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A Nonlinear Micromorphic Microcontinuum Theory with Nonlinear and Linear Microconstituent Kinematics for Thermoelastic Solids

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Abstract

This paper presents conservation and balance laws for nonlinear micromorphic microcontinuum for thermoelastic solid medium in which nonlinear elasticity is considered for the microconstituents, for the solid medium as well as for the interaction of the microconstituents with the solid medium. The theory is based on completely deformable microconstituents with classical rotations as rigid body rotations of the microconstituents and the use of balance of moment of moments balance law essential for thermodynamic equilibrium. A check on closure of the mathematical model consisting of conservation and balance laws and the constitutive theory reveals that additional nine equations are needed for the closure of the mathematical model. It is shown in the paper that six of the nine equations can be extracted from the balance of angular momenta. We present two alternatives for obtaining the remaining three equations needed for closure. In this first case, one could use Eringen's conservation of microinertia to obtain three additional equations; hence now we have closure. In the second alternative, if we only consider linear microconstituent kinematics, then we require only six additional equations; hence the mathematical model has closure with additional six equations extracted from the balance of angular momenta. Pros and cons of both approaches are discussed in the paper from the point of view of thermodynamic and mathematical consistency of the resulting theories. Since the microconstituents are deformable, we begin the derivation of the conservation and the balance laws for the microconstituents followed by integral-average definitions that facilitate the derivations of macro conservation and balance laws incorporating microconstituent kinematics. Constitutive theories are initiated using conjugate pairs in the entropy inequality in conjunction with axiom of causality and are derived using representation theorem, hence ensuring thermodynamic and mathematical consistency. Since classical rotation and the conjugate moments form a new kinematically conjugate pair in this theory, the balance of moment of moments balance law is necessitated by classical thermodynamics for thermodynamic equilibrium of the micromorphic solid medium and is used in the present work. In the derivation of conservation and balance laws for nonlinear micromorphic solid medium, we ensure that the modifications of the conservation and the balance laws of classical continuum mechanics are supported by classical thermodynamics. The thermodynamically and mathematically consistent nonlinear micromorphic theory presented here is compared with Eringen's work.

Keywords

Micromorphic, Micro, Macro, Finite Deformation, Finite Strain, Conservation and Balance Laws, Balance of Moment of Moments, Representation Theorem, Constitutive Theories, Integral-Average Definitions

1. Introduction

Literature review of published material presented here on micromorphic theories has also been presented in reference [1], but it is included here for the sake of completeness and for the convenience of the readers. This literature review contains the basic work mostly by Eringen and a systematic chronological presentation of the pertinent published work. The polar nature of solids, in particular crystalline solids was observed by Voigt in 1887 [2]. He presented equations of equilibrium for such solids including moment equilibrium. In 1909, Cosserat, E. and Cosserat, P. presented theory of elasticity by considering rotations about rigid directors using principle of virtual work [3]. They derived balance of momenta for dynamic case. This work lacked conservation and balance laws and had no infrastructure for constitutive theories; hence it remained dormant till 1960. Grad in 1952 [4], Gunther in 1958 [5] and Schaefer in 1967 [6] revisited Cosserat theory and established its connection to dislocation physics. Eringen in 1967 [7] [8] presented linear theory of micromorphic elastic solids and theory of micropolar plates. In 1964, Eringen and Suhibi [9] presented nonlinear theory for microelastic solids. Various works of Eringen [10]-[16] consider various aspects of micropolar continuum theories. Mechanics of micromorphic continua was introduced by Eringen in 1968 [17]. This was followed by subsequent works of Eringen, conditions for theory of micromorphic solids in 1969 [18], balance laws for micromorphic mechanics in 1970 [19], micromorphic materials with memory in 1972 [20], balance laws for micromorphic continua revisited in 1992 [20], application of micromorphic theory to dislocation physics in 1970 [21]. A compilation of Eringen's work on microcontinuum theories is published in the two books by Eringen [22] [23]. Since the publication of micromorphic theory and more generally microcontinuum theories, many papers have appeared on micropolar theories but not so many on micromorphic theory. Most published works on micropolar theory largely follow the balance laws and the constitutive theories derived by Eringen. We cite some more recent works in the following. Vernerey, Liu and Moran in 2003 [24] presented multiscale micromorphic theory for hierarchical materials. The authors employed the method for virtual power to derive the mathematical model for a particular scale with interaction with the macroscale. Chen, Lee and Xiong in 2009 [25] considered the concept of deformable material point, thus not taking a crystal without structure, hence not idealizing it as a mass point. The paper considers classical, micromorphic and generalized field theories and their applicability. Finite strain micromorphic elastoplasticity theory is presented by Regueiro in 2010 [26]. The basic foundations of the conservation and the balance laws and the constitutive theories in this work follow the approach presented by Eringen for micromorphic continua with extension to finite strain plasticity. Wang and Lee in 2010 presented micromorphic theory as a gateway to nano world [27]. This work uses basic micromorphic theory of Eringen but additionally introduces objective Eringen tensors in the kinematics and the balance laws are derived by requiring energy equations to be form-invariant under generalized Galilean transformation. Lee and Wang in 2011 [28] presented generalized micromorphic theory for solids and fluids based on the Eringen's micromorphic theory with adjustments similar to those in Lee's work in reference [27]. Isbuga and Regueiro in 2011 [29] presented three dimensional finite element analysis of finite deformation micromorphic linear elasticity. The conservation and the balance laws and the constitutive theories used in this work are same as those due to Eringen. Reges, Petangueira and Silva in 2024 [30] presented modeling of micromorphic continua based on a heterogeneous microscale. In ref [31], McAvoy presented consistent linearization technique for micromorphic continuum theories and its applicability to microcontinuum theories. They advocated that this linearization technique yields tractable linear theories for nonlinear elastic microstructured materials.

From the literature review, it is clear that most published works on micromorphic microcontinuum theories follow the conservation and the balance laws and the constitutive theories derived and presented by Eringen [7]-[23] [32].

2. Scope of Work

In this paper, we present derivation of conservation and balance laws and the constitutive theories for nonlinear micromorphic solid continua in which: 1) Finite deformation and finite strain physics is considered for the microconstituents, solid medium as well as for the interaction of the microconstituents with the solid medium. 2) The deformation/strain measures derived by Surana *et al.* [33] serve as basic measures of deformations and are utilized in the present work. 3) Derivation of the conservation and the balance laws are initiated for the microconstituents using finite strain, finite deformation physics of the microconstituents based on classical continuum mechanics. 4) This is followed by introduction of integral-average definitions that facilitate the derivation of macro conservation and the balance laws and the constitutive theories using well-established principles of thermodynamics and mathematics. 5) In deriving conservation and the balance laws and the constitutive theories, we maintain and adhere to the concept of classical rotations, Cauchy moment tensor, theory of isotropic tensor, etc introduced and used by Surana *et al.* [33]-[55] in conjunction with linear and nonlinear microcontinuum theories. 6) Care is taken in the derivations to ensure that the physics of rigid rotations of the microconstituents is treated identically in the same manner in all microcontinuum theories.

A check on the closure of the mathematical model consisting of conservation and the balance laws and the constitutive theories reveals that additional nine equations are needed for the closure of the mathematical model. In the paper, we show that six of these nine equations can be extracted from the balance of angular momenta, this we need additional three equations for closure of the mathematical model. We consider two alternative approaches to address the issue of additional three equations: 1) In this first approach, we can use conservation of microinertia advocated by Erigen to obtain additional three equations. Now we have closure of the mathematical model in which microconstituent deformation is nonlinear. The main problem with this approach is that conservation of microinertia conservation law is not supported by classical thermodynamics, thus the resulting nonlinear micromorphic theory is thermodynamically inconsistent. In view of the fact that all published microcontinuum theories (except those by the first author) are thermodynamically inconsistent and use conservation of microinertia conservation law whenever additional three equations are needed, this nonlinear micromorphic theory may be of some value to those that are already using conservation of microinertia as a conservation law. Our view is that thermodynamic inconsistency of this nonlinear theory suggests that this is not a valid microcontinuum theory, hence we do not advocate the use of this theory. 2) In the second approach, if we only consider linear microcontinuum kinematics, then only six additional equations are required for closure. These additional six equations are extracted from the balance of angular momenta, thus is this microcontinuum theory, the mathematical model has closure. This microcontinuum theory is thermodynamically and mathematically consistent. Constitutive theories in the paper are derived for the micromorphic theory in which microcontinuum kinematics is linear. The classical rotation ${}_{c}\boldsymbol{\Theta}^{(\alpha)}$ in the volume $\bar{V}^{(\alpha)} + \partial\bar{V}^{(\alpha)}$ remains as a free field, hence have no influence on the microcontinuum deformation physics.

The derivations of the constitutive theories follow derivations of micro and macro conservation and balance laws. Initial determination of the constitutive tensors and their argument tensors is made using conjugate pairs in the entropy inequality in conjunction with axiom of causality. The constitutive tensors and the argument tensors are modified and/or augmented as desired to accommodate the desire physics that may not have been considered in the derivation of the entropy inequality. The constitutive theories are derived using representation theorem and integrity, hence are always thermodynamically and mathematically consistent. Material coefficients are derived in all cases and are followed by simplified forms of the constitutive theories. The constitutive theories presented here are also compared with those of Eringen.

The section following the constitutive theories highlights significant aspects of the work presented here, various approaches used in the derivations that maintain thermodynamic and mathematical consistency of the micromorphic theory presented here. In the subsequent section, discussion of Eringen's micromorphic theory and various reasons for its thermodynamic and mathematical inconsistency are presented and discussed. Summary and conclusion are presented in the last section of the paper.

3. Preliminary Considerations

3.1. Micro and Macro Deformation Measures

Definition of classical rotation, micro and macro deformation gradient tensors, classical rotation gradient tensor, infinitesimal and finite deformation measures of stresses and strains, basic concept of a deformable material point in nonclassical theories, micro and macro deformation consideration are important to understand before we begin with the derivation of the conservation and the balance laws and the constitutive theories.

Macro deformation and displacement gradient tensors ($\boldsymbol{J}$ and ${}^{d}\boldsymbol{J}$) is Lagrangian description and their additive decomposition into symmetric and skew symmetric tensors are given in the following.

$$[J]=\left[\frac{\partial\{\bar{x}\}}{\partial\{x\}}\right]=\left[{}^{d}J\right]+[I];\quad \left[{}^{d}J\right]=\left[\frac{\partial\{u\}}{\partial\{x\}}\right] \tag{1}$$

$$\left[{}^{d}J\right]=\left[{}^{d}_{s}J\right]+\left[{}^{d}_{a}J\right];\ \left[{}^{d}_{s}J\right]=\frac{1}{2}\left(\left[{}^{d}J\right]+\left[{}^{d}J\right]^{T}\right);\ \left[{}^{d}_{a}J\right]=\frac{1}{2}\left(\left[{}^{d}J\right]-\left[{}^{d}J\right]^{T}\right) \tag{2}$$

Classical rotation (${}_{c}\boldsymbol{\Theta}$) are defined by

$$\nabla\times\boldsymbol{u}=\left({}_{c}\Theta_1\right)\boldsymbol{e}_1+\left({}_{c}\Theta_2\right)\boldsymbol{e}_2+\left({}_{c}\Theta_3\right)\boldsymbol{e}_3 \tag{3}$$

$${}_{c}\Theta_1=\left(\frac{\partial u_3}{\partial x_2}-\frac{\partial u_2}{\partial x_3}\right);\ {}_{c}\Theta_2=\left(\frac{\partial u_1}{\partial x_3}-\frac{\partial u_3}{\partial x_1}\right);\ {}_{c}\Theta_3=\left(\frac{\partial u_2}{\partial x_1}-\frac{\partial u_2}{\partial x_2}\right) \tag{4}$$

Skew symmetric tensor $\left[{}^{d}_{a}J\right]$ contains $\frac{{}_{c}\boldsymbol{\Theta}}{2}$. ${}_{c}\boldsymbol{\Theta}$ are the rotations about the axes of a triad at the material point, the axes being parallel to the *x*-frame. In classical continuum mechanics ${}_{c}\boldsymbol{\Theta}$ constitutes a free field, thus even though they are present in every deforming solid matter, but the classical continuum theory is not influenced by their presence as they constitute a free field. The presence of the microconstituents causes obstruction to the free field of ${}_{c}\boldsymbol{\Theta}$, thus in the presence of microconstituents the rotation field due to ${}_{c}\boldsymbol{\Theta}$ is no longer a free field and in fact describes the rotations of the microconstituents. A simple example illustrates this quite well. Consider 1D axial deformation of an unconstrained rod subjected to a force at the right end. The rigid body translations of the rod is a free field that has no affect on the deformation of the rod as all points of the rod are moving in the same direction by the same amount. If we constrain the left end of the rod from moving, then the displacement field is no longer a free field and is in fact the actual deformation field of the constrained rod with load on the right end. Thus, we see that the obstruction (constrained left end in this case) changes the free field to the actual deformation field of the constrained rod. Our situation of ${}_c\boldsymbol{\Theta}$ as a free field and the microconstituents obstructing this free field is exactly similar to the axial rod. That is, the free field ${}_c\boldsymbol{\Theta}$ in the presence of microconstituents becomes rotation field ${}_c\boldsymbol{\Theta}$ describing the rigid body rotations of the microconstituents, meaning ${}_c\boldsymbol{\Theta}$ are in fact the rigid rotations of the microconstituents. Thus, in micropolar theory in which microconstituents only experience rigid rotations referred to as ${}_\alpha\boldsymbol{\Theta}$ subsequently, ${}_c\boldsymbol{\Theta}$ suffice to be the rigid rotations of the microconstituents as known degrees of freedom, thus eliminating the need for ${}_\alpha\boldsymbol{\Theta}$ as unknown degrees of freedom for the microconstituents. Since rigid rotations of the microconstituents are based on ${}_c\boldsymbol{\Theta}$ in all microcontinuum theories ${}_c\boldsymbol{\Theta}$ serve as rigid body rotations of the microconstituents.

For macro deformation we also note the following.

$$\left[{}^{c\Theta}J\right]=\left[\frac{\partial\{{}_c\Theta\}}{\partial\{x\}}\right]=\left[{}^{c\Theta}_{s}J\right]+\left[{}^{c\Theta}_{a}J\right] \tag{5}$$

$$\left[{}^{c\Theta}_{s}J\right]=\frac{1}{2}\left(\left[{}^{c\Theta}J\right]+\left[{}^{c\Theta}J\right]^T\right);\ \left[{}^{c\Theta}_{a}J\right]=\frac{1}{2}\left(\left[{}^{c\Theta}J\right]-\left[{}^{c\Theta}J\right]^T\right) \tag{6}$$

in which $\left[{}^{c\Theta}J\right]$ is gradient tensor of classical rotations and $\left[{}^{c\Theta}_{s}J\right]$ and $\left[{}^{c\Theta}_{a}J\right]$ are its symmetric and skew symmetric decompositions. Next we consider the concept of deformable material point rationalized through deformable directors contained in a material point. Consider a volume of matter V enclosed by surface ∂V in the reference configuration, volume of a material point. Upon deformation, V and ∂V change to $\bar{V}$ and $\partial\bar{V}$ at time $t>0$. Let the volume $V+\partial V$ contain microconstituents uniformly dispersed in the volume. Let the volume V of material particle P contain N microconstituents. Let $V^{(\alpha)}$ and $\partial V^{(\alpha)}$ be volume and its closure for the α^{th} microconstituent with mass density $\rho_0^{(\alpha)}$. The center of mass of V has position coordinate $\boldsymbol{x}$ in $V+\partial V$. Let $\underset{\sim}{\boldsymbol{x}}^{(\alpha)}$ be the location of the microconstituent α with respect to the center of mass of $V+\partial V$ and let $\boldsymbol{x}^{(\alpha)}$ be its position coordinate in the x-frame. Upon deformation, in the current configuration, $\boldsymbol{x}$ changes to $\bar{\boldsymbol{x}}$, $\boldsymbol{x}^{(\alpha)}$ to $\bar{\boldsymbol{x}}^{(\alpha)}$ and $\underset{\sim}{\boldsymbol{x}}^{(\alpha)}$ to $\underset{\sim}{\bar{\boldsymbol{x}}}^{(\alpha)}$ (see Figure 1).

At this stage, we can possibly entertain two different methodologies in considering microconstituent deformation physics.

1) In the first case, we assume that each microconstituent is located at a different position in $\bar{V}$ and has its own deformation physics, implying that there are N different deformation physics within the volume $\bar{V}$ of the material point $\bar{P}$. We can assume that the material point $\bar{P}$ only sees the homogenized response of N microconstituents. Since the homogenization yields surrogate behavior, homogenization must include boundary conditions and the load so that the homogenized model with surrogate material properties is representative of the true physics. Clearly this homogenization is not practical for volume $\bar{V}$ of the material point.

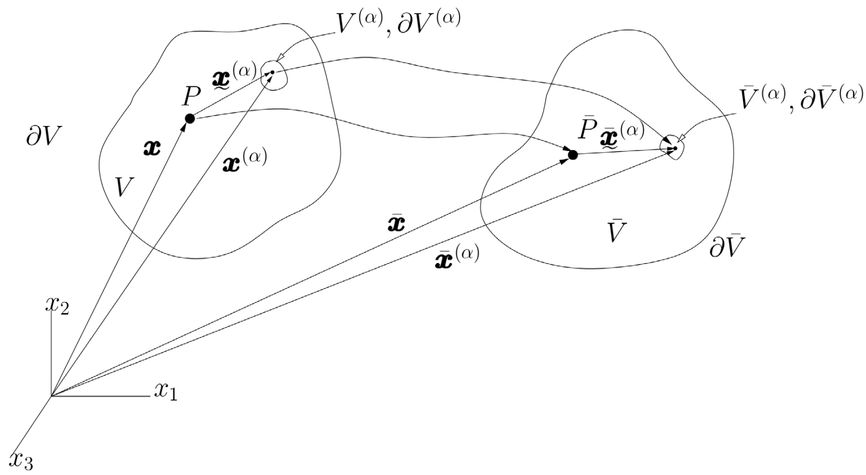

Figure 1. Undeformed and deformed configurations of material point volume.

2) In the second approach, we assume that the material point $\bar{P}$ only sees statistically averaged deformation physics of N microconstituents. This is more practical viewpoint of considering complex physics within the volume $\bar{V}$. To simplify the consideration of varied deformation physics of microconstituents in this approach, we assume that there is a surrogate configuration of microconstituents in $\bar{V}$ in which each of the N microconstituents has identical deformation physics. Thus, the average of the N microconstituent deformation physics in this case is same as the deformation physics of one microconstituent, which is assumed to be same as the statistically averaged deformation physics due to the original configuration of microconstituents in volume $\bar{V}$. Thus referring to **Figure 1**, $\underset{\sim}{\boldsymbol{x}}^{(\alpha)}$ is the director in the undeformed configuration and $\underset{\sim}{\bar{\boldsymbol{x}}}^{(\alpha)}$ is the director in the deformed configuration $\bar{V}$. The deformation of $\underset{\sim}{\boldsymbol{x}}^{(\alpha)}$ is assumed to represent the micro deformation of the microconstituents within the volume $\left(\bar{V}+\partial\bar{V}\right)$ of the material point with the director $\underset{\sim}{\boldsymbol{x}}^{(\alpha)}$. This concept is used in [33] for deriving linear and nonlinear deformation/strain measures for microcontinuum theories. In the following we only present the essential definition and concepts that are necessary in the derivation of the conservation and balance laws and the constitutive theories for nonlinear elastic micromorphic solid matter. Referring to **Figure 1**, the following relations hold:

$$\boldsymbol{x}^{(\alpha)} = \boldsymbol{x} + \underset{\sim}{\boldsymbol{x}}^{(\alpha)}; \ \bar{\boldsymbol{x}}^{(\alpha)} = \bar{\boldsymbol{x}} + \underset{\sim}{\bar{\boldsymbol{x}}}^{(\alpha)} \tag{7}$$

If we consider Lagrangian description, then $\bar{\boldsymbol{x}}^{(\alpha)}$ depends upon $\boldsymbol{x}$ and $\underset{\sim}{\boldsymbol{x}}^{(\alpha)}$ and we can write:

$$\underset{\sim}{\bar{\boldsymbol{x}}}^{(\alpha)} = \underset{\sim}{\bar{\boldsymbol{x}}}^{(\alpha)}\left(\boldsymbol{x}, \underset{\sim}{\boldsymbol{x}}^{(\alpha)}, t\right) \tag{8}$$

We note that $\underset{\sim}{\boldsymbol{x}}^{(\alpha)}$ and $\underset{\sim}{\bar{\boldsymbol{x}}}^{(\alpha)}$ are the undeformed and deformed coordinates. Hence,

$$\left\{\underset{\sim}{\bar{x}}^{(\alpha)}\right\} = \left[J^{(\alpha)}\right]\left\{\underset{\sim}{x}^{(\alpha)}\right\}; \ \left[J^{(\alpha)}\right] = \left[\frac{\partial\left\{\underset{\sim}{\bar{x}}^{(\alpha)}\right\}}{\partial\left\{\underset{\sim}{x}^{(\alpha)}\right\}}\right] \tag{9}$$

Substituting (9) into (7)

$$\left\{\bar{x}^{(\alpha)}\right\}=\{\bar{x}\}+\left[J^{(\alpha)}\right]\left\{\underset{\sim}{x}^{(\alpha)}\right\} \tag{10}$$

in which $\left[J^{(\alpha)}\right]$ is the micro deformation gradient tensor (similar to $[J]$ in macro physics). Additive decomposition of $\boldsymbol{J}^{(\alpha)}$ gives

$$\left[J^{(\alpha)}\right]=\left[{}_{s}J^{(\alpha)}\right]+\left[{}_{a}J^{(\alpha)}\right] \tag{11}$$

Symmetric tensor $\left[{}_{s}J^{(\alpha)}\right]$ contains the deformation physics of the microconstituent and the skew symmetric tensor $\left[{}_{a}J^{(\alpha)}\right]$ contains rigid rotations of the microconstituents *i.e.* it contains $\frac{{}_{\alpha}\boldsymbol{\Theta}}{2}$, ${}_{\alpha}\boldsymbol{\Theta}$ being rigid rotations of the microconstituents (${}_{\alpha}\boldsymbol{\Theta}$ being same as ${}_{c}\boldsymbol{\Theta}$). Using (8), (9) and (10) rest of the details regarding various micro deformation measure follow. For example, velocity $\boldsymbol{v}^{(\alpha)}$, velocity gradient tensor $\boldsymbol{L}^{(\alpha)}$ and $\boldsymbol{J}^{(\alpha)}$ in Lagrangian description are given by:

$$\left\{v^{(\alpha)}\right\}=\frac{D\left\{\underset{\sim}{\bar{x}}^{(\alpha)}\left(\underset{\sim}{\boldsymbol{x}}^{(\alpha)},t\right)\right\}}{Dt} \tag{12}$$

$$\left[L^{(\alpha)}\right]=\left[\frac{\partial \boldsymbol{v}^{(\alpha)}}{\partial \boldsymbol{x}^{(\alpha)}}\right] \tag{13}$$

$$\left[\dot{J}^{(\alpha)}\right]=\left[L^{(\alpha)}\right]\left[J^{(\alpha)}\right] \tag{14}$$

And

$$\left\{\dot{\bar{x}}^{(\alpha)}\right\}=\left\{v(\boldsymbol{x},t)\right\}+\left[\dot{J}^{(\alpha)}\left(\underset{\sim}{\boldsymbol{x}}^{(\alpha)}\right)\right]\left\{\underset{\sim}{x}^{(\alpha)}\right\} \tag{15}$$

