameter gives $q = -8.39 + 8.20a$. Hence with only this type of energy in the universe there is a transition from accelerated to decelerated expansion at the present time, $a \approx 1$, as we saw above by looking at the equation of state parameter. At this time the dark energy changes character and switches from being a source of repulsive gravity to being a source of attractive gravity. This is the reason for the behaviour of the expansion shown in **Figure 30**; that acceleration switches to deceleration while the dark energy still dominates the contents of the universe.

Dutra *et al.* [22] wrote: "Modelling ~20-year, multi-band optical light curves for 6992 active galactic nuclei (AGN), we find a tight relation linking the variability amplitude and characteristic timescale to their intrinsic luminosity. This empirical law enables us to construct an AGN-based Hubble diagram to z ~ 3.5. Joint inference with supernova distances reveals evidence for an evolving dark energy equation of state at the 3.8 - 3.9σ over constant-w models and 4.4 - 4.8σ over ΛCDM".

It is seen that this universe model has a small and positive value of the deceleration parameter indicating a decelerated expansion at the present time. According to this model we are now at a very special point of time with a rather sudden increase in the density of the dark energy and rather small deceleration parameter although the deceleration parameter varies rapidly in this universe model.

In this model the present density of cold matter is larger than the density of the dark energy. Hence there is presently decelerated expansion, and since the density of the dark energy decreases rather rapidly during the coming times, there will be a period with rather strong cosmic deceleration.

The question whether the universe is presently in a state of accelerated or decelerated cosmic expansion has not yet been further discussed in a published paper, but was commented by Riess in a mail to Koberlein, who wrote [23]: "Let's start with the central claims of the original paper. Based on observations of about 300 supernovae, the authors found a correlation between the peak brightness of Type-Ia supernovae and the age of its host galaxy. Basically, the younger the galaxy, the dimmer the supernova. As a result, the authors argue, our measure of galactic distances is wrong. Based on their results, the Universe is decelerating, which would also mean the standard ΛCDM is wrong. Although the paper is peer reviewed, Riess finds a couple of major flaws.

The first is on the issue of galactic ages. The authors emphasize that SN-Ia light curves don't take the age of their host galaxies into account. That's somewhat true, but they do take galactic mass into account. Determining the age of a galaxy is difficult to do. It's also model dependent, so the results can be a bit tweaked. Galactic mass, on the other hand, is much simpler to measure.

Studies have shown that the mass of a supernova's host galaxy should be considered. This is why modern catalogs such as Pantheon + adjust for mass. The reason they don't worry about galactic age is because the age of a galaxy and its mass correlate pretty strongly. Once you adjust for mass, adjusting for age buys you nothing.

Since around 2010, Type-Ia supernova catalogues all include the mass adjustment, which also serves as an age proxy. Since the authors wanted to focus on age directly, they used older databases without the mass adjustment. That's a bit of a red flag. If you want to disprove the current theory, don't use old data. But this leads to the second issue, which is the connection between galaxy age and progenitor age.

The authors focus on the measured age of the host galaxies, since that's something that can be measured. They don't focus on the age of a supernova's progenitor star because we don't have a good way to measure that. In the paper, the team uses galaxy age as a proxy for progenitor age, assuming that the progenitor formed when the galaxy formed. Thus, distant supernovae progenitors are young, while the progenitors of nearby supernovae are old. But local supernovae are typically found in young star-forming regions. In fact, studies suggest that Type-Ia supernovae occur less than a billion years after the formation of their progenitor star. So that very basis of their argument is shaky at best".

This discussion has presently just started. The coming year we will probably see a consensus on these matters. It is exciting times for the cosmologists right now.

9. The Age of the Proposed Universe Models

A test of how good these universe models are, is to compare the age of the models with the age of our universe, which has been determined by the Planck-measurements of temperature of the CMB-radiation to be 13.80 ± 0.02 billion years.

