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Long-Time Behavior of Solutions to the Classical Diffusion Equation with a Time-Dependent Memory Kernel

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Abstract

This paper investigates a class of classical reaction-diffusion equations with a time-dependent memory kernel, where the nonlinear term satisfies a supercritical growth condition. Under a new theoretical framework, we avoid the use of Sobolev embeddings in the treatment of the supercritical nonlinearity. Since the Sobolev control fails in the supercritical case, we establish the well-posedness of solutions by means of integral-type energy estimates combined with the intrinsic structure of the system. Furthermore, by employing a refined solution decomposition technique, we prove the existence and regularity of a time-dependent global attractor. Our results extend the existing conclusions for reaction–diffusion equations with subcritical or critical nonlinearities.

Keywords

Classical Reaction-Diffusion Equation, Time-Dependent Memory Kernel, Time-Dependent Global Attractors, Supercritical Nonlinearity

1. Introduction

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with smooth boundary $\partial\Omega$. The asymptotic dynamics of the following classical reaction-diffusion equation with time-dependent memory kernel

$$\begin{cases} \partial_t u - \Delta u - \int_0^\infty h_t(s)\Delta u(t-s)\,\mathrm{d}s + f(u) = g, & x \in \Omega, t \in (\tau, +\infty), \\ u(x,t) = 0, & x \in \partial\Omega, t \in (\tau, +\infty), \\ u(x,t) = u_\tau(x,t), & x \in \Omega, t \in (-\infty, \tau], \end{cases} \tag{1.1}$$

are investigated in this article.

Suppose that the time-dependent memory kernel function $\mu_t(s)$ is nonnegative, convex, and summable, with $h_t(s)=\int_s^\infty \mu_t(r)\mathrm{d}r$, $\forall s\in\mathbb{R}^+$, $t\in\mathbb{R}$. Assuming $\mu_t(s)=-\partial_s h_t(s)$, the mapping $(t,s)\mapsto\mu_t(s):\mathbb{R}\times\mathbb{R}^+\to\mathbb{R}^+$ is supposed to satisfy the following structural conditions:

(H$_1$) For every fixed $t\in\mathbb{R}$, the function $s\mapsto\mu_t(s)$ is nonincreasing, absolutely continuous, and summable. Moreover,

$$\kappa(t)=\int_0^\infty \mu_t(s)\mathrm{d}s,\ \inf_{t\in\mathbb{R}}\kappa(t)>0.$$

(H$_2$) For any $\tau\in\mathbb{R}$, there exists a continuous function $K_\tau:[\tau,\infty)\to\mathbb{R}^+$ such that

$$\mu_t(s)\le K_\tau(t)\mu_\tau(s),\ \forall t\ge\tau,\ \text{a.e. } s\in\mathbb{R}^+.$$

(H$_3$) For each fixed $s>0$, the mapping $t\mapsto\mu_t(s)$ is differentiable, and for any compact set $\mathcal{K}\subset\mathbb{R}\times\mathbb{R}^+$, we have

$$(t,s)\mapsto\mu_t(s)\in L^\infty(\mathcal{K}),\ (t,s)\mapsto\partial_t\mu_t(s)\in L^\infty(\mathcal{K}).$$

(H$_4$) There exists $\delta>0$ such that

$$\partial_t\mu_t(s)+\partial_s\mu_t(s)+\delta\kappa(t)\mu_t(s)\le 0,\ \forall t\in\mathbb{R}^+,\ \text{a.e. } s\in\mathbb{R}^+.$$

Assumptions (H1) - (H4) characterize the fundamental properties of time-dependent memory kernels arising from aging materials. Conditions (H1) - (H2) guarantee the integrability and boundedness of the memory effect, (H3) describes the smooth temporal variation of the memory intensity, while (H4) models the decay mechanism of memory and plays a crucial role in the derivation of the dissipative estimates.

Assume that $g\in L^2(\Omega)$ is the external forcing term, and $f(u)$ is the nonlinear term, satisfying the following supercritical growth condition

$$\gamma_1|s|^p-\beta_1\le f(s)s\le\gamma_2|s|^p+\beta_2,\ p\ge 2, \tag{1.2}$$

where $\gamma_1,\gamma_2,\beta_1,\beta_2>0$ are constants, and the dissipation condition

$$f'(s)\ge -l, \tag{1.3}$$

where l are all positive constants. Let $F(s)=\int_0^s f(v)\mathrm{d}v$. By assumption (1.2), there exist positive constants $\tilde{\alpha}_i,\tilde{\beta}_i>0$ ($i=1,2$) such that

$$\tilde{\alpha}_1|s|^p-\tilde{\beta}_1\le F(s)\le\tilde{\alpha}_2|s|^p+\tilde{\beta}_2. \tag{1.4}$$

In recent years, many scholars and experts are engaged in studying the asymptotic behavior of the solution of the classical reaction-diffusion equation see the literature [1]-[3]

$$\partial_t u-\Delta u+f(u)=g. \tag{1.5}$$

This equation describes the evolution behavior of the system under the joint action of diffusion and nonlinear reaction. It is found that the conduction properties also have an impact on the reaction-diffusion process, in actual materials, such as rubber, high molecular polymers, there are often “aging” phenomena, the elasticity will become weaker and weaker with the passage of time, the traditional memoryless model can not accurately describe such behavior. To this end, researchers have introduced the time-dependent memory term to describe the system's dependence on its historical states, see [4]-[7], and the existence of a time-dependent global attractor means that, despite the continuous degradation of memory caused by aging, the system still evolves towards a stable long-time regime, which provides a rigorous mathematical description of the asymptotic behavior of aging materials with time-varying memory effects.

When the equation contains a time independent memory kernel, the system is called a classical diffusion equation with fading memory. Under this framework, many scholars have conducted in-depth research on the long-time behavior of the system solution. For example, literature [8] studies the long-time dynamic behavior of the classical reaction-diffusion equation with fading memory when the nonlinear term satisfies the polynomial growth of any order, and proves the existence of the global attractor by using the abstract function theory and the semigroup method. Literature [9] proves the existence of the orbital attractor of the nonclassical diffusion equation when the nonlinear term satisfies the critical exponential growth. Reference [10] studied the well posedness of weak solutions of a nonlocal partial differential equation with long memory, and proved the existence of attractors.

In the past two decades, increasing attention has been paid to diffusion models with time-dependent memory kernels $\mu_t(s)$, since they are capable of describing more complex phenomena such as material aging, variations of memory intensity induced by environmental changes, and the evolution of dielectric properties over time. Representative works [11] have shown that such systems still possess dissipativity under time-dependent memory effects and, by constructing suitable families of time-dependent processes and establishing asymptotic compactness, have proved the existence of time-dependent global attractors. However, most of the available results are restricted to subcritical or critical reaction terms, see [12]. The main reason is that, in the supercritical case, the Sobolev embedding structure that is usually employed to control the nonlinear term is no longer applicable, which destroys many key a priori estimates and makes it impossible to obtain the dissipative and compactness properties of the solution process by standard methods.

Inspired by the above research, this paper studies a class of classical diffusion equations with time-dependent memory kernel for the first time, whose nonlinear term satisfies the supercritical growth condition. By combining integral estimation and solution decomposition techniques, the essential difficulties caused by the supercritical growth of nonlinear terms and time-dependent memory kernel are overcome, the well posedness, dissipative estimation and the existence and regularity of time-dependent global attractors are proved. The results in this paper generalize the conclusions in the literature on subcritical or critical cases, and thus systematically study the long-time dynamic behavior of such systems for the first time.

The structure of this paper is as follows: Section 2 recalls the function spaces and preliminary results; Section 3 discusses the well-posedness of solutions; Section 4 proves the existence and regularity of the time-dependent global attractor.

2. Notations and Preliminaries

2.1. Function Space and Memory Kernel Hypothesis

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with smooth boundary. Define $A = -\Delta$ with domain $D(A) = H^2(\Omega) \cap H_0^1(\Omega)$. For any $k \in \mathbb{R}$, define the family of Hilbert spaces $D\left(A^{k/2}\right)$, whose inner products and norms are respectively given by

$$\langle u, v\rangle_{D\left(A^{\frac{k}{2}}\right)} = \left\langle A^{\frac{k}{2}} u, A^{\frac{k}{2}} v\right\rangle, \quad \|u\|_{D\left(A^{\frac{k}{2}}\right)} = \left\|A^{\frac{k}{2}} u\right\|,$$

where $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$ denote the inner product and norm in $L^2(\Omega)$.

In particular, for $0 \le k < 3$, denote

$$V_k = D\left(A^{\frac{k}{2}}\right), \; H = L^2(\Omega), \; V_1 = H_0^1(\Omega), \; V_2 = H^2(\Omega) \cap H_0^1(\Omega).$$

For $0 \le k < 3$ and under the conditions on the memory kernel, define the memory space

$$\mathcal{M}_t^\sigma = L_{\mu_t}^2\left(\mathbb{R}^+; V_\sigma\right) = \left\{\xi^t : \mathbb{R}^+ \to V_\sigma \mid \int_0^\infty \mu_t(s) \left\|\xi^t(s)\right\|_\sigma^2 \mathrm{d}s < \infty\right\},$$

with inner product and norm

$$\left\langle \eta^t, \xi^t \right\rangle_{\mathcal{M}_t^\sigma} = \int_0^\infty \mu_t(s) \left\langle \eta^t(s), \xi^t(s)\right\rangle_\sigma \mathrm{d}s, \quad \left\|\xi^t\right\|_{\mathcal{M}_t^\sigma}^2 = \int_0^\infty \mu_t(s) \left\|\xi^t(s)\right\|_\sigma^2 \mathrm{d}s.$$

2.2. History Variable and Equation Transformation

Following the discussion in reference [13], we introduce the history variable

$$\eta^t(s) = \begin{cases} \int_0^s u(t-r)\,\mathrm{d}r, & 0 \le s \le t-\tau, \\ \eta_\tau(s-t+\tau) + \int_0^{t-\tau} u(t-r)\,\mathrm{d}r, & s > t-\tau, \end{cases} \tag{2.1}$$

with the corresponding initial-boundary conditions

$$\begin{cases} u(x,t) = 0, & x \in \partial\Omega, \; t > \tau, \\ \eta^t(x,s) = 0, & (x,s) \in \partial\Omega \times \mathbb{R}^+, \; t > \tau, \\ u(x,t) = u_\tau(x,t), & x \in \Omega, \; t \le \tau, \\ \eta^\tau(x,s) = \eta_\tau(x,s), & (x,s) \in \Omega \times \mathbb{R}^+. \end{cases} \tag{2.2}$$

Here, $u(\cdot)$ satisfies the following condition: there exist two positive constants $\mathcal{R}$ and $\varrho \le \delta$ such that

$$\int_0^\infty \mathrm{e}^{-\varrho s} \left\|\nabla u(-s)\right\|^2 \mathrm{d}s \le \mathcal{R},$$

where the constant δ is defined in assumption (H_4), and $\|\cdot\|$ denotes the norm in $L^2(\Omega)$.