In Eulerian description, we have the following:

$$\left[\bar{J}^{(\alpha)}\right]=\left[\frac{\partial\left\{\underset{\sim}{x}^{(\alpha)}\right\}}{\partial\left\{\underset{\sim}{\bar{x}}^{(\alpha)}\right\}}\right];\text{ inverse microdeformation gradient tensor} \tag{16}$$

$$\left[\bar{J}^{(\alpha)}\right]=\left[J^{(\alpha)}\right]^{-1};\ J^{(\alpha)}=\left[\bar{J}^{(\alpha)}\right]^{-1} \tag{17}$$

and

$$\left[J^{(\alpha)}\right]\left[\bar{J}^{(\alpha)}\right]=\left[\bar{J}^{(\alpha)}\right]\left[J^{(\alpha)}\right]=[J] \tag{18}$$

$$\left\{\dot{\bar{x}}^{(\alpha)}\right\}=\{\bar{v}\}+\left[L^{(\alpha)}\right]\left\{x^{(\alpha)}\right\} \tag{19}$$

$$\left[L^{(\alpha)}\right]=\left[\frac{\partial\left\{\bar{v}^{(\alpha)}\right\}}{\partial\left\{\underset{\sim}{\bar{x}}^{(\alpha)}\right\}}\right] \tag{20}$$

We also note the following useful relations

$$\left[\dot{\bar{J}}^{(\alpha)}\right]=\left[\bar{L}^{(\alpha)}\right]\left[J^{(\alpha)}\right] \tag{21}$$

$$\left[\dot{\bar{J}}^{(\alpha)}\right]=\left[\bar{J}^{(\alpha)}\right]\left[\bar{L}^{(\alpha)}\right] \tag{22}$$

$$\frac{D\left(\left|\boldsymbol{J}^{(\alpha)}\right|\right)}{Dt}=\left|\dot{\boldsymbol{J}}^{(\alpha)}\right|\operatorname{tr}\left(\bar{\boldsymbol{L}}^{(\alpha)}\right) \tag{23}$$

$$\left[\bar{L}^{(\alpha)}\right]=\left[\bar{D}^{(\alpha)}\right]+\left[\bar{W}^{(\alpha)}\right] \tag{24}$$

$$\frac{D}{Dt}\left\{d\bar{A}^{(\alpha)}\right\}=\operatorname{tr}\left(\bar{\boldsymbol{D}}^{(\alpha)}\right)[I]-\left[\bar{L}^{(\alpha)}\right]^{T}\left\{d\bar{A}^{(\alpha)}\right\} \tag{25}$$

$$\frac{D\left(d\bar{V}^{(\alpha)}\right)}{Dt}=d\dot{\bar{V}}^{(\alpha)}=\left(\operatorname{tr}\left[\bar{L}^{(\alpha)}\right]\right)d\bar{V}^{(\alpha)} \tag{26}$$

3.2. Micro and Macro Stress and Moment Tensors

Additionally, in case of finite deformation, finite strain deformation physics, we need appropriate measures of stresses and strains. First, let us consider a microconstituent with $\left(V^{(\alpha)}+\partial V^{(\alpha)}\right)$ and $\left(\bar{V}^{(\alpha)}+\partial\bar{V}^{(\alpha)}\right)$ as undeformed and deformed volumes. Consider a tetrahedron $T^{(\alpha)}$ in undeformed configuration such that its oblique plane is part of $\partial V^{(\alpha)}$ and its other three orthogonal planes are parallel to the planes of the fixed *x*-frame. Upon finite deformation, finite strain, tetrahedron $T^{(\alpha)}$ deforms into $\bar{T}^{(\alpha)}$. The oblique plane of $\bar{T}^{(\alpha)}$ and its orientation changes compared to $T^{(\alpha)}$, and the edges of $\bar{T}^{(\alpha)}$ become curvilinear. If we assume that the tangent vectors to the curvilinear edges, covariant base vectors $\tilde{\boldsymbol{g}}_i^{(\alpha)}$ approximate the edges of the deformed tetrahedron, then its edges are now straight and the faces are flat but not orthogonal to each other and are not parallel to the planes of the fixed *x*-frame. If we choose Green's strain $\boldsymbol{\varepsilon}_{[0]}^{(\alpha)}$ as the finite strain measure for the microconstituent, a covariant measure using $\boldsymbol{J}^{(\alpha)}$ whose columns are covariant base vectors, then we must use faces of the deformed tetrahedron to define contravariant Cauchy stress tensor $\boldsymbol{\sigma}^{(\alpha)}$. The lower case brackets imply that it is Cauchy stress tensor, α means a typical microconstituent. In the following $*$ implies that it is the first Piola-Kirchhoff stress tensor. Zero means derivation of order zero *i.e.* the tensor itself. This notation is necessary to accomodate derivatives of the stress tensor of higher order needed for rheology. We define first and second contravariant Piola-Kirchhoff stress tensor ($\boldsymbol{\sigma}^{(\alpha)*}$ or $\boldsymbol{\sigma}^{[\alpha,0]}$) acting on $T^{(\alpha)}$ using $\bar{\boldsymbol{\sigma}}^{(\alpha)}$ or $\boldsymbol{\sigma}^{(\alpha)}$. Following references [56] [57], we can write the following.

$$\bar{\boldsymbol{P}}^{(\alpha)}=\left(\bar{\boldsymbol{\sigma}}^{(\alpha)}\right)^{T}\cdot\bar{\boldsymbol{n}}^{(\alpha)} \tag{27}$$

Using correspondence rules:

$$\left\{d\bar{F}^{(\alpha)}\right\}=\left\{dF^{(\alpha)}\right\} \quad \text{for } \boldsymbol{\sigma}^{(\alpha)*} \tag{28}$$

$$\left\{d\bar{F}^{(\alpha)}\right\}=\left[J^{(\alpha)}\right]\left\{dF^{(\alpha)}\right\} \quad \text{for } \boldsymbol{\sigma}^{[\alpha,0]} \tag{29}$$

We can write

$$\left[\sigma^{(\alpha)*}\right]^{T}=\left|\boldsymbol{J}^{(\alpha)}\right|\left[\sigma^{(\alpha)}\right]^{T}\left[\left[J^{(\alpha)}\right]^{T}\right]^{-1} \tag{30}$$

$$\left[\sigma^{[\alpha,0]}\right]^{T}=\left|\boldsymbol{J}^{(\alpha)}\right|\left[J^{(\alpha)}\right]^{-1}\left[\sigma^{(\alpha)}\right]^{T}\left[\left[J^{(\alpha)}\right]^{T}\right]^{-1} \tag{31}$$

and

$$\left[\sigma^{(\alpha)*}\right]=\left[\sigma^{(\alpha)}\right]\left[J^{(\alpha)}\right]^T \tag{32}$$

In deriving $\boldsymbol{\sigma}^{(\alpha)*}$ the corresponding rule is $\{d\bar{F}\}=\{dF\}$, it implies that

$$\left(\boldsymbol{\sigma}^{(\alpha)}\right)^T\cdot\boldsymbol{n}^{(\alpha)}=\left(\boldsymbol{\sigma}^{(\alpha)*}\right)^T\cdot\boldsymbol{n}^{(\alpha)} \tag{33}$$

Stress and moment tensors $\boldsymbol{\sigma}^{(0)},\boldsymbol{\sigma}^{[0]},\boldsymbol{\sigma}^{*},\boldsymbol{m}^{*},\boldsymbol{m}^{(0)}$ and $\boldsymbol{m}^{[0]}$ are used as measures in macro deformation for which we use the following. Using corresponding rules [56] [57]

$$\{d\bar{F}\}=\{dF\};\ \{d\bar{M}\}=\{dM\}\ \left(\text{for } \boldsymbol{\sigma}^{*} \text{ and } \boldsymbol{m}^{*}\right) \tag{34}$$

using

$$\{d\bar{F}\}=[J]\{dF\};\ \{d\bar{M}\}=[J]\{dM\}\ \left(\text{for } \boldsymbol{\sigma}^{[0]} \text{ and } \boldsymbol{m}^{[0]}\right) \tag{35}$$

Following details in reference [56] [57] we can derive the following relation

$$\left[\sigma^{*}\right]^T=|\boldsymbol{J}|\left[\sigma^{(0)}\right]^T\left[[J]^T\right]^{-1};\ \left[m^{*}\right]^T=|\boldsymbol{J}|\left[m^{(0)}\right]^T\left[[J]^T\right]^{-1} \tag{36}$$

$$\left[\sigma^{[0]}\right]^T=|\boldsymbol{J}|[J]^{-1}\left[\sigma^{(0)}\right]\left[[J]^T\right]^{-1};\ \left[m^{[0]}\right]^T=|\boldsymbol{J}|[J]^{-1}\left[m^{(0)}\right]\left[[J]^T\right]^{-1} \tag{37}$$

and

$$\left[\sigma^{*}\right]^T=[J]\left[\sigma^{[0]}\right]^T;\ \left[m^{*}\right]^T=[J]\left[m^{[0]}\right]^T \tag{38}$$

We note that $\boldsymbol{\sigma}^{(0)}$ is not symmetric, $\boldsymbol{\sigma}^{*}$ and $\boldsymbol{\sigma}^{[0]}$ are non symmetric as well. When balance of moment of moments is used as a balance law $\boldsymbol{m}^{(0)}$ is symmetric, hence $\boldsymbol{m}^{[0]}$ is symmetric but $\boldsymbol{m}^{*}$ remains not symmetric.

3.3. Microconstituent Stress Tensor S Due to Micro Cauchy Stress Tensor $\sigma^{(\alpha)}$

In the derivation of the conservation and the balance laws, we use the following integral-average definitions.

$$\int_{\bar{V}^{(\alpha)}(t)}\bar{\sigma}_{mk}^{(\alpha)}d\bar{V}^{(\alpha)}\overset{\text{def}}{=}\bar{S}_{mk}d\bar{V} \tag{39}$$

or alternatively (Equation (89))

$$\int_V\sigma_{mk}^{(\alpha)*}dV^{(\alpha)}\overset{\text{def}}{=}S_{mk}^{*}dV \tag{40}$$

In which $\bar{\boldsymbol{\sigma}}^{(\alpha)}$ is the total Cauchy stress tensor and $\boldsymbol{\sigma}^{*}$ is the total first Piola-Kirchhoff stress tensor. Thus $\bar{S}_{mk}$ and S_{mk}^{*} are total stress tensors. In this process there is no concept of additive decomposition of $\bar{\boldsymbol{\sigma}}^{(\alpha)}$ into equilibrium and deviatoric stress tensors, hence volumetric and distortional physics are not considered explicitly. Secondly, microconstituent density is eliminated through integral-average definitions. But $\rho^{(\alpha)}$ is needed if we were to consider constitutive theory for equilibrium stress for the microconstituents. Both of these consideration help us in concluding that the stress tensor $\bar{\boldsymbol{S}}$ or $\boldsymbol{S}^*$ is due to mechanical loading, hence is a function of work conjugate strain tensor and elastic properties of the microconstituent. Henceforth, we do not consider any additive decomposition of $\boldsymbol{S}$ but consider work conjugate strain tensor and temperature as its argument tensors in deriving the constitutive theory for it.

4. Degrees of Freedom in Microconstituent Kinematics

We consider nonlinear deformation physics of microconstituents. This requires that second Piola-Kirchhoff stress tensor and rate of Green's strain tensor must be rate of work conjugate pair in the microconstituent deformation physics. Green's strain tensor for the microconstituent α requires $\boldsymbol{J}^{(\alpha)}$ so that we can define Green's strain tensor $\boldsymbol{\varepsilon}_{[0]}^{(\alpha)}$ as $\boldsymbol{\varepsilon}_{[0]}^{(\alpha)} = \frac{1}{2}\left(\left(\boldsymbol{J}^{(\alpha)}\right)^T \cdot \boldsymbol{J}^{(\alpha)} - \boldsymbol{I}\right)$. Components of $\boldsymbol{J}^{(\alpha)}$ in fact are gradients of microconstituent displacements $\boldsymbol{u}^{(\alpha)}$, thus in principle, only three microconstituent displacements are needed to define $\boldsymbol{J}^{(\alpha)}$ (just like only $\boldsymbol{u}$ are needed for $\boldsymbol{J}$, macro physics). Unfortunately $\boldsymbol{u}^{(\alpha)}$ are of the microconstituent α are not monitored, hence are not available in the microcontinuum theories, thus we are left with no alternatives but to consider all nine components of $\boldsymbol{J}^{(\alpha)}$ as unknown microconstituent deformational degrees of freedom in addition to ${}_c\boldsymbol{\Theta}$ as three unknown rigid rotational degrees of freedom for the microconstituents. Thus, the nonlinear deformation physics of microconstituents plus their rigid rotations require a total of twelve degrees of freedom. Eringen also uses nine components of $\boldsymbol{J}^{(\alpha)}$ as deformational degrees of freedom, but in his work, rigid rotations of the microconstituents are described by ${}_\alpha\boldsymbol{\Theta}$, three unknown rotational degrees of freedom about the axes of a triad with axes being parallel to x-frame. It has been shown in reference [1] [33]-[60] that the use of ${}_\alpha\boldsymbol{\Theta}$ leads to thermodynamically inconsistent theory as the entropy inequality is not satisfied in this case. This is not the case when ${}_c\boldsymbol{\Theta}$ are used as rigid body rotations of the microconstituents.

5. Conservation and Balance Laws for Nonlinear Micromorphic Continua

We present derivation of conservation and balance laws for nonlinear microconstituent kinematics: conservation of mass, balance of linear momenta, balance of angular momenta, balance of moment of moments, first and second laws of thermodynamics in Eulerian as well as Lagrangian description for nonlinear micromorphic solid continua. The two descriptions can be derived from each other when displacements are kinematic variables in both. We always begin derivations of the conservation and the balance laws for micro deformation of the microconstituent and show that a valid thermodynamic law is possible to derive using classical continuum theory. This is followed by introduction of integral-average definitions that hold at macro level and are used to derive valid conservation and balance laws for macro deformation physics. The conservation and balance laws of classical continuum mechanics are used for the micro deformation physics. Due to use of integral-average definitions at the macro level, the conservation and the balance laws of classical continuum mechanics get modified for the macro physics. Introduction of new kinematic conjugate pair, rotations and moments in addition to displacements and forces requires additional balance law, balance of moment of moments at the macro level only [45] [55] [61]. Contravariant second Piola-Kirchhoff stress tensor and the contravariant second Piola-Kirchhoff moment tensor are approximate measures of stresses and moments for finite deformation physics. Since the first and second Piola-Kirchhoff stress tensors and the moment tensors are related to each other it is more convenient to derive the conservation and the balance laws using first Piola-Kirchhoff tensors. In the end we make substitution of first Piola-Kirchhoff tensors in terms of second Piola-Kirchhoff tensors. Finite strain/deformation measures derive in [33] are used in the present work.

5.1. Conservation of Mass

5.1.1. Conservation of Micro Mass

For the microconstituent in the reference and the deformed configurations, conservation of mass can be expressed as:

$$\int_{V^{(\alpha)}} \rho_0^{(\alpha)} dV^{(\alpha)} = \int_{\bar{V}^{(\alpha)}(t)} \bar{\rho}^{(\alpha)} d\bar{V}^{(\alpha)} \tag{41}$$

If microconstituent mass is conserved, then

$$\frac{D}{Dt} \int_{\bar{V}^{(\alpha)}(t)} \bar{\rho}^{(\alpha)} d\bar{V}^{(\alpha)} = 0 \tag{42}$$

Using transport theorem [56] [57], we can write the following for (42)

$$\int_{\bar{V}^{(\alpha)}(t)} \left(\frac{D}{Dt} \bar{\rho}^{(\alpha)}\left(\bar{\boldsymbol{x}}^{(\alpha)},t\right) + \bar{\rho}^{(\alpha)}\left(\bar{\boldsymbol{x}}^{(\alpha)},t\right) \frac{\partial \bar{v}_l^{(\alpha)}\left(\bar{\boldsymbol{x}}^{(\alpha)},t\right)}{\partial \bar{x}_l^{(\alpha)}} \right) d\bar{V}^{(\alpha)} = 0 \tag{43}$$

Using localization theorem, we obtain the following from (43)

$$\frac{D}{Dt} \bar{\rho}^{(\alpha)} + \bar{\rho}^{(\alpha)} \frac{\partial \bar{v}_l^{(\alpha)}}{\partial \bar{x}_l^{(\alpha)}} = 0 \tag{44}$$

Equation (44) is the differential form of the conservation of mass in Eulerian description for the microconstituent based on classical continuum mechanics, the continuity equation.

In Lagrangian description, using (41)

$$\int_{V^{(\alpha)}} \rho_0^{(\alpha)} dV^{(\alpha)} = \int_{V^{(\alpha)}} \rho^{(\alpha)} \left|\boldsymbol{J}^{(\alpha)}\right| dV^{(\alpha)} \tag{45}$$

Equation (45) implies that

$$\rho_0^{(\alpha)} = \rho^{(\alpha)} \left|\boldsymbol{J}^{(\alpha)}\right| \tag{46}$$

Equation (46) is the continuity equation resulting from the conservation of mass for microconstituent in Lagrangian description based on classical continuum mechanics.

5.1.2. Conservation of Macro Mass

Consider Eulerian description in (41) and integration over $\bar{V}$ to obtain

$$\int_{\bar{V}(t)} \int_{\bar{V}^{(\alpha)}(t)} \bar{\rho}^{(\alpha)} d\bar{V}^{(\alpha)} \tag{47}$$

Introducing integral-average density

$$\int_{\bar{V}^{(\alpha)}(t)} \bar{\rho}^{(\alpha)} d\bar{V}^{(\alpha)} \stackrel{\text{def}}{=} \bar{\rho} d\bar{V} \tag{48}$$

Substituting (48) in (47) and setting its material derivative to zero (as mass in conserved for volume $\bar{V}$).

$$\frac{D}{Dt} \int_{\bar{V}(t)} \bar{\rho} d\bar{V} = 0 \tag{49}$$

Using transport theorem [56] [57], we obtain the following from (49)

$$\int_{\bar{V}(t)} \left(\frac{D\bar{\rho}(\bar{x},t)}{Dt} + \bar{\rho}(\bar{x},t) \bar{\boldsymbol{\nabla}} \cdot \bar{\boldsymbol{v}}(\bar{x},t) \right) d\bar{V} = 0 \tag{50}$$

Using localization theorem

$$\frac{D\bar{\rho}}{Dt} + \bar{\rho} \bar{\boldsymbol{\nabla}} \cdot \bar{\boldsymbol{v}} = 0 \tag{51}$$

Equation (51) is the "continuity equation" at the macro level in Eulerian description.

In Lagrangian description we can obtain the continuity equation using the following.

$$\int_{\bar{V}(t)} \bar{\rho} d\bar{V} = \int_{V} \rho_0 dV \tag{52}$$

or

$$\int_{V} \rho |\boldsymbol{J}| dV = \int_{V} \rho_0 V \tag{53}$$

Using localization theorem

$$\rho_0 = |\boldsymbol{J}| \rho(\boldsymbol{x},t) \tag{54}$$

Equation (54) is the continuity equation resulting from the conservation of mass at macro level in Lagrangian description.

5.2. Balance of Linear Momenta

5.2.1. For the Microconstituents: Lagrangian Description

If $\bar{a}_k^{(\alpha)}, {}^b\bar{F}_k^{(\alpha)}$ and $\bar{\sigma}_{lk}^{(\alpha)}$ are microconstituent acceleration, body forces per unit mass and contravariant Cauchy stress tensor, then using balance of linear momenta of classical continuum mechanics for a microconstituent in Eulerian description, we can write the following

$$\int_{\bar{V}^{(\alpha)}(t)} \left(\bar{\rho}^{(\alpha)}\bar{a}_k^{(\alpha)} - {}^b\bar{F}_k^{(\alpha)}\bar{\rho}^{(\alpha)}\right) d\bar{V}^{(\alpha)} - \int_{\partial\bar{V}^{(\alpha)}(t)} \bar{\sigma}_{lk}^{(\alpha)}\bar{n}_l^{(\alpha)} d\bar{A}^{(\alpha)} = 0 \quad (55)$$

Using

$$\bar{\sigma}_{lk}^{(\alpha)}\bar{n}_l^{(\alpha)} d\bar{A}^{(\alpha)} = \sigma_{lk}^{(\alpha)*} n_l^{(\alpha)} dA^{(\alpha)} \quad (56)$$

We can write (55) in Lagrangian description as follows

$$\int_{V^{(\alpha)}} \left(\rho^{(\alpha)} a_k^{(\alpha)} - \rho^{(\alpha)}\left({}^b F_k^{(\alpha)}\right)\right) dV^{(\alpha)} - \int_{\partial V^{(\alpha)}} \sigma_{lk}^{(\alpha)*} n_l^{(\alpha)} dA^{(\alpha)} = 0 \quad (57)$$

or

$$\int_{V^{(\alpha)}} \left(\rho^{(\alpha)} a_k^{(\alpha)} - \rho^{(\alpha)}\left({}^b F_k^{(\alpha)}\right) - \sigma_{lk,l}^{(\alpha)*}\right) dV^{(\alpha)} = 0 \quad (58)$$

Using localization theorem [56] [57],

$$\rho_0^{(\alpha)} a_k^{(\alpha)} - {}^b F_k^{(\alpha)} \rho_0^{(\alpha)} - \sigma_{lk,l}^{*} = 0 \quad (59)$$

Equation (59) is balance of linear momenta for the microconstituent in the Lagrangian description for finite deformation physics. $\boldsymbol{\sigma}^{(\alpha)*}$ is the first Piola-Kirchhoff stress tensor for the microconstituents.

5.2.2. Balance of Macro Linear Momenta

Define the following integral-average relations

$$\int_{V^{(\alpha)}} \rho^{(\alpha)} a_k^{(\alpha)} dV^{(\alpha)} \stackrel{\text{def}}{=} \rho_0 a_k dV \quad (60)$$

$$\int_{V^{(\alpha)}} {}^b F_k^{(\alpha)} \rho^{(\alpha)} dV^{(\alpha)} \stackrel{\text{def}}{=} {}^b F_k \rho_0 dV \quad (61)$$

$$\int_{\partial V^{(\alpha)}} \sigma_{lk}^{(\alpha)} n_l^{(\alpha)} dA^{(\alpha)} \stackrel{\text{def}}{=} \sigma_{lk}^{*} n_l dA \quad (62)$$

Using (60) - (62) in (57) and integrating over V and ∂V

$$\int_V \left(\rho_0 a_k - {}^b F_k \rho_0\right) dV - \int_{\partial V} \sigma_{lk}^{*} n_l dA = 0 \quad (63)$$

or

$$\int_V \left(\rho_0 a_k - {}^b F_k \rho_0 - \sigma_{lk,l}^{*}\right) dV = 0 \quad (64)$$

Using localization theorem

$$\rho_0 a_k - \rho_0\left({}^b F_k\right) - \sigma_{lk,l}^{*} = 0 \quad (65)$$

Equation (65) is balance of macro linear momenta in Lagrangian description in terms of macro first Piola-Kirchhoff stress tensor.

5.3. Balance of Macro Angular Momenta

We consider micro balance of linear momenta for volume $\bar{V}^{(\alpha)} + \partial\bar{V}^{(\alpha)}$ and multiply it by $\epsilon_{mkn}\bar{x}_m^{(\alpha)}$ and integrate the resulting expression over $\bar{V}^{(\alpha)}$ and $\partial\bar{V}^{(\alpha)}$ and then integrate over $\bar{V}$ and $\partial\bar{V}$. We also include $\bar{M}^{(\alpha)}$ acting on

$d\bar{A}^{(\alpha)}$. This balance of angular momenta can be written in three different forms, all three are conceptually identical, but there are some differences. We label these as Case 1, Case 2 and Case 3.