The age of the models is

$$t_0 = \hat{t}\int_0^1 \left[\frac{a}{A + a^3 f_{de}(a)}\right]^{1/2} \mathrm{d}a = \hat{t}\int_\infty^0 \left[A(1+z)^3 + g_{de}(z)\right]^{-1/2} (1+z)\,\mathrm{d}z\,, \tag{79}$$

with

$$\hat{t} = \frac{1}{H_0\sqrt{\Omega_{de0}}}\,. \tag{80}$$

The value of $\hat{t}$ is determined by using that the age of the ΛCDM-universe model is $t_{\Lambda CDM0} = 13.8\times10^9$ years. This universe model has $f_{de}(a) = 1$. Hence,

$$\hat{t} = \frac{t_{\Lambda CDM0}}{\int_0^1 \left[\frac{a}{A + a^3}\right]^{1/2} \mathrm{d}a}\,. \tag{81}$$

Performing this integration (see Appendix) we arrive at

$$\hat{t} = \frac{3t_{\Lambda CDM0}}{2\operatorname{arsinh}\left(1/\sqrt{A}\right)}\,. \tag{82}$$

Inserting $t_{\Lambda CDM0} = 13.8\times10^9$ years and $A = 0.43$ gives $\hat{t} = 17.1\times10^9$ years. Inserting the $f_{de}(a)$ function in Equation (13) gives

$$t_0 = \hat{t}\int_0^1 \left[a\Big/\left\{A + a^3 \exp\left[-3\int_1^a \frac{1+w(a')}{a'}\mathrm{d}a'\right]\right\}\right]^{1/2} \mathrm{d}a\,. \tag{83}$$

With the density factor (15) corresponding to the equation of state (8) Equation (83) gives

$$t_0 = \hat{t}\int_0^1 \left[\frac{a}{A + a^{-3(w_0+w_a)}\mathrm{e}^{-3w_a(1-a)}}\right]^{1/2} \mathrm{d}a\,, \tag{84}$$

Inserting the DESI-values, $A = 0.43$, $w_0 = -0.7$, $w_a = -1.0$, gives $t_0 = 13.6\times10^9$ years. With the corrected values, $A = 0.56$, $w_0 = -0.3$, $w_a = -1.9$, the age of the universe is $t_0 = 12.0\times10^9$ years. The more extreme Dutra-values, $A = 1.27$, and $w_0 = -0.46$, $w_a = -16.4$, give $t_0 = 9.5\times10^9$ years, which is an unrealistically short age for the universe.

The $P^w(01)$ dark energy has equation of state

$$w(a)=\frac{w_0 a}{(1-b_1)a+b_1}, \tag{85}$$

This gives

$$\frac{1+w}{a}=\frac{1}{a}+\frac{w_0}{(1-b_1)a+b_1}. \tag{86}$$

Hence

$$\int_1^a \frac{1+w}{a'}\mathrm{d}a'=\ln\left\{a\left[(1-b_1)a+b_1\right]^{\frac{w_0}{1-b_1}}\right\}. \tag{87}$$

Then

$$g_{de}(a)=\mathrm{e}^{-3\int_1^a \frac{1+w}{a'}\mathrm{d}a'}=a^{-3}\left[(1-b_1)a+b_1\right]^{-\frac{3w_0}{1-b_1}}. \tag{88}$$

Inserting this into Equation (79) leads to

$$t_0=\hat{t}\int_0^1\left[\frac{a}{A+\left[(1-b_1)a+b_1\right]^{-\frac{3w_0}{1-b_1}}}\right]^{1/2}\mathrm{d}a. \tag{89}$$

Integration with $A=0.43$, $w_0=-1.05$, $b_1=3.7\times10^{-4}$ gives $t_0=13.88$ years.

The $P^w(11)$ dark energy has equation of state

$$w(a)=\frac{(w_0-a_1)a+a_1}{(1-b_1)a+b_1}, \tag{90}$$

with $w_0=-0.98$, $a_1=-0.5$, $b_1=0.3$. This gives

$$\frac{1+w}{a}=\frac{b_1+a_1}{b_1}\frac{1}{a}+\frac{w_0 b_1-a_1}{b_1}\frac{1}{(1-b_1)a+b_1}. \tag{91}$$

Hence

$$\int_1^a \frac{1+w}{a'}\mathrm{d}a'=\ln\left\{a^{\frac{b_1+a_1}{b_1}}\left[(1-b_1)a+b_1\right]^{\frac{w_0 b_1-a_1}{b_1(1-b_1)}}\right\}. \tag{92}$$