Let $\sigma \in \mathbb{R}$, $T > \tau \in \mathbb{R}$, $u \in C\left([\tau, T]; V_\sigma\right)$, $\eta_\tau \in \mathcal{M}_\tau^\sigma$, where $[\tau, T] \in \mathbb{R}$. Moreover,

$$\partial_t \eta^t(s) = -\partial_s \eta^t(s) + u(t). \tag{2.3}$$

Then, the original equation can be transformed into the following system:

$$\begin{cases} \partial_t u - \Delta u - \int_0^\infty \mu_t(s) \Delta \eta^t(s) \mathrm{d}s + f(u) = g, \\ \partial_t \eta^t = -\partial_s \eta^t + u(t), \\ u|_{\partial\Omega} = 0, \ \eta^t|_{\partial\Omega} = 0, \\ u(x,t) = u_\tau(x,t), \ \eta^\tau(x,s) = \eta_\tau(x,s). \end{cases} \tag{2.4}$$

From (H_2), we obtain

$$\left\|\eta^t\right\|^2_{\mathcal{M}_t^\sigma} \le K_\tau(t) \left\|\eta^t\right\|^2_{\mathcal{M}_\tau^\sigma}, \ \forall t \ge \tau. \tag{2.5}$$

Furthermore, there is a continuous embedding

$$\mathcal{M}_\tau^\sigma \subset \mathcal{M}_t^\sigma, \ \forall t \ge \tau.$$

Consider the linear operator

$$\mathbb{T}_t : D(\mathbb{T}_t) \subset \mathcal{M}_t^\sigma \to \mathcal{M}_t^\sigma, \ \mathbb{T}_t \eta^t = -\partial_s \eta^t,$$

which represents the weak derivative. Its domain in $\mathcal{H}^\sigma$ is

$$D(\mathbb{T}_t) = \left\{ \eta^t \in \mathcal{M}_t^\sigma : \partial_s \eta^t \in \mathcal{M}_t^\sigma, \lim_{s \to 0} \eta^t(s) = 0 \right\}.$$

As in reference [14], $\mathbb{T}_t$ is the infinitesimal generator of the right-translation contraction semigroup on $\mathcal{M}_t^\sigma$, and thus is a dissipative operator. More precisely, the following estimate holds:

$$\left\langle \mathbb{T}_t \eta^t, \eta^t \right\rangle_{\mathcal{M}_t^\sigma} = \frac{1}{2} \int_0^\infty \partial_s \mu_t(s) \left\|\eta^t(s)\right\|_\sigma^2 \mathrm{d}s, \ \forall \eta^t \in D(\mathbb{T}_t), \tag{2.6}$$

Then, by condition (H_1), we have

$$\left\langle \mathbb{T}_t \eta^t, \eta^t \right\rangle_{\mathcal{M}_t^\sigma} \le 0, \ \forall \eta^t \in D(\mathbb{T}_t).$$

From (2.3), it follows that

$$\mathbb{T}_\tau \subset \mathbb{T}_t, \ \forall t \ge \tau. \tag{2.7}$$

Define the time-dependent space

$$\mathcal{H}_t^\sigma = V_{\sigma-1} \times \mathcal{M}_t^\sigma,$$

with norm

$$\|z\|^2_{\mathcal{H}_t^\sigma} = \left\|(u, \eta^t)\right\|^2_{\mathcal{H}_t^\sigma} = \|u\|^2_{\sigma-1} + \left\|\eta^t\right\|^2_{\mathcal{M}_t^\sigma}.$$

2.3. The Solution Process and Attractor Definitions

Definition 2.1 ([6]). For every $t \in \mathbb{R}$, let X_t be a family of normed spaces. We consider a two-parameter family of operators

$$U(t,\tau) : X_\tau \to X_t,$$

depending on $t \ge \tau \in \mathbb{R}$, and satisfying the following properties:

(i) $U(\tau,\tau)$ is the identity map on X_τ;

(ii) For any $\sigma \in \mathbb{R}$ and any $t \ge \tau \ge \sigma$, it holds that $U(t,\tau)U(\tau,\sigma) = U(t,\sigma)$.

The family $U(t,\tau)$ will still be called a process.

Definition 2.2. A family of sets $\mathfrak{B} = \{B_t\}_{t\in\mathbb{R}}$ is called uniformly bounded if

$$\sup_{t\in\mathbb{R}} \|B_t\|_{X_t} = \sup_{t\in\mathbb{R}} \sup_{\xi\in B_t} \|\xi\|_{X_t} < +\infty,$$

and if for every $R > 0$, there exists a constant $\tau_e = \tau_e(R) \ge 0$ such that

$$U(t,\tau)\mathbb{B}_\tau(R) \subset B_t,\ \forall t-\tau \ge \tau_e,$$

where $\mathbb{B}_\tau(R) = \left\{\xi \in X_\tau \mid \|\xi\|_{X_\tau} \le R\right\}$. In this case, $\mathfrak{B} = \{B_t\}_{t\in\mathbb{R}}$ is called a time-dependent absorbing set.

Definition 2.3. A family $C = \{C_t\}_{t\in\mathbb{R}}$ of bounded sets $C_t \subset X_t$ is said to be uniformly bounded if for each $t \in \mathbb{R}$, there exists a constant $R > 0$ such that

$$C_t \subset \left\{z \in X_t : \|z\|_{X_t} \le R\right\} = \mathbb{B}_t(R),\ \forall t \in \mathbb{R}.$$

Definition 2.4. A family $\mathcal{A} = \{A_t\}_{t\in\mathbb{R}}$ is called a time-dependent global attractor for the process $U(t,\tau)$ if it satisfies the following properties:

(i) For each $t \in \mathbb{R}$, the set A_t is compact in X_t;

(ii) $\mathcal{A}$ is pullback attracting, that is, it is uniformly bounded, and for every uniformly bounded family $\mathcal{C} = \{C_t\}_{t\in\mathbb{R}}$, we have

$$\lim_{\tau\to-\infty} \operatorname{dist}_{X_t}\left(U(t,\tau)C_\tau, A_t\right) = 0;$$

holds for every uniformly bounded family $\mathfrak{C} = \{C_t\}_{t\in\mathbb{R}}$ and every $t \in \mathbb{R}$.

Definition 2.5. A function $t \to Z(t) \in X_t$ is a complete bounded trajectory (CBT) of the process $U(t,\tau)$, if and only if

(i) $\sup_{t\in\mathbb{R}} \|Z(t)\|_{X_t} < \infty$;

(ii) $Z(t) = U(t,\tau)Z(\tau)$ for all $\tau \le t$, $\tau \in \mathbb{R}$.

Definition 2.6. A time-dependent attractor $\mathcal{A} = \{A_t\}_{t\in\mathbb{R}}$ is invariant, if for all $t \ge \tau$,

$$U(t,\tau)A_\tau = A_t,\ \forall t \ge \tau.$$

2.4. Fundamental Lemmas

Lemma 2.1 ([15]). Let X, Y, Z be three Banach spaces. For $T > 0$, if $X \hookrightarrow\hookrightarrow Y \hookrightarrow Z$, and

$$V = \left\{u \in L^p([0,T];X) \mid \partial_t u \in L^r([0,T];Z)\right\}$$

$$V_1 = \left\{u \in L^\infty([0,T];X) \mid \partial_t u \in L^r([0,T];Z)\right\},$$

where $r > 1$, $1 \le p < \infty$, then

$$V \hookrightarrow\hookrightarrow L^p([0,T];Y),\ V_1 \hookrightarrow\hookrightarrow C([0,T];Y).$$

Lemma 2.2 ([5]).(Gronwall-type Lemma in Integral Form) Let $\Lambda : [\tau,\infty) \to \mathbb{R}$ be a continuous function. Suppose that for some $\epsilon > 0$ and any $b > a \ge \tau$, the following integral inequality holds:

$$\Lambda(b)+2\epsilon\int_a^b \Lambda(y)\mathrm{d}y \le \Lambda(a)+\int_a^b q_1(y)\Lambda(y)\mathrm{d}y+\int_a^b q_2(y)\mathrm{d}y,$$

where $q_1,q_2 \ge 0$ and $q_i \in L^1_{loc}[\tau,\infty)(i=1,2)$ satisfy: there exist $c_1,c_2 \ge 0$ such that

$$\int_a^b q_1(y)\mathrm{d}y \le \epsilon(b-a)+c_1,\ \sup_{t\ge\tau}\int_t^{t+1} q_2(y)\mathrm{d}y \le c_2,$$

then

$$\Lambda(t) \le \mathrm{e}^{c_1}\left[\left|\Lambda(\tau)\right|\mathrm{e}^{-\epsilon(t-\tau)}+\frac{c_2\mathrm{e}^{\epsilon}}{1-\mathrm{e}^{-\epsilon}}\right],\ \forall t\ge\tau.$$

Lemma 2.3 ([3]). Assume that $\mu \in C^1(\mathbb{R}^+)\cap L^1(\mathbb{R})$ is a nonnegative function and satisfies: if there exists $s_0 \in \mathbb{R}^+$ such that $\mu(s_0)=0$, then $\mu(s)=0$ for all $s \ge s_0$. Moreover, let B_0,B_1,B_2 be Banach spaces, where B_0,B_1 are reflexive and satisfy $B_0 \hookrightarrow\hookrightarrow B_1 \hookrightarrow B_2$. If $\mathfrak{C} \subset L^2_\mu(\mathbb{R}^+;B_1)$ satisfies

1) $\mathfrak{C} \subset L^2_\mu(\mathbb{R}^+;B_0)\cap H^1_\mu(\mathbb{R}^+;B_2)$;

2) $\sup_{\eta\in\mathfrak{C}}\left\|\eta(s)\right\|^2_{B_1} \le h(s)$ for all $s\in\mathbb{R}^+$, where $h(s)\in L^1_\mu(\mathbb{R}^+)$,

then $\mathfrak{C} \subset L^2_\mu(\mathbb{R}^+;B_1)$ is relatively compact.

Lemma 2.4 ([16]). Let (M,d) be a metric space, and let $U(t,\tau)$ be a Lipschitz continuous process on M, i.e., there exist constants C and K, independent of m_i, τ, and t, such that

$$d\left(U(t,\tau)m_1,U(t,\tau)m_2\right) \le C\mathrm{e}^{K(t-\tau)}d(m_1,m_2).$$

Assume further that there exist subsets $M_1,M_2,M_3 \subset M$ such that

$$\mathrm{dist}_M\left(U(t,\tau)M_1,U(t,\tau)M_2\right) \le L_1\mathrm{e}^{-\nu_1(t-\tau)},\ \nu_1,L_1>0,$$

$$\mathrm{dist}_M\left(U(t,\tau)M_2,U(t,\tau)M_3\right) \le L_2\mathrm{e}^{-\nu_2(t-\tau)},\ \nu_2,L_2>0,$$

then

$$\mathrm{dist}_M\left(U(t,\tau)M_1,U(t,\tau)M_3\right) \le L\mathrm{e}^{-\nu(t-\tau)},$$

where $\nu=\dfrac{\nu_1\nu_2}{K+\nu_1+\nu_2}$ and $L=CL_1+L_2$.

Lemma 2.5 ([6]). Let $Z(t)$ be a complete bounded trajectory (CBT) of the process $U(t,\tau)$. If the time-dependent global attractor $\mathcal{A}=\{A(t)\}_{t\in\mathbb{R}}$ of the process $U(t,\tau)$ is invariant, then $\mathcal{A}=\{Z \mid t\to Z(t)\in X_t\}$.

3. Well-Posedness and Regularity of Solutions

In order to obtain the well-posedness of solutions and perform dissipative estimates, we need to rely on the following results.