Case 1

$$\begin{aligned}&\int_{\bar{V}(t)}\int_{\bar{V}^{(\alpha)}(t)} \epsilon_{mkn}\bar{x}_m^{(\alpha)}\left(\bar{\rho}^{(\alpha)}\bar{a}_k^{(\alpha)}-\bar{\rho}^{(\alpha)}\left({}^b\bar{F}_k^{(\alpha)}\right)\right)d\bar{V}^{(\alpha)}\\&-\int_{\partial\bar{V}(t)}\int_{\partial\bar{V}^{(\alpha)}(t)} \epsilon_{mkn}\bar{x}_m^{(\alpha)}\bar{\sigma}_{lk}^{(\alpha)}\bar{n}_l^{(\alpha)}d\bar{A}^{(\alpha)}-\int_{\partial\bar{V}(t)}\int_{\partial\bar{V}^{(\alpha)}(t)}\bar{M}_n^{(\alpha)}d\bar{A}^{(\alpha)}=0\end{aligned} \tag{66}$$

Case 2

$$\begin{aligned}&\int_{\bar{V}(t)}\int_{\bar{V}^{(\alpha)}(t)} \epsilon_{mkn}\bar{x}_m^{(\alpha)}\left(\bar{\rho}^{(\alpha)}\bar{a}_k^{(\alpha)}-\bar{\rho}^{(\alpha)}\left({}^b\bar{F}_k^{(\alpha)}\right)\right)d\bar{V}^{(\alpha)}\\&-\int_{\bar{V}(t)}\int_{\bar{V}^{(\alpha)}(t)} \epsilon_{mkn}\bar{x}_m^{(\alpha)}\bar{\sigma}_{lk,l}^{(\alpha)}d\bar{V}^{(\alpha)}-\int_{\partial\bar{V}(t)}\int_{\partial\bar{V}^{(\alpha)}(t)}\bar{M}_n^{(\alpha)}d\bar{A}^{(\alpha)}=0\end{aligned} \tag{67}$$

Case 3

We consider the following identity

$$\left(\bar{x}_m^{(\alpha)}\bar{\sigma}_{lk}^{(\alpha)}\right)_{,l}=\bar{x}_{m,l}^{(\alpha)}\bar{\sigma}_{lk}^{(\alpha)}+\bar{x}_m^{(\alpha)}\bar{\sigma}_{lk,l}^{(\alpha)} \tag{68}$$

$$\therefore\ \bar{x}_m^{(\alpha)}\bar{\sigma}_{lk,l}^{(\alpha)}=\left(\bar{x}_m^{(\alpha)}\bar{\sigma}_{lk}^{(\alpha)}\right)_{,l}-\bar{x}_{m,l}^{(\alpha)}\bar{\sigma}_{lk}^{(\alpha)} \tag{69}$$

We substitute for (69) in the second term of (67) to obtain

$$\begin{aligned}&\int_{\bar{V}(t)}\int_{\bar{V}^{(\alpha)}(t)} \epsilon_{mkn}\bar{x}_m^{(\alpha)}\left(\bar{\rho}^{(\alpha)}\bar{a}_k^{(\alpha)}-\bar{\rho}^{(\alpha)}\left({}^b\bar{F}_k^{(\alpha)}\right)\right)d\bar{V}^{(\alpha)}\\&-\int_{\bar{V}(t)}\int_{\bar{V}^{(\alpha)}(t)}\left(\epsilon_{mkn}\left(\bar{x}_m^{(\alpha)}\bar{\sigma}_{lk}^{(\alpha)}\right)_{,l}-\epsilon_{mkn}\bar{x}_{m,l}^{(\alpha)}\bar{\sigma}_{lk}^{(\alpha)}\right)d\bar{V}^{(\alpha)}\\&-\int_{\partial\bar{V}(t)}\int_{\partial\bar{V}^{(\alpha)}(t)}\bar{M}_n^{(\alpha)}d\bar{A}^{(\alpha)}=0\end{aligned} \tag{70}$$

This Equation (70) is the third possible form that can be used to derive macro balance of angular momenta. The reason for using this identity is explained in Section 5.8.2.

Integral terms that are common in Case 1, Case 2 and Case 3

In all these three forms; (66), (67) and (70), the first and the third terms are identically the same.

Thus, we consider the first and the third terms appearing in (66), (67) and (70), first and then provide individual details of the second term in (66), (67) and (68).

Consider first term (Say T1) in (66), (67) and (70).

$$\mathrm{T1}=\int_{\bar{V}(t)}\int_{\bar{V}^{(\alpha)}(t)} \epsilon_{mkn}\bar{x}_m^{(\alpha)}\left(\bar{\rho}^{(\alpha)}\bar{a}_k^{(\alpha)}-\bar{\rho}^{(\alpha)}\left({}^b\bar{F}_k^{(\alpha)}\right)\right)d\bar{V}^{(\alpha)} \tag{71}$$

Let

$$\bar{x}_m^{(\alpha)}=\bar{x}_m+\underset{\sim}{x}_m^{(\alpha)} \tag{72}$$

Substitute from (72) in (71)

$$\begin{aligned}\mathrm{T1} = &\int_{\bar{V}(t)} \int_{\bar{V}^{(\alpha)}(t)} \epsilon_{mkn} \bar{x}_m \left(\bar{\rho}^{(\alpha)} \bar{a}_k^{(\alpha)} - \bar{\rho}^{(\alpha)} \left({}^b \bar{F}_k^{(\alpha)} \right) \right) d\bar{V}^{(\alpha)} \\ &+ \int_{\bar{V}(t)} \int_{\bar{V}^{(\alpha)}(t)} \epsilon_{mkn} \underset{\sim}{\bar{x}}_m^{(\alpha)} \left(\bar{\rho}^{(\alpha)} \bar{a}_k^{(\alpha)} - \bar{\rho}^{(\alpha)} \left({}^b \bar{F}_k^{(\alpha)} \right) \right) d\bar{V}^{(\alpha)}\end{aligned} \tag{73}$$

$$\begin{aligned}\mathrm{T1} = &\int_{\bar{V}(t)} \epsilon_{mkn} \bar{x}_m \int_{\bar{V}^{(\alpha)}(t)} \left(\bar{\rho}^{(\alpha)} \bar{a}_k^{(\alpha)} - \bar{\rho}^{(\alpha)} \left({}^b \bar{F}_k^{(\alpha)} \right) \right) d\bar{V}^{(\alpha)} \\ &+ \int_{\bar{V}(t)} \int_{\bar{V}^{(\alpha)}(t)} \epsilon_{mkn} \underset{\sim}{\bar{x}}_m^{(\alpha)} \left(\bar{\rho}^{(\alpha)} \bar{a}_k^{(\alpha)} - \bar{\rho}^{(\alpha)} \left({}^b \bar{F}_k^{(\alpha)} \right) \right) d\bar{V}^{(\alpha)}\end{aligned} \tag{74}$$

Define

$$\int_{\bar{V}^{(\alpha)}(t)} \left(\bar{\rho}^{(\alpha)} \bar{a}_k^{(\alpha)} - \bar{\rho}^{(\alpha)} \left({}^b \bar{F}_k^{(\alpha)} \right) \right) d\bar{V}^{(\alpha)} \overset{\text{def}}{=} \left(\bar{\rho} \bar{a}_k - \bar{\rho} \left({}^b \bar{F}_k \right) \right) d\bar{V} \tag{75}$$

Using (75) in (74), we can write

$$\begin{aligned}\mathrm{T1} = &\int_{\bar{V}(t)} \epsilon_{mkn} \bar{x}_m \left(\bar{\rho} \bar{a}_k - \bar{\rho} \left({}^b \bar{F}_k \right) \right) d\bar{V} \\ &+ \int_{\bar{V}(t)} \int_{\bar{V}^{(\alpha)}(t)} \epsilon_{mkn} \underset{\sim}{\bar{x}}_m^{(\alpha)} \left(\bar{\rho}^{(\alpha)} \bar{a}_k^{(\alpha)} - \bar{\rho}^{(\alpha)} \left({}^b \bar{F}_k^{(\alpha)} \right) \right) d\bar{V}^{(\alpha)}\end{aligned} \tag{76}$$

In Lagrangian description

$$\begin{aligned}\mathrm{T1} = &\int_V \epsilon_{mkn} x_m \left(\rho_0 a_k - \rho_0 \left({}^b F_k \right) \right) dV \\ &+ \int_V \int_{V^{(\alpha)}} \epsilon_{mkn} \underset{\sim}{x}_m^{(\alpha)} \left(\rho^{(\alpha)} a_k^{(\alpha)} - \rho^{(\alpha)} \left({}^b F_k^{(\alpha)} \right) \right) dV^{(\alpha)}\end{aligned} \tag{77}$$

Consider third term (Say T3) in (66), (67) and (70)

$$\mathrm{T3} = \int_{\partial\bar{V}(t)} \int_{\partial\bar{V}^{(\alpha)}(t)} \bar{M}_n^{(\alpha)} d\bar{A}^{(\alpha)} \tag{78}$$

$$\mathrm{T3} = \int_{\partial\bar{V}(t)} \int_{\partial\bar{V}^{(\alpha)}(t)} \bar{m}_{ln}^{(\alpha)} \bar{n}_l^{(\alpha)} d\bar{A}^{(\alpha)} \tag{79}$$

Define

$$\int_{\partial\bar{V}^{(\alpha)}(t)} \bar{m}_{ln}^{(\alpha)} \bar{n}_l^{(\alpha)} d\bar{A}^{(\alpha)} = \int_{\partial V^{(\alpha)}} m_{ln}^{(\alpha)*} n_l^{(\alpha)} dA^{(\alpha)} \overset{\text{def}}{=} m_{ln}^{*} n_l dA \tag{80}$$

Using (80) in (79) we can write (79) as follows

$$\mathrm{T3} = \int_{\partial V} m_{ln}^{*} n_l dA = \int_V m_{ln,l}^{*} dV \tag{81}$$

Integral terms that are not common in Case 1, Case 2 and Case 3

Case 1: Consider second term (Say T2C1) in (66)

$$\mathrm{T2C1} = \int_{\partial\bar{V}(t)} \int_{\partial\bar{V}^{(\alpha)}(t)} \epsilon_{mkn} \bar{x}_m^{(\alpha)} \bar{\sigma}_{lk}^{(\alpha)} \bar{n}_l^{(\alpha)} d\bar{A}^{(\alpha)} \tag{82}$$

Substitute $\bar{x}_m^{(\alpha)}$ from (72) in (82)

$$\mathrm{T2C1} = \int_{\partial\bar{V}(t)} \int_{\partial\bar{V}^{(\alpha)}(t)} \epsilon_{mkn} \left(\bar{x}_m + \underset{\sim}{\bar{x}}_m^{(\alpha)} \right) \bar{\sigma}_{lk}^{(\alpha)} \bar{n}_l^{(\alpha)} d\bar{A}^{(\alpha)} \tag{83}$$

$$\begin{aligned}\text{T2C1} &= \int_{\partial\bar{V}(t)} \int_{\partial\bar{V}^{(\alpha)}(t)} \epsilon_{mkn}\bar{x}_m \bar{\sigma}_{lk}^{(\alpha)} \bar{n}_l^{(\alpha)} d\bar{A}^{(\alpha)} + \int_{\partial\bar{V}(t)} \int_{\partial\bar{V}^{(\alpha)}(t)} \epsilon_{mkn} \bar{\underset{\sim}{x}}_m^{(\alpha)} \bar{\sigma}_{lk}^{(\alpha)} \bar{n}_l^{(\alpha)} d\bar{A}^{(\alpha)} \\ &= \int_{\partial\bar{V}(t)} \epsilon_{mkn}\bar{x}_m \int_{\partial\bar{V}^{(\alpha)}(t)} \bar{\sigma}_{lk}^{(\alpha)} \bar{n}_l^{(\alpha)} d\bar{A}^{(\alpha)} + \int_{\partial\bar{V}(t)} \int_{\partial\bar{V}^{(\alpha)}(t)} \epsilon_{mkn} \bar{\underset{\sim}{x}}_m^{(\alpha)} \bar{\sigma}_{lk}^{(\alpha)} \bar{n}_l^{(\alpha)} d\bar{A}^{(\alpha)}\end{aligned} \tag{84}$$

We note that

$$\bar{\sigma}_{lk}^{(\alpha)} \bar{n}_l^{(\alpha)} d\bar{A}^{(\alpha)} = \sigma_{lk}^{(\alpha)*} n_l^{(\alpha)} dA^{(\alpha)} \tag{85}$$

Using (85) in (84)

$$\text{T2C1} = \int_{\partial V} \epsilon_{mkn} x_m \int_{\partial V^{(\alpha)}} \sigma_{lk}^{(\alpha)*} n_l^{(\alpha)} dA^{(\alpha)} + \int_{\partial V} \int_{\partial V^{(\alpha)}} \epsilon_{mkn} \underset{\sim}{x}_m^{(\alpha)} \sigma_{lk}^{(\alpha)*} n_l^{(\alpha)} dA^{(\alpha)} \tag{86}$$

Define

$$\int_{\partial V^{(\alpha)}} \sigma_{lk}^{(\alpha)*} n_l^{(\alpha)} dA^{(\alpha)} \overset{\text{def}}{=} \sigma_{lk}^* n_l dA \tag{87}$$

Using (87) in (86)

$$\begin{aligned}\text{T2C1} &= \int_{\partial V} \epsilon_{mkn} x_m \sigma_{lk}^* n_l dA + \int_V \int_{V^{(\alpha)}} \epsilon_{mkn} \left(\underset{\sim}{x}_m^{(\alpha)} \sigma_{lk}^{(\alpha)*} \right)_{,l} dV^{(\alpha)} \\ &= \int_V \epsilon_{mkn} \left(x_m \sigma_{lk}^* \right)_{,l} dV + \int_V \int_{V^{(\alpha)}} \epsilon_{mkn} \left(\underset{\sim}{x}_{m,l}^{(\alpha)} \sigma_{lk}^{(\alpha)*} + \underset{\sim}{x}_m^{(\alpha)} \sigma_{lk,l}^{(\alpha)*} \right) dV^{(\alpha)} \\ &= \int_V \epsilon_{mkn} \left(x_{m,l} \sigma_{lk}^* + x_m \sigma_{lk,l}^* \right) dV + \int_V \epsilon_{mkn} \int_{V^{(\alpha)}} \left(\sigma_{mk}^{(\alpha)*} + \underset{\sim}{x}_m^{(\alpha)} \sigma_{lk,l}^{(\alpha)*} \right) dV^{(\alpha)} \\ &= \int_V \left(\epsilon_{mkn} \sigma_{mk}^* + x_m \sigma_{lk,l}^* \right) dV + \int_V \epsilon_{mkn} \int_{V^{(\alpha)}} \left(\sigma_{mk}^{(\alpha)*} + \underset{\sim}{x}_m^{(\alpha)} \sigma_{lk,l}^{(\alpha)*} \right) dV^{(\alpha)}\end{aligned} \tag{88}$$

Define

$$\int_{V^{(\alpha)}} \sigma_{mk}^{(\alpha)*} dV^{(\alpha)} \overset{\text{def}}{=} S_{mk}^* dV \tag{89}$$

Using (89) in (88)

$$\text{T2C1} = \int_V \epsilon_{mkn} \left(\sigma_{mk}^* + x_m \sigma_{lk,l}^* + S_{mk}^* \right) dV + \int_V \int_{V^{(\alpha)}} \epsilon_{mkn} \underset{\sim}{x}_m^{(\alpha)} \sigma_{lk,l}^{(\alpha)*} dV^{(\alpha)} \tag{90}$$

This is the final form of the second term in (66) (T2C1) for Case 1.

Case 2: Consider second term (say T2C2) in (67)

$$\text{T2C2} = \int_{\bar{V}(t)} \int_{\bar{V}^{(\alpha)}(t)} \epsilon_{mkn} \bar{x}_m^{(\alpha)} \bar{\sigma}_{lk,l}^{(\alpha)} d\bar{V}^{(\alpha)} \tag{91}$$

Substitutes $\bar{x}_m^{(\alpha)}$ from (72) in (91)

$$\begin{aligned}\text{T2C2} &= \int_{\bar{V}(t)} \int_{\bar{V}^{(\alpha)}(t)} \epsilon_{mkn} \left(\bar{x}_m + \bar{\underset{\sim}{x}}_m^{(\alpha)} \right) \bar{\sigma}_{lk,l}^{(\alpha)} d\bar{V}^{(\alpha)} \\ &= \int_{\bar{V}(t)} \epsilon_{mkn} \bar{x}_m \int_{\bar{V}^{(\alpha)}(t)} \bar{\sigma}_{lk,l}^{(\alpha)} d\bar{V}^{(\alpha)} + \int_{\bar{V}(t)} \int_{\bar{V}^{(\alpha)}(t)} \epsilon_{mkn} \bar{\underset{\sim}{x}}_m^{(\alpha)} \bar{\sigma}_{lk,l}^{(\alpha)} d\bar{V}^{(\alpha)} \\ &= \int_{\partial\bar{V}(t)} \epsilon_{mkn} \bar{x}_m \int_{\partial\bar{V}^{(\alpha)}(t)} \bar{\sigma}_{lk}^{(\alpha)} \bar{n}_l^{(\alpha)} d\bar{A}^{(\alpha)} + \int_{\bar{V}(t)} \epsilon_{mkn} \bar{\underset{\sim}{x}}_m^{(\alpha)} \int_{\bar{V}^{(\alpha)}(t)} \bar{\sigma}_{lk,l}^{(\alpha)} d\bar{V}^{(\alpha)}\end{aligned} \tag{92}$$

Define

$$\int_{\partial\bar{V}^{(\alpha)}(t)} \bar{\sigma}_{lk}^{(\alpha)}\bar{n}_l^{(\alpha)}d\bar{A}^{(\alpha)} = \int_{\partial V^{(\alpha)}} \sigma_{lk}^{(\alpha)*}n_l^{(\alpha)}dA^{(\alpha)} \overset{\text{def}}{=} \sigma_{lk}^{*}n_l dA \tag{93}$$

Using (93) in (92)

$$\begin{aligned}
\text{T2C2} &= \int_{\partial V} \epsilon_{mkn}x_m\sigma_{lk}^{*}n_l dA + \int_{\partial\bar{V}(t)} \epsilon_{mkn}\bar{x}_m^{(\alpha)} \int_{\partial\bar{V}^{(\alpha)}(t)} \bar{\sigma}_{lk}^{(\alpha)}\bar{n}_l^{(\alpha)}d\bar{A}^{(\alpha)} \\
&= \int_V \epsilon_{mkn}\left(x_m\sigma_{lk}^{*}\right)_{,l}dV + \int_{\partial V} \epsilon_{mkn}\underset{\sim}{x}_m^{(\alpha)} \int_{\partial V^{(\alpha)}} \sigma_{lk}^{(\alpha)*}n_l^{(\alpha)}dA^{(\alpha)} \\
&= \int_V \epsilon_{mkn}\left(x_{m,l}\sigma_{lk}^{*} + x_m\sigma_{lk,l}^{*}\right)dV + \int_{\partial V} \epsilon_{mkn} \int_{\partial V^{(\alpha)}} \underset{\sim}{x}_m^{(\alpha)}\sigma_{lk}^{(\alpha)*}n_l^{(\alpha)}dA^{(\alpha)}
\end{aligned} \tag{94}$$

$$\begin{aligned}
\text{T2C2} &= \int_V \epsilon_{mkn}\left(\sigma_{mk}^{*} + x_m\sigma_{lk,l}^{*}\right)dV + \int_V \epsilon_{mkn} \int_{V^{(\alpha)}} \left(\underset{\sim}{x}_m^{(\alpha)}\sigma_{lk}^{(\alpha)*}\right)_{,l}dV^{(\alpha)} \\
&= \int_V \epsilon_{mkn}\left(\sigma_{mk}^{*} + x_m\sigma_{lk,l}^{*}\right)dV + \int_V \epsilon_{mkn} \int_{V^{(\alpha)}} \left(\underset{\sim}{x}_{m,l}^{(\alpha)}\sigma_{lk}^{(\alpha)*} + \underset{\sim}{x}_m^{(\alpha)}\sigma_{lk,l}^{(\alpha)*}\right)dV^{(\alpha)} \\
&= \int_V \epsilon_{mkn}\left(\sigma_{mk}^{*} + x_m\sigma_{lk,l}^{*}\right)dV + \int_V \epsilon_{mkn} \int_{V^{(\alpha)}} \sigma_{mk}^{(\alpha)*}dV^{(\alpha)} + \int_V\int_{V^{(\alpha)}} \epsilon_{mkn}\underset{\sim}{x}_m^{(\alpha)}\sigma_{lk,l}^{(\alpha)*}dV^{(\alpha)}
\end{aligned} \tag{95}$$

Define

$$\int_{V^{(\alpha)}} \sigma_{mk}^{(\alpha)*}dV^{(\alpha)} \overset{\text{def}}{=} S_{mk}^{*}dV \tag{96}$$

Using (96) in (95)

$$\text{T2C2} = \int_V \epsilon_{mkn}\left(\sigma_{mk}^{*} + x_m\sigma_{lk,l}^{*} + S_{mk}^{*}\right)dV + \int_V\int_{V^{(\alpha)}} \epsilon_{mkn}\underset{\sim}{x}_m^{(\alpha)}\sigma_{lk,l}^{(\alpha)*}dV^{(\alpha)} \tag{97}$$

As expected, T2C2 in (97) for Case 2 is exactly same as T2C1 in (90) for Case 1.