Then

$$g_{de}(a)=\mathrm{e}^{-3\int_1^a \frac{1+w}{a'}\mathrm{d}a'}=a^{-3\frac{b_1+a_1}{b_1}}\left[(1-b_1)a+b_1\right]^{-3\frac{w_0 b_1-a_1}{b_1(1-b_1)}}. \tag{93}$$

Inserting this into Equation (78) leads to

$$t_0=\hat{t}\int_0^1\left[\frac{a}{A+a^{-\frac{3a_1}{b_1}}\left[(1-b_1)a+b_1\right]^{-3\frac{w_0 b_1-a_1}{b_1(1-b_1)}}}\right]^{1/2}\mathrm{d}a. \tag{94}$$

For this model the integration with $A=0.43$, $w_0=-0.98$, $a_1=-0.5$, $b_1=0.3$ gives $t_0=13.85$ years.

The $P^q(01)$ dark energy has equation of state

$$w(a) = -\frac{1}{3}\left(1 + \frac{2a}{a - 0.048}\right), \tag{95}$$

with $w_0 = -0.98$, $a_1 = -0.5$, $b_1 = 0.3$. This gives

$$\frac{1+w}{a} = \frac{2}{3}\left(\frac{1}{a} - \frac{1}{a - 0.048}\right). \tag{96}$$

Hence

$$\int_1^a \frac{1+w}{a'} \mathrm{d}a' = \ln\left(\frac{0.952a}{a - 0.048}\right)^{2/3}. \tag{97}$$

Then

$$a^3 g_{de}(a) = a^3 \mathrm{e}^{-3\int_1^a \frac{1+w}{a'}\mathrm{d}a'} = a(1.05a - 0.05)^2. \tag{98}$$

Inserting this into Equation (79) leads to

$$t_0 = \hat{t}\int_0^1 \left[\frac{a}{A + a(1.05a - 0.05)^2}\right]^{1/2} \mathrm{d}a. \tag{99}$$

Inserting $A = 0.43$ also this model has the age $t_0 = 13.85$ years. The conclusion of the calculations in this section is that there are no age-problems for the Padé-universe models.

We then consider the model considered by Zhao *et al.* [24]. It has an equation of state of the dark energy with

$$w_a = -\frac{a^6}{a^6 + \alpha}. \tag{100}$$

with $\alpha = -5 \times 10^{-4}$. It reduces to the ΛCDM-model for $\alpha = 0$. Hence with their small value of α this model does not differ much from the ΛCDM-model.

Inserting Equation (100) into Equation (13) gives the density factor of the dark energy

$$f_{de}(a) = \sqrt{\frac{a^6 + \alpha}{a^6(1+\alpha)}}. \tag{101}$$

Hence, the age of this model is

$$t_0 = \hat{t}\int_1^1 \sqrt{\frac{a}{A + \sqrt{\frac{a^6 + \alpha}{1+\alpha}}}} \mathrm{d}a. \tag{102}$$

With $A = 0.43$ this gives $t_0 = 13.68 \times 10^9$ years.

We shall finally calculate the age of the universe models where the dark energy has the equations of state given in Equations (41) - (43). Inserting the density factors (46) - (48) into Equation (79) gives

$$t_{BA0} = \hat{t}\int_0^1 a^{1/2}\left[A + \left[a^2 + (1-a)^2\right]^{3w_a/2} a^{3(2+w_0+w_a)}\right]^{-1/2} \mathrm{d}a, \tag{103}$$

$$t_{PL0} = \hat{t}\int_0^1 a^{1/2}\left[A + \mathrm{e}^{\frac{3w_0}{\alpha}\left(a^{-\alpha}-1\right)}\right]^{-1/2} \mathrm{d}a. \tag{104}$$

$$t_{MPL0} = \hat{t}\int_0^1 a^{1/2}\left[A + \left(\frac{1+a^{-\alpha}}{2}\right)^{6w_0/\alpha}\right]^{-1/2} \mathrm{d}a \,. \tag{105}$$

Inserting the constants $A = 0.43$, $w_0 = -0.9$, $w_a = -0.4$ and $\alpha = 0.6$ for t_{PL0} and $\alpha = 2$ for t_{MPL0} gives $t_{BA0} = 12.3\times10^9$ years, $t_{PL0} = 13.9\times10^9$ years and $t_{MPL0} = 14.0\times10^9$ years.