Lemma 3.1. Let

$$\Phi(u,\eta_\tau)=(2+t-\tau)\left[(t-\tau)\kappa(\tau)\|u\|^2_{L^\infty([\tau,T];\mathcal{H}^\sigma)}\right]+2\|\eta_\tau\|^2_{\mathcal{M}^\sigma_\tau}$$

and assume $\eta^t \in \mathcal{M}_\tau^\sigma \subset \mathcal{M}_t^\sigma$, where

$$\left\|\eta^t\right\|_{\mathcal{M}_\tau^\sigma}^2 \le \Phi\left(u,\eta_\tau\right),\ \forall t \in [\tau,T], \tag{3.1}$$

and

$$\left\|\eta^t\right\|_{\mathcal{M}_t^\sigma}^2 \le \Phi\left(u,\eta_\tau\right)K_\tau(t) \in L^1\left([\tau,T]\right). \tag{3.2}$$

Lemma 3.2. Assume $\eta_\tau \in D\left(\mathbb{T}_\tau\right)$. Then for any $t \in [\tau,T]$, we have $\eta^t \in D\left(\mathbb{T}_\tau\right)$, and the equality

$$\partial_t \eta^t = \mathbb{T}_\tau \eta^t + u(t) \tag{3.3}$$

holds in the space $\mathcal{M}_\tau^\sigma$.

Remark 3.3. Due to the embedding $\mathcal{M}_\tau^\sigma \hookrightarrow \mathcal{M}_t^\sigma$ and from formula (2.12), we know that for any fixed t, the equality

$$\partial_t \eta^t = \mathbb{T}_t \eta^t + u(t)$$

holds in the space $\mathcal{M}_t^\sigma$.

Remark 3.4. When $\eta^t \in D\left(\mathbb{T}_\tau\right)$, it follows from (2.10) and (3.3) that

$$\left\|\partial_s \eta^t\right\|_{\mathcal{M}_t^\sigma}^2 \le \Psi\left(u,\eta_\tau\right)K_\tau(t),\ \forall t \in [\tau,T], \tag{3.4}$$

where $\Psi\left(u,\eta_\tau\right) = \kappa(\tau)\|u\|_{L^\infty([\tau,T];V_\sigma)}^2 + \left\|\partial_s \eta_\tau\right\|_{\mathcal{M}_\tau^\sigma}^2$.

Lemma 3.5. Let $I = [\tau,T]$. Assume $u \in C\left(I;V_\sigma\right)$ and $\eta_\tau \in C^1\left(\mathbb{R}^+,V_\sigma\right) \cap D\left(\mathbb{T}_\tau\right)$. Then, for any $\tau \le a \le b \le T$, the following inequality holds:

$$\begin{aligned}&\left\|\eta^b\right\|_{\mathcal{M}_b^\sigma}^2 - \int_a^b \int_0^\infty \left[\partial_t \mu_t(s) + \partial_s \mu_t(s)\right]\left\|\eta^t(s)\right\|_\sigma^2 \mathrm{d}s\mathrm{d}t \\ &\le \left\|\eta^a\right\|_{\mathcal{M}_a^\sigma}^2 + 2\int_a^b \left\langle u(t),\eta^t\right\rangle_{\mathcal{M}_t^\sigma} \mathrm{d}t.\end{aligned}$$

Lemma 3.6 ([11]). For all $\tau \le a \le b \le T$, the following estimate holds:

$$\begin{aligned}&\left\|\eta^b\right\|_{\mathcal{M}_b^\sigma}^2 + \delta\int_a^b \kappa(t)\left\|\eta^t(s)\right\|_{\mathcal{M}_t^\sigma}^2 \mathrm{d}t \\ &\le \left\|\eta^b\right\|_{\mathcal{M}_b^\sigma}^2 - \int_a^b \int_0^\infty \left(\partial_t \mu_t(s) + \partial_s \mu_t(s)\right)\left\|\eta^t(s)\right\|_\sigma^2 \mathrm{d}s\mathrm{d}t \\ &\le \left\|\eta^a\right\|_{\mathcal{M}_a^\sigma}^2 + 2\int_a^b \left\langle u(t),\eta^t\right\rangle_{\mathcal{M}_t^\sigma} \mathrm{d}t.\end{aligned} \tag{3.5}$$

Definition 3.7. Let $g \in L^2(\Omega)$ and also let $T > \tau \in \mathbb{R}$. A binary $z(t) = \left(u(t),\eta^t\right)$ is said to be a

- strong solution of the problem (2.4) on the interval $[\tau,T]$, if

(i) $\left(u,\eta^t\right) \in L^\infty\left([\tau,T];\mathcal{H}_t^2\right)$;

(ii) The function η^t satisfies the formula (2.1);

(iii) For every $\phi \in V_1$ and almost every $t \in [\tau,T]$,

$$\left\langle \partial_t u,\phi\right\rangle + \left\langle u,\phi\right\rangle_1 + \int_0^\infty \mu_t(s)\left\langle \eta^t(s),\phi\right\rangle_1 \mathrm{d}s + \left\langle f(u),\phi\right\rangle = \left\langle g,\phi\right\rangle. \tag{3.6}$$

- weak solution of the problem (2.4) on an interval $[\tau,T]$,

(i) if there exists a sequence of regular data $\left(u_\tau^n,\eta_\tau^n\right) \in \mathcal{H}_\tau^2$ such that

$$\left(u_\tau^n,\eta_\tau^n\right)\to\left(u_\tau,\eta_\tau\right)\ \text{in}\ \mathcal{H}_\tau^1,\ \left(u,\eta^t\right)\in L^\infty\left(\left[\tau,T\right];\mathcal{H}_t^1\right),\ \text{and}$$

$$u^n\to u\ \text{in}\ C\left(\left[\tau,T\right];V_1\right),$$

where, $\left(u_\tau^n,\eta_\tau^n\right)$ is the sequence of the strong solution of the problem (2.4) with initial data $z_\tau^n=\left(u_\tau^n,\eta_\tau^n\right)\in\mathcal{H}_\tau^2$;

(ii) The function η^t satisfies the formula (2.1);

(iii) For every $\phi\in V_1$ and almost every $t\in\left[\tau,T\right]$, Eq. (2.4) satisfies (3.6).

Theorem 3.8. (Well-posedness and Regularity) Let $I=\left[\tau,T\right]$ for any $T>\tau$. Assume that (1.3), (1.2) and conditions (H$_1$) - (H$_4$) hold, and $g\in L^2\left(\Omega\right)$. Then,

(i) For any $\left(u_\tau,\eta_\tau\right)\in\mathcal{H}_\tau^1$, problem (2.7) admits a unique weak solution $\left(u,\eta^t\right)$, satisfying

$$\sup_{t\ge\tau}\left\|z\left(t\right)\right\|_{\mathcal{H}_t^1}^2+\int_\tau^t\kappa\left(r\right)\left\|\eta^r\right\|_{\mathcal{M}_r^1}^2\mathrm{d}r+\int_\tau^t\left\|u\left(r\right)\right\|_1^2\mathrm{d}r+\int_\tau^t\left\|\partial_t u\left(r\right)\right\|^2\mathrm{d}r\le Q,$$

where $Q=\max\left\{Q_1,Q_3\right\}$. In addition, if there exists a sequence of regular initial data $\left(u_\tau^n,\eta_\tau^n\right)\in\mathcal{H}_\tau^2$ such that

$$\left(u_\tau^n,\eta_\tau^n\right)\to\left(u_\tau,\eta_\tau\right)\ \text{in}\ \mathcal{H}_\tau^1,$$

then $u^n\to u$ in $C\left(\left[\tau,T\right];V_1\right)$.

(ii) For any $\left(u_\tau,\eta_\tau\right)\in\mathcal{H}_\tau^2$, problem (2.7) admits a unique strong solution $\left(u,\eta^t\right)$, satisfying

$$\sup_{t\ge\tau}\left\|z\left(t\right)\right\|_{\mathcal{H}_t^2}^2+\int_\tau^t\left\|u\left(r\right)\right\|_2^2\mathrm{d}r+\int_\tau^t\kappa\left(r\right)\left\|\eta^r\right\|_{M_r^2}^2\mathrm{d}r\le Q_2.$$

(iii) Moreover, the solutions of problem (2.7) depend continuously on the initial data, that is,

$$\left\|z_1\left(t\right)-z_2\left(t\right)\right\|_{\mathcal{H}_t^1}^2\le C\mathrm{e}^{C\left(R,\lambda_1\right)\left(t-\tau\right)}\left\|z_1\left(\tau\right)-z_2\left(\tau\right)\right\|_{\mathcal{H}_\tau^1}^2,$$

where $z_{1_\tau},z_{2_\tau}\in\mathcal{H}_\tau^1$, $t\in\left[\tau,T\right]$, and

$$\left\|z_1\left(t\right)-z_2\left(t\right)\right\|_{\mathcal{H}_t^2}^2\le C\mathrm{e}^{C\left(R,\lambda_1\right)\left(t-\tau\right)}\left\|z_1\left(\tau\right)-z_2\left(\tau\right)\right\|_{\mathcal{H}_\tau^2}^2,$$

where $z_1\left(t\right)$, $z_2\left(t\right)$ are two weak solutions of problem (2.7) corresponding to the initial data $z_{1_\tau}=\left(u_{1_\tau},\eta_{1_\tau}\right)$, $z_{2_\tau}=\left(u_{2_\tau},\eta_{2_\tau}\right)$, respectively.

Proof. Taking the inner product of equation (2.7) with u, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\|u\right\|^2+2\left\|u\right\|_1^2-2\int_0^\infty\mu_t\left(s\right)\left\langle\Delta\eta^t\left(s\right),u\right\rangle\mathrm{d}s+2\left\langle f\left(u\right),u\right\rangle=2\left\langle g,u\right\rangle.\tag{3.7}$$

From (1.2), we have

$$\left\langle f\left(u\right),u\right\rangle\ge\gamma_1\left\|u\right\|_{L^p}^p-\beta_1\left|\Omega\right|.$$

Furthermore, by Young's inequality and Poincaré's inequality, it holds that

$$2\left\langle g,u\right\rangle\le\theta\left\|u\right\|_1^2+\frac{1}{\lambda_1\theta}\left\|g\right\|^2,$$

where $\theta\in\left(0,1\right)$ and λ_1 is a constant.

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\|u\right\|^2+\left(2-\theta\right)\left\|u\right\|_1^2+2\left\langle u,\eta^t\right\rangle_{\mathcal{M}_t^1}+2\gamma_1\left\|u\right\|_{L^p}^p\le\frac{1}{\lambda_1\theta}\left\|g\right\|^2+2\beta_1\left|\Omega\right|.$$

Therefore, integrating the above inequality over $[\tau,t]$, we get

$$\begin{aligned}&\|u(t)\|^2+(2-\theta)\int_\tau^t\|u(r)\|_1^2\,\mathrm{d}r+2\int_\tau^t\langle u(r),\eta^r\rangle_{\mathcal{M}_r^1}\,\mathrm{d}r+2\gamma_1\int_\tau^t\|u(r)\|_{L^p}^p\,\mathrm{d}r\\&\le\|u(\tau)\|^2+Q_0(t-\tau),\ \forall t\ge\tau,\end{aligned}\tag{3.8}$$

where $Q_0=\dfrac{1}{\lambda_1\theta}\|g\|^2+2\beta_1|\Omega|$.