***Case 3*: *Consider second term* (*say T2C3*) *in* (*70*)**

$$\text{T2C3} = \int_{\bar{V}(t)}\int_{\bar{V}^{(\alpha)}(t)} \epsilon_{mkn}\left(\bar{x}_m^{(\alpha)}\bar{\sigma}_{lk}^{(\alpha)}\right)_{,l}d\bar{V}^{(\alpha)} - \int_{\bar{V}(t)}\int_{\bar{V}^{(\alpha)}(t)} \epsilon_{mkn}\bar{x}_{m,l}^{(\alpha)}\bar{\sigma}_{lk}^{(\alpha)}d\bar{V}^{(\alpha)} \tag{98}$$

$$\text{T2C3} = \int_{\partial\bar{V}(t)}\int_{\partial\bar{V}^{(\alpha)}(t)} \epsilon_{mkn}\bar{x}_m^{(\alpha)}\bar{\sigma}_{lk}^{(\alpha)}\bar{n}_l^{(\alpha)}d\bar{A}^{(\alpha)} - \int_{\bar{V}(t)}\int_{\bar{V}^{(\alpha)}(t)} \epsilon_{mkn}\bar{\sigma}_{mk}^{(\alpha)}d\bar{V}^{(\alpha)} \tag{99}$$

Substitute for $\bar{x}_m^{(\alpha)}$ from (72) in (99)

$$\text{T2C3} = \int_{\partial\bar{V}(t)}\int_{\partial\bar{V}^{(\alpha)}(t)} \epsilon_{mkn}\left(\bar{x}_m + \underset{\sim}{x}_m^{(\alpha)}\right)\bar{\sigma}_{lk}^{(\alpha)}\bar{n}_l^{(\alpha)}d\bar{A}^{(\alpha)} - \int_{\bar{V}(t)}\int_{\bar{V}^{(\alpha)}(t)} \epsilon_{mkn}\bar{\sigma}_{mk}^{(\alpha)}d\bar{V}^{(\alpha)} \tag{100}$$

Let

$$\bar{\sigma}_{lk}^{(\alpha)}\bar{n}_l^{(\alpha)}d\bar{A}^{(\alpha)} = \sigma_{lk}^{(\alpha)*}n_l^{(\alpha)}dA^{(\alpha)} \tag{101}$$

$$\bar{\sigma}_{mk}^{(\alpha)}d\bar{V}^{(\alpha)} = \sigma_{mk}^{(\alpha)*}dV^{(\alpha)} \tag{102}$$

Using (101) and (102) in (100)

$$\begin{aligned}
\text{T2C3} &= \int_{\partial V} \epsilon_{mkn}x_m \int_{\partial V^{(\alpha)}} \sigma_{lk}^{(\alpha)*}n_l^{(\alpha)}dA^{(\alpha)} + \int_{\partial V} \epsilon_{mkn} \int_{\partial V^{(\alpha)}} \underset{\sim}{x}_m^{(\alpha)}\sigma_{lk}^{(\alpha)*}n_l^{(\alpha)}dA^{(\alpha)} \\
&\quad - \int_V\int_{V^{(\alpha)}} \epsilon_{mkn}\sigma_{mk}^{(\alpha)*}dV^{(\alpha)}
\end{aligned} \tag{103}$$

Define

$$\int_{V^{(\alpha)}} \sigma_{mk}^{(\alpha)*} dV^{(\alpha)} \overset{\text{def}}{=} S_{mk}^{*} dV \tag{104}$$

$$\int_{\partial V^{(\alpha)}} \sigma_{lk}^{(\alpha)*} n_l^{(\alpha)} dA^{(\alpha)} \overset{\text{def}}{=} \sigma_{lk}^{*} n_l dA \tag{105}$$

Using (104) and (105) in (103)

$$\begin{aligned} \text{T2C3} &= \int_{\partial V} \epsilon_{mkn} x_m \sigma_{lk}^{*} n_l dA + \int_{\partial V} \epsilon_{mkn} \underset{\sim}{x}_m^{(\alpha)} \int_{\partial V^{(\alpha)}} \sigma_{lk}^{(\alpha)*} n_l^{(\alpha)} dA^{(\alpha)} - \int_V \epsilon_{mkn} S_{mk}^{*} dV \\ &= \int_V \epsilon_{mkn} (x_m \sigma_{lk}^{*})_{,l} dV + \int_V \epsilon_{mkn} \int_{V^{(\alpha)}} \underset{\sim}{x}_m^{(\alpha)} \sigma_{lk,l}^{(\alpha)*} dV^{(\alpha)} - \int_V \epsilon_{mkn} S_{mk}^{*} dV \\ &= \int_V \epsilon_{mkn} \left(x_{m,l} \sigma_{lk}^{*} + x_m \sigma_{lk,l}^{*} \right) dV + \int_V \int_{V^{(\alpha)}} \epsilon_{mkn} \underset{\sim}{x}_m^{(\alpha)} \sigma_{lk,l}^{(\alpha)*} dV^{(\alpha)} - \int_V \epsilon_{mkn} S_{mk}^{*} dV \\ &= \int_V \epsilon_{mkn} \left(\sigma_{mk}^{*} + x_m \sigma_{lk,l}^{*} \right) dV + \int_V \int_{V^{(\alpha)}} \epsilon_{mkn} \underset{\sim}{x}_m^{(\alpha)} \sigma_{lk,l}^{(\alpha)*} dV^{(\alpha)} - \int_V \epsilon_{mkn} S_{mk}^{*} dV \end{aligned} \tag{106}$$

Or

$$\text{T2C3} = \int_V \epsilon_{mkn} \left(\sigma_{mk}^{*} + x_m \sigma_{lk,l}^{*} - S_{mk}^{*} \right) dV + \int_V \int_{V^{(\alpha)}} \epsilon_{mkn} \underset{\sim}{x}_m^{(\alpha)} \sigma_{lk,l}^{(\alpha)*} dV^{(\alpha)} \tag{107}$$

Differential forms of balance of angular momenta for Case 1, Case 2 and Case 3

We note that T2C3 *i.e.* (107) is exactly same as in T2C1 and T2C2 except here in (107) $\boldsymbol{S}^{*}$ has a negative sign. This of course as well to introduce identity to replace $\bar{x}_m^{(\alpha)} \bar{\sigma}_{lk,l}^{(\alpha)}$ term by where terms obtained due to identity.

By using T2C1, T2C2 and T2C3 in Case 1, Case 2 and Case 3 and also using (77) (81) that are common in all three cases, we can derive the final expressions for balance of angular momenta for Case 1, Case 2 and Case 3.

Balance of angular momenta*: *Case 1.

$$\begin{aligned} &\int_V \epsilon_{mkn} x_m \left(\rho_0 a_k - \rho_0 \left({}^b F_k \right) \right) dV + \int_V \int_{V^{(\alpha)}} \epsilon_{mkn} \underset{\sim}{x}_m^{(\alpha)} \left(\rho_0^{(\alpha)} a_k^{(\alpha)} - \rho^{(\alpha)} \left({}^b F_k^{(\alpha)} \right) \right) dV^{(\alpha)} \\ &- \int_V \epsilon_{mkn} \left(\sigma_{mk}^{*} + x_m \sigma_{lk,l}^{*} + S_{mk}^{*} \right) dV - \int_V \int_{V^{(\alpha)}} \epsilon_{mkn} \underset{\sim}{x}_m^{(\alpha)} \sigma_{lk,l}^{(\alpha)*} dV^{(\alpha)} - \int_V m_{ln,l}^{*} dV = 0 \end{aligned} \tag{108}$$

Grouping terms in (108)

$$\begin{aligned} &\int_V \epsilon_{mkn} x_m \left(\rho_0 a_k - \rho_0 \left({}^b F_k \right) - \sigma_{lk,l}^{*} \right) dV \\ &+ \int_V \int_{V^{(\alpha)}} \epsilon_{mkn} \underset{\sim}{x}_m^{(\alpha)} \left(\rho_0^{(\alpha)} a_k^{(\alpha)} - \rho^{(\alpha)} \left({}^b F_k^{(\alpha)} \right) - \sigma_{lk,l}^{(\alpha)*} \right) dV^{(\alpha)} \\ &- \int_V \left(\epsilon_{mkn} \left(\sigma_{mk}^{*} + S_{mk}^{*} \right) + m_{ln,l}^{*} \right) dV = 0 \end{aligned} \tag{109}$$

The first and second terms in (109) are zero due to macro and micro balance of linear momenta, thus (109) reduces to

$$\int_V \left(\epsilon_{mkn} \left(\sigma_{mk}^{*} + S_{mk}^{*} \right) + m_{ln,l}^{*} \right) dV = 0 \tag{110}$$

Using localization theorem (110) yields

$$\epsilon_{mkn} \left(\sigma_{mk}^{*} + S_{mk}^{*} \right) + m_{ln,l}^{*} = 0 \tag{111}$$

Equation (111) is the final form of balance of macro angular momenta for Case 1.

Balance of angular momenta*: *Case 2.

Since the final form resulting from T2C2 for Case 2 (Equation (97)) is same as the final form resulting for Case 1 for T2C1 (Equation (90)), thus the final form of the balance of macro angular momenta for this case is same as that for Case 1.

$$\epsilon_{mkn}\left(\sigma^*_{mk}+S^*_{mk}\right)+m^*_{lk,l}=0 \tag{112}$$

Balance of angular momenta*: *Case 3.

$$\begin{aligned}&\int_V \epsilon_{mkn}x_m\left(\rho_0 a_k-\rho_0\left({}^bF_k\right)\right)dV+\int_V\int_{V^{(\alpha)}}\epsilon_{mkn}\underset{\sim}{x}^{(\alpha)}_m\left(\rho_0^{(\alpha)}a_k^{(\alpha)}-\rho_0^{(\alpha)}\left({}^bF_k^{(\alpha)}\right)\right)d\bar{V}^{(\alpha)}\\&-\int_V\epsilon_{mkn}\left(\sigma^*_{mk}+x_m\sigma^*_{lk,l}-S^*_{mk}\right)dV-\int_V\int_{V^{(\alpha)}}\epsilon_{mkn}\underset{\sim}{x}^{(\alpha)}_m\sigma^{(\alpha)*}_{lk,l}dV^{(\alpha)}-\int_V m^*_{ln,l}dV=0\end{aligned} \tag{113}$$

Collecting terms in (113)

$$\begin{aligned}&\int_V\epsilon_{mkn}x_m\left(\rho_0a_k-\rho_0\left({}^bF_k\right)-\sigma^*_{lk,l}\right)dV\\&+\int_V\int_{V^{(\alpha)}}\epsilon_{mkn}\underset{\sim}{x}^{(\alpha)}_m\left(\rho_0^{(\alpha)}a_k^{(\alpha)}-\rho^{(\alpha)}\left({}^bF_k^{(\alpha)}\right)-\sigma^{(\alpha)*}_{lk,l}\right)dV^{(\alpha)}\\&-\int_V\left(\epsilon_{mkn}\left(\sigma^*_{mk}-S^*_{mk}\right)+m^*_{ln,l}\right)dV=0\end{aligned} \tag{114}$$

The first two terms in (114) are zero due to balance of macro and micro balance of linear momenta. Thus, (114) reduces to

$$\int_V\left(\epsilon_{mkn}\left(\sigma^*_{mk}-S^*_{mk}\right)+m^*_{ln,l}\right)dV=0 \tag{115}$$

Using localization theorem (115) yields

$$\epsilon_{mkn}\left(\sigma^*_{mk}-S^*_{mk}\right)+m^*_{ln,l}=0 \tag{116}$$

This is the final form of balance of macro angular momenta for Case 3.

Remarks

We note that Case 1 and Case 2 use actual balance of micro linear momenta in the derivation. Whereas in Case 3, the micro gradient stress term in the balance of micro linear momenta is altered using an identity. The result is the negative sign for $\boldsymbol{S}^*$ in the balance of angular momenta (Equation (116)). Equation (116) is what is derived by Eringen using a weighting function $\Phi^{(\alpha)}$ for the balance of micro linear momenta. The derivation presented here for Case 3 shows that weighting function is not needed as the end result of using weighting function is same as what have presented. The answer to the question of whether the correct form of balance of angular momenta is (111) (or (112)) or (116) is important. Based on the consideration of correct physics (*i.e.* not using identity) Case 1 or Case 2 is the obvious choice.

The derivation presented for Case 1 and Case 2 is a straight forward use of balance of angular momenta that leads to a positive sign for $\boldsymbol{S}^*$ in balance of angular momenta equations. We have seen that in Case 3, use of identity results in negative sign for $\boldsymbol{S}^*$ in the balance of angular momenta equation. At this point, Case 1 and Case 2 may be more convincing at a first glance as these are based on usual standard derivations. For now, we maintain both positive and negative signs for $\boldsymbol{S}^*$ terms in balance of angular momenta. We wait until the derivations of all balance laws and equations to decide on the choice of the sign for $\boldsymbol{S}^*$. Thus, we write balance of angular momenta as

$$\epsilon_{mkn}\left(\sigma^*_{mk} \pm S^*_{mk}\right) + m^*_{ln,l} = 0 \tag{117}$$

$\boldsymbol{\sigma}^*$ and $\boldsymbol{S}^*$ are nonsymmetric tensor. Even after considering them to contravariant second Piola-Kirchhoff stresses, the tensors remain nonsymmetric, thus we continue using first Piola-Kirchhoff stress in (117).

We note that due to finite deformation physics of microconstituent we have nonsymmetric $\boldsymbol{S}^*$ instead of symmetric $\boldsymbol{S}^{(\alpha)}$ (Cauchy stress).

5.4. First Law of Thermodynamics

Since the conservation and the balance laws of classical continuum mechanics hold for micro deformation of the microconstituents, we can begin with the energy equation for the microconstituents over volume $\bar{V}^{(\alpha)} + \partial\bar{V}^{(\alpha)}$ and integrate them over $\bar{V}$.

$$\begin{aligned} &\int_{\bar{V}(t)}\int_{\bar{V}^{(\alpha)}(t)} \bar{\rho}^{(\alpha)}\dot{\bar{e}}^{(\alpha)} d\bar{V}^{(\alpha)} - \int_{\bar{V}(t)}\int_{\bar{V}^{(\alpha)}(t)} \bar{\sigma}^{(\alpha)}_{kl}\bar{v}^{(\alpha)}_{l,k} d\bar{V}^{(\alpha)} + \int_{\bar{V}(t)}\int_{\bar{V}^{(\alpha)}(t)} \bar{q}^{(\alpha)}_{k,k} d\bar{V}^{(\alpha)} \\ &- \int_{\partial\bar{V}(t)}\int_{\partial\bar{V}^{(\alpha)}(t)} \bar{M}^{(\alpha)}_k \left({}_c^r\bar{\Theta}\right)_k d\bar{A}^{(\alpha)} = 0 \end{aligned} \tag{118}$$

in which $\bar{e}^{(\alpha)}$ is specific internal energy, $\bar{\boldsymbol{q}}^{(\alpha)}$ is heat flux and ${}_c^r\bar{\boldsymbol{\Theta}}$ are classical rotation rates (due to $\bar{\boldsymbol{\nabla}}^{(\alpha)} \times \bar{\boldsymbol{v}}^{(\alpha)}$). We consider each term in (118).

Consider first term (say t1) in (118)

$$\mathrm{t1} = \int_{\bar{V}(t)}\int_{\bar{V}^{(\alpha)}(t)} \bar{\rho}^{(\alpha)}\dot{\bar{e}}^{(\alpha)} d\bar{V}^{(\alpha)} = \int_V\int_{V^{(\alpha)}} \rho^{(\alpha)}_0 \dot{e}^{(\alpha)} dV^{(\alpha)} = \int_V \frac{D}{Dt}\int_{V^{(\alpha)}} \rho^{(\alpha)}_0 e^{(\alpha)} dV^{(\alpha)} \tag{119}$$

Define

$$\int_{V^{(\alpha)}} \rho^{(\alpha)}_0 e^{(\alpha)} dV^{(\alpha)} \overset{\text{def}}{=} \rho_0 e dV \tag{120}$$

Substituting (120) in (119)

$$\mathrm{t1} = \int_V \frac{D}{Dt}\left(\rho_0 e\right) dV = \int_V \rho_0 \dot{e} dV \tag{121}$$

Consider second term in (118), say t2, $\bar{\boldsymbol{\sigma}}^{(\alpha)}$ is symmetric *i.e.* $\bar{\sigma}^{(\alpha)}_{lk} = \bar{\sigma}^{(\alpha)}_{kl}$

$$\begin{aligned} \mathrm{t2} &= \int_{\bar{V}(t)}\int_{\bar{V}^{(\alpha)}(t)} \bar{\sigma}^{(\alpha)}_{lk}\bar{v}^{(\alpha)}_{l,k} d\bar{V}^{(\alpha)} = \int_{\bar{V}(t)}\int_{\bar{V}^{(\alpha)}(t)} \left(\left(\bar{\sigma}^{(\alpha)}_{lk}\bar{v}^{(\alpha)}_l\right)_{,k} - \bar{\sigma}^{(\alpha)}_{kl,k}\bar{v}^{(\alpha)}_l\right) d\bar{V}^{(\alpha)} \\ &= \int_{\partial\bar{V}(t)}\int_{\partial\bar{V}^{(\alpha)}(t)} \bar{\sigma}^{(\alpha)}_{lk}\bar{v}^{(\alpha)}_l\bar{n}^{(\alpha)}_k d\bar{A}^{(\alpha)} - \int_{\bar{V}(t)}\int_{\bar{V}^{(\alpha)}(t)} \bar{\sigma}^{(\alpha)}_{kl,k}\bar{v}^{(\alpha)}_l d\bar{V}^{(\alpha)} \end{aligned} \tag{122}$$

We note that

$$\bar{v}_l^{(\alpha)} = \bar{v}_l + \bar{L}_{lm}^{(\alpha)} \bar{\underset{\sim}{x}}_m^{(\alpha)} \tag{123}$$

and

$$\bar{\sigma}_{kl,k}^{(\alpha)} = \bar{\rho}^{(\alpha)} \bar{a}_l^{(\alpha)} - \bar{\rho}^{(\alpha)}\, {}^{b}\bar{F}_l^{(\alpha)} \tag{124}$$

Substituting (123) and (124) in (122)

$$\begin{aligned} t2 = & \int_{\partial \bar{V}(t)} \int_{\partial \bar{V}^{(\alpha)}(t)} \bar{\sigma}_{lk}^{(\alpha)} \left(\bar{v}_l + \bar{L}_{lm}^{(\alpha)} \bar{\underset{\sim}{x}}_m^{(\alpha)} \right) \bar{n}_k^{(\alpha)} d\bar{A}^{(\alpha)} \\ & - \int_{\bar{V}(t)} \int_{\bar{V}^{(\alpha)}(t)} \left(\bar{\rho}^{(\alpha)} \bar{a}_l^{(\alpha)} - \bar{\rho}^{(\alpha)}\, {}^{b}\bar{F}_l^{(\alpha)} \right) \left(\bar{v}_l + \bar{L}_{lm}^{(\alpha)} \bar{\underset{\sim}{x}}_m^{(\alpha)} \right) d\bar{V}^{(\alpha)} \end{aligned} \tag{125}$$

$$\begin{aligned} t2 = & \int_{\partial \bar{V}(t)} \bar{v}_l \int_{\partial \bar{V}^{(\alpha)}(t)} \bar{\sigma}_{kl}^{(\alpha)} \bar{n}_k^{(\alpha)} d\bar{A}^{(\alpha)} + \int_{\partial \bar{V}(t)} \int_{\partial \bar{V}^{(\alpha)}(t)} \bar{\sigma}_{kl}^{(\alpha)} \bar{L}_{lm}^{(\alpha)} \bar{\underset{\sim}{x}}_m^{(\alpha)} \bar{n}_k^{(\alpha)} d\bar{A}^{(\alpha)} \\ & - \int_{\bar{V}(t)} \bar{v}_l \int_{\bar{V}^{(\alpha)}(t)} \left(\bar{\rho}^{(\alpha)} \bar{a}_l^{(\alpha)} - \bar{\rho}^{(\alpha)}\, {}^{b}\bar{F}_l^{(\alpha)} \right) d\bar{V}^{(\alpha)} \\ & - \int_{\bar{V}(t)} \int_{\bar{V}^{(\alpha)}(t)} \left(\bar{\rho}^{(\alpha)} \bar{a}_l^{(\alpha)} - \bar{\rho}^{(\alpha)} \left({}^{b}\bar{F}_k^{(\alpha)} \right) \right) \bar{L}_{lm}^{(\alpha)} \bar{\underset{\sim}{x}}_m^{(\alpha)} d\bar{V}^{(\alpha)} \end{aligned} \tag{126}$$

Define

$$\int_{\partial \bar{V}^{(\alpha)}(t)} \bar{\sigma}_{kl}^{(\alpha)} \bar{n}_k^{(\alpha)} d\bar{A}^{(\alpha)} = \int_{\partial V^{(\alpha)}} \sigma_{kl}^{(\alpha)*} n_k^{(\alpha)} dA^{(\alpha)} \overset{\text{def}}{=} \sigma_{kl}^{*} n_k dA \tag{127}$$

and

$$\int_{\bar{V}^{(\alpha)}(t)} \left(\bar{\rho}^{(\alpha)} \bar{a}_l^{(\alpha)} - \bar{\rho}^{(\alpha)}\, {}^{b}\bar{F}_l^{(\alpha)} \right) d\bar{V}^{(\alpha)} \overset{\text{def}}{=} \left(\rho_0 a_l - \rho_0 \left({}^{b}F_l \right) \right) dV \tag{128}$$

Substituting (127) and (128) in (126)

$$\begin{aligned} t2 = & \int_{\partial V} v_l \sigma_{kl}^{*} n_k dA + \int_{\partial \bar{V}(t)} \int_{\partial \bar{V}^{(\alpha)}(t)} \bar{\sigma}_{kl}^{(\alpha)} \bar{L}_{lm}^{(\alpha)} \bar{\underset{\sim}{x}}_m^{(\alpha)} \bar{n}_k^{(\alpha)} d\bar{A}^{(\alpha)} - \int_V \left(\rho_0 a_l - \rho_0 \left({}^{b}F_l \right) \right) dV \\ & - \int_{\bar{V}(t)} \int_{\bar{V}^{(\alpha)}(t)} \left(\bar{\rho}^{(\alpha)} \bar{a}_l^{(\alpha)} - \bar{\rho}^{(\alpha)} \left({}^{b}\bar{F}_l^{(\alpha)} \right) \right) \bar{L}_{lm}^{(\alpha)} \bar{\underset{\sim}{x}}_m^{(\alpha)} d\bar{V}^{(\alpha)} \end{aligned} \tag{129}$$

We note that

$$\begin{aligned} \int_{\partial V} \sigma_{kl}^{*} v_l n_k dA & = \int_V \left(\sigma_{kl}^{*} v_l \right)_{,k} dV \\ & = \int_V \left(v_l \sigma_{lk,k}^{*} + \sigma_{kl}^{*} v_{l,k} \right) dV \end{aligned} \tag{130}$$

And

$$\begin{aligned} \int_{\partial \bar{V}(t)} \int_{\partial \bar{V}^{(\alpha)}(t)} \bar{L}_{lm}^{(\alpha)} \bar{\underset{\sim}{x}}_m^{(\alpha)} \bar{\sigma}_{kl}^{(\alpha)} \bar{n}_k^{(\alpha)} d\bar{A}^{(\alpha)} & = \int_{\partial V} \int_{\partial V^{(\alpha)}} L_{lm}^{(\alpha)} \underset{\sim}{x}_m^{(\alpha)} \sigma_{kl}^{(\alpha)*} n_k^{(\alpha)} dA^{(\alpha)} \\ & = \int_{\partial V} L_{lm}^{(\alpha)} \int_{\partial V^{(\alpha)}} \underset{\sim}{x}_m^{(\alpha)} \sigma_{kl}^{(\alpha)*} n_k^{(\alpha)} dA^{(\alpha)} \\ & = \int_V \int_{V^{(\alpha)}} L_{lm}^{(\alpha)} \left(\sigma_{kl}^{(\alpha)*} \underset{\sim}{x}_m^{(\alpha)} \right)_{,k} dV^{(\alpha)} \\ & = \int_V \int_{V^{(\alpha)}} L_{lm}^{(\alpha)} \left(\sigma_{kl,k}^{(\alpha)*} \underset{\sim}{x}_m^{(\alpha)} + \sigma_{kl}^{(\alpha)*} \underset{\sim}{x}_{m,k}^{(\alpha)} \right) dV^{(\alpha)} \\ & = \int_V \int_{V^{(\alpha)}} L_{lm}^{(\alpha)} \left(\sigma_{kl,k}^{(\alpha)*} \underset{\sim}{x}_m^{(\alpha)} + \sigma_{ml}^{(\alpha)*} \right) dV^{(\alpha)} \end{aligned} \tag{131}$$

Define

$$\int_{V^{(\alpha)}} L_{lm}^{(\alpha)} \sigma_{ml}^{(\alpha)*} dV^{(\alpha)} \stackrel{\text{def}}{=} S_{ml}^{*} L_{lm}^{(\alpha)} dV \tag{132}$$

Substituting (130) in (129) and (132) in (131) and then (131) in (129) and converting last term in (129) to Lagrangian description.

$$\begin{aligned} \text{t2} = & \int_V \left(v_l \sigma_{kl,k}^{*} + \sigma_{kl}^{*} v_{l,k} \right) dV + \int_V \left(S_{ml}^{*} L_{lm}^{(\alpha)} \right) dV + \int_V \int_{V^{(\alpha)}} L_{lm}^{(\alpha)} \sigma_{kl,k}^{(\alpha)*} \underset{\sim}{x}_m^{(\alpha)} dV^{(\alpha)} \\ & - \int_V v_l \left(\rho_0 a_l - \rho_0 \left({}^{b}F_l \right) \right) dV - \int_V \int_{V^{(\alpha)}} \left(\rho_0^{(\alpha)} a_l^{(\alpha)} - \rho_0^{(\alpha)} \, {}^{b}F_l^{(\alpha)} \right) L_{lm}^{(\alpha)} \underset{\sim}{x}_m^{(\alpha)} dV^{(\alpha)} \end{aligned} \tag{133}$$

Collecting terms

$$\begin{aligned} \text{t2} = & -\int_V v_l \left(\rho_0 a_l - \rho_0 \left({}^{b}F_l \right) - \sigma_{kl,k}^{*} \right) dV \\ & - \int_V \int_{V^{(\alpha)}} \left(\rho_0^{(\alpha)} a_l^{(\alpha)} - \rho_0^{(\alpha)} \left({}^{b}F_l^{(\alpha)} \right) - \sigma_{kl,k}^{(\alpha)*} \right) L_{lm}^{(\alpha)} \underset{\sim}{x}_m^{(\alpha)} dV^{(\alpha)} \\ & + \int_V \left(S_{ml}^{*} L_{lm}^{(\alpha)} + \sigma_{kl}^{*} v_{l,k} \right) dV \end{aligned} \tag{134}$$