10. Can the Proposed New Universe Models Solve the Hubble Tension?

The Hubble tension is that determination of the Hubble constant from early universe observations of the temperature fluctuations of the cosmic microwave background and late time universe observation using among other supernovae of type Ia to determine cosmic distances, have given results that differ by more than the uncertainty in the determined values. There exist a large number of proposals to eliminate this problem, but there is still no generally accepted solution to it.

I will here investigate whether the introduction of a dynamical dark energy can solve this problem.

The idea in this section is to use the value of the Hubble parameter, *H*, at recombination as determined by the Planck-observations of the temperature fluctuations in the cosmic microwave background radiation, as initial value, and then calculate the Hubble constant, *i.e.* the present value of the Hubble parameter, first according to the ΛCDM-universe model, and then according to the $w_0 w_a$CDM-universe model, and determine the difference, ΔH_0. If $\Delta H_0/H_0$ is larger than about 10^{-2} the introduction of the $w_0 w_a$CDM-universe model can possibly solve the Hubble tension.

According to the ΛCDM-universe model the Hubble parameter can be expressed in terms of the Hubble constant and the cosmic redshift as

$$H = H_{0\Lambda CDM}\sqrt{\frac{1+A(1+z)^3}{1+A}} \,. \tag{106}$$

It may be noted that Z.F. Wang and co-workers recently (12. January 2026) published a preprint [25] with title: "New $H(z)$ measurement at Redshift = 0.12 with DESI Data Release 1" where they found that the favoured value of the Hubble parameter at $z = 0.12$ is $H(0.12) = 71.33 \pm 4.20$ km·s^{-1}·Mpc^{-1}. Inserting this into Equation (106) with $A = 0.43$ gives the following value for the Hubble constant $H_{0\Lambda CDM} = 68.7$ km·s^{-1}·Mpc^{-1} in good agreement with value coming from the Planck measurements of the temperature fluctuation of the cosmic microwave radiation.

According to the $w_0 w_a$CDM-universe model the corresponding expression for the Hubble parameter is

$$H = H_{0w_0w_aCDM}\sqrt{\frac{g_{de}(z)+A(1+z)^3}{1+A}} \,. \tag{107}$$

Hence,

$$H_{0w_0w_aCDM} = H_{0\Lambda CDM}\sqrt{\frac{1+A(1+z)^3}{g_{de}(z)+A(1+z)^3}} \,. \tag{108}$$