Define

$$E_0(t)=\|u(t)\|^2+\|\eta^t\|_{\mathcal{M}_t^1}^2.$$

Applying Lemma 3.6, we obtain

$$\begin{aligned}&E_0(t)+(2-\theta)\int_\tau^t\|u(r)\|_1^2\,\mathrm{d}r+\delta\int_\tau^t\kappa(r)\|\eta^r\|_{\mathcal{M}_r^1}^2\,\mathrm{d}r+2\gamma_1\int_\tau^t\|u(r)\|_{L^p}^p\,\mathrm{d}r\\&\le E_0(\tau)+Q_0(t-\tau),\end{aligned}\tag{3.9}$$

that is,

$$\sup_{t\ge\tau}\|z(t)\|_{\mathcal{H}_t^1}^2+\int_\tau^t\|u(r)\|_1^2\,\mathrm{d}r+\int_\tau^t\kappa(r)\|\eta^r\|_{\mathcal{M}_r^1}^2\,\mathrm{d}r+\int_\tau^t\|u(r)\|_{L^p}^p\,\mathrm{d}r\le Q_1.\tag{3.10}$$

where $Q_1=C\left(R,T,\delta,2-\theta,\theta\lambda_1,2\gamma_1,\|g\|,\beta_1|\Omega|\right)$.

Taking the inner product of equation (2.7) with Au, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|_1^2+2\|u\|_2^2+2\langle u,\eta^t\rangle_{\mathcal{M}_t^2}+2\langle f(u),-\Delta u\rangle-2\langle g,-\Delta u\rangle=0.\tag{3.11}$$

From (1.3), we know that

$$-2\langle f(u),-\Delta u\rangle=-2\int_\Omega f'(u)|\nabla u|^2\,\mathrm{d}x\le 2l\|u(t)\|_1^2.\tag{3.12}$$

Obviously, we have

$$2\langle g,-\Delta u\rangle\le\|u\|_2^2+\|g\|^2.$$

Thus we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|_1^2+\|u\|_2^2+2\langle u,\eta^t\rangle_{\mathcal{M}_t^2}\le 2l\|u(t)\|_1^2+\|g\|^2.\tag{3.13}$$

Integrating (3.13) over $[\tau,t]$ yields

$$\begin{aligned}&\|u(t)\|_1^2+\int_\tau^t\|u(r)\|_2^2\,\mathrm{d}r+2\int_\tau^t\langle u(r),\eta^r\rangle_{\mathcal{M}_r^2}\,\mathrm{d}r\\&\le\|u(\tau)\|_1^2+2l\int_\tau^t\|u(r)\|_1^2\,\mathrm{d}r+\|g\|^2(t-\tau),\end{aligned}\tag{3.14}$$

Applying Lemma 3.6, we have for any $t>\tau$,

$$\begin{aligned}&\|u(t)\|_1^2+\int_\tau^t\|u(r)\|_2^2\,\mathrm{d}r+\|\eta^t\|_{\mathcal{M}_t^2}^2+\delta\int_\tau^t\kappa(r)\|\eta^r\|_{\mathcal{M}_r^2}^2\,\mathrm{d}r\\&\le\|u(\tau)\|_1^2+\|\eta^\tau\|_{\mathcal{M}_\tau^2}^2+2l\int_\tau^t\|u(r)\|_1^2\,\mathrm{d}r+\|g\|^2(t-\tau).\end{aligned}\tag{3.15}$$

Define

$$E_1(t)=\|u(t)\|_1^2+\|\eta^t\|_{\mathcal{M}_t^2}^2,$$

then we have

$$\left\|z(t)\right\|_{\mathcal{H}_t^2}^2 \le E_1(t) \le \left(1+\frac{1}{\lambda_1}\right)\left\|z(t)\right\|_{\mathcal{H}_t^2}^2. \tag{3.16}$$

Therefore, for any $t \ge \tau$, we have

$$\begin{aligned} &E_1(t)+\int_\tau^t \left\|u(r)\right\|_2^2 \mathrm{d}r+\delta\int_\tau^t \kappa(r)\left\|\eta^r\right\|_{\mathcal{M}_r^2}^2 \mathrm{d}r \\ &\le E_1(\tau)+2l\int_\tau^t \left\|u(r)\right\|_1^2 \mathrm{d}r+\left\|g\right\|^2(t-\tau). \end{aligned} \tag{3.17}$$

Applying Gronwall's inequality, we deduce

$$\begin{aligned} &\sup_{t\ge\tau}\left\|z(t)\right\|_{\mathcal{H}_t^2}^2+\int_\tau^t \left\|u(r)\right\|_2^2 \mathrm{d}r+\int_\tau^t \kappa(r)\left\|\eta^r\right\|_{M_\tau^2}^2 \mathrm{d}r \\ &\le C\left(\left\|z(\tau)\right\|_{\mathcal{H}_\tau^2},T,\left\|g\right\|^2,\theta,\delta,\lambda_1,l,\beta_1\left|\Omega\right|\right):=Q_2. \end{aligned} \tag{3.18}$$

Taking the inner product of equation (2.7) with $\partial_t u$, we obtain

$$\left\|\partial_t u\right\|^2+\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|u\right\|_1^2-\int_0^\infty \mu_t(s)\left\langle\Delta\eta^t(s),\partial_t u\right\rangle\mathrm{d}s+\left\langle f(u),\partial_t u\right\rangle=\left\langle g,\partial_t u\right\rangle, \tag{3.19}$$

From (1.4), we have

$$\left\langle f(u),\partial_t u\right\rangle=\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega F(u)\mathrm{d}x.$$

By Hölder's inequality, it holds that

$$\left\langle g,\partial_t u\right\rangle \le \left\|g\right\|\left\|\partial_t u\right\|,$$

and from condition (H₂), we obtain

$$\begin{aligned} \left|\int_0^\infty \mu_t(s)\left\langle\Delta\eta^t(s),\partial_t u\right\rangle\mathrm{d}s\right| &\le \left\|\partial_t u\right\|\int_0^\infty \mu_t(s)\left\|\Delta\eta^t(s)\right\|\mathrm{d}s \\ &\le \left\|\partial_t u\right\|\left(\int_0^\infty \mu_t(s)\mathrm{d}s\right)^{\frac{1}{2}}\left(\int_0^\infty \mu_t(s)\left\|\eta^t(s)\right\|_2^2\mathrm{d}s\right)^{\frac{1}{2}} \\ &\le \left\|\partial_t u\right\|\sqrt{K_\tau(t)}\sqrt{\kappa(\tau)}\left\|\eta^t\right\|_{\mathcal{M}_t^2}. \end{aligned} \tag{3.20}$$

From (3.20), it follows that

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\left\|u\right\|_1^2+\int_\Omega F(u)\mathrm{d}x\right)+\left\|\partial_t u\right\|^2 \\ &\le \left\|\partial_t u\right\|\left(\left\|g\right\|+\sqrt{K_\tau(t)}\sqrt{\kappa(\tau)}\left\|\eta^t\right\|_{\mathcal{M}_t^2}\right) \\ &\le \frac{1}{2}\left\|\partial_t u\right\|^2+\frac{1}{2}\left(\left\|g\right\|+\sqrt{K_\tau(t)}\sqrt{\kappa(\tau)}\left\|\eta^t\right\|_{\mathcal{M}_t^2}\right)^2 \\ &\le \frac{1}{2}\left\|\partial_t u\right\|^2+\frac{1}{2}\left(2\left\|g\right\|^2+2K_\tau(t)\kappa(\tau)\left\|\eta^t\right\|_{\mathcal{M}_t^2}^2\right), \end{aligned} \tag{3.21}$$

Rearranging gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\left\|u\right\|_1^2+\int_\Omega F(u)\mathrm{d}x\right)+\frac{1}{2}\left\|\partial_t u\right\|^2 \le \left\|g\right\|^2+K_\tau(t)\kappa(\tau)\left\|\eta^t\right\|_{\mathcal{M}_t^2}^2. \tag{3.22}$$

Define the energy functional

$$E(t)=\frac{1}{2}\left\|u\right\|_1^2+\int_\Omega F(u)\mathrm{d}x,$$

then we have

$$\frac{\mathrm{d}}{\mathrm{d}t}E(t)+\frac{1}{2}\|\partial_t u\|^2 \le \|g\|^2 + K_\tau(t)\kappa(\tau)\|\eta^t\|^2_{\mathcal{M}_t^2}. \tag{3.23}$$

Integrating (3.23) over $[\tau,t]$, we get

$$E(t)+\frac{1}{2}\int_\tau^t \|\partial_t u(r)\|^2\,\mathrm{d}s \le E(\tau)+\|g\|^2(t-\tau)+\int_\tau^t \kappa(r)\|\eta^r\|^2_{\mathcal{M}_r^2}\,\mathrm{d}r, \tag{3.24}$$

hence,

$$\sup_{t\ge\tau} E(t)+\int_\tau^t \|\partial_t u(r)\|^2\,\mathrm{d}r \le Q_3. \tag{3.25}$$

Let $\{w_j\}_{j=1}^\infty$ be an orthonormal basis of $L^2(\Omega)$ which is also orthonormal in V_1, and satisfies $-\Delta w_j = \lambda_j w_j$, $j=1,2,\cdots$. Let $\{\chi_n\}_{n=1}^\infty$ be an orthonormal basis of $L^2_{\mu_t}(\mathbb{R}_+;V_1)$, and satisfies $-\Delta\chi_j=\lambda_j\chi_j$, $j=1,2,\cdots$. For each $n\in\mathbb{N}$, define the finite-dimensional subspaces

$$H_n=\operatorname{span}\{w_1,\cdots,w_n\}\subset V_1,\ M_n=\operatorname{span}\{\chi_1,\cdots,\chi_n\}\subset L^2_{\mu_t}(\mathbb{R}^+;V_1).$$

Denote by $P_n: V_1\to H_n$ the orthogonal projection onto H_n, and by $Q_n: L^2_{\mu_t}(\mathbb{R}^+;V_1)\to M_n$ the orthogonal projection onto M_n. Approximate the initial datum $z_\tau=(u_\tau,\eta_\tau)$ with a sequence $\{z_{\tau n}=(u_{\tau n},\eta_{\tau n})\}\subset\mathcal{H}_t^2$, such that

$$u_{\tau n}=P_n u_\tau \to u_\tau\in V_1,\ \eta_{\tau n}=Q_n\eta_\tau\to\eta_\tau\in M_t^1. \tag{3.26}$$

For each $n\in\mathbb{N}$, let $z_n=(u_n,\eta_n^t)$ be the approximation solutions, where

$$u_n=\sum_{j=1}^n T_j^n(t)w_j,\ \eta_n^t=\sum_{j=1}^n A_j^n(t)\chi_j,$$

with $T_j^n, A_j^n\in C^1([\tau,T])$. Then, for every test function $\psi\in H_n$ and every $t\in[\tau,T]$, $z_n=(u_n,\eta_n^t)$ satisfies

$$\langle\partial_t u_n,\psi\rangle+\langle u_n,\psi\rangle_1+\int_0^\infty \mu_t(s)\langle\eta_n^t(s),\psi\rangle_1\,\mathrm{d}s+\langle f(u_n),\psi\rangle=\langle g,\psi\rangle, \tag{3.27}$$

and

$$\eta_n^t(s)=\begin{cases}\int_0^s u_n(t-r)\,\mathrm{d}r, & 0<s\le t-\tau,\\ \eta_{\tau_n}(s-t+\tau)+\int_0^{t-\tau}u_n(t-r)\,\mathrm{d}r, & s>t-\tau.\end{cases} \tag{3.28}$$

