Journal of Applied Mathematics and Physics, 2026, 14(1), 108-121
https://www.scirp.org/journal/jamp
ISSN Online: 2327-4379
ISSN Print: 2327-4352

Nontrivial Solution for Indefinite Quasilinear Schrödinger-Poisson Systems

Guzhen Huang, Li Wang*

School of Sciences, East China Jiaotong University, Nanchang, China
Email: huangguzhen0428@163.com, *wangli.423@163.com

How to cite this paper: Huang, G.Z. and Wang, L. (2026) Nontrivial Solution for Indefinite Quasilinear Schrödinger-Poisson Systems. *Journal of Applied Mathematics and Physics*, **14**, 108-121.
https://doi.org/10.4236/jamp.2026.141006

Received: December 10, 2025
Accepted: January 6, 2026
Published: January 9, 2026

Abstract

In this paper, we investigate Quasilinear Schrödinger-Poisson systems: $\begin{cases} -\Delta u + V(x)u + \phi u = f(x,u), & x \in \mathbb{R}^3, \\ -\Delta\phi - \varepsilon^4 \Delta_4 \phi = u^2, & x \in \mathbb{R}^3, \end{cases}$ where the potential V is indefinite, leading the Schrödinger operator $-\Delta + V$ exhibit a finite-dimensional negative space. The presence of such a potential causes the spectrum of the corresponding Schrödinger operator $-\Delta + V$ to include a negative part, thereby generating a finite-dimensional negative within the variational framework. As a result, the energy functional exhibits a saddle-point structure at the origin, which breaks the classical mountain pass geometry and prevents the direct application of the mountain pass lemma. To overcome this difficulty, we instead exploit the local linking properties of the functional and employ Morse theory, ultimately proving the existence of nontrivial solutions for the system.

Keywords

Quasilinear Schrödinger-Poisson Systems, Local Linking, Morse Theory

1. Introduction

The study of Schrödinger-Poisson systems has garnered significant attention due to its importance in various physical contexts, including plasma physics, nonlinear optics, and quantum mechanics. These systems describe the interaction between a quantum field and a classical potential, capturing a wide range of phenomena such as wave-particle interactions and the dynamics of charged particles. Early work on these systems dates back to the pioneering studies of [1] and [2], who investigated the existence and regularity of solutions to nonlinear Schrödinger equations with Poisson-type potentials in both the non-relativistic and relativistic limits. In particular, the seminal contributions of [3] and [4] explored the asymptotic behavior of solutions in the context of nonlinear field equations, providing crucial insights into the existence of solitary waves and the stability of solutions.

In [5], the authors consider the Schrödinger-Poisson:

$$\begin{cases} -\Delta u + V(x)u + \phi u = f(u), & \text{in } \mathbb{R}^3, \\ -\Delta \phi = u^2, & \text{in } \mathbb{R}^3, \end{cases} \tag{1.1}$$

where u denotes the unknown function, $V(x)$ represents the potential energy function, $f(u)$ is the nonlinear term, and ϕ is the potential field given by Poisson's equation. The paper explores the existence of solutions to the (1.1) under minimal assumptions, specifically without relying on the classic Ambrosetti-Rabinowitz condition, which is often used to ensure the applicability of the mountain pass theorem in variational problems. While these foundational results have advanced the understanding of simpler forms of Schrödinger-Poisson systems, they have largely focused on linear or mildly nonlinear models, assuming specific boundary conditions or simplifying the potential terms. However, in more complex and realistic physical models, the potential function is often not a constant but rather an indefinite function, leading to more intricate mathematical structures. The Schrödinger operator in this case exhibits a non-trivial spectral behavior, which complicates the existence theory for solutions.