Defining

$$\Delta H_0 = H_{0w_0w_aCDM} - H_{0\Lambda CDM}, \tag{109}$$

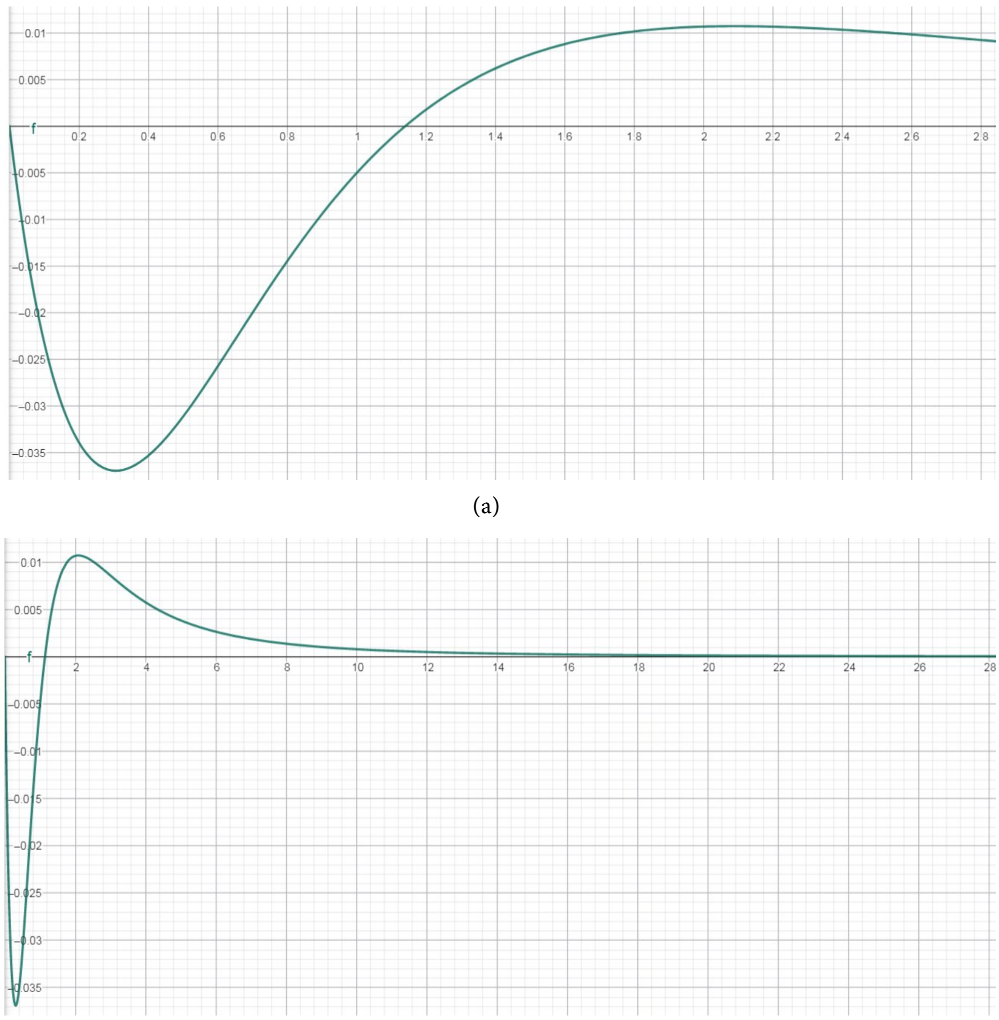
we have

$$\frac{\Delta H_0}{H_{0\Lambda CDM}} = \sqrt{\frac{A+(1+z)^{-3}}{A+(1+z)^{-3} g_{de}(z)}} - 1. \tag{110}$$

Inserting the expression (17) for the function $g_{de}(z)$ of the $w_0 w_a$CDM-universe model we finally get

$$\frac{\Delta H_0}{H_{0\Lambda CDM}} = \sqrt{\frac{1+A^{-1}(1+z)^{-3}}{1+A^{-1}(1+z)^{3(w_0+w_a)}\exp\left(-\frac{3w_0 z}{1+z}\right)}} - 1. \tag{111}$$

$\Delta H_0 / H_{0\Lambda CDM}$ is plotted as a function of z with the favoured values $w_0 = -0.7$ and $w_a = -1$ in **Figure 31(a)** and **Figure 31(b)**.



(a)

(b)

Figure 31. (a) and (b): The relative change of the Hubble constant when calculating it from a given common initial value of the Hubble parameter for a source with redshift z either by using the variation of the Hubble parameter in the ΛCDM-universe model or in the $w_0 w_a$-universe model.

It should be noted that $\Delta H_0/H_{0\Lambda CDM}$ approaches a very small value for a large redshift.

The redshift of radiation observed now, and emitted 380,000 years after Big Bang, at the time when the Hubble parameter was determined by the Planck-observations of temperature fluctuations in the cosmic microwave background radiation, is of the order $z \sim 1100$. For this redshift $A(1+z)^3 \gg 1$. Further we note that with values $w_0 = -0.7$ and $w_a = -1$ we have

$$(1+z)^{3(w_0+w_a)} \exp\left(-\frac{3w_0 z}{1+z}\right) = (1+z)^{-5.1} \mathrm{e}^{\frac{3z}{1+z}}. \tag{112}$$

which approaches $\mathrm{e}^3 z^{-5.1}$ for large values of *z*. This means that with $z = 1100$ we can with sufficient accuracy use a series expansion of the expression (111) to first order in $A^{-1}(1+z)^{-3}$ which gives

$$\frac{\Delta H_0}{H_{0\Lambda CDM}} \approx \left(2Az^3\right)^{-1} \approx 8.7 \times 10^{-10}. \tag{113}$$

This shows that the introduction of CPL-dark energy instead of LIVE together with dust in a flat universe model gives much too small change of the Hubble constant to have any significance for the Hubble tension.