Assume that $\psi\in H_m$ is fixed. Then for every $n\ge m$, equation (4.1) holds. Multiplying (4.1) by an arbitrary $\varphi\in C_0^\infty([\tau,T])$ and integrating over $[\tau,T]$, we obtain

$$\begin{aligned}&\int_\tau^T\varphi\langle\partial_t u_n(r),\psi\rangle\,\mathrm{d}r+\int_\tau^T\varphi\langle u_n(r),\psi\rangle_1\,\mathrm{d}r\\&+\int_\tau^T\varphi\int_0^\infty\mu_r(s)\langle\eta_n^r(s),\psi\rangle_1\,\mathrm{d}s\mathrm{d}r+\int_\tau^T\varphi\langle f(u_n),\psi\rangle\,\mathrm{d}r\\&=\int_\tau^T\varphi\langle g,\psi\rangle\,\mathrm{d}r.\end{aligned} \tag{3.29}$$

Hence, we have the following results:

$$\partial_t u_n \text{ is bounded in } L^2([\tau,T];L^2(\Omega)),$$

$$u_n \text{ is bounded in } L^\infty([\tau,T];H_0^1(\Omega)),$$

$$u_n \text{ is bounded in } L^2\left(\left[\tau,T\right];H_0^1(\Omega)\right),$$

$$\eta_n^t \text{ is bounded in } L^\infty\left(\left[\tau,T\right];\mathcal{M}_t^2\right),$$

$$f(u_n) \text{ is bounded in } L^{\frac{p}{p-1}}\left(\left[\tau,T\right];L^{\frac{p}{p-1}}(\Omega)\right).$$

Using the Galerkin approximation method, we know that there exists $z=\left(u,\eta^t\right)\in L^\infty\left(\left[\tau,T\right];\mathcal{H}_t^2\right)$ such that

$$\partial_t u_n \to \partial_t u \text{ weakly in } L^2\left(\left[\tau,T\right];L^2(\Omega)\right), \tag{3.30}$$

$$u_n \to u \text{ weakly* in } L^\infty\left(\left[\tau,T\right];H_0^1(\Omega)\right), \tag{3.31}$$

$$u_n \to u \text{ weakly in } L^2\left(\left[\tau,T\right];H_0^1(\Omega)\right), \tag{3.32}$$

$$\eta_n^t \to q^t \text{ weakly* in } L^\infty\left(\left[\tau,T\right];\mathcal{M}_t^2\right), \tag{3.33}$$

$$f(u_n)\to f(u) \text{ weakly in } L^{\frac{p}{p-1}}\left(\left[\tau,T\right];L^{\frac{p}{p-1}}(\Omega)\right). \tag{3.34}$$

By Lemma 2.1, we have

$$u_n \to u \text{ strongly in } C\left(\left[\tau,T\right];L^2(\Omega)\right),$$

and $u_n \to u$ almost everywhere in $\left[\tau,T\right]\times\Omega$. Due to the continuity of f, we obtain

$$f(u_n)\to f(u) \text{ almost everywhere in } \left[\tau,T\right]\times\Omega.$$

Let

$$\bar{\eta}_n^t=\eta_n^t-\eta^t,\ \bar{u}_n=u_n-u,\ \bar{\eta}_{\tau_n}=\eta_{\tau_n}-\eta_\tau,\ \bar{u}_{\tau_n}=u_{\tau_n}-u_\tau.$$

From condition (H$_2$) and Lemma 3.1, we obtain

$$\begin{aligned}&\left\|\bar{\eta}_n^t\right\|_{\mathcal{M}_t^1}^2\le K_\tau(t)\left\|\bar{\eta}_n^t\right\|_{\mathcal{M}_\tau^1}^2\\&\le C(T)\left((2+T-\tau)(T-\tau)\left\|\bar{u}_n\right\|_{C\left([\tau,T];H_0^1\right)}^2\kappa(\tau)+2\left\|\bar{\eta}_{\tau n}\right\|_{\mathcal{M}_\tau^1}^2\right)\to 0,\end{aligned}$$

hence $\eta_n^t\to\eta^t$ strongly in $\mathcal{M}_t^1$. By the uniqueness of the limit, we have $q^t=\eta^t$.

Since

$$\bar{\eta}_n^t(s)=\begin{cases}\int_0^s\bar{u}_n(t-r)\,\mathrm{d}r, & 0\le s\le t-\tau,\\ \bar{\eta}_\tau(s-t+\tau)+\int_\tau^t\bar{u}_n(r)\,\mathrm{d}r, & s>t-\tau,\end{cases}$$

then

$$\begin{aligned}&\int_0^\infty\mu_t(s)\left\langle\bar{\eta}_n^t(s),\varphi\right\rangle_1\mathrm{d}s\\&=\int_0^{t-\tau}\mu_t(s)\int_0^s\left\langle\bar{u}_n(t-r),\varphi\right\rangle_1\mathrm{d}r\mathrm{d}s+\int_0^\infty\mu_t(s+t-\tau)\left\langle\bar{\eta}_{\tau n}(s),\varphi\right\rangle_1\mathrm{d}s\\&\quad+\int_{t-\tau}^\infty\mu_t(s)\int_\tau^t\left\langle\bar{u}_n(r),\varphi\right\rangle_1\mathrm{d}r\mathrm{d}s.\end{aligned}$$

Furthermore, by (H_2), for $\forall t \in [\tau, T]$, we have almost everywhere

$$\int_0^{t-\tau} \mu_t(s) \int_0^s \langle \overline{u}_n(t-r), \psi \rangle_1 \,\mathrm{d}r\mathrm{d}s \le \int_0^{t-\tau} \mu_t(s) \int_0^s \|\overline{u}_n(t-r)\|_1 \|\psi\|_1 \,\mathrm{d}r\mathrm{d}s$$
$$\le \|\overline{u}_n\|_{L^\infty([\tau,T];\mathcal{H}^1)} \|\psi\|_1 (T-\tau)^2 K_\tau(t)\kappa(\tau) \to 0,$$

$$\int_{t-\tau}^{\infty} \mu_t(s) \int_\tau^t \langle \overline{u}_n(r), \psi \rangle_1 \,\mathrm{d}r\mathrm{d}s \le \int_{t-\tau}^{\infty} \mu_t(s) \int_\tau^t \|\overline{u}_n(r)\|_1 \|\psi\|_1 \,\mathrm{d}r\mathrm{d}s$$
$$\le \|\overline{u}_n\|_{L^\infty([\tau,T];\mathcal{H}^1)} \|\psi\|_1 (T-\tau)^2 K_\tau(t)\kappa(\tau) \to 0,$$

Therefore,

$$\int_0^\infty \mu_t(s+t-\tau) \langle \overline{\eta}_{\tau_n}(s), \psi \rangle_1 \,\mathrm{d}s \le \|\psi\|_1 K_\tau(t) \sqrt{\kappa(\tau)} \|\overline{\eta}_{\tau_n}\|_{M_\tau^2} \to 0,$$

$$\lim_{n\to\infty} \int_0^\infty \mu_t(s) \langle \overline{\eta}_n^t(s), \psi \rangle_1 \,\mathrm{d}s = 0,$$

$$\left| \int_0^\infty \mu_t(s) \langle \overline{\eta}_n^t(s), \psi \rangle_1 \,\mathrm{d}s \right| \le \int_0^\infty \mu_t(s) \|\overline{\eta}_n^t(s)\|_1 \|\psi\|_1 \,\mathrm{d}s$$
$$\le \|\psi\|_1 \sqrt{K_\tau(t)\kappa(\tau)} \|\overline{\eta}_n^t\|_{M_t^1} \in L^1([\tau, T]).$$

By the Dominated Convergence Theorem, we obtain

$$\lim_{n\to\infty} \int_\tau^T \varphi \int_0^\infty \mu_\tau(s) \langle \overline{\eta}_n^t(s), \psi \rangle_1 \,\mathrm{d}s\mathrm{d}\tau = 0.$$

Finally, we obtain the weak solution $z = (u, \eta^t)$ of equation (2.4).

Now, we prove the uniqueness of the solution. Assume $z_1 = (u_1(t), \eta_1^t)$ and $z_2 = (u_2(t), \eta_2^t)$ are two weak solutions. Then $\overline{z}(t) = z_1(t) - z_2(t) = (\overline{u}(t), \overline{\eta}^t)$ satisfies

$$\partial_t \overline{u} - \Delta \overline{u} - \int_0^\infty \mu_t(s) \Delta \overline{\eta}^t(s) \,\mathrm{d}s + f(u_1) - f(u_2) = 0, \tag{3.37}$$

where

$$\overline{\eta}^t(s) = \begin{cases} \int_0^s \overline{u}(t-r)\,\mathrm{d}r, & 0 \le s \le t-\tau, \\ \overline{\eta}_\tau(s-t+\tau) + \int_0^{t-\tau} \overline{u}(r)\,\mathrm{d}r, & s > t-\tau. \end{cases}$$

Taking the inner product of equation (3.35) with $\overline{u}$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\overline{u}\|^2 + 2\int_0^\infty \mu_t(s) \langle \overline{\eta}^t(s), \overline{u}(t) \rangle \,\mathrm{d}s = -2\|\overline{u}\|_1^2 - 2\langle f(u_1) - f(u_2), \overline{u}(t) \rangle$$
$$\le -2\lambda_1 \|\overline{u}\|^2 - 2|f'(\xi)| \|\overline{u}\|^2$$
$$\le C(\lambda_1, R) \|\overline{u}\|^2.$$

Integrating the above inequality over $[\tau, t]$, we get

$$\|\overline{u}(t)\|^2 + 2\int_\tau^t \langle \overline{u}(r), \overline{\eta}^r \rangle_{\mathcal{M}_r^1} \,\mathrm{d}r \le \|\overline{u}_\tau\|^2 + C(\lambda_1, R) \int_\tau^t \|\overline{u}(r)\|^2 \,\mathrm{d}r, \ t \in [\tau, T]. \tag{3.36}$$

By Lemma 3.6, we know

$$\|\overline{\eta}^t\|_{\mathcal{M}_t^1}^2 + \delta \int_\tau^t \kappa(r) \|\overline{\eta}^r(s)\|_{\mathcal{M}_r^1}^2 \,\mathrm{d}r \le \|\overline{\eta}_\tau\|_{\mathcal{M}_\tau^1}^2 + 2\int_\tau^t \langle \overline{u}, \overline{\eta}^r \rangle_{\mathcal{M}_r^1} \,\mathrm{d}r. \tag{3.37}$$

Set $F(t)=\|\bar{u}\|_1^2+\|\bar{\eta}^t\|^2_{\mathcal{M}_t^1}$, we have

$$\|\bar{z}(t)\|^2_{\mathcal{H}_t^1}\le F(t)\le C\|\bar{z}(t)\|^2_{\mathcal{H}_t^1}.$$

Then, from the above results, we get

$$F(t)\le F(\tau)+C(\lambda_1,R)\int_\tau^t\|\bar{u}(r)\|^2\,\mathrm{d}r.$$

Applying Lemma 2.2, we obtain

$$\|\bar{z}(t)\|^2_{\mathcal{H}_t^1}\le C\mathrm{e}^{C(\lambda_1,R)(t-\tau)}\|\bar{z}(\tau)\|^2_{\mathcal{H}_\tau^1},\ t\in[\tau,T],$$

where $\|\bar{z}_\tau\|^2_{\mathcal{H}_\tau^1}\le R$. This completes the proof.