The first and second terms in (134) are zero due to balance of macro and micro linear momenta, hence (134) reduces to

$$\text{t2} = \int_V \left(S_{ml}^{*} \dot{J}_{lm}^{(\alpha)} + \sigma_{kl}^{*} \dot{J}_{lk} \right) dV \tag{135}$$

Consider the third term (say t3) in (118)

$$\text{t3} = \int_{\bar{V}(t)} \int_{\bar{V}^{(\alpha)}(t)} \bar{q}_{k,k}^{(\alpha)} dV^{(\alpha)} = \int_{\bar{V}(t)} \int_{\partial \bar{V}^{(\alpha)}(t)} \bar{q}_k^{(\alpha)} \bar{n}_k^{(\alpha)} d\bar{A}^{(\alpha)} \tag{136}$$

Define

$$\int_{\partial \bar{V}^{(\alpha)}(t)} \bar{q}_k^{(\alpha)} \bar{n}_k^{(\alpha)} d\bar{A}^{(\alpha)} \stackrel{\text{def}}{=} \bar{q}_k \bar{n}_k d\bar{A} \tag{137}$$

Using (137) in (136)

$$\text{t3} = \int_{\partial \bar{V}(t)} \bar{q}_k \bar{n}_k d\bar{A} = \int_{\bar{V}(t)} \bar{q}_{k,k} d\bar{V} = \int_V q_{k,k} dV \tag{138}$$

Consider the fourth term (say t4) in (118)

$$\begin{aligned} \text{t4} & = \int_{\partial \bar{V}(t)} \int_{\partial \bar{V}^{(\alpha)}(t)} \overline{M}_k^{(\alpha)} \left({}_c^r \Theta \right)_k d\bar{A}^{(\alpha)} \\ & = \int_{\partial \bar{V}(t)} \left({}_c^r \Theta \right)_k \int_{\partial \bar{V}^{(\alpha)}(t)} \bar{m}_{lk}^{(\alpha)} \bar{n}_l^{(\alpha)} d\bar{A}^{(\alpha)} \\ & = \int_{\partial V} {}_c\dot{\Theta}_k \int_{\partial V^{(\alpha)}} m_{lk}^{(\alpha)*} n_l^{(\alpha)} dA^{(\alpha)} \end{aligned} \tag{139}$$

Define

$$\int_{\partial V^{(\alpha)}} m_{lk}^{(\alpha)*} n_l^{(\alpha)} dA^{(\alpha)} \stackrel{\text{def}}{=} m_{lk}^{*} n_l dA \tag{140}$$

Using (140) in (139)

$$\begin{aligned} \text{t4} &= \int_{\partial V} {}_c\dot{\Theta}_k m^*_{lk} n_l dA = \int_{\partial V} {}_c\dot{\boldsymbol{\Theta}} \cdot \left(\boldsymbol{m}^*\right)^T dA \\ &= \int_V \nabla \cdot \left({}_c\dot{\boldsymbol{\Theta}} \cdot \left(\boldsymbol{m}^*\right)^T \right) dV \end{aligned} \tag{141}$$

A simple calculation shows [50] [54]

$$\nabla \cdot \left({}_c\dot{\boldsymbol{\Theta}} \cdot \left(\boldsymbol{m}^*\right)^T \right) = {}_c\dot{\boldsymbol{\Theta}} \cdot \left(\nabla \cdot \boldsymbol{m}^* \right) + \boldsymbol{m}^* : {}^{c\Theta}\dot{\boldsymbol{J}} \tag{142}$$

Using (142) in (141)

$$\text{t4} = \int_V \left({}_c\dot{\boldsymbol{\Theta}} \cdot \left(\nabla \cdot \boldsymbol{m}^* \right) + \boldsymbol{m}^* : {}^{c\Theta}\dot{\boldsymbol{J}} \right) dV \tag{143}$$

Substituting t1, t2, t3 and t4 from (121), (135), (138), and (143) in (118) we can write (118) as follows

$$\begin{aligned} &\int_V \rho_0 \dot{e} dV - \int_V \left(S^*_{ml} \dot{J}^{(\alpha)}_{lm} + \sigma^*_{kl} \dot{J}_{lk} \right) dV + \int_V q_{k,k} dV \\ &- \int_V \left({}_c\dot{\boldsymbol{\Theta}} \cdot \left(\nabla \cdot \boldsymbol{m}^* \right) + \boldsymbol{m}^* : {}^{c\Theta}\dot{\boldsymbol{J}} \right) dV = 0 \end{aligned} \tag{144}$$

or

$$\int_V \left(\rho_0 \dot{e} - \boldsymbol{S}^* : \dot{\boldsymbol{J}}^{(\alpha)} - \boldsymbol{\sigma}^* : \dot{\boldsymbol{J}} + q_{k,k} - \left({}_c\dot{\boldsymbol{\Theta}} \cdot \left(\nabla \cdot \boldsymbol{m}^* \right) + \boldsymbol{m}^* : {}^{c\Theta}\dot{\boldsymbol{J}} \right) \right) dV = 0 \tag{145}$$

Using localization theorem

$$\rho_0 \dot{e} - \boldsymbol{S}^* : \dot{\boldsymbol{J}}^{(\alpha)} - \boldsymbol{\sigma}^* : \dot{\boldsymbol{J}} + q_{k,k} - \left({}_c\dot{\boldsymbol{\Theta}} \cdot \left(\nabla \cdot \boldsymbol{m}^* \right) + \boldsymbol{m}^* : {}^{c\Theta}\dot{\boldsymbol{J}} \right) = 0 \tag{146}$$

This is the macro energy equation in Lagrangian description.

5.5. Second Law of Thermodynamics

The rate of increase of entropy for a microconstituent volume $\bar{V}^{(\alpha)} + \partial\bar{V}^{(\alpha)}$ due to entropy imparted to it by contacting or noncontacting sources is given by

$$\frac{D}{Dt} \int_{\bar{V}^{(\alpha)}(t)} \bar{\eta}^{(\alpha)} \bar{\rho}^{(\alpha)} d\bar{V}^{(\alpha)} \geq \int_{\partial\bar{V}^{(\alpha)}(t)} \bar{h}^{(\alpha)} d\bar{A}^{(\alpha)} + \int_{\bar{V}^{(\alpha)}(t)} \bar{s}^{(\alpha)} \bar{\rho}^{(\alpha)} d\bar{V}^{(\alpha)} \tag{147}$$

$\bar{\eta}^{(\alpha)}$ is the entropy density of the volume $\bar{V}^{(\alpha)}$ of the microconstituent, $\bar{h}^{(\alpha)}$ is the entropy flux imparted to volume $\bar{V}^{(\alpha)}$ through $\partial\bar{V}^{(\alpha)}$ by the surrounding medium through contact and $\bar{s}^{(\alpha)}$ is the source of entropy in $\bar{V}^{(\alpha)}$ due to non-contacting sources or bodies. Integrating (147) over $\bar{V}$

$$\int_{\bar{V}(t)} \frac{D}{Dt} \int_{\bar{V}^{(\alpha)}(t)} \bar{\eta}^{(\alpha)} d\bar{V}^{(\alpha)} \geq \int_{\partial\bar{V}(t)} \int_{\partial\bar{V}^{(\alpha)}(t)} \bar{h}^{(\alpha)} d\bar{A}^{(\alpha)} + \int_{\bar{V}(t)} \int_{\bar{V}^{(\alpha)}(t)} \bar{s}^{(\alpha)} \bar{\rho}^{(\alpha)} d\bar{V}^{(\alpha)} \tag{148}$$

Using

$$\bar{h}^{(\alpha)} = -\bar{\Psi}^{(\alpha)}_k \bar{n}^{(\alpha)}_k \tag{149}$$

$$\bar{\Psi}^{(\alpha)}_k = \frac{\bar{q}^{(\alpha)}_k}{\bar{\theta}} \tag{150}$$

$$\therefore\ \bar{h}^{(\alpha)} = -\frac{\bar{q}^{(\alpha)}_k \bar{n}^{(\alpha)}_k}{\bar{\theta}} \tag{151}$$

$$\bar{s}^{(\alpha)} = \frac{\bar{r}^{(\alpha)}}{\bar{\theta}} \tag{152}$$

Using (151) and (152) in (148)

$$\int_{\bar{V}(t)} \frac{D}{Dt} \int_{\bar{V}^{(\alpha)}(t)} \bar{\eta}^{(\alpha)} \bar{\rho}^{(\alpha)} d\bar{V}^{(\alpha)} \geq \int_{\partial\bar{V}(t)} - \int_{\partial\bar{V}^{(\alpha)}(t)} \frac{\bar{q}_k^{(\alpha)} \bar{n}_k^{(\alpha)}}{\bar{\theta}} d\bar{A}^{(\alpha)} + \int_{\bar{V}(t)} \int_{\bar{V}^{(\alpha)}(t)} \frac{\bar{r}^{(\alpha)} \bar{\rho}^{(\alpha)}}{\bar{\theta}} d\bar{V}^{(\alpha)} \tag{153}$$

Define

$$\frac{D}{Dt} \int_{\bar{V}^{(\alpha)}(t)} \bar{\eta}^{(\alpha)} \bar{\rho}^{(\alpha)} d\bar{V}^{(\alpha)} \stackrel{\text{def}}{=} \dot{\bar{\eta}} \bar{\rho} d\bar{V} \tag{154}$$

$$\int_{\partial\bar{V}^{(\alpha)}(t)} \frac{\bar{q}_k^{(\alpha)} \bar{n}_k^{(\alpha)}}{\bar{\theta}} d\bar{A}^{(\alpha)} \stackrel{\text{def}}{=} \frac{\bar{q}_k \bar{n}_k d\bar{A}}{\bar{\theta}} \tag{155}$$

$$\int_{\bar{V}^{(\alpha)}(t)} \frac{\bar{r}^{(\alpha)} \bar{\rho}^{(\alpha)}}{\bar{\theta}} d\bar{V}^{(\alpha)} \stackrel{\text{def}}{=} \frac{\bar{r}\bar{\rho}}{\bar{\theta}} d\bar{V} \tag{156}$$

Substituting (154) - (156) in (153)

$$\int_{\bar{V}(t)} \bar{\rho} \dot{\bar{\eta}} d\bar{V} \geq - \int_{\partial\bar{V}(t)} \frac{\bar{q}_k}{\bar{\theta}} \bar{n}_k d\bar{A} + \int_{\bar{V}(t)} \frac{\bar{r}\bar{\rho}}{\bar{\theta}} d\bar{V} \tag{157}$$

Converting (157) to Lagrangian description

$$\int_V \rho_0 \dot{\eta} dV \geq - \int_{\partial V} \frac{q_k}{\theta} n_k dA + \int_V \frac{r\rho_0}{\theta} dV \tag{158}$$

$$\int_V \rho_0 \dot{\eta} dV \geq - \int_V \left(\frac{q_k}{\theta} \right)_{,k} dV + \int_V \frac{r\rho_0}{\theta} dV \tag{159}$$

or

$$\int_V \left(\rho_0 \dot{\eta} + \frac{q_{k,k}}{\theta} - \frac{q_k}{\theta^2} \theta_{,k} - \frac{r\rho_0}{\theta} \right) dV \geq 0 \tag{160}$$

Using localization theorem

$$\rho_0 \dot{\eta} + \frac{q_{k,k}}{\theta} - \frac{q_k}{\theta^2} \theta_{,k} - \frac{r\rho_0}{\theta} \geq 0 \tag{161}$$

Multiply (161) throughout by θ

$$\rho_0 \dot{\eta} \theta + q_{k,k} - \frac{q_k}{\theta} \theta_{,k} - r\rho_0 \geq 0 \tag{162}$$

Let

$$\Phi = e - \eta\theta \tag{163}$$

$$\dot{\Phi} = \dot{e} - \dot{\eta}\theta - \eta\dot{\theta} \tag{164}$$

$$\therefore \; \bar{\rho}\bar{\theta}\dot{\bar{\eta}} = \bar{\rho}\dot{\bar{e}} - \bar{\rho}\dot{\bar{\Phi}} - \bar{\rho}\bar{\eta}\dot{\bar{\theta}} \tag{165}$$

Substituting (165) in (162)

$$-\rho_0\left(\dot{\Phi}+\eta\dot{\theta}\right)+\rho_0\dot{e}+q_{k,k}-\frac{q_k\theta_{,k}}{\theta}-r\rho_0\geq 0 \tag{166}$$

Substituting $\rho\dot{e}$ from energy Equation (146) after inserting $\rho_0 r$ in it

$$-\rho_0\left(\dot{\Phi}+\eta\dot{\theta}\right)+\boldsymbol{S}^*:\dot{\boldsymbol{J}}^{(\alpha)}+\boldsymbol{\sigma}^*:\dot{\boldsymbol{J}}-q_{k,k}+\left({}_c\dot{\boldsymbol{\Theta}}\cdot\left(\nabla\cdot\boldsymbol{m}^*\right)+\boldsymbol{m}^*:{}^{c\Theta}\dot{\boldsymbol{J}}\right)+r\rho_0 \\ +q_{k,k}-\frac{q_k\theta_{,k}}{\theta}-r\rho_0\geq 0 \tag{167}$$

Since $q_{k,k}$ and $r\rho_0$ terms cancel, we can write (167) as follows after changing the sign.

$$\rho_0\left(\dot{\Phi}+\eta\dot{\theta}\right)-\boldsymbol{\sigma}^*:\dot{\boldsymbol{J}}-\boldsymbol{S}^*:\dot{\boldsymbol{J}}^{(\alpha)}-\frac{q_k\theta_{,k}}{\theta}-\left({}_c\dot{\boldsymbol{\Theta}}\cdot\left(\nabla\cdot\boldsymbol{m}^*\right)+\boldsymbol{m}^*:{}^{c\Theta}\dot{\boldsymbol{J}}\right)\leq 0 \tag{168}$$

Inequality (168) is the macro entropy inequality in Lagrangian description resulting from the second law of thermodynamics.

5.6. Balance of Moment of Moments Balance Law

In classical continuum mechanics for solid medium based on classical thermodynamics, the displacements and forces coexist as a kinematically conjugate pair. Displacements are kinematics variables and forces are conjugate quantities to the kinematic variables. For a kinematically conjugate pair, the classical thermodynamics requires two balance laws for the thermodynamic equilibrium of the deforming volume of matter. The first balance laws is the balance of conjugate quantities *i.e.* forces and the second balance law is the balance of moment of the conjugate quantities *i.e.* balance of moment of the forces. These two balance law are of course balance of linear momenta and the balance of angular momenta. In the absence of either one of these balance laws, the deforming volume of matter is not in thermodynamic equilibrium.

In microcontinuum theories for solid matter, in addition to displacements and forces as a kinematically conjugate pair, we also have classical rotations ${}_c\boldsymbol{\Theta}$ and moments $\boldsymbol{M}$ as a second kinematically conjugate pair. Thus, based on classical thermodynamics, each kinematically conjugate pair requires two balance laws: balance of conjugate quantities and balance of moment of the conjugate quantity. Thus, for the two kinematically conjugate pair, we need: 1) balance of forces and balance of moment of forces due to displacements and forces as a kinematically conjugate pair. These are balance of linear momenta and balance of angular momenta. 2) Balance of moments and balance of moment of moments due to classical rotations ${}_c\boldsymbol{\Theta}$ and moments $\boldsymbol{M}$ as a second kinematically conjugate pair. Balance of moments is same as balance of angular momenta that already exists due to displacements and forces as a kinematically conjugate pair, hence can be modified to include nonclassical physics. This modification is obviously supported by classical thermodynamics. The second balance law, balance of moment of moments is a new balance law needed for thermodynamic equilibrium of the deforming microcontinuum volume of matter. This balance law was first proposed by Yang *et al.* [61] as a statement of static equilibrium. Surana *et al.* [45]

[55] presneted derivation of this balance law based on rate considerations for solid and fluent media. They showed that the outcome of this balance law is that the Cauchy moment tensor is symmetric in microcontinuum theories.

$$\epsilon_{ijk}\bar{m}_{ij} = 0 \text{ or } \epsilon_{ijk}m_{ij} = 0 \tag{169}$$

Details of the derivation are omitted here but are given in reference [45] [55]. We remark that in the absence of this balance law (case for almost all published works on microcontinuum theories):

1) The deforming solid microcontinuum is not in thermodynamic equilibrium as this balance law is a requirement for thermodynamic equilibrium based on classical thermodynamics.

2) The outcome of this balance law (169) is that Cauchy moment tensor is symmetric. The absence of the balance law has serious consequences in the derivation of the constitutive theories for the moment tensor. When this balance law is used, the constitutive theory is required only for symmetric Cauchy moment tensor.

3) In the absence of this balance law, the Cauchy moment tensor is nonsymmetric. This results in spurious conjugate pairs in the entropy inequality that necessitate nonphysical and invalid constitutive theories as demonstrated by Surana *et al.* [45] [55].

In the absence of this balance law, a thermodynamically and mathematically consistent, physically valid microcontinuum theory is not possible.

5.7. Summary of Macro Conservation and Balance Laws in Lagrangian Description

Continuity equation, balance of linear momenta, balance of angular momenta, energy equation and entropy inequality and the balance of moment of moments are given in the following:

$$\rho_0(\boldsymbol{x}) = |\boldsymbol{J}|\rho(\boldsymbol{x},t) \tag{170}$$

$$\rho_0 a_k - \rho_0 {}^bF_k - \sigma_{lk,l} = 0 \tag{171}$$

$$\epsilon_{mkn}\left(\sigma^*_{mk} \pm S^*_{mk}\right) + m^*_{lk,l} = 0 \tag{172}$$

$$\rho_0\dot{e} - \boldsymbol{\sigma}^* : \dot{\boldsymbol{J}} - \boldsymbol{S}^* : \dot{\boldsymbol{J}}^{(\alpha)} - \nabla\cdot\boldsymbol{q} - \left({}_c\dot{\boldsymbol{\Theta}}\cdot\left(\nabla\cdot\boldsymbol{m}^*\right) + \boldsymbol{m}^* : {}^{c\Theta}\dot{\boldsymbol{J}}\right) = 0 \tag{173}$$

$$\rho_0\left(\dot{\Phi} + \eta\dot{\theta}\right) - \boldsymbol{\sigma}^* : \dot{\boldsymbol{J}} - \boldsymbol{S}^* : \dot{\boldsymbol{J}}^{(\alpha)} + \frac{\boldsymbol{q}\cdot\boldsymbol{g}}{\theta} - \left({}_c\dot{\boldsymbol{\Theta}}\cdot\left(\nabla\cdot\boldsymbol{m}^*\right) + \boldsymbol{m}^* : {}^{c\Theta}\dot{\boldsymbol{J}}\right) \le 0 \tag{174}$$

$$\epsilon_{ijk}m_{ij} = 0 \tag{175}$$

This mathematical model consists of seven partial differential equations: balance of linear momenta (3), balance of angular momenta (3) and energy Equation (1) in thirty seven dependent variables: $\boldsymbol{u}(3)$, $\boldsymbol{\sigma}(9)$, $\boldsymbol{S}(6)$, $\boldsymbol{m}(6)$, $\boldsymbol{q}(3)$, $\theta(1)$, $\boldsymbol{J}^{(\alpha)}(9)$. Thus, additional thirty equations are needed for closure. Constitutive theories provides twenty one equations: $\boldsymbol{\sigma}(6)$, $\boldsymbol{S}(6)$, $\boldsymbol{m}(6)$, $\boldsymbol{q}(3)$. Thus, additional nine equations are needed for closure. These are discussed in Section 5.8.

5.8. Additional Six Equations in the Mathematical Model in Lagrangian Descriptions

We need nine additional equation for closure when the microconstituent deformation is nonlinear. For the linear deformation of microconstituents, $\boldsymbol{S}^* = \boldsymbol{S}$ is advantageous in discussing additional equation. From the conservation and the balance laws, we note that microconstituent stress $\boldsymbol{S}$ only appears in the energy equation and the entropy inequality. This of course implies that if we were to solve a boundary value problem for isothermal physics in which case energy equation is not part of the mathematical model, then the microconstituent stress $\boldsymbol{S}$ is completely absent from the mathematical model. This certainly is not physical as the microconstituent deformation contributes to macro physics for stationary processes as well as evolutions. Thus, we must have another relationship which considers symmetric part of $\boldsymbol{S}$ and symmetric part of $\boldsymbol{\sigma}^*$. We return back to $\boldsymbol{S}^*$.

Eringen [7]-[23] [32] and those following his work suggest that in the derivations of balance of angular momenta (to obtain additional six equation), the permutation tensor must be dropped to obtain another balance law, moments of symmetric parts of $\boldsymbol{S}^*$ and $\boldsymbol{\sigma}^*$ (nonsymmetric) that must balance with gradients of the symmetric part of the moment tensor. In Eringen's work, the non-symmetric moment tensor also contains permutation tensor in balance of angular momenta hence yields three equations. Sum of symmetric parts of $\boldsymbol{S}^*$ and $\boldsymbol{\sigma}^*$ are balanced by gradient of the symmetric part of third rank moment tensor yields another six equations in Eringen's work. Thus, Eringen [7]-[23] [32] suggests that balance of angular momenta, containing skew symmetric part of $\boldsymbol{\sigma}^*$ and gradients of skew symmetric third rank moment tensor providing three equations and other six are due to moments of symmetric part of $\boldsymbol{S}^*$ and symmetric part of $\boldsymbol{\sigma}^*$, balanced by the gradients of the symmetric part of the moment tensor, providing a total of nine equations that are suitable for the nine components of micro deformation gradient tensor $\boldsymbol{J}^{(\alpha)}$. In Section 6.6, Eringen's linear and nonlinear micromorphic theories are discussed and compared to the micromorphic theory presented in this paper to point out differences between the two and evaluate their thermodynamic and mathematical consistency. It is shown in this section that this approach of obtaining additional six equations is not valid.

First, we note that the definition of moment tensor is in error [7]-[23] [32] as it can not be defined using $\boldsymbol{\sigma}^{(\alpha)}$ that is due to classical mechanics. Nonsymmetry of Cauchy moment tensor because of not using balance of moment of moments balance law results in spurious constitutive theories for the moment tensor as well as invalid balance of angular momenta and balance of moments of ${}_s\boldsymbol{\sigma}^*$ and ${}_s\boldsymbol{S}^*$ balanced by gradients of the symmetric parts of the moment tensor.

In the work presented in this paper, we show that the additional six equations needed for closure of the mathematical model are already present implicitly in the balance of angular momenta. In the following we show how to explicitly extract these from the balance of angular momenta. Thus, in our work presented here, no additional balance law is necessary as advocated by Eringen.