One may wonder whether CPL-dark energy with other equations of state parameters may solve the Hubble tension. In order to investigate this we require that $\Delta H_0/H_{0\Lambda CDM} > \varepsilon$ (say $\varepsilon = 0.05$). From Equation (111) we then get the requirement

$$\sqrt{\frac{1+K_1}{1+K_2}} > 1+\varepsilon, \quad K_1 = A^{-1}(1+z)^{-3}, \quad K_2 = A^{-1}(1+z)^{3(w_0+w_a)} \exp\left(-\frac{3w_0 z}{1+z}\right). \tag{114}$$

This gives to 1. order in ε

$$K_2 < \frac{K_1 - 2\varepsilon}{1+2\varepsilon}. \tag{115}$$

Inserting the numerical values we get $K_1 = 1.7 \times 10^{-9}$. Hence Equation (115) requires K_2 to be negative. But the expression for K_2 in Equation (114) shows that his is impossible. Hence none of the $w_0 w_a$CDM-universe models can solve the Hubble tension.

We can obtain a more general conclusion. Taking Equation (110) as a point of departure instead of Equation (111) gives

$$K_2 = A^{-1}(1+z)^{-3} g_{de}(z). \tag{116}$$

with an unspecified density factor of the dark energy. Then we arrive at the following result: In order that a dark dynamical dark energy with a time varying equation of state shall be able to solve the Hubble tension, its density factor $g_{de}(z)$ must be negative. As seen from the definition of $g_{de}(z)$ in Equation (17) this is not possible. Hence, *the introduction of a dynamical dark energy defined by letting it have a time dependent equation of state, cannot solve the Hubble tension*.

14. December 2025 Jing-Ya Zhao and co-workers published a preprint [24] with

title: "A parameterized equation of state for dark energy and Hubble Tension". They applied an equation of state of the dark energy with

$$w_z = -\left[1+\alpha\left(1+z\right)^6\right] \tag{117}$$

to the Hubble tension, and wrote in the abstract "the model exhibits a high degree of consistency with astronomical observations and provides a promising parameterized method for addressing the Hubble tension". In the final lines of their conclusion the authors wrote: "These results show excellent agreement with Cepheid-calibrated supernova observations, but exhibit a significant discrepancy (exceeding reported uncertainties) compared to the 2018 Planck CMB measurements. This inconsistency, known as the Hubble tension, arises primarily from the elevated value of the Hubble constant H_0 obtained from late Universe probes compared to the value inferred from the standard ΛCDM model applied to early Universe data. In the model we proposed, this tension has been alleviated". However, they did not include the Planck-data in their analysis, writing: "The model was further validated against Hubble, Pantheon +, BAO, and DESI DR2 datasets, and the best-fit parameter values are constrained to $H_0 = 73.96 \pm 0.16$ km·s^{-1}·Mpc^{-1}, $\Omega_m = 0.2434 \pm 0.0079$ and $\alpha = -0.00049 \pm 0.00092$, indicating the dark energy behaves like phantom". Hence, there was no solution of the Hubble tension in this work, in agreement with the general conclusion above.

11. Comparison of the Models

1) Theoretical motivation

Chevallier and Polarski [3], and independently Linder [4], introduced the equation of state parameter (8) as a toy-model of dark energy in order to consider how far such a model can be distinguished from a model with constant w. They introduced $\alpha = 1 + w_0$ as a measure of the deviation from the LIVE-dark energy of the ΛCDM-universe mode of the considered dark energy, while w_a measures the rate of change with time of w.

These authors found that at redshifts $z \sim (1 - 2)$, measurements with an accuracy at the percentage level might be able to distinguish the varying equation of state with $w_a = 0.5$ from a constant equation of state (and same value today).

Linder [4] wrote that the equation of state parameter (8) has several advantages: 1) reduction to the old linear redshift behaviour at low redshift, 2) well behaved, bounded behaviour for high redshift, 3) high accuracy in reconstructing many scalar field equations of state and the resulting distance-redshift relations, 4) good sensitivity to observational data, 5) simple physical interpretation.

In the DESI-report [2] five toy-models of dark energy with different equations of state were mentioned (see **Figure 6** above). None of them are justified physically, and all of them introduce two arbitrary constants in the equation of state. From a physical point of view this is a weakness of these models compared to the ΛCDM-model which have dark energy without any arbitrary constant in the equation of state. Although this model is a special case of the $w_0 w_a$-models, the equation of state of LIVE is deduced from a reasonable theoretical requirement; that the properties of the dark energy shall be Lorentz invariant in order not to be in conflict with the principle of relativity. The new types of dark energy have an energy-momentum tensor with components that are not Lorentz invariant. Hence in principle these types of dark energy violate the principle of relativity. Also due to the arbitrary constants in the equation of state of the dark energy in these models they are more difficult to falsify than the ΛCDM-model.