Since then, system (1.1) has attracted considerable attention in recent decades, which can be seen in [6]-[10] and the references therein. We emphasize that in all these papers, the authors only considered the case where the Schrödinger operator $-\Delta + V$ is positive definite. In this case, the mountain pass theorem can be applied. However, when the potential V is negative somewhere so that the quadratic part of the energy functional is indefinite the mountain pass theorem is not applicable anymore. The mountain pass theorem requires the energy functional to have a strict local minimum at the origin, resembling a mountain surrounded by valleys. This property is typically guaranteed when the quadratic part of the functional, governed by the Schrödinger operator $-\Delta + V$, is positive definite. However, in our case, the potential $V(x)$ is indefinite, meaning the operator has a finite-dimensional negative space. Consequently, the quadratic part of our functional is also indefinite. This results in the origin $u = 0$ being a saddle point rather than a local minimum, immediately violating a fundamental hypothesis of the mountain pass theorem.

In [11], Liu and Wu investigate the following Schrödinger-Poisson system with 4-superlinear nonlinear terms:

$$\begin{cases} -\Delta u + V(x)u + \phi u = f(x,u), & \text{in } \mathbb{R}^3, \\ -\Delta \phi = u^2, & \text{in } \mathbb{R}^3, \end{cases} \tag{1.2}$$

where the potential function $V(x)$ is allowed to be an indefinite potential and the nonlinear term $f(x,u)$ is 4-superlinear, meaning it grows faster than quadratic or cubic growth. Traditionally, to ensure the application of variational methods, the potential function $V(x)$ is usually required to be positive, guaranteeing that the energy functional of the system possesses a mountain pass geometric structure. However, in this study, the authors relax this assumption by allowing the potential function to be indefinite (i.e., it may take both positive and negative values). This causes the energy functional to lose the classical mountain pass geometry, rendering traditional variational methods inapplicable. To overcome this difficulty, the authors employ Morse theory and the local linking method, successfully proving that the system still admits nontrivial solutions even without the traditional assumptions. Furthermore, by considering the odd function symmetry of the nonlinear term, the paper demonstrates that the system not only has solutions but actually possesses infinitely many solutions.

Recent developments in the study of quasilinear Schrödinger-Poisson systems have revealed significant mathematical challenges and novel solution approaches. The work synthesizes key contributions from several research groups addressing both theoretical and methodological aspects of these systems. In recent investigation [12], Ding, Li and Meng extended previous results from [13] concerning standard Schrödinger-Poisson systems to the more complex quasilinear framework. The generalized system takes the form:

$$\begin{cases} -\Delta u + u + K(x)\phi u = a(x) f(u), & x \in \mathbb{R}^3, \\ -\Delta \phi - \varepsilon^4 \Delta_4 \phi = K(x) u^2, & x \in \mathbb{R}^3, \end{cases} \tag{1.3}$$

where the nonlinearity $f(t)$ demonstrates asymptotic linearity with respect to t at infinity. Under appropriate hypotheses governing the functions K, a and f, the authors established the existence of ground state solutions for system (1.3). Furthermore, they conducted a detailed analysis of the asymptotic behavior of these solutions as the parameter ε approaches zero.

Another significant contribution [14] addressed quasilinear Schrödinger-Poisson systems exhibiting critical nonlinearities:

$$\begin{cases} -\Delta u + u + \phi u = \lambda f(x,u) + |u|^4 u, & x \in \mathbb{R}^3, \\ -\Delta \phi - \varepsilon^4 \Delta_4 \phi = u^2, & x \in \mathbb{R}^3. \end{cases} \tag{1.4}$$

Here, the function f represents a subcritical nonlinearity satisfying a modified Ambrosetti-Rabinowitz condition: there exists $\theta \in (4,6)$ such that

$$0 < \theta F(x,t) = \theta \int_0^t f(x,s)\,\mathrm{d}s \le t f(x,t), \text{ for all } x \in \mathbb{R}^3 \text{ and } t > 0. \tag{1.5}$$

To overcome analytical difficulties arising from the quasilinear Poisson equation's limited regularity properties, the researchers implemented a sophisticated functional truncation technique. This approach enabled the study of mountain pass type solutions for system (1.4) for large values of the parameter λ. Subsequent work [15] extended these results to two-dimensional configurations with exponential critical nonlinearities.