5.8.1. Derivation of Additional Six Equations

The derivation of balance of angular momenta leading to Equation (172) we note that both $\boldsymbol{\sigma}^*$ and $\boldsymbol{S}^*$ have nine independent components, three skew symmetric and six symmetric. However, the presence of permutation tensor on the left side of Equation (172) forces us to discard six symmetric components of $\boldsymbol{\sigma}^*$ and $\boldsymbol{S}^*$. This observation suggests that the relationship between ${}_s\boldsymbol{\sigma}^*$ and ${}_s\boldsymbol{S}^*$ is present in (172), but we are not able to obtain it explicitly due to the permutation tensor. This necessitates that we eliminate ϵ_{mkn} from the left side of Equation (172). This can be accomplished by premultiplying (172) by $(\epsilon_{mkn})^{-1}$, inverse of ϵ_{mkn}, symbolically we can write

$$(\epsilon_{mkn})^{-1}(\epsilon_{mkn})\left(\sigma^*_{mk} \pm S^*_{mk}\right) + (\epsilon_{mkn})^{-1} m^*_{ln,l} = 0 \tag{176}$$

or

$$\sigma^*_{mk} \pm S^*_{mk} + (\epsilon_{mkn})^{-1} m^*_{ln,l} = 0 \tag{177}$$

Consider additive decompositions of σ^*_{mk} and S^*_{mk} and note that inverse of ϵ_{mkn} (for values 1, 2, 3 for mkn) is ϵ_{mkn}, we can write (177) as

$$\left({}_a\sigma^*_{mk} \pm {}_aS^*_{mk}\right) + {}_s\sigma^*_{mk} \pm {}_sS^*_{mk} + \epsilon_{mkn} m^*_{ln,l} = 0 \tag{178}$$

since

$${}_a\sigma^*_{mk} \pm {}_aS^*_{mk} + \epsilon_{mkn} m^*_{ln,l} = 0 \tag{179}$$

Equation (178) reduces to

$${}_s\sigma^*_{mk} \pm {}_sS^*_{mk} = 0 \tag{180}$$

At this point, the choice of negative sign in (180) is physical as it would suggest that the symmetric part of $\boldsymbol{\sigma}^*$ and $\boldsymbol{S}^*$ are equal *i.e.* they balance. Thus, we can write the following for (180)

$${}_s\sigma^*_{mk} - {}_sS^*_{mk} = 0 \tag{181}$$

5.8.2. Further Explanation of Using Identity for Case 3 in Balance of Angular Momenta and the Negative Sign for S^* in (181)

In Section 5.3, the third form of balance of angular momenta (case 3) makes use of identity (69). The consequence of this is that in the resulting balance of angular momenta $\boldsymbol{S}^*$ has negative sign. Even though the rationale for (181) given above is perfectly valid, but a stronger support in favor of (181) is more desirable. In classical continuum mechanics, the balance of angular momenta (consider infinitesimal theory for illustration purposes) $\epsilon_{ijk}\bar{\sigma}_{ij} = \epsilon_{ijk}\sigma_{ij} = 0$ is simply a statement that establishes symmetry of the Cauchy stress tensor. When this balance law is considered for microcontinuum theories, its modifications require addition of ${}_a\boldsymbol{\sigma}$ and $\boldsymbol{m}$ in the balance law, both of which are due to microcontinuum physics. This addition to the balance law is supported by classical thermodynamics. This, as we see from case 1 and case 2 in Section 5.3, the balance of angular momenta would yield (using infinitesimal deformation)

$$\epsilon_{ijk}\left(\sigma_{mk}+S_{mk}\right)+m_{ln,l}=0 \tag{182}$$

We note that in (182) ${}_s\boldsymbol{\sigma}$ and ${}_s\boldsymbol{S}=\boldsymbol{S}$ are absent due to the presence of permutation tensor. We also note that (182) is purely for nonclassical physics whereas ${}_s\boldsymbol{\sigma}$ and $\boldsymbol{S}$ are due to classical continuum physics, thus their absence in (182) is natural. In the derivation presented in 5.8.1, we are extracting information related to classical continuum physics from a balance law that is purely derived for nonclassical physics. Following details of Section 5.8.1 and using (182), we will obtain the following equation containing purely classical continuum physics.

$${}_s\sigma_{mk}+S_{mk}=0 \tag{183}$$

This should not be a surprise, as (182) that holds for nonclassical physics is not sensitive to the precise relationship between ${}_s\boldsymbol{\sigma}$ and $\boldsymbol{S}$ that are related to classical physics. At this point, we realize that (183) may require modifications to describe the physics related to ${}_s\boldsymbol{\sigma}$ and $\boldsymbol{S}$. Negative sign for S_{mk} giving ((181) or the following)

$${}_s\sigma_{mk}-S_{mk}=0 \tag{184}$$

is the correct balance equation between ${}_s\boldsymbol{\sigma}$ and $\boldsymbol{S}$ between the microconstituent and the medium. The use of identity for case 3 is in fact motivated by realizing that the physics described by (184) is in fact can be derived by using the identity in case 3. We note that balance of angular momenta remains (182) or (185) is case of nonlinear kinematics of the microconstituents.

$$\epsilon_{ijk}\left(\sigma^*_{mk}+S^*_{mk}\right)+m^*_{ln,l}=0 \tag{185}$$

We reiterate that the balance of angular momenta in Section 5.3 that is for nonclassical physics, hence case 1 and case 2, are totally insensitive to the physics in ${}_s\boldsymbol{\sigma}$ and $\boldsymbol{S}$ and the relationship between them as these are due to classical continuum physics. Thus case 3 using identity is initiated to recover (184) from the balance of angular momenta. Hence, Equation (184) or (181) is a thermodynamic requirement and not ad-hoc imposed condition as it is extracted from a thermodynamic law, balance of angular momenta, case 3.

5.8.3. Consideration of Additional Three Equation

We still need additional three equation for closure of the mathematical model. Before we consider these, let us consider Eringen's nonlinear micromorphic theory. In his theory, $\boldsymbol{\sigma},\boldsymbol{m}$ are nonsymmetric tensors and the constitutive theories for $\boldsymbol{\sigma}$ and $\boldsymbol{m}$ also consider them to be nonsymmetric. In Eringen's theory, we have 40 equations: balance of linear momenta (3), balance of angular momenta (3), energy Equation (1) and constitutive theories: $\boldsymbol{\sigma}(9)$, $\boldsymbol{m}(9)$, $\boldsymbol{S}(6)$, $\boldsymbol{q}(3)$, balance of momentum (6) in 43 variables: $\boldsymbol{u}(3)$, $\boldsymbol{\sigma}(9)$, $\boldsymbol{m}(9)$, $\boldsymbol{S}(6)$, $\boldsymbol{q}(3)$, $\boldsymbol{J}^{(\alpha)}(9)$, ${}_\alpha\Theta(3)$, $\theta(1)$, hence three additional equations are needed for closure.

Eringen proposes conservation of micro inertia as a new conservation law to obtain additional three equations that provide closure to the mathematical model.

This mathematical model is what is currently used in the published works for nonlinear micromorphic microcontinuum theories. First comment related to this mathematical model is that balance of momentum and conservation of microinertia are not supported by classical thermodynamics *i.e.* classical thermodynamic has no such balance and conservation laws. Thus, appending these two laws to actual valid laws of classical thermodynamics leads to a mathematical framework that is no longer a valid thermodynamic framework. Thus, Eringen's nonlinear micromorphic theory is not a valid and thermodynamically consistent microcontinuum theory. Tensors $\boldsymbol{m}$, $\boldsymbol{\sigma}$ being nonsymmetric constitutive tensors is in violation of representation theorem, hence this microcontinuum theory contains nonphysical and mathematically invalid constitutive theories.

Returning back to our quest for obtaining additional three equations needed for closure, we find that the classical thermodynamic framework is unable to provide any further mechanism for obtaining additional three equations. At this stage, we can consider two possibilities:

1) In this first case, we can use Eringen's conservation of microinertia conservation law to obtain additional three equations needed for closure of the mathematical model. The main problem in this approach is that this conservation law is not supported by classical thermodynamics, hence the resulting mathematical model is thermodynamically inconsistent. We point out that in published work, this conservation law is routinely used when additional three equation is needed for closure. Our view is that the thermodynamic inconsistency of the resulting theory rules out the theory to be valid microcontinuum theory when this conservation law is used, hence we do not support this approach. If the thermodynamic consistency of the resulting theory is of no concern (as the case is in majority of published works), then we have a mathematical model with closure in which the microconstituents have nonlinear kinematics.

2) In the second approach, we look for an alternative in which the thermodynamic consistency of the resulting theory is preserved and the mathematical model also has closure. Since the requirement of additional nine equation is due to nonlinear kinematics of microconstituents, in this approach we only consider linear microconstituent kinematics that requires only additional six equations, thus eliminating the need for additional three equations. The linear microconstituent kinematics only requires six independent components of ${}_{s}^{d}\boldsymbol{J}^{(\alpha)}$ as microconstituent deformational degrees of freedom. The other three degrees of freedom ${}_{c}\boldsymbol{\Theta}^{(\alpha)}$ in ${}_{a}^{d}\boldsymbol{J}^{(\alpha)}$ and ${}_{a}\boldsymbol{J}^{(\alpha)}$, the classical rotations within the microconstituent volume remain as free field, hence not influencing the microconstituent deformation. Thus, now we have a nonlinear micromorphic microcontinuum theory in which the microconstituent deformation is linear, the solid medium deformation is nonlinear, and the interaction of the microconstituents with the solid medium is nonlinear. Another way to rationalize this linear deformation of microconstituents is to realize that the nonlinear deformation of microconstituents will require very high forces on their surfaces that must be generated by the sur-

rounding medium. This may not be physically possible without generating a very high strain field in the medium that may not be supported by elastic deformation of the solid medium. Perhaps the lack of means in the classical thermodynamics to obtain these three additional equations is an indication that nonlinear deformation of microconstituents is not possible in a physical theory supported by classical thermodynamics. Equation (184) clearly demonstrates the very high value of $\boldsymbol{S}$ needed for nonlinear deformation of the microconstituent must come from the surrounding medium through ${}_s\boldsymbol{\sigma}$ which the medium may not be able to support.

5.8.4. Linear Microconstituent Kinematics

Essentially to eliminate the need for three additional equations, we need to eliminate three degrees of freedom from the microconstituent deformational degrees of freedom that consists of all nine components of $\boldsymbol{J}^{(\alpha)}$. Let us assume that the microconstituents can only undergo linear deformation. In this case we have

$$ {}^{d}\boldsymbol{J}^{(\alpha)} = {}^{d}_{s}\boldsymbol{J}^{(\alpha)} + {}^{d}_{a}\boldsymbol{J}^{(\alpha)} \tag{186} $$

${}^{d}_{s}\boldsymbol{J}^{(\alpha)}$ contains the linear strain measures of the deforming microconstituents and ${}^{d}_{a}\boldsymbol{J}^{(\alpha)}$ contains three classical rigid rotations ${}_c\boldsymbol{\Theta}^{(\alpha)}$ within the volume of the microconstituents that constitutive a free field as the microconstituents isotropic and homogeneous volume of matter offers no restriction to the rotation field due to ${}_c\boldsymbol{\Theta}^{(\alpha)}$. Thus, now the microconstituent deformation is completely derived by six independent components of ${}^{d}_{s}\boldsymbol{J}^{(\alpha)}$, hence requiring only six additional equation which we have already extracted from balance of angular momenta. With this assumption, $\boldsymbol{S}^*$ changes back to $\boldsymbol{S}$, symmetric macro Cauchy stress tensor obtained using micro Cauchy stress tensor $\boldsymbol{\sigma}^{(\alpha)}$ through integral-average definitions. Balance of angular momenta now contains $\boldsymbol{S}$ (symmetric) instead of $\boldsymbol{S}^*$, hence is eliminated from it due to permutation tensor and the additional six equations (181) get modified as

$$ {}_s\sigma^*_{mk} - S_{mk} = 0 \tag{187} $$

Likewise, the conjugate pair in energy equation and entropy inequality containing $\boldsymbol{S}^* : \dot{\boldsymbol{J}}^{(\alpha)}$ is modified as $\boldsymbol{S} : \dot{\boldsymbol{\varepsilon}}^{(\alpha)}$ using

$$ \boldsymbol{\varepsilon}^{(\alpha)} = {}^{d}_{s}\boldsymbol{J}^{(\alpha)} = \frac{1}{2}\left({}^{d}\boldsymbol{J}^{(\alpha)} + \left({}^{d}\boldsymbol{J}^{(\alpha)} \right)^T \right) \tag{188} $$

The conservation and the balance laws (170) - (175) and additional equations can now be written as:

$$ \rho_0(\boldsymbol{x}) = |\boldsymbol{J}|\rho(\boldsymbol{x},t) \tag{189} $$

$$ \rho_0 a_k - \rho_0 {}^{b}F_k - \sigma_{lk,l} = 0 \tag{190} $$

$$ \epsilon_{mkn}\left(\sigma^*_{mk} \pm S^*_{mk} \right) + m^*_{ln,l} = 0 \tag{191} $$

$$ \rho_0 \dot{e} - \boldsymbol{\sigma}^* : \dot{\boldsymbol{J}} - \boldsymbol{S} : \dot{\boldsymbol{\varepsilon}}^{(\alpha)} - \nabla\cdot\boldsymbol{q} - \left({}_c\dot{\boldsymbol{\Theta}} \cdot \left(\nabla\cdot\boldsymbol{m}^* \right) + \boldsymbol{m}^* : {}^{c\Theta}\dot{\boldsymbol{J}} \right) = 0 \tag{192} $$

$$ \rho_0\left(\dot{\Phi} + \eta\dot{\theta} \right) - \boldsymbol{\sigma}^* : \dot{\boldsymbol{J}} - \boldsymbol{S} : \dot{\boldsymbol{\varepsilon}}^{(\alpha)} + \frac{\boldsymbol{q}\cdot\boldsymbol{g}}{\theta} - \left({}_c\dot{\boldsymbol{\Theta}} \cdot \left(\nabla\cdot\boldsymbol{m}^* \right) + \boldsymbol{m}^* : {}^{c\Theta}\dot{\boldsymbol{J}} \right) \le 0 \tag{193} $$

$$\epsilon_{ijk} m_{ij} = 0 \tag{194}$$

$$_s\sigma^*_{mk} - S_{mk} = 0 \tag{195}$$

in which $S_{mk} = S_{km}$.

Remarks

1) In the nonlinear micromorphic microcontinuum theory, the microconstituent deformation is linear because with nonlinear microconstituent deformation, a thermodynamically valid nonlinear micromorphic theory is not possible.

2) There is another physical reasoning related to microconstituent deformation that strongly supports consideration of linear microdeformation in the theory. For microconstituents to have nonlinear deformation, the microconstituents must have very high unrealistic forces acting on these surfaces that are exerted by the surrounding medium that may in general be unrealistic for the surrounding medium, thus lack of support for the nonlinear deformation of the microconstituents.

6. Constitutive Theories for Nonlinear Micromorphic Elastic Solid

6.1. Constitutive Tensors and Their Argument Tensors

The initial determination of constitutive tensors and their argument tensor is made by considering conjugate pair in the entropy inequality and the axiom of causality. This choice may be augmented or altered depending upon the desired physics. Once the constitutive tensor and the argument tensors are established, we follow theory of isotropic tensors *i.e.* representation theorem [36] [62]-[73] for deriving the constitutive theories and the standard procedure based on Taylor series expansion for determining the material coefficients from the linear combinations of the combined generators. Consider entropy inequality (174)

$$\rho_0\left(\dot{\Phi} + \eta\dot{\theta}\right) - \boldsymbol{\sigma}^* : \dot{\boldsymbol{J}} - \boldsymbol{S} : \dot{\boldsymbol{\varepsilon}}^{(\alpha)} + \frac{\boldsymbol{q}\cdot\boldsymbol{g}}{\theta} - \left({}_c\dot{\boldsymbol{\Theta}} \cdot \left(\nabla\cdot\boldsymbol{m}^*\right) + \boldsymbol{m}^* : {}^{c\Theta}\dot{\boldsymbol{J}}\right) \leq 0 \tag{196}$$

In (196) we realize that $\boldsymbol{\sigma}^*, \boldsymbol{J}, \boldsymbol{S}^*, \boldsymbol{m}^*, {}^{c\Theta}\boldsymbol{J}$ are not valid measures for finite deformation, finite strain physics, but we continue to illustrate some important points. First, these are all nonsymmetric tensors of rank two hence can neither be constitutive tensors, nor argument tensors, but $\boldsymbol{q}$ and $\boldsymbol{g}$ are admissible as these are tensors of rank one. Thus,

$$\boldsymbol{q} = \boldsymbol{q}(\boldsymbol{g},\theta); \boldsymbol{\sigma}^* \neq \boldsymbol{\sigma}^*(\boldsymbol{J},\theta); \boldsymbol{S} \neq \boldsymbol{S}\left(\boldsymbol{\varepsilon}^{(\alpha)},\theta\right); \boldsymbol{m}^* \neq \boldsymbol{m}^*\left({}^{c\Theta}\boldsymbol{J},\theta\right) \tag{197}$$

Through additive decomposition, all nonsymmetric tensors must expressed as sum of symmetric and skew symmetric tensor, followed by simplification so that valid conjugate pairs can be established. Additionally, the last term in (196) must also be addressed. We present details in the following. From balance of linear momenta

$$\nabla\cdot\boldsymbol{m}^* = -\epsilon : \boldsymbol{\sigma}^* \tag{198}$$

Using (198), the last term in (196) can be expressed as

$$ {}_{c}\dot{\Theta}\cdot\left(\nabla\cdot\boldsymbol{m}^{*}\right)=-{}_{c}\dot{\Theta}\cdot\left(\epsilon:\boldsymbol{\sigma}^{*}\right) \tag{199}$$

A simple calculation shows that

$$ {}_{c}\dot{\Theta}\cdot\left(\epsilon:\boldsymbol{\sigma}^{*}\right)={}_{a}\boldsymbol{\sigma}^{*}:\dot{\boldsymbol{J}}=\boldsymbol{\sigma}^{*}:{}_{a}\dot{\boldsymbol{J}} \tag{200}$$

Using (200) in (199)

$$ {}_{c}\dot{\Theta}\cdot\left(\nabla\cdot\boldsymbol{m}^{*}\right)=-\boldsymbol{\sigma}^{*}:{}_{a}\dot{\boldsymbol{J}} \tag{201}$$

Following reference [58], we have

$$ \boldsymbol{\sigma}^{*}:\dot{\boldsymbol{J}}={}_{s}\boldsymbol{\sigma}^{[0]}:\dot{\boldsymbol{\varepsilon}}_{[0]}+\boldsymbol{\sigma}^{*}:{}_{a}\dot{\boldsymbol{J}}={}_{s}\boldsymbol{\sigma}^{[0]}:\dot{\boldsymbol{\varepsilon}}_{[0]}+{}_{a}\boldsymbol{\sigma}^{*}:{}_{a}\dot{\boldsymbol{J}} \tag{202}$$

Substituting (201) and (202) in (196) and using $\boldsymbol{m}^{*}=\boldsymbol{m}^{[0]}\cdot\boldsymbol{J}^{T}$

$$ \begin{aligned}&\rho_0\left(\dot{\Phi}+\eta\dot{\theta}\right)+\frac{\boldsymbol{q}\cdot\boldsymbol{g}}{\theta}-{}_{s}\boldsymbol{\sigma}^{[0]}:\dot{\boldsymbol{\varepsilon}}_{[0]}-\boldsymbol{S}:\dot{\boldsymbol{\varepsilon}}^{(\alpha)}-\boldsymbol{m}^{[0]}:\left(\boldsymbol{J}^{T}\cdot\left({}^{c\Theta}\dot{\boldsymbol{J}}\right)\right)\\&-{}_{a}\boldsymbol{\sigma}^{*}:{}_{a}\dot{\boldsymbol{J}}-\left(-{}_{a}\boldsymbol{\sigma}^{*}:{}_{a}\dot{\boldsymbol{J}}\right)\le 0\end{aligned} \tag{203}$$

Or

$$ \rho_0\left(\dot{\Phi}+\eta\dot{\theta}\right)+\frac{\boldsymbol{q}\cdot\boldsymbol{g}}{\theta}-{}_{s}\boldsymbol{\sigma}^{[0]}:\dot{\boldsymbol{\varepsilon}}_{[0]}-\boldsymbol{S}:\dot{\boldsymbol{\varepsilon}}^{(\alpha)}-\boldsymbol{m}^{[0]}:\left(\boldsymbol{J}^{T}\cdot\left({}^{c\Theta}\dot{\boldsymbol{J}}\right)\right)\le 0 \tag{204}$$

We further note that volumetric deformation and distortion deformation of the solid medium that are naturally exclusive, hence cannot be described by a single constitutive theory for ${}_{s}\boldsymbol{\sigma}^{[0]}$. Thus, we must consider additive decomposition of ${}_{s}\boldsymbol{\sigma}^{[0]}$ into equilibrium and deviatoric stresses (${}^{e}_{s}\boldsymbol{\sigma}^{[0]}$ and ${}^{d}_{s}\boldsymbol{\sigma}^{[0]}$). Constitutive theory for ${}^{e}_{s}\boldsymbol{\sigma}^{[0]}$ describes volumetric deformation and the constitutive theory for ${}^{d}_{s}\boldsymbol{\sigma}^{[0]}$ describes distortional deformation.

$$ {}_{s}\boldsymbol{\sigma}^{[0]}={}^{e}_{s}\boldsymbol{\sigma}^{[0]}+{}^{d}_{s}\boldsymbol{\sigma}^{[0]} \tag{205}$$

Using (205) in (204)

$$ \rho_0\left(\dot{\Phi}+\eta\dot{\theta}\right)+\frac{\boldsymbol{q}\cdot\boldsymbol{g}}{\theta}-{}^{e}_{s}\boldsymbol{\sigma}^{[0]}:\dot{\boldsymbol{\varepsilon}}_{[0]}-{}^{d}_{s}\boldsymbol{\sigma}^{[0]}:\dot{\boldsymbol{\varepsilon}}_{[0]}-\boldsymbol{S}:\dot{\boldsymbol{\varepsilon}}^{(\alpha)}-\boldsymbol{m}^{[0]}:\left(\boldsymbol{J}^{T}\cdot\left({}^{c\Theta}\dot{\boldsymbol{J}}\right)\right)\le 0 \tag{206}$$

Since $\boldsymbol{m}^{[0]}$ is symmetric, last term in (206) can be simplified

$$ \begin{aligned}\boldsymbol{m}^{[0]}:\left(\boldsymbol{J}^{T}\cdot\left({}^{c\Theta}\dot{\boldsymbol{J}}\right)\right)&=\frac{1}{2}\boldsymbol{m}^{[0]}:\left(\boldsymbol{J}^{T}\cdot\left({}^{c\Theta}\dot{\boldsymbol{J}}\right)+\left({}^{c\Theta}\dot{\boldsymbol{J}}\right)^{T}\cdot\boldsymbol{J}\right)\\&=\boldsymbol{m}^{[0]}:{}^{c\Theta}\dot{\boldsymbol{\varepsilon}}_{[0]}\end{aligned} \tag{207}$$

where

$$ {}^{c\Theta}\dot{\boldsymbol{\varepsilon}}_{[0]}=\frac{1}{2}\left(\left[\boldsymbol{J}\right]^{T}\cdot\left[{}^{c\Theta}\dot{\boldsymbol{J}}\right]+\left[{}^{c\Theta}\dot{\boldsymbol{J}}\right]^{T}\cdot\boldsymbol{J}\right) \tag{208}$$

$$ \begin{aligned}{}^{c\Theta}\dot{\boldsymbol{\varepsilon}}_{[0]}&=\frac{1}{2}\left({}_{s}\boldsymbol{J}-{}_{a}\boldsymbol{J}\right)\cdot\left({}^{c\Theta}_{s}\dot{\boldsymbol{J}}+{}^{c\Theta}_{a}\dot{\boldsymbol{J}}\right)+\left({}^{c\Theta}_{s}\dot{\boldsymbol{J}}-{}^{c\Theta}_{a}\dot{\boldsymbol{J}}\right):\left({}_{s}\boldsymbol{J}+{}_{a}\boldsymbol{J}\right)\\&={}_{s}\boldsymbol{J}\cdot{}^{c\Theta}_{s}\dot{\boldsymbol{J}}-{}_{a}\boldsymbol{J}\cdot{}^{c\Theta}_{a}\dot{\boldsymbol{J}}\end{aligned} \tag{209}$$

Now we can rewrite (206) using (209)

$$ \rho_0\left(\dot{\Phi}+\eta\dot{\theta}\right)+\frac{\boldsymbol{q}\cdot\boldsymbol{g}}{\theta}-{}^{e}_{s}\boldsymbol{\sigma}^{[0]}:\dot{\boldsymbol{\varepsilon}}_{[0]}-{}^{d}_{s}\boldsymbol{\sigma}^{[0]}:\dot{\boldsymbol{\varepsilon}}_{[0]}-\boldsymbol{S}:\dot{\boldsymbol{\varepsilon}}^{(\alpha)}-\boldsymbol{m}^{[0]}:\left({}^{c\Theta}\dot{\boldsymbol{\varepsilon}}_{[0]}\right)\le 0 \tag{210}$$

in which ${}^{c\Theta}\dot{\boldsymbol{\varepsilon}}_{[0]}$ is given by (209).