However, at the present time is seems that the ΛCDM-model is violated by the DESI-observations in combination with other observation. Hence the introduction of new models with more flexible properties than LIVE seems necessary although these new models lack the theoretical strength of the ΛCDM-model.

2) Ability to fit the DESI data

The DESI-team gave several values of w_0 and w_a depending upon which types of measurement that were included in the analysis. In the last line of their table V for a flat $w_0 w_a$-universe they report the results from combined CMB and DESI observational data: $\Omega_{m0} = 0.32$, $w_0 = -0.7t$, $w_a = -0.82$. However with other data they had values of w_a down to $w_a = -1.75$. The main results presented in [1] where the analysis was based on the equation of state (8) is shown in their Figure 11 which is shown as **Figure 32** here.

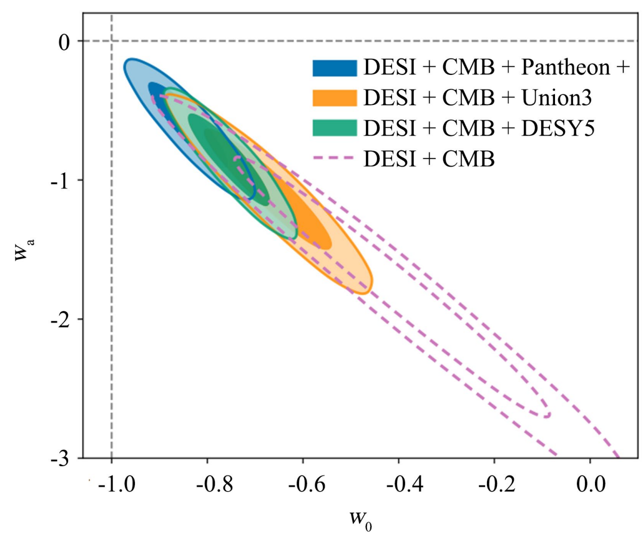


Figure 32. Results for the distributions of w_0 and w_a as presented in the DESI-report [1]. There is a significant conflict with the ΛCDM-universe model which has $w_0 = -1$, $w_a = 0$.

In [2] the authors found that extending ΛCDM to include a two parameter equation of state parameter $w(z)$ were sufficient to capture the trends present in the data. In particular, they examined three dark energy classes with distinct dynamics, including quintessence scenarios satisfying $w \geq -1$, to explore what underlying physics could explain such deviations. The observation data indicated a preference for models that feature a phantom crossing. The authors concluded that their analysis found that the evidence for dynamical dark energy, particularly at low redshift ($z \lesssim 0.3$), was robust and stable under different choices of dark energy models.

In their Figure 4 (**Figure 6** here) the authors have shown alternative expressions containing the constants w_0 and w_a for the equation of state parameter w as functions of the scale factor. In this figure they have also shown how w depends upon the redshift for the different expressions. But they have not told which values of w_0 and w_a they have used in the figure. Looking at **Figure 6** here, we see that the values $w_0 = -0.7$, $w_a = -1$ are near the preferred region, so according to [1] and [2] one is tempted to conclude that the values of w_0 and w_a must be close to these values.

However, in several papers published as response to the DESI-reports the authors have obtained good agreement with the large set of observation for models of the dark energy with other values of w_0 and w_a. For example Son *et al.* [4] introduced a "corrected" analysis of the observational data based upon the CPL-model and obtained "corrected" values $w_0 = -0.3$, $w_a = -1.9$. With these values the universe is presently in a state with decelerated cosmic expansion. In the BA-model [16], [18] the preferred values are $w_0 = -0.9$ and $w_a = -0.4$. In the LOG-model $w_{0C} = -1$, $w_{aC} = -0.2$ was used [19]. The universe model of Dutra *et al.* [22] has $A = 1.27$, and $w_0 = -0.46$, $w_a = -16.4$.

Obviously, there is not a single preferred universe model coming out of the observational data. Different models of the dark energy can be made compatible with the observational data.