According to Theorem 3.8, we can define the solution process for problem (2.4) on $\mathcal{H}_t^1$ as

$$U(t,\tau):\mathcal{H}_\tau^1\to\mathcal{H}_t^1,\ U(t,\tau)z_\tau=z(t),\forall z_\tau\in\mathcal{H}_\tau^1,t\ge\tau, \tag{3.38}$$

and $\{U(t,\tau)\}$ is a family of processes acting on $\{\mathcal{H}_t^1\}_{t\in\mathbb{R}}$.

4. Time-Dependent Attractors

4.1. Existence of Time-Dependent Absorbing Sets

Theorem 4.1. (Dissipativity) Assume that conditions (2.6), (2.7) and conditions (H₁) - (H₄) hold, and $g\in L^2(\Omega)$. Let $U(t,\tau)$, $t\ge\tau$, $\tau\in\mathbb{R}$ be the solution process defined by formula (3.38). For any initial value $z(\tau)\in\mathbb{B}_\tau(R)\subset\mathcal{E}_\tau^1$, there exist $\epsilon>0$ and $R_0>0$ such that the process $U(t,\tau)$ possesses a time-dependent absorbing set, namely, the family

$$\mathfrak{B}_t=\{\mathbb{B}_t(R_0)\}_{t\in\mathbb{R}}.$$

Proof. Using Poincaré's inequality and from formula (3.9), we obtain

$$\begin{aligned}&\|u(t)\|^2+\|\eta^t\|^2_{\mathcal{M}_t^1}+(2-\theta)\lambda_1\int_\tau^t\|u(r)\|^2\,\mathrm{d}r+\delta\int_\tau^t\kappa(r)\|\eta^r\|^2_{\mathcal{M}_r^1}\,\mathrm{d}r\\&+2\gamma_1\int_\tau^t\|u(r)\|_p^p\,\mathrm{d}r\le\|u(\tau)\|^2+\|\eta^\tau\|^2_{\mathcal{M}_\tau^1}+Q_0(t-\tau),\end{aligned}\tag{4.1}$$

Define

$$E(t)=\|u(t)\|^2+\|\eta^t\|^2_{M_t^1},$$

that is,

$$E(t)+2\varepsilon\int_\tau^tE(r)\,\mathrm{d}r\le E(\tau)+\varepsilon\int_\tau^tE(r)\,\mathrm{d}r+Q_0(t-\tau),$$

where $\varepsilon=\min\{(2-\theta)\lambda_1,\delta\inf_{y\in[\tau,t]}\kappa(r)\}$. Applying Lemma 2.2, we get

$$E(t)\le E(\tau)\mathrm{e}^{-\varepsilon(t-\tau)}+\frac{Q_0\mathrm{e}^{\varepsilon}}{1-\mathrm{e}^{-\varepsilon}}.$$

Furthermore,

$$\|z(t)\|^2_{\mathcal{H}_t^1}\le E(t)\le\left(1+\frac{1}{\lambda_1}\right)\|z(\tau)\|^2_{\mathcal{H}_\tau^1}\mathrm{e}^{-\varepsilon(t-\tau)}+\frac{R_0}{2}, \tag{4.2}$$

where $R_0 = \frac{2Q_0 e^{\varepsilon}}{1-e^{-\varepsilon}}$. For every $R>0$, there exist $t_0 = t_0(R) = \frac{1}{\varepsilon}\ln\frac{2\left(1+\frac{1}{\lambda_1}\right)R}{R_0} \le t$ and $R_0 > 0$ such that

$$\tau \le t - t_0 \Rightarrow U(t,\tau)\mathbb{B}_\tau(R) \subset \mathbb{B}_t(R_0).$$

The proof is complete.

4.2. Existence of Time-Dependent Attractors

To prove the asymptotic compactness of the solution process $U(t,\tau)$, we employ the solution decomposition technique, decomposing the solution into two parts

$$U(t,\tau)z_\tau = U_0(t,\tau)z_\tau + U_1(t,\tau)z_\tau, \tag{4.3}$$

where $U_0(t,\tau)z_\tau = (v(t),\xi^t)$ and $U_1(t,\tau)z_\tau = (w(t),\zeta^t)$.

By (1.3), let $f_0(u) = f(u) + lu$, where $l>0$ is chosen such that $f_0'(u) \ge 0$.

Define $U_0(t,\tau)z_\tau = (v(t),\xi^t)$ and $U_1(t,\tau)z_\tau = (w(t),\zeta^t)$ as satisfying the following systems respectively:

$$\begin{cases} v_t - \Delta v - \int_0^\infty \mu_t(s)\Delta\xi^t(s)\mathrm{d}s + f_0(u) - f_0(w) = 0, \\ \xi_t^t = -\partial_s \xi^t + v, \\ (v(\tau),\xi^\tau) = (u_\tau,\eta_\tau), \end{cases} \tag{4.4}$$

where,

$$\xi^t(s) = \begin{cases} \int_0^s v(t-r)\mathrm{d}r, & 0 \le s \le t-\tau, \\ \xi_\tau(s-t+\tau) + \int_0^{t-\tau} v(t-r)\mathrm{d}r, & s > t-\tau, \end{cases}$$

and

$$\begin{cases} w_t - \Delta w - \int_0^\infty \mu_t(s)\Delta\zeta^t(s)\mathrm{d}s + f_0(w) - lu = g, \\ \zeta_t^t = -\partial_s \zeta^t + w, \\ (w(\tau),\zeta^\tau) = (0,0), \end{cases} \tag{4.5}$$

where,

$$\zeta^t(s) = \begin{cases} \int_0^s w(t-r)\mathrm{d}r, & 0 \le s \le t-\tau, \\ \int_0^{t-\tau} w(t-r)\mathrm{d}r, & s > t-\tau. \end{cases}$$

Lemma 4.2. If there exists a sequence of regular data $(u_\tau^n,\eta_\tau^n) \in \mathcal{H}_\tau^2$ such that

$$(u_\tau^n,\eta_\tau^n) \to (u_\tau,\eta_\tau) \in \mathbb{B}_\tau(R) \subset \mathcal{H}_\tau^1,$$

and (H$_1$) - (H$_4$) hold, then the solutions of (4.4) satisfy the estimate:

$$\|z_1(t)\|_{\mathcal{H}_t^1}^2 \le C(R)e^{-\varepsilon_1(t-\tau)}. \tag{4.9}$$

Proof. Taking the inner product of the first equation in (4.5) with v in $L^2(\Omega)$, and the second equation with ξ^t in $\mathcal{M}_t^1$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\|v\|^2 + 2\|v\|_1^2 + 2\langle v, \eta^t\rangle_{\mathcal{M}_t^1} + 2\langle f_0(u) - f_0(w), v\rangle = 0. \tag{4.8}$$

$$\langle \xi_t^t, \xi^t\rangle_{\mathcal{M}_t^1} = \langle -\xi_s^t, \xi^t\rangle_{\mathcal{M}_t^1} + \langle v, \xi^t\rangle_{\mathcal{M}_t^1}.$$

where

$$\begin{aligned}\langle f_0(u) - f_0(w), v\rangle &= \langle f_0(u) - f_0(w), u - w\rangle \\ &= \int_\Omega \left(f_0(u(x)) - f_0(w(x))\right)\left(u(x) - w(x)\right)\mathrm{d}x \\ &= \int_\Omega f_0'\left(\theta w(x) + (1-\theta)u(x)\right)\left(u(x) - w(x)\right)^2 \mathrm{d}x \ge 0.\end{aligned}$$

That is,

$$\frac{\mathrm{d}}{\mathrm{d}t}\|v\|^2 + 2\|v\|_1^2 + 2\langle v, \xi^t\rangle_{\mathcal{M}_t^1} \le 0. \tag{4.9}$$

Integrating (4.9) over $[\tau, t]$ yields

$$\|v(t)\|^2 + 2\int_\tau^t \|v(r)\|_1^2 \,\mathrm{d}r + 2\int_\tau^t \langle v(r), \xi^t\rangle_{\mathcal{M}_r^1} \mathrm{d}r \le \|v(\tau)\|^2.$$

By Lemma 3.6, for any $t > \tau$, we have

$$\|v(t)\|^2 + 2\int_\tau^t \|v(r)\|_1^2 \,\mathrm{d}r + \|\xi^t\|_{\mathcal{M}_t^1}^2 + \delta\int_\tau^t \kappa(r)\|\xi^r(s)\|_{\mathcal{M}_r^1}^2 \mathrm{d}r \le \|v(\tau)\|^2 + \|\xi^\tau\|_{\mathcal{M}_\tau^1}^2. \tag{4.10}$$

Define

$$E_3(t) = \|v(t)\|^2 + \|\xi^t\|_{\mathcal{M}_t^1}^2,$$

Clearly,

$$\|z_1(t)\|_{\mathcal{H}_t^1}^2 \le E_3(t) = \|v(t)\|^2 + \|\xi^t\|_{\mathcal{M}_t^1}^2 \le \left(1 + \frac{1}{\lambda_1}\right)\|z_1(t)\|_{\mathcal{H}_t^1}^2.$$

Therefore, for any $t \ge \tau$, we have

$$E_3(t) + 2\int_\tau^t \|v(r)\|_1^2 \,\mathrm{d}r + \delta\int_\tau^t \kappa(r)\|\xi^r(s)\|_{\mathcal{M}_r^1}^2 \mathrm{d}r \le E_3(\tau). \tag{4.11}$$

That is,

$$E_3(t) + 2\varepsilon_1\int_\tau^t E_3(r)\,\mathrm{d}r \le E_3(\tau) + \varepsilon_1\int_\tau^t E_3(r)\,\mathrm{d}r,$$

where $\varepsilon_1 = \min\left\{1, \lambda_1, \delta\inf_{r\in[\tau,t]}\kappa(r)\right\}$. Applying Lemma 2.2, we obtain

$$E_3(t) \le C\mathrm{e}^{-\varepsilon_1(t-\tau)}.$$

Furthermore,

$$\|z_1(t)\|_{\mathcal{H}_t^1}^2 \le E_3(t) \le \left(1 + \frac{1}{\lambda_1}\right)\|z(\tau)\|_{\mathcal{H}_\tau^1}^2 \mathrm{e}^{-\varepsilon_1(t-\tau)} \le C(R)\mathrm{e}^{-\varepsilon_1(t-\tau)}, \tag{4.15}$$

where $\|z(\tau)\|_{\mathcal{H}_\tau^1} \le R$. This completes the proof.