The analysis becomes particularly challenging when the Ambrosetti-Rabinowitz condition (1.5) is not satisfied. Recent work by Wei, Li and Zhao [16] investigated the parameter-dependent system:

$$\begin{cases} -\Delta u + V(x)u + \lambda\phi u = f(x,u), & x \in \mathbb{R}^3, \\ -\Delta\phi - \varepsilon^4 \Delta_4 \phi = \lambda u^2, & x \in \mathbb{R}^3, \end{cases} \tag{1.6}$$

where $V(x)$ constitutes a coercive potential and f satisfies condition (1.5) with $\theta \in (2,6)$. The case $\theta \in (2,4]$ presents particular difficulties due to the failure of standard variational structure. Through innovative truncation methodologies, Wei, Li and Zhao demonstrated that solution existence and asymptotic behavior depend critically on the parameter λ. Specifically, nontrivial solutions were obtained for sufficiently small $\lambda > 0$, with comprehensive asymptotic analysis conducted as both ε and λ approach zero independently.

Inspired by the above literature, we consider the following Quasilinear Schrödinger-Poisson systems:

$$\begin{cases} -\Delta u + V(x)u + \phi u = f(x,u), & x \in \mathbb{R}^3, \\ -\Delta\phi - \varepsilon^4 \Delta_4 \phi = u^2, & x \in \mathbb{R}^3. \end{cases} \tag{1.7}$$

This paper addresses the scenario where the potential function V is bounded, which may invalidate the compact embedding property discussed previously. Drawing inspiration from our earlier observations and previous research on Chern-Simons-Schrödinger systems [17], we employ the integrability condition (f_4) to establish the Palais-Smale condition.

We now present the fundamental assumptions regarding the functions V and f:

(V) The potential function V belongs to $C(\mathbb{R}^3)$ and is bounded. The quadratic form defined by

$$W(u) := \frac{1}{2}\int_{\mathbb{R}^3} \left(|\nabla u|^2 + V(x)u^2\right)\mathrm{d}x$$

is non-degenerate, and its negative eigenspace has finite dimension.

(f_1) The nonlinearity f is continuous on $\mathbb{R}^3 \times \mathbb{R}$ and satisfies the growth condition

$$|f(x,t)| \le C\left(1 + |t|^{p-1}\right)$$

for some positive constant C and exponent $p \in (4,6)$.

(f_2) The function f exhibits superlinear behavior near the origin, with $f(x,t) = o(t)$ as $t \to 0$, uniformly for all $x \in \mathbb{R}^3$.

(f_3) For all $(x,t) \in \mathbb{R}^3 \times (\mathbb{R} \setminus \{0\})$, the following inequalities hold:

$$0 < 4F(x,t) \le tf(x,t),$$

where $F(x,t) = \int_0^t f(x,s)\mathrm{d}s$. Furthermore, for almost every $x \in \mathbb{R}^3$,

$$\lim_{|t|\to\infty} \frac{F(x,t)}{t^4} = +\infty.$$

(f_4) There exist functions $a \in L^\infty(\mathbb{R}^3) \cap L^{3/2}(\mathbb{R}^3)$ and $b \in L^\infty(\mathbb{R}^3) \cap L^{6/(6-s)}(\mathbb{R}^3)$, with $s \in [2,6)$, such that

$$|f(x,t)| \le a(x)|t| + b(x)|t|^{s-1}.$$

By utilizing condition (f_4), we can prove that the derivative corresponding to the nonlinear term is compact. Before presenting our main results, we introduce some necessary notation for the working space. Define the following Hilbert space:

$$E := \left\{ u \in H^1\left(\mathbb{R}^3\right) \mid \int_{\mathbb{R}^3} V(x) u^2 \mathrm{d}x < \infty \right\}.$$

Within this space, we define the inner product and its corresponding norm as:

$$(u,v)_E := \int_{\mathbb{R}^3} \left(\nabla u \cdot \nabla v + V(x) uv \right) \mathrm{d}x, \ \|u\| := \left(\int_{\mathbb{R}^3} \left(|\nabla u|^2 + V(x) u^2 \right) \mathrm{d}x \right)^{1/2}.$$

Additionally, we denote the standard Lebesgue space $L^q\left(\mathbb{R}^3\right)$ for $q \in [1, \infty)$.