3) Implications for the Hubble tension

None of the models are able to solve the Hubble tension. They are in fact very far from being able to solve the Hubble tension.

4) Predictions for the future evolution of the universe

The different models of the dark energy predict different future evolutions of the universe.

The ΛCDM-universe model predicts eternal accelerated expansion. The CPL-model with equation of state (8) of the dark energy and the DESI-values $w_0 = -0.7$, $w_a = -1$ predicts a deceleration parameter as shown by the red curve in **Figure 26**. With the corrected values, $w_0 = -0.3$, $w_a = -1.9$, of Son *et al.* [8] the universe is presently in an era with decelerated cosmic expansion. The universe may reach a time when the expansion stops and the universe begins to collapse.

Hence predicting the future evolution of the universe is not possible as long as we do not know which universe model is the most reliable one.

12. Conclusions

For the last 25 years, the ΛCDM universe model has been the standard model of our universe. During the year 2025, it has been increasingly clear that the ΛCDM universe model may be in conflict with new observational data, in particular the DESI-data in combination with observations of CMP temperature fluctuations, supernova data, gravitational lens data and observations of the large scale properties of the cosmic mass distribution.

I have here reviewed the problem revealed by the DESI-data [1] and presented universe models which seem to be preferred by the observational data at the end of the year 2025.

The main object of this paper has been to make the reader familiar with the physical properties of different types of dark energy that have been proposed in several efforts to obtain a flat universe model containing dust and dark energy which is in agreement with the available observational data.

In particular, I have discussed how different types of dark energy can evolve so that at one stage it causes repulsive gravity and accelerated expansion, while at another stage it causes attractive gravity and decelerated expansion.

Also, the density of dynamical dark energy changes during the evolution of the universe, and in combination with a decreasing density of dust, this determines the deceleration parameter and the age of these universe models. We have seen how such universe models with dust and different types of dark energy have different expansion histories. Most of the results have been calculated analytically and illustrated graphically.

I have also applied such modifications of the ΛCDM-universe model to the Hubble tension, and calculated the differences of the calculated value of the Hubble constant for universe models with different types of dark energy with the same set of observational data. The calculation showed that the differences were far too small to have any possibility of solving the Hubble tension.

My conclusion as to the possibility of solving the Hubble tension by introducing a new type of dark energy, may be formulated in the following way: It is a general feature of all dynamical dark energy models constrained by low redshift data, that the introduction of a dynamical dark energy defined by letting it have a time dependent equation of state, cannot solve the Hubble tension.

In light of the DESI- and other observational data, it is natural to ask: Do we need a new standard model for the universe? At the present time, it seems that the most realistic answer to this question is: Yes, we need a modification of the ΛCDM-universe model, namely a modification of the dark energy it contains.

As mentioned earlier, I think the most realistic modification is not to remove LIVE, but to add a dynamical dark energy to LIVE, in other words, to construct a universe model described mathematically by Einstein's field equations with a cosmological constant, but also containing a realistic type of dynamical dark energy. Since this dark energy may be a remnant from the inflationary era, I suggest that one try out universe models with some type of quintessence energy which have been used to make models of the inflationary era, together with LIVE and dust.

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The graphs are made using Geogebra.

The integrals in the expressions for the ages of the universe model are calculated by

means of the Integral Calculator on the page https://www.integral-calculator.com/.

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Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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Appendix: Deduction of Equation (54)

We shall calculate the integral

$$I = \int_0^1 \sqrt{\frac{a}{a^3 + A}}\,\mathrm{d}a \,. \tag{A1}$$

Introducing the variable

$$u = \sqrt{\frac{a^3}{A}} \,, \tag{A2}$$

we get

$$x = A^{\frac{1}{3}} u^{\frac{2}{3}} \,, \quad \mathrm{d}x = \frac{2}{3} A^{\frac{1}{3}} u^{-\frac{1}{3}} \mathrm{d}u \,. \tag{A3}$$

Inserting this into the integral (A1) gives

$$I = \frac{2}{3} \int_0^{1/\sqrt{A}} \frac{1}{\sqrt{u^2 + 1}}\,\mathrm{d}u \,. \tag{A4}$$

Hence, we get

$$I = \frac{2}{3} \operatorname{arsinh} \frac{1}{\sqrt{A}} \,, \tag{A5}$$

which immediately leads to Equation (54).