Lemma 4.3. Assume that the nonlinearity f satisfy (1.2) - (1.4). If $g \in L^2(\Omega)$ and (H$_1$) - (H$_4$) hold, and there exists a sequence of regular data $(u_\tau^n, \eta_\tau^n) \in \mathcal{H}_\tau^2$ such that

$$\left(u_\tau^n,\eta_\tau^n\right)\to\left(u_\tau,\eta_\tau\right)\in\mathbb{B}_\tau\left(R\right)\subset\mathcal{H}_\tau^1,$$

then for each time $T>0$, there exists a positive constant $I=I\left(\|g\|,\|z_\tau\|_{\mathcal{H}_\tau^1},T,\lambda_1\right)$, such that the solutions of (4.8) satisfy:

$$\left\|U_2\left(T+\tau,\tau\right)z_2\left(\tau\right)\right\|_{\mathcal{H}_\tau^2}^2=\left\|z_2\left(T+\tau\right)\right\|_{\mathcal{H}_\tau^2}^2\le I. \tag{4.12}$$

Proof. Taking the inner product of the first equation in (4.5) with $-\Delta w$ in $H_0^1\left(\Omega\right)$, and the second equation with ζ^t in $\mathcal{M}_t^2$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\|w\|_1^2+2\|w\|_2^2+2\left\langle w,\zeta^t\right\rangle_{\mathcal{M}_t^2}+2\left\langle f_0\left(w\right)-lu,-\Delta w\right\rangle-2\left\langle g,-\Delta w\right\rangle=0. \tag{4.13}$$

$$\left\langle\zeta_t^t,\zeta^t\right\rangle_{\mathcal{M}_t^2}=\left\langle-\zeta_s^t,\zeta^t\right\rangle_{\mathcal{M}_t^2}+\left\langle w,\zeta^t\right\rangle_{\mathcal{M}_t^2}. \tag{4.14}$$

where

$$2\left\langle f_0\left(w\right),-\Delta w\right\rangle=2\left\langle\nabla f_0\left(w\right),\nabla w\right\rangle=2\int_\Omega f_0'\left(w\right)\left|\nabla w\right|^2\mathrm{d}x\ge0,$$

$$2\left\langle g,-\Delta w\right\rangle\le\frac{1}{2}\|w\|_2^2+2\|g\|^2,$$

$$2l\left\langle u,-\Delta w\right\rangle\le\frac{1}{2}\|w\|_2^2+2l^2\|u\|^2.$$

That is,

$$\frac{\mathrm{d}}{\mathrm{d}t}\|w\|_1^2+\|w\|_2^2+2\left\langle w,\zeta^t\right\rangle_{\mathcal{M}_t^2}\le2\|g\|^2+2l\|u\|^2. \tag{4.15}$$

Integrating over $\left[\tau,T+\tau\right]$ yields

$$\begin{aligned}&\left\|w\left(T+\tau\right)\right\|_1^2+\int_\tau^{T+\tau}\left\|w\left(r\right)\right\|_2^2\mathrm{d}r+2\int_\tau^{T+\tau}\left\langle w\left(r\right),\zeta^t\right\rangle_{\mathcal{M}_r^2}\mathrm{d}r\\&\le\left\|w\left(\tau\right)\right\|_1^2+2\|g\|^2T+2l\int_\tau^{T+\tau}\left\|u\left(r\right)\right\|^2\mathrm{d}r.\end{aligned}$$

By Lemma 3.6, for any $t>\tau$, we have

$$\begin{aligned}&\left\|w\left(T+\tau\right)\right\|_1^2+\int_\tau^{T+\tau}\left\|w\left(r\right)\right\|_2^2\mathrm{d}r+\left\|\zeta^T+\tau\right\|_{\mathcal{M}_t^2}^2+\delta\int_\tau^{T+\tau}\kappa\left(r\right)\left\|\zeta^r\left(s\right)\right\|_{\mathcal{M}_r^2}^2\mathrm{d}r\\&\le\left\|w\left(\tau\right)\right\|_1^2+2\|g\|^2T+2l\int_\tau^{T+\tau}\left\|u\left(r\right)\right\|^2\mathrm{d}r+\left\|\zeta^\tau\right\|_{\mathcal{M}_\tau^2}^2.\end{aligned} \tag{4.16}$$

Define

$$E_4\left(t\right)=\left\|w\left(t\right)\right\|_1^2+\left\|\zeta^t\right\|_{\mathcal{M}_t^2}^2,$$

Therefore, we have

$$\begin{aligned}&E_4\left(T+\tau\right)+\int_\tau^{T+\tau}\left\|w\left(r\right)\right\|_2^2\mathrm{d}r+\delta\int_\tau^{T+\tau}\kappa\left(r\right)\left\|\zeta^r\left(s\right)\right\|_{\mathcal{M}_r^2}^2\mathrm{d}r\\&\le E_4\left(\tau\right)+2\|g\|^2T+2l\int_\tau^{T+\tau}\left\|u\left(r\right)\right\|^2\mathrm{d}r.\end{aligned} \tag{4.17}$$

That is,

$$E_4\left(T+\tau\right)+2\varepsilon_1\int_\tau^{T+\tau}E_4\left(r\right)\mathrm{d}r\le C+\varepsilon_1\int_\tau^{T+\tau}E_4\left(r\right)\mathrm{d}r,$$

where $\varepsilon_1=\min\left\{1,\lambda_1,\delta\inf_{r\in[\tau,t]}\kappa\left(r\right)\right\}$, and $C=E_4\left(\tau\right)+2\|g\|^2T$. Applying Lemma 2.2, we obtain

$$E_4(T+\tau)\le C\mathrm{e}^{-\varepsilon_1 T}.$$

Similarly,

$$\left\|z_2(T+\tau)\right\|^2_{\mathcal{H}^2_{T+\tau}}\le E_4(T+\tau)\le I.$$

The proof is complete.

For any $\zeta_\tau\in L^2_{\mu_\tau}(\mathbb{R}^+;V_1)$, the Cauchy problem

$$\begin{cases}\partial_t\zeta^t=-\partial_s\zeta^t+w,\ t>\tau,\\ \zeta^\tau=\zeta_\tau\end{cases}\tag{4.19}$$

admits a unique solution $\zeta^t\in C\left([\tau,+\infty);L_{\mu_\tau}(\mathbb{R}^+;V_1)\right)$, with the explicit expression

$$\zeta^t(s)=\begin{cases}\int_0^s w(t-y)\,\mathrm{d}y, & 0\le s\le t-\tau,\\ \int_0^{t-\tau}w(t-y)\,\mathrm{d}y, & s>t-\tau.\end{cases}\tag{4.20}$$

Denote by $\mathfrak{B}_t$ the obtained time-dependent absorbing set. Let

$$\mathcal{K}_T=\Pi U_1(T,\tau)\mathfrak{B}_\tau,$$

where $\Pi:\mathcal{H}^1\times L_{\mu_t}(\mathbb{R}^+;\mathcal{H}^1)\to L_{\mu_t}(\mathbb{R}^+;\mathcal{H}^1)$ is a projection operator. This completes the proof.

Lemma 4.4. Let $z_2(t)=(w(t),\zeta^t)$ be the solution of problem (4.5). Assume that (1.2), (1.3) and conditions (H₁) - (H₄) hold, and $g\in L^2(\Omega)$. For any given $T>\tau$, there exists a positive constant $P_1=P_1\left(\|\mathfrak{B}_\tau\|_{\mathcal{H}^1_\tau}\right)$ such that

(i) K_T is bounded in $L^2_{\mu_\tau}(\mathbb{R}^+;V_2)\cap H^1_{\mu_\tau}(\mathbb{R}^+;V_1)$;

(ii) $\sup_{\eta^T\in K_T}\left\|\zeta^T(s)\right\|_1^2\le P_1$.

Proof. From expression (4.5), we have

$$\partial_s\zeta^t(s)=\begin{cases}w(t-s), & 0\le s\le t-\tau,\\ 0, & s>t-\tau.\end{cases}$$

By Lemma 4.3, (i) holds.

Next, it is easy to see that

$$\left\|\zeta^T(s)\right\|_1\le\begin{cases}\int_0^s\left\|w(T-y)\right\|_1\mathrm{d}y\le\int_0^{T-\tau}\left\|w(T-y)\right\|_1\mathrm{d}y, & 0\le s\le T-\tau,\\ \int_0^{T-\tau}\left\|w(T-y)\right\|_1\mathrm{d}y, & s>T-\tau.\end{cases}$$

Therefore, it follows from (4.12) that (ii) holds. The proof is complete.

Lemma 4.5. Assume Lemma 4.4 holds. Then for any given $T>\tau$, $U_1(T,\tau)\mathfrak{B}_\tau$ is relatively compact in $\mathcal{H}^1_\tau$.

Proof. Indeed, applying Lemma 2.3, we know that K_T is relatively compact in $L_{\mu_\tau}(\mathbb{R}^+;V_1)$. Using condition (H₂) once again, we obtain that K_T is relatively compact in $L_{\mu_\tau}(\mathbb{R}^+;V_1)$. Furthermore, from the compact embedding $V_2\hookrightarrow\hookrightarrow V_1$, we conclude that $U_1(T,\tau)\mathfrak{B}_\tau$ is relatively compact in $\mathcal{H}^1_T$. The proof is complete.

Theorem 4.6. Assume that (1.2), (1.3) and conditions (H₁)–(H₄) hold, $g \in L^2(\Omega)$, and $z_\tau = (u_\tau, \eta_\tau) \in \mathcal{H}_\tau^1$, satisfying $\|z_\tau\|_{\mathcal{H}_\tau^1} \le R$. Then the process $U(t,\tau)$ possesses a time-dependent global attractor $\mathcal{A} = \{A_t\}_{t\in\mathbb{R}}$. Moreover, the attractor is invariant, i.e., $U(t,\tau)A_\tau = A_t$, $\forall t \ge \tau$.

Proof. Theorem 3.8 shows that $U(t,\tau)$ possesses a time-dependent absorbing set $\mathfrak{B}_t = \{\mathbb{B}_t(R_0)\}_{t\in\mathbb{R}}$. Lemma 4.3 indicates that, for a sufficiently large positive constant R_1, the family $B_t = \{\mathbb{B}_t(R_1)\}_{t\in\mathbb{R}}$ is pullback attracting, where

$$B_t(R_1) = \left\{\zeta \mid \|\zeta\|_{\mathcal{H}_t^2} \le R_1\right\}.$$

Combining (4.7) and (4.12), we have

$$\begin{aligned}\operatorname{dist}_{\mathcal{H}_t^1}\left(U(t,\tau)\mathfrak{B}_\tau, B_t\right) &\le \operatorname{dist}_{\mathcal{H}_t^1}\left(U_0(t,\tau)\mathfrak{B}_\tau + U_1(t,\tau)\mathfrak{B}_\tau, B_t\right)\\ &= \operatorname{dist}_{\mathcal{H}_t^1}\left(U_0(t,\tau)\mathfrak{B}_\tau, B_t\right)\\ &\le C\left(\|\mathfrak{B}_\tau\|_{\mathcal{H}_\tau^1}\right)\mathrm{e}^{-\varepsilon_1(t-\tau)},\end{aligned}$$

where $\varepsilon_1 = \min\left\{1, \lambda_1, \delta \inf_{y\in[\tau,t]} \kappa(y)\right\}$.