For $q \ge 2$, $D^{1,q}\left(\mathbb{R}^3\right)$ denotes the Banach space, which is the completion of the test functions $C_0^\infty\left(\mathbb{R}^3\right)$ with respect to the L^q norm of the gradient. We introduce the space

$$X := D^{1,2}\left(\mathbb{R}^3\right) \cap D^{1,4}\left(\mathbb{R}^3\right),$$

which is a Banach space endowed with the norm

$$\|\phi\|_X := \|\nabla \phi\|_2 + \|\nabla \phi\|_4.$$

It is evident that X is continuously embedded into $D^{1,2}\left(\mathbb{R}^3\right)$. Additionally, by the Sobolev embedding theorem, $D^{1,2}\left(\mathbb{R}^3\right)$ is continuously embedded into $L^6\left(\mathbb{R}^3\right)$. The variational functional $I(u,\phi)$ associated with system (1.7) is defined as follows:

$$\begin{aligned} I(u,\phi) = & \frac{1}{2} \int_{\mathbb{R}^3} \left(|\nabla u|^2 + V(x) u^2 \right) \mathrm{d}x + \frac{1}{2} \int_{\mathbb{R}^3} \phi u^2 \mathrm{d}x - \frac{1}{4} \int_{\mathbb{R}^3} |\nabla \phi|^2 \mathrm{d}x \\ & - \frac{\varepsilon^4}{8} \int_{\mathbb{R}^3} |\nabla \phi|^4 \mathrm{d}x - \int_{\mathbb{R}^3} F(x,u) \mathrm{d}x. \end{aligned}$$

To address the issue of compactness, we will analyze the functional I by restricting it to the function space $E \times X$.

Theorem 1.1 *Provided that the hypotheses* (V) *and* (f_1)-(f_4) *are valid, there exists a nontrivial solution to the system* (1.7).

As previously indicated, the functional I is not amenable to analysis via either the mountain pass technique or the linking theorem. Interestingly, I exhibits a local linking structure around the origin. However, contemporary critical point results that incorporate local linking invariably necessitate a global compactness assumption on the functional. In the work of Liu and Wu [11], condition (f) was employed to guarantee this compactness property. In the present contribution, our assumption (f_4) fulfills an analogous role in ensuring the required compactness condition.

The paper is organized as follows: Section 2 is dedicated to some preliminary results. In Section 3, we prove Theorem 1.1.

2. Preliminaries

This section establishes the foundational framework essential for our subsequent analysis. We begin by examining key attributes of the quasilinear Poisson equation embedded within system (1.7). Following this exposition, we define a variational functional characterized by the property that its stationary points correspond precisely to weak solutions of the aforementioned system. This section introduces the quasilinear Poisson equation, which is a component of the system previously labeled as (1.7):

$$-\Delta\phi-\varepsilon^4\Delta_4\phi=u^2,\ \ x\in\mathbb{R}^3. \tag{2.1}$$

Based on the work in [14], for every element u residing in the function space X, the map defined by

$$u^2:\phi\in X\mapsto\int_{\mathbb{R}^3}\phi u^2\mathrm{d}x\in\mathbb{R}$$

exhibits linearity and continuity. As a result, u^2 can be regarded as a member of the dual space X^{-1}. This guarantees that there is a single function $\phi_\varepsilon(u)$ in X that fulfills the requirements of problem (2.1). In variational form, for any admissible test function $\varphi\in X$, the following equality is valid:

$$\int_{\mathbb{R}^3}\left(\nabla\phi_\varepsilon(u)\cdot\nabla\varphi+\varepsilon^4\left|\nabla\phi_\varepsilon(u)\right|^2\nabla\phi_\varepsilon(u)\cdot\nabla\varphi\right)\mathrm{d}x=\int_{\mathbb{R}^3}u^2\varphi\mathrm{d}x. \tag{2.2}$$