In (210) all tensors of rank two in the rate of work conjugate pairs are symmetric tensors, hence conjugate pairs in (210) are suitable for representation theorem. Choice of ${}_s^e\boldsymbol{\sigma}^{[0]}, {}_s^d\boldsymbol{\sigma}^{[0]}, \boldsymbol{S}^{[0]}, \boldsymbol{q}, \boldsymbol{m}^{[0]}$ as constitutive tensors is admissible based on axiom of causality and their possible choices for argument tensors based on conjugate pairs in (210) can made as follows:

$$ {}_s^e\boldsymbol{\sigma}^{[0]} = {}_s^e\boldsymbol{\sigma}^{[0]}(\rho,\theta) \tag{211} $$

$$ {}_s^d\boldsymbol{\sigma}^{[0]} = {}_s^d\boldsymbol{\sigma}^{[0]}\left(\boldsymbol{\varepsilon}_{[0]},\theta\right) \tag{212} $$

$$ \boldsymbol{S} = \boldsymbol{S}\left(\boldsymbol{\varepsilon}^{(\alpha)},\theta\right) \tag{213} $$

$$ \boldsymbol{m}^{[0]} = \boldsymbol{m}^{[0]}\left({}^{c\Theta}\boldsymbol{\varepsilon}_{[0]},\theta\right) \tag{214} $$

$$ \boldsymbol{q} = \boldsymbol{q}(\boldsymbol{g},\theta) \tag{215} $$

Even though we do not need constitutive theories for Φ and η but their argument tensors are necessary as Φ is used to simplify entropy inequality (210). Except ρ and θ, we do not have a basis for choosing argument tensors of Φ and η, so we use axiom of equipresence.

$$ \Phi = \Phi\left(\rho,\boldsymbol{\varepsilon}_{[0]},\boldsymbol{\varepsilon}^{(\alpha)},{}^{c\Theta}\boldsymbol{\varepsilon}_{[0]},\boldsymbol{q},\theta\right) \tag{216} $$

$$ \eta = \eta\left(\rho,\boldsymbol{\varepsilon}_{[0]},\boldsymbol{\varepsilon}^{(\alpha)},{}^{c\Theta}\boldsymbol{\varepsilon}_{[0]},\boldsymbol{q},\theta\right) \tag{217} $$

From the physics of pure volumetric deformation we know that equilibrium stress must be a function of density and temperature, hence the choice of the argument tensor in (211). In Lagrangian description ρ is not a dependent variable, hence in general ρ can not be used as an argument tensor. The use of ρ as an argument tensor in (211) is symbolic.

6.2. Constitutive Theory for Equilibrium Cauchy Stress Tensor ${}_s^e\boldsymbol{\sigma}^{[0]}$

Compressibility, hence density in solids is controlled by $|\boldsymbol{J}|$ and the density in the current configuration is deterministic through conservation of mass when $\boldsymbol{J}$ is known. Thus, density is not a dependent variable in the conservation and balance laws in Lagrangian description for solid matter. The equation of state in solid matter is a consequence of density change caused due to $|\boldsymbol{J}|$ *i.e.* there is a pressure field associated with density change. The presence of this pressure field through equilibrium stress in the balance of linear momenta is essential for correct force balance. Thus, in compressible solid matter one could determine solution of the mathematical model without using the equation of state, but such solution would be in error due to incorrect force balance as the balance of linear momenta. Since the compressibility physics depends upon density and temperature the constitutive theory for ${}_s^e\boldsymbol{\sigma}^{[0]}$ must be obtained using the constitutive theory for ${}^e\boldsymbol{\sigma}^{(0)}$, equilibrium Cauchy stress tensor. Details of this derivation can be found in a recent paper [2] and references [56] [57], the final form of the constitutive theories for ${}^{e}_{s}\boldsymbol{\sigma}^{[0]}$ for compressible and incompressible non isothermal physics of solid medium are given by

$$ {}^{e}_{s}\boldsymbol{\sigma}^{[0]} = |\boldsymbol{J}|(\boldsymbol{J})^{-1}\cdot p(\rho,\theta)\boldsymbol{\delta}\cdot\left((\boldsymbol{J})^{-1}\right)^{T} = |\boldsymbol{J}|\,p(\rho,\theta)\boldsymbol{\delta}\left(\boldsymbol{J}^{T}\cdot\boldsymbol{J}\right)^{-1} \quad \text{compressible} \tag{218} $$

$$ {}^{d}_{s}\boldsymbol{\sigma}^{[0]} = |\boldsymbol{J}|(\boldsymbol{J})^{-1}\cdot p(\theta)\boldsymbol{\delta}\cdot\left((\boldsymbol{J})^{-1}\right)^{T} = |\boldsymbol{J}|\,p(\theta)\boldsymbol{\delta}\left(\boldsymbol{J}^{T}\cdot\boldsymbol{J}\right)^{-1} \quad \text{incompressible} \tag{219} $$

in which $p(\rho,\theta)$ and $p(\theta)$ are thermodynamic and mechanical pressures. In (219) we could have used $\boldsymbol{J}\simeq\boldsymbol{I}$ and $|\boldsymbol{J}|\simeq 1$, but leave the expression in (219) as they are. The reduced form of the entropy inequality (210) (after deriving constitutive theory for ${}^{e}_{s}\boldsymbol{\sigma}^{[0]}$) in Lagrangian description can be written as

$$ \frac{\boldsymbol{q}\cdot\boldsymbol{g}}{\theta} - {}^{d}_{s}\boldsymbol{\sigma}^{[0]} : \dot{\boldsymbol{\varepsilon}}_{[0]} - \boldsymbol{S} : \dot{\boldsymbol{\varepsilon}}_{[0]}^{(\alpha)} - \boldsymbol{m}^{[0]} : \left({}^{c}{}^{\Theta}\dot{\boldsymbol{\varepsilon}}_{[0]}\right) \le 0 \tag{220} $$

6.3. Constitutive Theory for ${}^{d}_{s}\boldsymbol{\sigma}^{[0]}$

Constitutive theory for ${}^{d}_{s}\boldsymbol{\sigma}^{[0]}$ must address distortional deformation physics (without volumetric change). Using (212) with $\boldsymbol{\varepsilon}_{[0]}$ and θ as argument tensors of ${}^{d}_{s}\boldsymbol{\sigma}^{[0]}$, we can derive the constitutive theory for ${}^{d}_{s}\boldsymbol{\sigma}^{[0]}$ using representation theorem. Let ${}^{\sigma}\underset{\sim}{\boldsymbol{G}}^{i}; i=1,2,\cdots,N^{\sigma}$ be the combined generators of the argument tensors of ${}^{d}_{s}\boldsymbol{\sigma}^{[0]}$ in (212) that are symmetric tensors of rank two. Then, $\boldsymbol{I}, {}^{\sigma}\underset{\sim}{\boldsymbol{G}}^{i}; i=1,2,\cdots,N^{\sigma}$ constitute the basis (integrity) of the space of tensor ${}^{d}_{s}\boldsymbol{\sigma}^{[0]}$. Thus, ${}^{d}_{s}\boldsymbol{\sigma}^{[0]}$ can be expressed as a linear combination of the basis (in the current configuration).

$$ {}^{d}_{s}\boldsymbol{\sigma}^{[0]} = {}^{\sigma}\alpha^{0}\boldsymbol{I} + \sum_{i=1}^{N^{\sigma}} {}^{\sigma}\alpha^{i}\left({}^{\sigma}\underset{\sim}{\boldsymbol{G}}^{i}\right) \tag{221} $$

$$ {}^{\sigma}\alpha^{i} = {}^{\sigma}\alpha^{i}\left({}^{\sigma}\underset{\sim}{I}^{j},\theta\right); i=0,1,\cdots,N^{\sigma}; j=1,2,\cdots,M^{\sigma} \tag{222} $$

in which ${}^{\sigma}\underset{\sim}{I}^{j}; j=1,2,\cdots,M^{\sigma}$ are combined invariants of the argument tensors of ${}^{d}_{s}\boldsymbol{\sigma}^{[0]}$ in (212). The material coefficients in (221) are determined by expanding ${}^{\sigma}\underset{\sim}{\alpha}^{i}; i=0,1,\cdots,N^{\sigma}$ in Taylor series in ${}^{\sigma}\underset{\sim}{I}^{j}; j=1,2,\cdots,M^{\sigma}$ and the temperature θ about a known configuration $\underset{\sim}{\Omega}$ and only retaining up to linear terms in ${}^{\sigma}\underset{\sim}{I}^{j}; j=1,2,\cdots,M^{\sigma}$ and temperature θ. (for the sake of simplicity of the resulting theory)

$$ {}^{\sigma}\alpha^{i} = {}^{\sigma}\alpha^{i}\Big|_{\underset{\sim}{\Omega}} + \sum_{j=1}^{M^{\sigma}} \frac{\partial\,{}^{\sigma}\alpha^{i}}{\partial\,{}^{\sigma}\underset{\sim}{I}^{j}}\Bigg|_{\underset{\sim}{\Omega}} \left({}^{\sigma}\underset{\sim}{I}^{j} - {}^{\sigma}\underset{\sim}{I}^{j}\Big|_{\underset{\sim}{\Omega}}\right) + \frac{\partial\,{}^{\sigma}\alpha^{i}}{\partial\theta}\Bigg|_{\underset{\sim}{\Omega}} \left(\theta - \theta\big|_{\underset{\sim}{\Omega}}\right); i=0,1,\cdots,N^{\sigma} \tag{223} $$

We substitute ${}^{\sigma}\alpha^{0}$ and ${}^{\sigma}\alpha^{i}; i=1,2,\cdots,N^{\sigma}$ from (223) into (221)

$$ \begin{aligned} {}^{d}_{s}\boldsymbol{\sigma}^{[0]} = {} & \left({}^{\sigma}\alpha^{0}\Big|_{\underset{\sim}{\Omega}} + \sum_{j=1}^{M^{\sigma}} \frac{\partial\,{}^{\sigma}\alpha^{0}}{\partial\,{}^{\sigma}\underset{\sim}{I}^{j}}\Bigg|_{\underset{\sim}{\Omega}} \left({}^{\sigma}\underset{\sim}{I}^{j} - {}^{\sigma}\underset{\sim}{I}^{j}\Big|_{\underset{\sim}{\Omega}}\right) + \frac{\partial\,{}^{\sigma}\alpha^{0}}{\partial\theta}\Bigg|_{\underset{\sim}{\Omega}} \left(\theta - \theta\big|_{\underset{\sim}{\Omega}}\right)\right)\boldsymbol{I} \\ & + \sum_{i=1}^{N^{\sigma}} \left({}^{\sigma}\alpha^{i}\Big|_{\underset{\sim}{\Omega}} + \sum_{j=1}^{M^{\sigma}} \frac{\partial\,{}^{\sigma}\alpha^{i}}{\partial\,{}^{\sigma}\underset{\sim}{I}^{j}}\Bigg|_{\underset{\sim}{\Omega}} \left({}^{\sigma}\underset{\sim}{I}^{j} - {}^{\sigma}\underset{\sim}{I}^{j}\Big|_{\underset{\sim}{\Omega}}\right) + \frac{\partial\,{}^{\sigma}\alpha^{i}}{\partial\theta}\Bigg|_{\underset{\sim}{\Omega}} \left(\theta - \theta\big|_{\underset{\sim}{\Omega}}\right)\right) {}^{\sigma}\underset{\sim}{\boldsymbol{G}}^{i} \end{aligned} \tag{224} $$

Collecting coefficients of $\boldsymbol{I}, {}^{\sigma}\underset{\sim}{I}^{j}\boldsymbol{I}, {}^{\sigma}\underset{\sim}{\boldsymbol{G}}^{i}, {}^{\sigma}\underset{\sim}{I}^{j}\,{}^{\sigma}\underset{\sim}{\boldsymbol{G}}^{i}, \left(\theta-\theta\big|_{\underset{\sim}{\Omega}}\right){}^{\sigma}\underset{\sim}{\boldsymbol{G}}^{i}$ and $\left(\theta-\theta\big|_{\underset{\sim}{\Omega}}\right)\boldsymbol{I}$

in (224), we can write (224) as follows:

$$\begin{aligned} {}_s^d\boldsymbol{\sigma}^{[0]} &= \sigma_0 \boldsymbol{I} + \sum_{j=1}^{M^\sigma} {}^\sigma \underset{\sim}{a}_j \left({}^\sigma \underset{\sim}{I}^j \right) \boldsymbol{I} + \sum_{i=1}^{N^\sigma} {}^\sigma \underset{\sim}{b}_i \left({}^\sigma \underset{\sim}{\boldsymbol{G}}^i \right) + \sum_{j=1}^{M^\sigma} \sum_{i=1}^{N^\sigma} {}^\sigma \underset{\sim}{c}_{ij} \left({}^\sigma \underset{\sim}{I}^j \right) \left({}^\sigma \underset{\sim}{\boldsymbol{G}}^i \right) \\ &- \sum_{i=1}^{N^\sigma} {}^\sigma \underset{\sim}{d}_i \left(\theta - \theta|_{\underset{\sim}{\Omega}} \right) \left({}^\sigma \underset{\sim}{\boldsymbol{G}}^i \right) - {}^\sigma \alpha_{tm} \left(\theta - \theta|_{\underset{\sim}{\Omega}} \right) \boldsymbol{I} \end{aligned} \tag{225}$$

The material coefficients ${}^\sigma \underset{\sim}{a}_j, {}^\sigma \underset{\sim}{b}_i, {}^\sigma \underset{\sim}{c}_{ij}, {}^\sigma \underset{\sim}{d}_i$ and ${}^\sigma \alpha_{tm}$ are defined in the following:

$$\begin{aligned} &\sigma_0 = {}^\sigma \alpha^0 \big|_{\underset{\sim}{\Omega}} - \sum_{j=1}^{M^\sigma} \left. \frac{\partial \left({}^\sigma \alpha^0 \right)}{\partial \left({}^\sigma \underset{\sim}{I}^j \right)} \right|_{\underset{\sim}{\Omega}} \left(- {}^\sigma \underset{\sim}{I}^j \big|_{\underset{\sim}{\Omega}} \right); \quad {}^\sigma \underset{\sim}{a}_j = \left. \frac{\partial \left({}^\sigma \alpha^0 \right)}{\partial \left({}^\sigma \underset{\sim}{I}^j \right)} \right|_{\underset{\sim}{\Omega}} \\ &{}^\sigma \underset{\sim}{b}_i = {}^\sigma \alpha^i \big|_{\underset{\sim}{\Omega}} + \sum_{j=1}^{M^\sigma} \left. \frac{\partial \left({}^\sigma \alpha^i \right)}{\partial \left({}^\sigma \underset{\sim}{I}^j \right)} \right|_{\underset{\sim}{\Omega}} \left(- {}^\sigma \underset{\sim}{I}^j \big|_{\underset{\sim}{\Omega}} \right); \quad {}^\sigma \underset{\sim}{c}_{ij} = \left. \frac{\partial \left({}^\sigma \alpha^i \right)}{\partial \left({}^\sigma \underset{\sim}{I}^j \right)} \right|_{\underset{\sim}{\Omega}} \\ &{}^\sigma \underset{\sim}{d}_i = - \left. \frac{\partial \left({}^\sigma \alpha^i \right)}{\partial \theta} \right|_{\underset{\sim}{\Omega}}; \quad {}^\sigma \alpha_{tm} = - \left. \frac{\partial \left({}^\sigma \alpha^0 \right)}{\partial \theta} \right|_{\underset{\sim}{\Omega}} \\ &i = 0, 1, \cdots, N^\sigma; \quad j = 1, 2, \cdots, M^\sigma \end{aligned} \tag{226}$$

The constitutive theory (225) is based on integrity (complete basis of the space of constitutive tensor ${}_s^d\boldsymbol{\sigma}^{[0]}$) and requires ($2N^\sigma + \left(M^\sigma\right)\left(N^\sigma\right) + M^\sigma + 1$) material coefficients. Various simplified forms of this constitutive theory can be derived from (225) by choosing desired generators and invariants. Based on (215), in this constitutive theory $N^\sigma = 2$, $M^\sigma = 3$ and ${}^\sigma\boldsymbol{G}^1 = \boldsymbol{\varepsilon}_{[0]}$, ${}^\sigma\boldsymbol{G}^2 = \left(\boldsymbol{\varepsilon}_{[0]}\right)^2$ and the three invariants are $I_{\varepsilon_{[0]}}$, $II_{\varepsilon_{[0]}}$, and $III_{\varepsilon_{[0]}}$. Most simplified form of the linear constitutive theory is obtained for $N^\sigma = 1$ (after redefining material coefficients)

$${}_s^d\boldsymbol{\sigma}^{[0]} = \sigma^0 \boldsymbol{I} + 2\mu^\sigma \boldsymbol{\varepsilon}_{[0]} + \lambda^\sigma \left(\mathrm{tr}\boldsymbol{\varepsilon}_{[0]} \right) \boldsymbol{I} - {}^\sigma \alpha_{tm} \left(\theta - \theta|_{\underset{\sim}{\Omega}} \right) \boldsymbol{I} \tag{227}$$

The material coefficients could be functions of all three invariant and θ in a known configuration $\underset{\sim}{\Omega}$.

6.4. Constitutive Theory for Micro Stress Tensor $\boldsymbol{S}$

Consider (213), *i.e.*

$$\boldsymbol{S} = \boldsymbol{S}\left(\boldsymbol{\varepsilon}^{(\alpha)}, \theta\right) \tag{228}$$

in which $\boldsymbol{\varepsilon}^{(\alpha)}$ is the Green's strain tensor for the microconstituent (explained at the end of the derivation). Let ${}^s\underset{\sim}{\boldsymbol{G}}^i; i = 1, 2, \cdots, N^s$ be the combined generators of the argument tensors of $\boldsymbol{S}$ in (228) that are symmetric tensors of tank two and ${}^s\underset{\sim}{I}^j; j = 1, 2, \cdots, M^s$ be the combined invariants of the same argument tensors of $\boldsymbol{S}$ in (228), then $\boldsymbol{I}, {}^s\underset{\sim}{\boldsymbol{G}}^i; i = 1, 2, \cdots, N^s$ constitutes the basis of the space of constitutive tensor $\boldsymbol{S}$, hence we can express $\boldsymbol{S}$ as a linear combination of the basis (integrity).

$$\boldsymbol{S} = {}^s\alpha^0 \boldsymbol{I} + \sum_{i=1}^{N^s} {}^s\alpha^i \left({}^s\underset{\sim}{\boldsymbol{G}}^i \right) \tag{229}$$

in which the coefficient ${}^s\alpha^i; i = 1,2,\cdots,N^s$ is the linear combination (229) can be functions of ${}^s\underset{\sim}{I}^j; j = 1,2,\cdots,M^s$ and temperature θ. Material coefficients in (229) are determined using exactly same approach as described and used for ${}^d_s\boldsymbol{\sigma}$ in Section 6.2. Expanding ${}^s\alpha^i; i = 1,2,\cdots,N^s$ in Taylor series in ${}^s\underset{\sim}{I}^j; j = 1,2,\cdots,M^s$ and θ about a known configuration $\underset{\sim}{\Omega}$ and retaining only up to linear terms in ${}^s\underset{\sim}{I}^j; j = 1,2,\cdots,M^s$ and θ, substituting these in (229) and collecting coefficients of $\boldsymbol{I}, {}^s\underset{\sim}{I}^j\boldsymbol{I}, {}^s\underset{\sim}{\boldsymbol{G}}^i, {}^s\underset{\sim}{I}^j\left({}^s\underset{\sim}{\boldsymbol{G}}^i\right), \left(\theta-\theta|_{\underset{\sim}{\Omega}}\right){}^s\underset{\sim}{\boldsymbol{G}}^i$ and $\left(\theta-\theta|_{\underset{\sim}{\Omega}}\right)\boldsymbol{I}$, we can obtain.

$$\begin{aligned}\boldsymbol{S} = {} & S_0\boldsymbol{I} + \sum_{j=1}^{M^s} {}^s\underset{\sim}{a}_j\left({}^s\underset{\sim}{I}^j\right)\boldsymbol{I} + \sum_{i=1}^{N^s} {}^s\underset{\sim}{b}_i\left({}^s\underset{\sim}{\boldsymbol{G}}^i\right) + \sum_{i=1}^{N^s}\sum_{j=1}^{M^s} {}^s\underset{\sim}{c}_{ij}\left({}^sI^j\right)\left({}^s\underset{\sim}{\boldsymbol{G}}^i\right) \\ & - \sum_{i=1}^{N^s} {}^s\underset{\sim}{d}_i\left(\theta-\theta|_{\underset{\sim}{\Omega}}\right){}^s\underset{\sim}{\boldsymbol{G}}^i - {}^s\alpha_{tm}\left(\theta-\theta|_{\underset{\sim}{\Omega}}\right)\boldsymbol{I}\end{aligned} \tag{230}$$

The Taylor series expansion of ${}^s\alpha^i; j = 0,1,\cdots,N^s$ can be obtained from (223) by replacing ${}^\sigma\alpha^i$ with ${}^s\alpha^i$, ${}^\sigma\underset{\sim}{I}^j$ with ${}^s\underset{\sim}{I}^j$, N^σ and M^σ with N^s and M^s. The material coefficients ${}^s\underset{\sim}{a}_j, {}^s\underset{\sim}{b}_i, {}^s\underset{\sim}{c}_{ij}, {}^s\underset{\sim}{d}_i$ and ${}^s\alpha_{tm}$ can be obtained using (226) be replacing ${}^\sigma\alpha^i$ with ${}^s\alpha^i$, ${}^\sigma\underset{\sim}{I}^j$ into ${}^s\underset{\sim}{I}^j$, N^σ and M^σ with N^s and M^s. Also $\sigma_0, {}^\sigma\underset{\sim}{a}_j, {}^\sigma\underset{\sim}{b}_i, {}^\sigma\underset{\sim}{c}_{ij}, {}^\sigma\underset{\sim}{d}_i$ and ${}^\sigma\alpha_{tm}$ are replaced by $S_0, {}^s\underset{\sim}{a}_j, {}^s\underset{\sim}{b}_i, {}^s\underset{\sim}{c}_{ij}, {}^s\underset{\sim}{d}_i$ and ${}^s\alpha_{tm}$. The material coefficients can be functions of ${}^s\underset{\sim}{I}^j; j = 1,2,\cdots,M^s$ and θ in a known configuration $\underset{\sim}{\Omega}$.

The constitutive theory (230) is based on integrity (complete basis of the space of $\boldsymbol{S}$). Simplified form of (230) can be obtained by choosing desired generators and invariants. A constitutive theory for $\boldsymbol{S}$ that is linear in the components of $\boldsymbol{\varepsilon}^{(\alpha)}$ and θ is given by (after redefining the material coefficients)

$$\boldsymbol{S} = S_0\boldsymbol{I} + 2\mu^s\left(\boldsymbol{\varepsilon}^{(\alpha)}\right) + \lambda^s\left(\operatorname{tr}\left(\boldsymbol{\varepsilon}^{(\alpha)}\right)\right)\boldsymbol{I} - {}^s\alpha_{tm}\left(\theta-\theta|_{\underset{\sim}{\Omega}}\right)\boldsymbol{I} \tag{231}$$

It is instructive to examine $\boldsymbol{\varepsilon}^{(\alpha)}$, the Green's strain tensor for the microconstituents.

$$\boldsymbol{\varepsilon}^{(\alpha)} = {}^d_s\boldsymbol{J}^{(\alpha)} \tag{232}$$

in which ${}^d\boldsymbol{J}^{(\alpha)}$ is the displacement gradient tensor for the microconstituents. In (232), six components of ${}^d_s\boldsymbol{J}^{(\alpha)}$ and three rotations ${}_c\boldsymbol{\Theta}$ in ${}^d_a\boldsymbol{J}^{(\alpha)}$ are the degrees of freedom for the microconstituent, thus only six unknown degrees of freedom as ${}_c\boldsymbol{\Theta}$ are known (same as linear micromorphic theory).