For any bounded set $\mathfrak{B}_\tau = \{\mathbb{B}_\tau(R)\}_{\tau\in\mathbb{R}}$ in $\mathcal{H}_\tau^1$, there exists $t_0 = t_0(R)$ such that

$$\tau \le t - t_0 \Rightarrow U(t,\tau)\mathbb{B}_\tau(R) \subset \mathbb{B}_t(R_0).$$

Therefore,

$$\operatorname{dist}_{\mathcal{H}_t^1}\left(U(t,\tau)B_\tau, \mathfrak{B}_t\right) \le \varpi \mathrm{e}^{\varepsilon_1 t_0}\mathrm{e}^{-\varepsilon_1(t-\tau)},$$

where $\varpi = \sup_{0\le t-\tau\le t_0} \|U(t,\tau)B_\tau\|_{\mathcal{H}_t^1}$

Applying Lemma 2.4, we obtain

$$\operatorname{dist}_{\mathcal{H}_t^1}\left(U(t,\tau)B_\tau, B_t\right) \le C\left(\|B_\tau\|_{\mathcal{H}_\tau^1}\right)\mathrm{e}^{-\varepsilon_1(t-\tau)}.$$

We know that the solution process $U(t,\tau)$ of problem (2.7) is asymptotically compact in $\mathcal{H}_t^1$. Consequently, there exists a time-dependent attractor $\mathcal{A} = \{A_t\}_{t\in\mathbb{R}}$ in $\mathcal{H}_t^1$, and $\mathcal{A}$ is invariant, i.e.,

$$U(t,\tau)A_\tau = A_t,$$

and

$$\mathcal{A} = \left\{Z \mid t \to Z(t) \in \mathcal{H}_t^1 \text{ and } Z(t) \text{ is a CBT of the process } U(t,\tau)\right\}.$$

The proof is complete.

4.3. Regularity of the Attractors

For any fixed $\tau \in \mathbb{R}$ and $z_\tau \in A_\tau$, decompose $U(t,\tau)z_\tau$ as

$$U(t,\tau)z_\tau = U_2(t,\tau)z_\tau + U_3(t,\tau)z_\tau = \left(p(t), \varrho^t\right) + \left(q(t), \varsigma^t\right),$$

From (1.3), let $f_0(u) = f(u) + lu$, where $l > 0$ is chosen such that $f_0'(u) \ge 0$. $U_2(t,\tau)z_\tau$ satisfies

$$\begin{cases}\partial_t p + Ap + \int_0^\infty \mu_t(s) A\varrho^t(s)\,\mathrm{d}s = 0,\\ \partial_t\varrho^t + \partial_s\varrho^t = p(t),\\ p(x,t)\big|_{\partial\Omega} = 0,?p(x,t,\tau) = u_\tau(x),\\ \varrho^t(x,s)\big|_{\partial\Omega} = 0,\ \varrho^t(x,s) = \varphi_\tau(x,s),\end{cases} \tag{4.21}$$

and

$$\varrho^t(s) = \begin{cases}\int_0^s p(t-r)\,\mathrm{d}r, & 0 \le s \le t-\tau,\\ \varrho_\tau(s-t+\tau) + \int_0^{t-\tau} p(t-r)\,\mathrm{d}r, & s > t-\tau,\end{cases}$$

$U_3(t,\tau)z_\tau$ satisfies

$$\begin{cases}\partial_t q + Aq + \int_0^\infty \mu_t(s) A\varsigma^t(s)\,\mathrm{d}s + f_0(u) - lu = g,\\ \partial_t\varsigma^t + \partial_s\varsigma^t = q(t),\\ q(x,t)\big|_{\partial\Omega} = 0,\ q(x,t,\tau) = 0,\\ \varsigma^t(x,s)\big|_{\partial\Omega} = 0,\ \varsigma^\tau(x,s) = 0,\end{cases} \tag{4.22}$$

and

$$\varsigma^t(s) = \begin{cases}\int_0^s q(t-r)\,\mathrm{d}r, & 0 \le s \le t-\tau,\\ \int_0^{t-\tau} q(t-r)\,\mathrm{d}r, & s > t-\tau.\end{cases}$$

Here, $\varphi_\tau = \varrho_\tau + \varsigma_\tau$.

Multiplying equation (4.21) by p, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\|p\|^2 + 2\|p\|_1^2 + 2\langle p, \varrho^t\rangle_{\mathcal{M}_t^2} = 0.$$

Integrating the above over $[\tau, t]$, we have

$$\|p(t)\|^2 + 2\int_\tau^t \|p(r)\|_1^2\,\mathrm{d}r + 2\int_\tau^t \langle p, \varrho^t\rangle_{\mathcal{M}_r^1}\,\mathrm{d}r \le \|p(\tau)\|^2,\ \forall t \ge \tau.$$

By Lemma 3.6, we get

$$\|p(t)\|^2 + \|\varrho^t\|_{\mathcal{M}_t^1}^2 + 2\int_\tau^t \|p(r)\|_1^2\,\mathrm{d}r + \delta\int_\tau^t \kappa(r)\|\varrho^r(s)\|_{\mathcal{M}_r^1}^2\,\mathrm{d}r$$
$$\le \|p(\tau)\|^2 + \|\varrho_\tau\|_{\mathcal{M}_\tau^1}^2,\ \forall t \ge \tau.$$

Clearly,

$$\|z_1(t)\|_{\mathcal{H}_t^1}^2 \le E_5(t) = \|p(t)\|^2 + \|\varrho^t\|_{\mathcal{M}_t^1}^2 \le \left(1 + \frac{1}{\lambda_1}\right)\|z_1(t)\|_{\mathcal{H}_t^1}^2.$$

Therefore,

$$E_5(t) + 2\int_\tau^t \|p(r)\|_1^2\,\mathrm{d}r + \delta\int_\tau^t \kappa(r)\|\varrho^r(s)\|_{\mathcal{M}_r^1}^2\,\mathrm{d}r \le E_5(\tau).$$

That is,

$$E_5(t) + 2\varepsilon_1\int_\tau^t E_5(r)\,\mathrm{d}r \le E_5(\tau) + \varepsilon_1\int_\tau^t E_5(r)\,\mathrm{d}r,$$

where $\varepsilon_1 = \min\left\{1, \lambda_1, \delta \inf_{r\in[\tau,t]} \kappa(r)\right\}$. Applying Lemma 2.2, we obtain

$$E_5(t) \le E_5(\tau) \mathrm{e}^{-\varepsilon_1(t-\tau)}.$$

Furthermore,

$$\left\| U_0(t,\tau) z_\tau \right\|_{\mathcal{H}_t^1} \le C \mathrm{e}^{-\varepsilon_1(t-\tau)}. \tag{4.23}$$

Theorem 4.7 Assume that (1.2) - (1.3) and conditions (H_1) - (H_4) hold, $g \in L^2(\Omega)$. Let $z_2(t)$ be the solution of equation (3.43) with initial value $z_2(\tau) \in A_\tau$. Then the time-dependent global attractor $\mathcal{A} = \{A_t\}_{t\in\mathbb{R}}$ is bounded in the space $\mathcal{H}_t^2$, and the bound is independent of t.

Proof Multiplying equation (4.22) by Aq, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\|q\|_1^2 + 2\|q(t)\|_2^2 + 2\left\langle q, \varsigma^t \right\rangle_{\mathcal{M}_t^2} = 2\left\langle g, Aq \right\rangle - 2\left\langle f_0(u) - lu, Aq \right\rangle, \tag{4.24}$$

Since $-f_0'(u) \le 0 \le \delta$, we have

$$\begin{aligned}\left|-2\left\langle f_0(u), Aq \right\rangle\right| &= \left|-2\left\langle \nabla f_0(u), \nabla q \right\rangle\right| \\ &\le \left|-2\int_\Omega f_0'(u) \nabla u \nabla q \mathrm{d}x\right| \\ &\le 3\delta^2 \|u\|^2 + \frac{1}{3}\|q\|_2^2,\end{aligned}$$

$$\left|2l\left\langle u, Aq \right\rangle\right| \le 3l^2 \|u\|^2 + \frac{1}{3}\|q\|_2^2,$$

$$2\left\langle g, Aq \right\rangle \le 3\|g\|^2 + \frac{1}{3}\|q\|_2^2,$$

Substituting the above into (4.24), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|q\|_2^2 + \|q(t)\|_2^2 + 2\left\langle q, \varsigma^t \right\rangle_{\mathcal{M}_t^2} \le \left(3\delta^2 + 3l^2\right)\|u\|^2 + 3\|g\|^2, \tag{4.25}$$

Integrating (4.25) over $[\tau, t]$ yields

$$\|q(t)\|_1^2 + \int_\tau^t \|q(r)\|_2^2 \mathrm{d}r + 2\int_\tau^t \left\langle \varsigma^r, q(r) \right\rangle_{\mathcal{M}_r^2} \mathrm{d}r \le \|q(\tau)\|_1^2 + C(t-\tau), \tag{4.26}$$

Let

$$E_6(t) = \|q(t)\|_1^2 + \left\|\varsigma^t\right\|_{\mathcal{M}_t^2}^2.$$

By Lemma 3.6, we know

$$E_6(t) + \delta\int_\tau^t \kappa(r)\left\|\varsigma^r(s)\right\|_{\mathcal{M}_r^2}^2 \mathrm{d}r + \int_\tau^t \|q(r)\|_2^2 \mathrm{d}r \le E_6(\tau) + C(t-\tau), \tag{4.27}$$

That is,

$$E_6(t) + 2\varepsilon_2\int_\tau^t E_6(r)\mathrm{d}r \le E_6(\tau) + \varepsilon_2\int_\tau^t E_6(r)\mathrm{d}r + C(t-\tau),$$

where $\varepsilon_2 = \min\left\{\frac{1}{2}, \frac{\lambda_1}{2}, \delta \inf_{r\in[\tau,t]} \kappa(r)\right\}$.

Applying Lemma 2.2, we obtain

$$E_6(t) \le E_6(\tau)\mathrm{e}^{-\varepsilon_2(t-\tau)} + \frac{C\mathrm{e}^{\varepsilon_2}}{1-\mathrm{e}^{-\varepsilon_2}},$$

Furthermore,

$$\left\|z_2(t)\right\|^2_{\mathcal{H}_t^2} \le E_6(t) \le \left(1+\frac{1}{\lambda_1}\right)\left\|z_2(t)\right\|^2_{\mathcal{H}_t^2},$$

Thus,

$$\left\|z_2(t)\right\|^2_{\mathcal{H}_t^2} \le \left(1+\frac{1}{\lambda_1}\right)\left\|z_2(\tau)\right\|^2_{\mathcal{H}_\tau^2} \mathrm{e}^{-\varepsilon_2(t-\tau)} + \frac{C\mathrm{e}^{\varepsilon_2}}{1-\mathrm{e}^{-\varepsilon_2}} = \frac{C\mathrm{e}^{\varepsilon_2}}{1-\mathrm{e}^{-\varepsilon_2}} \le P_1. \tag{4.28}$$

Then, $\left\|U_1(t,\tau)z_\tau\right\|_{\mathcal{H}_t^2}$ is uniformly bounded with respect to t.
From (4.23) and (4.28), we get

$$\lim_{\tau\to-\infty} \operatorname{dist}_{\mathcal{H}_t^1}\left(U(t,\tau)A_\tau, K_t^2\right) = 0, \forall t \in \mathbb{R}.$$

By the invariance of the time-dependent attractor, we know

$$\operatorname{dist}_{\mathcal{H}_t^1}\left(A_t, K_t^2\right) = 0, \forall t \in \mathbb{R}.$$

Therefore, $A_t \subset \overline{K_t^2} = K_t^2$. The proof is complete.

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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