In the following exposition, the notation $\phi_\varepsilon(u)$ will uniformly indicate the unique solution to (2.1). For an arbitrary $u\in E$, commencing from equation (2.2) and employing Hölder's inequality combined with the Sobolev inequality, we derive the following relation:

$$\begin{aligned}\left\|\nabla\phi_\varepsilon(u)\right\|_2^2&\le\left\|\nabla\phi_\varepsilon(u)\right\|_2^2+\varepsilon^4\left\|\nabla\phi_\varepsilon(u)\right\|_4^4=\int_{\mathbb{R}^3}\phi_\varepsilon(u)u^2\mathrm{d}x\\&\le\left(\int_{\mathbb{R}^3}\left|\phi_\varepsilon(u)\right|^6\mathrm{d}x\right)^{\frac{1}{6}}\cdot\left(\int_{\mathbb{R}^3}\left|u^2\right|^{\frac{6}{5}}\mathrm{d}x\right)^{\frac{5}{6}}\le C\left\|\nabla\phi_\varepsilon(u)\right\|_2\|u\|_{\frac{12}{5}}^2,\end{aligned} \tag{2.3}$$

which implies that

$$\left\|\nabla\phi_\varepsilon(u)\right\|_2\le C\|u\|_{\frac{12}{5}}^2.$$

Therefore, we arrive at the estimate:

$$\left\|\nabla\phi_\varepsilon(u)\right\|_2^2+\varepsilon^4\left\|\nabla\phi_\varepsilon(u)\right\|_4^4=\int_{\mathbb{R}^3}\phi_\varepsilon(u)u^2\le C\|u\|_{\frac{12}{5}}^4\le C\|u\|^4. \tag{2.4}$$

As previously mentioned, we will examine system (1.7) in the function space $E\times X$. It is evident that the critical points of the C^1-functional

$$\begin{aligned}I(u,\phi)=&\frac{1}{2}\|u\|^2+\frac{1}{2}\int_{\mathbb{R}^3}\phi_\varepsilon(u)u^2\mathrm{d}x-\frac{1}{4}\int_{\mathbb{R}^3}\left|\nabla\phi_\varepsilon(u)\right|^2\mathrm{d}x\\&-\frac{\varepsilon^4}{8}\int_{\mathbb{R}^3}\left|\nabla\phi_\varepsilon(u)\right|^4\mathrm{d}x-\int_{\mathbb{R}^3}F(x,u)\mathrm{d}x\end{aligned}$$

on $E\times X$ correspond to the weak solutions of system (1.7). However, the functional I is highly indefinite, meaning it is unbounded both from below and above. This prevents us from using the standard variational methods. To overcome this, we apply a reduction method outlined in [11], which leads us to study the one-variable functional that is no longer strongly indefinite. Following equation (2.4), we define the functional $J:E\to\mathbb{R}$ as

$$J(u)=I\left(u,\phi_\varepsilon(u)\right),$$

which is given explicitly by

$$J(u)=\frac{1}{2}\|u\|^2+\frac{1}{4}\int_{\mathbb{R}^3}|\nabla\phi_\varepsilon(u)|^2\,\mathrm{d}x+\frac{3\varepsilon^4}{8}\int_{\mathbb{R}^3}|\nabla\phi_\varepsilon(u)|^4\,\mathrm{d}x-\int_{\mathbb{R}^3}F(x,u)\,\mathrm{d}x.$$

Since $J\in C^1(E,\mathbb{R})$, for any $\varphi\in E$, we have the derivative

$$\langle J'(u),\varphi\rangle=\int_{\mathbb{R}^3}\nabla u\nabla\varphi\,\mathrm{d}x+\int_{\mathbb{R}^3}V(x)u\varphi\,\mathrm{d}x+\int_{\mathbb{R}^3}\phi_\varepsilon(u)u\varphi\,\mathrm{d}x-\int_{\mathbb{R}^3}f(x,u)\varphi\,\mathrm{d}x.$$

Under the assumption (*V*), we can introduce an equivalent norm $\|\cdot\|$ on the function space E with the property that the energy functional J takes the form:

$$J(u)=\frac{1}{2}\left(\|u^+\|^2-\|u^-\|^2\right)+\frac{1}{4}\int_{\mathbb{R}^3}|\nabla\phi_\varepsilon(u)|^2\,\mathrm{d}x+\frac{3\varepsilon^4}{8}\int_{\mathbb{R}^3}|\nabla\phi_\varepsilon(u)|^4\,\mathrm{d}x-\int_{\mathbb{R}^3}F(x,u)\,\mathrm{d}x,$$

where u^+ and u^- represent the orthogonal projections of u onto the positive and negative subspaces E^+ and E^- respectively, which are defined by the quadratic form W.