6.5. Constitutive Theory for Moment Tensor $\boldsymbol{m}^{[0]}$

Consider Equation (214)

$$\boldsymbol{m}^{[0]} = \boldsymbol{m}^{[0]}\left({}^{c\Theta}\boldsymbol{\varepsilon}_{[0]}, \theta\right) \tag{233}$$

Let ${}^m\underset{\sim}{\boldsymbol{G}}^i; i = 1,2,\cdots,N^m$ be combined generators of the argument tensors of $\boldsymbol{m}^{[0]}$ in (233) that are symmetric tensors of rank two and let ${}^mI^j; j = 1,2,\cdots,M^m$ be the combined invariants of the same argument tensors $\boldsymbol{m}^{[0]}$ in (233). Then $\boldsymbol{I}, {}^m\underset{\sim}{\boldsymbol{G}}^i; i = 1,2,\cdots,N^m$ constitutes the basis of the space of tensor $\boldsymbol{m}^{[0]}$, hence we can express $\boldsymbol{m}^{[0]}$ as a linear combination of the basis (integrity).

$$\boldsymbol{m}^{[0]} = {}^{m}\alpha^{0}\boldsymbol{I} + \sum_{i=1}^{N^m} {}^{m}\alpha^{i}\left({}^{m}\underset{\sim}{\boldsymbol{G}}^{i}\right) \tag{234}$$

in which the coefficients ${}^{m}\alpha^{i}; i = 0,1,\cdots,N^{m}$ in the linear combination can be functions of ${}^{m}I^{j}; j = 1,2,\cdots,M^{m}$ and temperature θ. The material coefficients in (234) are determined using the same approach as used in Section 6.2 for ${}_{s}^{d}\boldsymbol{\sigma}^{[0]}$. We expand ${}^{m}\alpha^{i}; i = 0,1,\cdots,N^{m}$ in Taylor series in ${}^{m}I^{j}; j = 1,2,\cdots,M^{m}$ and θ about a known configuration $\underline{\Omega}$ and retain only up to linear term in ${}^{m}I^{j}; j = 1,2,\cdots,M^{m}$ and θ. The resulting expression can be obtained from (223) by replacing ${}^{\sigma}\alpha^{i}$ with ${}^{m}\alpha^{i}$, ${}^{\sigma}\underset{\sim}{I}^{j}$ with ${}^{m}\underset{\sim}{I}^{j}$, N^{σ} and M^{σ} with N^{m} and M^{m} and so on. Substituting these ${}^{m}\alpha^{i}; i = 0,1,\cdots,N^{m}$ in (234) and collecting coefficients of $\boldsymbol{I}, {}^{m}\underset{\sim}{I}^{j}\boldsymbol{I}, {}^{m}\underset{\sim}{\boldsymbol{G}}^{i}, {}^{m}\underset{\sim}{I}^{j}\,{}^{m}\underset{\sim}{\boldsymbol{G}}^{i}, \left(\theta - \theta|_{\underline{\Omega}}\right){}^{m}\underset{\sim}{\boldsymbol{G}}^{i}$ and $\left(\theta - \theta|_{\underline{\Omega}}\right)\boldsymbol{I}$ we can write the following.

$$\begin{aligned} \boldsymbol{m}^{[0]} = {} & m_0\boldsymbol{I} + \sum_{j=1}^{M^m} {}^{m}\underset{\sim}{a}_{j}\,{}^{m}\underset{\sim}{I}^{j}\boldsymbol{I} + \sum_{i=1}^{N^m} {}^{m}\underset{\sim}{b}_{i}\,{}^{m}\underset{\sim}{\boldsymbol{G}}^{i} + \sum_{i=1}^{N^m}\sum_{j=1}^{M^m} {}^{m}\underset{\sim}{c}_{ij}\,{}^{m}\underset{\sim}{I}^{j}\,{}^{m}\underset{\sim}{\boldsymbol{G}}^{i} \\ & - \sum_{i=1}^{N^m} {}^{m}\underset{\sim}{d}_{i}\left(\theta - \theta|_{\underline{\Omega}}\right){}^{m}\underset{\sim}{\boldsymbol{G}}^{i} - {}^{m}\alpha_{tm}\left(\theta - \theta|_{\underline{\Omega}}\right)\boldsymbol{I} \end{aligned} \tag{235}$$

the material coefficients ${}^{m}\underset{\sim}{a}_{j}, {}^{m}\underset{\sim}{b}_{i}, {}^{m}\underset{\sim}{c}_{ij}, {}^{m}\underset{\sim}{d}_{i}$ and ${}^{m}\alpha_{tm}$ as well as m_0 can be obtained by replacing N^{σ}, M^{σ} with N^{m}, M^{m}, σ_0 by m_0 and ${}^{\sigma}\underset{\sim}{a}_{j}, {}^{\sigma}\underset{\sim}{b}_{i}, {}^{\sigma}\underset{\sim}{c}_{ij}, {}^{\sigma}\underset{\sim}{d}_{i}, {}^{\sigma}\alpha_{tm}$ by ${}^{m}\underset{\sim}{a}_{j}, {}^{m}\underset{\sim}{b}_{i}, {}^{m}\underset{\sim}{c}_{ij}, {}^{m}\underset{\sim}{d}_{i}, {}^{m}\alpha_{tm}$ and ${}^{\sigma}\alpha^{i}$ by ${}^{m}\alpha^{i}$. The constitutive theory (235) is based on complete basis (integrity) of the space of constitutive tensor $\boldsymbol{m}^{[0]}$. Simplified form of the constitutive theories for $\boldsymbol{m}^{[0]}$ can be obtained from (235) by retaining desired generators and invariants. A constitutive theory for $\boldsymbol{m}^{[0]}$ that is linear in the components of its argument tensors is given by

$$\boldsymbol{m}^{[0]} = m_0\boldsymbol{I} + 2\left(\mu^{m}\right)\left({}^{c\Theta}\boldsymbol{\varepsilon}_{[0]}\right) + \left(\lambda^{m}\right)\left(\mathrm{tr}\left({}^{c\Theta}\boldsymbol{\varepsilon}_{[0]}\right)\right)\boldsymbol{I} - {}^{m}\alpha_{tm}\left(\theta - \theta|_{\underline{\Omega}}\right)\boldsymbol{I} \tag{236}$$

6.6. Constitutive Theory for $\boldsymbol{q}$

Consider

$$\boldsymbol{q} = \boldsymbol{q}\left(\boldsymbol{g}, \theta\right) \tag{237}$$

Following reference [56] [57] we can derive the following constitutive theory for $\boldsymbol{q}$ using representation theorem

$$\boldsymbol{q} = -k\boldsymbol{g} - k_1\left(\{g\}^{T}\{g\}\right)\boldsymbol{g} - k_2\left(\theta - \theta|_{\underline{\Omega}}\right)\boldsymbol{g} \tag{238}$$

k, k_1 and k_2 are material coefficients. These can be functions of $\left(\boldsymbol{g}\cdot\boldsymbol{g}\right)|_{\underline{\Omega}}$ and $\theta|_{\underline{\Omega}}$. In (238) $\left(\boldsymbol{g}\cdot\boldsymbol{g}\right)$ is the invariant of the argument tensor $\boldsymbol{g}$. Simplified form of (238), the Fourier heat conduction law is given by

$$\boldsymbol{q} = -k\boldsymbol{g} \tag{239}$$

7. Thermodynamic and Mathematical Consistency of the Micromorphic Theory Presented in This Paper

The laws of classical thermodynamics used in classical continuum mechanics are well-founded and accepted laws. Microcontinuum theories contain new physics beyond classical continuum mechanics, hence may require new considerations. For establishing conservation and balance laws for microcontinuum theories in general, we must begin with classical thermodynamics, but can only make changes in them and incorporate new conservation and balance laws if the classical thermodynamics framework supports these. The resulting microcontinuum theory will be referred to as thermodynamically consistent with the law of classical thermodynamics *i.e.* classical continuum mechanics. We list important features of the present work that establish thermodynamical and mathematical consistency of the nonlinear micromorphic theory presented in this paper.

1) If we consider nonlinear deformation of the microconstituents, then we have nine deformational degrees of freedom for the microconstituent, all nine components of $\boldsymbol{J}^{(\alpha)}$ or six independent components of ${}^{d}_{s}\boldsymbol{J}^{(\alpha)}$ plus three classical rigid rotation ${}_{c}\boldsymbol{\Theta}^{(\alpha)}$ in the microconstituent volume. In Sections 5.8.1 and 5.8.2, six additional equations have already been derived out of nine needed, thus, we need additional three equations for closure of the mathematical models. The classical thermodynamics has no mechanism for obtaining these from the existing balance laws and also does not provide any means of deriving them otherwise. Thus, we are left with no choice but to only consider linear deformation of the microconstituent in which case, these additional three equations are not needed and we have thermodynamically consistent theory in which the mathematical model has closure.

2) When the microconstituent deformation is nonlinear, Eringen advocates using conservation of microinertia as a conservation law to obtain additional three equations. We have discussed that the use of this conservation law is not supported by classical thermodynamic, hence the resulting theory is thermodynamically inconsistent. However, in view of the fact that all published works follow Eringen's works, if one wishes to use the conservation of microinertia as additional conservation law to obtain three additional equations, going back to nonlinear microconstituent deformation only requires that we use

$$\boldsymbol{S}^{*}:\dot{\boldsymbol{J}}^{(\alpha)}=\boldsymbol{S}^{[0]}:\boldsymbol{\varepsilon}_{[0]}^{(\alpha)} \tag{240}$$

in the energy equations and entropy inequality, then constitutive theory for $\boldsymbol{S}^{[0]}$ can be derived using the following and representation theorem.

$$\boldsymbol{S}^{[0]}=\boldsymbol{S}^{[0]}\left(\boldsymbol{\varepsilon}_{[0]}^{(\alpha)},\theta\right) \tag{241}$$

for thermoelastic micromorphic solid. In (241), $\boldsymbol{\varepsilon}_{[0]}^{(\alpha)}$ is given by

$$\boldsymbol{\varepsilon}_{[0]}^{(\alpha)}=\frac{1}{2}\left(\left(\boldsymbol{J}^{(\alpha)}\right)^{T}\cdot\boldsymbol{J}^{(\alpha)}-\boldsymbol{I}\right) \tag{242}$$

in this approach we must keep in mind that conservation of microinertia conservation law is not supported by classical thermodynamics. Hence, with the use of this conservation law, the resulting microcontinuum theory is thermodynamically inconsistent.

3) Existence of moment independent of forces that is conjugate to rotations is a result of the resistance offered by the medium to the rigid rotations of the microconstituents. Balance of angular momenta, a statement of balance of moments (of forces in classical continuum mechanics) permits inclusion of the moment tensor in the balance of angular momenta. Thus, this modification of the balance law of classical thermodynamics, balance of angular momenta is supported by classical thermodynamics.

4) In classical thermodynamics, a kinematically conjugate pair requires two balance laws. Kinematically conjugate pair of displacements and forces require two balance laws: balance of forces and balance of moment of forces *i.e.* balance of linear momenta and balance of angular momenta. Based on this, the classical thermodynamics will permit two additional balance laws for each new kinematically conjugate pair. Thus, for the kinematically conjugate pair of rotations and moments in the microcontinuum theories, we need two new balance laws: balance of moments which already exists as balance of angular momenta and can be modified to include moment tensor as discussed in (1) and balance of moment of moments which is a new balance law needed in the microcontinuum theories. Consequence of this balance law is that Cauchy moment tensor becomes symmetric. In the absence of this balance law dynamic equilibrium of moment of moments is violated, hence thermodynamic consistency is violated.

5) It has been shown by Surana *et al.* that if classical rotations are not used as rigid rotations of the microconstituents, entropy inequality is violated. That is, a microcontinuum theory based ${}_{\alpha}\boldsymbol{\Theta}$ as unknown rigid rotations of the microconstituents or ${}_{c}\boldsymbol{\Theta} + {}_{\alpha}\boldsymbol{\Theta}$ as rigid rotations of the microconstituents results in violation of entropy inequality. These choices produce additional terms in the entropy inequality that cannot be accounted for, thus, resulting in the violation of thermodynamic inconsistency of the theory.

6) Since rotations and moments are a new kinematically conjugate pair in microcontinuum theories that does not exist in classical continuum mechanics, therefore, the integral-average definition of moment tensor cannot be derived using microconstituent Cauchy stress tensor $\bar{\boldsymbol{\sigma}}^{(\alpha)}$ or $\boldsymbol{\sigma}^{(\alpha)}$ as this stress is due to classical continuum mechanics. Insistence in doing so will result in a theory that is thermodynamically inconsistent.

7) In micropolar microcontinuum theories, (1) - (4) that are supported by classical thermodynamics are sufficient to yield a microcontinuum theory that is thermodynamically consistent and has closure when the constitutive theories are included.

8) When the microconstituents are deformable, (1) - (4) are not sufficient (along with constitutive theories) to provide closure to the mathematical model. In case of micromorphic theory, six additional equations are needed and in case of microdilation theory only one additional equation is needed for closure. We have shown that balance of angular momenta in fact contains nine equations, six of these are eliminated due to presence of permutation tensor with the stress terms.

We have shown that by premultiplying balance of angular momenta with the inverse of the permutation tensor, we can recover the six additional equations needed for closure. This part of the derivation is related to balance of angular momenta, hence obviously does not violate thermodynamic consistency.

9) Thus, we note that the use of (2) - (5) or (2) - (5) and (7) that are supported by classical thermodynamics yield conservation and balance laws of all three microcontinuum theories, confirming that the conservation and the balance laws in these theories derived using the approach presented in this paper are thermodynamically consistent.

10) In case of constitutive theories, we must use conjugate pairs in the entropy inequality and axiom of causality to determine constitutive tensors and their argument tensors that are supported by the theory of isotropic tensors (as done in the present work). A violation of this results in thermodynamic inconsistency as well as mathematical inconsistency of the resulting theory.

11) Constitutive theories must be derived strictly using representation theorem (as done in the present work) to ensure mathematical consistency of the resulting constitutive theories. If the constitutive theories are derived using any other means such as potentials and energy functionals, then we must show that the same theories can also be derived using representation theorem, otherwise the constitutive theories derived without using representation theorem are mathematically inconsistent. Clearly the constitutive theories presented in the paper are mathematically and thermodynamically consistent.

12) The two new conservation and the balance laws introduced by Eringen: 1) Conservation of microinertia and 2) balance of moment of symmetric parts of the stress tensors with the gradients of the symmetric part of the moment tensor are not supported by the classical thermodynamics *i.e.* classical continuum mechanics, hence can only be viewed as phenomenological or ad-hoc. Inclusion of these in the laws of classical thermodynamics used in deriving conservation and the balance laws for microcontinuum theories will result in a thermodynamically inconsistent microcontinuum theory.

13) It has been shown that the microcontinuum theory presented here requires linear deformation of microconstituent for the mathematical model to have closure. When the microconstituent deformation is nonlinear, additional three equations are needed for closure of the mathematical model. It has been shown that the classical thermodynamic has no means of obtaining these, hence suggesting that nonlinear deformation is not permissibly valid physics in a thermodynamically consistent framework.

8. Micromorphic Theories of Eringen

We summarize some aspects of Eringen's theories that have lead to their thermodynamic and mathematical inconsistencies. These are applicable to microcontinuum theories in general, hence also hold for the nonlinear micromorphic theory presented in this paper.

1) Use of ${}_{\alpha}\boldsymbol{\Theta}$ or ${}_{\alpha}\boldsymbol{\Theta} + {}_{c}\boldsymbol{\Theta}$ as rigid rotations of the microconstituents results in violation of entropy inequality, hence thermodynamic inconsistency of the resulting theory.

2) Including rigid rotations in the strain measures in Eringen's work results in tensors that cannot be used in the constitutive theories without violating physics of deformation.

3) Eringen's work defines integral-average moment tensor (nonclassical physics) using microconstituent Cauchy stress tensor $\bar{\boldsymbol{\sigma}}^{(\alpha)}$ or $\boldsymbol{\sigma}^{(\alpha)}$ that is due to classical continuum mechanics. This is obviously wrong. The origin of moment is due to resistance offered to the rigid rotations of the microconstituents by the medium and not $\boldsymbol{\sigma}^{(\alpha)}$. Due to this wrong definition the balance laws such as balance of angular momenta that uses this moment tensor is of concern.

4) Use of weighted integral of balance of micro linear momenta using a weight function $\tilde{\phi}^{(\alpha)}\left(\bar{\underset{\sim}{x}}_m^{(\alpha)}\right)$ with three different choices for balance of linear momenta, balance of angular momenta and the new balance law proposed has no thermodynamic foundation. Our work in the paper shows that this is neither needed nor used.

5) Use of nonsymmetric tensors of rank two as constitutive tensors and the nonsymmetric tensors of rank two as their argument tensor is not supported by the theory of isotropic tensor. It results in constitutive theories that are mathematical inconsistent and are nonphysical.

6) Constitutive theories derived using potentials or energy functional (as in Eringen's work) are nonphysical, not valid and mathematically inconsistent if the same theories cannot be derived using representation theorem.

7) Due to not using balance of moment of moment balance law, the dynamic equilibrium is not satisfied in the Eringen's mathematical model. Another consequence of not using this balance law is that moment tensor is non-symmetric resulting in spurious constitutive theories.

8) Use of principle of equipresence introduces nonphysical coupling between classical and nonclassical physics and results in nonphysical material coefficients.

9) Lack of various additive decompositions of the stress tensors leads to nonphysical and non-valid constitutive tensors. For example, ${}_{a}\boldsymbol{\sigma}$ must be eliminated from $\boldsymbol{\sigma}$ as it is defined by balance of angular momenta hence cannot be part of constitutive theory. Further decomposition of ${}_{s}\boldsymbol{\sigma} = {}_{s}^{e}\boldsymbol{\sigma} + {}_{s}^{d}\boldsymbol{\sigma}$ is necessary to address volumetric and distortional physics correctly as these are mutually exclusive. None of these decompositions are used in Eringen's work, hence the constitutive theories in Eringen's work are of concern.

10) In micromorphic theories additional nine equations are needed for closure. Eringen proposes a new balance law to obtain six of these using balance of moments of symmetric stresses with gradients of the symmetric part of moment tensor. This law is not supported by classical thermodynamics, hence its use will yield thermodynamically inconsistent theory.

11) Eringen proposes conservation of inertia to obtain the remaining three

equations needed for closure. There is no such conservation law in classical thermodynamics, hence its use will lead to thermodynamically inconsistent theory.

We have presented plenty of evidence based on thermodynamics and well-established principles of mathematics that Eringen's microcontinuum theories are thermodynamically and mathematically inconsistent, hence are not valid microcontinuum theories.

9. Summary and Conclusions

A nonlinear micromorphic continuum theory has been presented with nonlinear and linear microconstituent kinematics in which mechanism of elasticity is considered for the microconstituents, for the solid medium and for the interaction of the microconstituents with the solid medium. In the following, we summarize the work presented in the paper and draw some conclusions.

1) In the present micromorphic theory, linear deformation of microconstituents requires six deformational degrees of freedom and three rigid rotations ${}_{c}\boldsymbol{\Theta}$, a total of nine as in the case of Eringen's theory, but the degrees of freedom are completely different. In our work, rotations of the microconstituents (described by the classical rotations ${}_{c}\boldsymbol{\Theta}$, hence known) and the six independent components of the symmetric part of the micro displacement gradient tensor (unknown) are nine degrees of freedom. In Eringen's work all nine components of the micro deformation gradient tensor are considered unknown deformational degrees of freedom in addition to three unknown rigid rotations ${}_{\alpha}\boldsymbol{\Theta}$, a total of twelve.

2) In the theory presented here, care is taken to ensure that the rigid body rotation physics of microconstituent that is common to all three microcontinuum theories is incorporated in identical manner in all three microcontinuum theories.

3) Our work recognizes that rotations ${}_{c}\boldsymbol{\Theta}$ and Cauchy moment tensor are a new kinematically conjugate pair in all three microcontinuum theories, hence it requires two balance laws just as displacements and forces kinematic pair does in classical continuum mechanics. This necessitates new balance law in all microcontinuum theories [44] [55] [61], balance of moment of moments. This balance law is never used in Eringen's work; the consequence of this is spurious conjugate pairs in the entropy inequality and spurious constitutive theories.

4) Varying rotations ${}_{c}\boldsymbol{\Theta}$ in the deforming solid medium when resisted, create moments. Our derivation shows that the Cauchy moment tensor and the symmetric part of the gradients of ${}_{c}\boldsymbol{\Theta}$ are kinematically work conjugate. This physics is purely due to nonclassical mechanics, hence has no interaction or any connection to classical continuum theory. Based on this, the 'integral-average' definition of moment tensor by Eringen's work is incorrect as it is based on $\bar{\boldsymbol{\sigma}}^{(\alpha)}$ which is purely due to classical continuum mechanics.

5) Our derivation in this paper shows that the use of weight function $\bar{\phi}^{(\alpha)}\left(\bar{\underset{\sim}{x}}_{m}^{(\alpha)}\right)$ in the derivation of macro balance of linear momenta, balance of angular momenta and moment of momentum has no basis and has no thermodynamic basis. Our work shows that the use of $\bar{\phi}^{(\alpha)}\left(\bar{\underset{\sim}{x}}_{m}^{(\alpha)}\right)$ as advocated by Eringen is not justi-

fied and lead to balance laws different than without using it.

6) In our work, all constitutive tensors of rank two are symmetric tensors and their argument tensors of rank two are also symmetric tensors, hence permitting the use of representation theorem in deriving constitutive theory that are naturally mathematically consistent. This is in contrast with published works in which the constitutive tensors of rank two are non symmetric tensors with non symmetric argument tensors. Such constitutive theories derived using assumed potentials are non physical and not justified based on representation theorem.

7) Conservation of micro inertia advocated by Eringen to be necessary in microcontinuum theories is neither needed in the present work nor used. This conservation law is not supported by classical thermodynamics. The need for this law is primarily due to ${}_{\alpha}\boldsymbol{\Theta}$ being unknown degrees of freedom, whereas in our work ${}_{\alpha}\boldsymbol{\Theta}$ are in fact ${}_{c}\boldsymbol{\Theta}$, hence are known. Other significant differences are that in Eringen's work $\boldsymbol{\sigma}$ and $\boldsymbol{m}$ are non symmetric and nine constitutive equations are considered for $\boldsymbol{\sigma}$ as well as $\boldsymbol{m}$. In our work, $\boldsymbol{\sigma} = {}_{s}\boldsymbol{\sigma} + {}_{a}\boldsymbol{\sigma}$ decomposition is used and there are only six constitutive equations needed for ${}_{s}\boldsymbol{\sigma}$ as ${}_{a}\boldsymbol{\sigma}$ is defined by balance of angular momenta. $\boldsymbol{m}$ is symmetric due to balance of moment of moments balance law, hence only six constitutive equations are needed for $\boldsymbol{m}$ as well. Eringen's micromorphic theory does not have closure without conservation of micro inertia conservation law primarily due to ${}_{\alpha}\boldsymbol{\Theta}$.

8) Thermodynamic and mathematical consistency of the nonlinear micromorphic theory presented in this paper has been established in Section 7. The lack of thermodynamic and mathematical consistency of Eringen's linear micromorphic theory has been discussed and illustrated in Section 8.

9) It is established that with nonlinear microconstituent kinematics a thermodynamically and mathematically consistent micromorphic theory is not possible as in this case classical thermodynamics has no means of providing three additional equations needed for closure of the mathematical model. This is perhaps an indication that nonlinear deformation of the microconstituents is not physical. We do not advocate using conservation of microinertia conservation law as this law proposed by Eringen has no thermodynamic basis, hence the resulting theory is not a valid theory. By assuming linear microconstituent kinematics but nonlinear deformation for the solid medium and for the interaction of the microconstituents with the solid medium, we have shown that the resulting micromorphic theory is thermodynamically and mathematically consistent without the need of ad-hoc conservation and/or balance laws that are not supported by classical thermodynamics.

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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