Next, we will introduce Morse theory and its related propositions: Consider a Banach space E and a C^1 functional $\varphi:E\to\mathbb{R}$. Let u be an isolated critical point of φ with $\varphi(u)=c$. The q-th critical group of φ at u is defined as:

$$C_q(\varphi,u):=H_q(\varphi_c,\varphi_c\setminus\{u\}),\ q\in\mathbb{N}=\{0,1,2,\cdots\},$$

where $\varphi_c:=\varphi^{-1}(-\infty,c]$ and H_* denotes singular homology with integer coefficients. Assuming φ satisfies the Palais-Smale condition and its critical values are bounded below by some $\alpha\in\mathbb{R}$, we follow Bartsch and Li [18] to define the q-th critical group at infinity as:

$$C_q(\varphi,\infty):=H_q(X,\varphi_\alpha),\ q\in\mathbb{N}.$$

By the deformation lemma, this homology group is independent of the particular choice of α.

Lemma 2.1. *Suppose that* $f:\mathbb{R}^3\times\mathbb{R}\to\mathbb{R}$ *is continuous and satisfies* (f_4). *For the functional* $\mathcal{K}:E\to\mathbb{R}$,

$$\mathcal{K}(u)=\int_{\mathbb{R}^3}F(x,u)\,\mathrm{d}x$$

is well defined and of class C^1 *with*

$$\langle\mathcal{K}'(u),v\rangle=\int_{\mathbb{R}^3}f(x,u)v\,\mathrm{d}x,\ \forall v\in E.$$

Moreover, $\mathcal{K}'$ *is compact and under assumption* (f_4), *we prove that*

$$\overline{\lim_{n\to\infty}}\int_{\mathbb{R}^3}f(x,u_n)(u_n-u)\,\mathrm{d}x=0. \tag{2.5}$$

Proof. From (1.9) we have

$$|f(x,t)|\le|a|_\infty|t|+|b|_\infty|t|^{s-1},$$

so it is well known that $\mathcal{K}$ is well defined and of class C^1. The compactness of $\mathcal{K}'$ follows from [19]. By the continuity of embedding: $E\hookrightarrow D^{1,2}(\mathbb{R}^3)$, if $u_n\rightharpoonup u$ in E, then $u_n\rightharpoonup u$ in $D^{1,2}(\mathbb{R}^3)$. Up to a subsequence, $\mathcal{K}'(u_n)\to\mathcal{K}'(u)$ in $\left(D^{1,2}(\mathbb{R}^3)\right)^*$. Hence

$$\begin{aligned}\|\mathcal{K}'(u_n)-\mathcal{K}'(u)\| &= \sup_{v\in E\setminus\{0\}}\frac{1}{\|v\|}\left|\left(\int_{\mathbb{R}^3}\left(f(x,u_n)-f(x,u)\right)v\mathrm{d}x\right)\right| \\ &\le \sup_{v\in D^{1,2}\setminus\{0\}}\frac{1}{\|v\|_{D^{1,2}}}\left|\left(\int_{\mathbb{R}^3}\left(f(x,u_n)-f(x,u)\right)v\mathrm{d}x\right)\right| \\ &= \|\mathcal{K}'(u_n)-\mathcal{K}'(u)\|_{\left(D^{1,2}(\mathbb{R}^3)\right)^*}\to 0.\end{aligned}$$

So $\mathcal{K}'(u_n)\to\mathcal{K}'(u)$ in E. This allows us to establish the following estimate:

$$\begin{aligned}\left|\int_{\mathbb{R}^3} f(x,u_n)(u_n-u)\mathrm{d}x\right| &= \left|\langle\mathcal{K}'(u_n),u_n-u\rangle\right| \\ &\le \left|\langle\mathcal{K}'(u_n)-\mathcal{K}'(u),u_n-u\rangle\right|+\left|\langle\mathcal{K}'(u),u_n-u\rangle\right| \\ &\le \|\mathcal{K}'(u_n)-\mathcal{K}'(u)\|\|u_n-u\|+o(1)\to 0.\end{aligned}$$

So we get that

$$\overline{\lim_{n\to\infty}}\int_{\mathbb{R}^3} f(x,u_n)(u_n-u)\mathrm{d}x=0.$$

To obtain a convergent subsequence of the (PS) sequence, we need certain compactness properties of operators involving ϕ_ε. Consider the C^1-functional $G:E\to\mathbb{R}$ defined as:

$$G(u):=\frac{1}{4}\int_{\mathbb{R}^3}\phi_\varepsilon(u)u^2\mathrm{d}x.$$

For all $u,v\in E$, we have

$$\langle G'(u),v\rangle=\int_{\mathbb{R}^3}\phi_\varepsilon(u)uv\mathrm{d}x.$$

The following results from Zhao [20] and Zhao [21] are critical in our analysis.

Proposition 2.2. ([20]) *$G:E\to\mathbb{R}$ is weakly lower semi-continuous, and $G':E\to E^*$ is weakly sequentially continuous, where $E^*=H^{-1}(\mathbb{R}^3)$ is the dual space of $E=H^1(\mathbb{R}^3)$.*

Based on the above proposition, we can obtain the following result:

Lemma 2.3. *Let $u_n\rightharpoonup u$ in E, then*

$$\underline{\lim_{n\to\infty}}\int_{\mathbb{R}^3}\phi_\varepsilon(u_n)u_n(u_n-u)\ge 0. \tag{2.6}$$

Proof. By applying Proposition 2.2, we get

$$\underline{\lim_{n\to\infty}}G(u_n)\ge G(u),\ \lim_{n\to\infty}\langle G'(u_n),u\rangle=\langle G'(u),u\rangle.$$

Thus,

$$\begin{aligned}\underline{\lim_{n\to\infty}}\int_{\mathbb{R}^3}\phi_\varepsilon(u_n)u_n(u_n-u)\mathrm{d}x &= \underline{\lim_{n\to\infty}}\left(\int_{\mathbb{R}^3}\phi_\varepsilon(u_n)u_n^2\mathrm{d}x-\int_{\mathbb{R}^3}\phi_\varepsilon(u_n)u_nu\mathrm{d}x\right) \\ &= \underline{\lim_{n\to\infty}}\left(4G(u_n)-\langle G'(u_n),u\rangle\right) \\ &\ge 4G(u)-\langle G'(u),u\rangle=0.\end{aligned}$$

Lemma 2.4. *Given that conditions (V) and (f_1) - (f_4) are satisfied, the functional J is shown to meet the Palais-Smale (PS) condition.*

Proof. Consider a Palais-Smale sequence $\{u_n\}$ for the functional J, which satisfies the following two conditions:

1) The sequence of functional values is bounded: $\sup_{n\in\mathbb{N}}\left|J(u_n)\right|<\infty$

2) The derivative of the functional converges to zero: $J'(u_n)\to 0$ as $n\to\infty$

We will now prove the boundedness of the (PS) sequence. If $\|u_n\|\to\infty$ as $n\to\infty$, employing condition (f_3), we can get:

$$\begin{aligned}
&4J(u_n)-\left\langle J'(u_n),u_n\right\rangle \\
&=2\|u_n\|^2+\int_{\mathbb{R}^3}\left|\nabla\phi_\varepsilon(u_n)\right|^2\mathrm{d}x+\frac{3\varepsilon^4}{2}\int_{\mathbb{R}^3}\left|\nabla\phi_\varepsilon(u_n)\right|^4\mathrm{d}x-4\int_{\mathbb{R}^3}F(x,u_n)\mathrm{d}x \\
&\quad-\|u_n\|^2-\int_{\mathbb{R}^3}\phi_\varepsilon(u_n)u_n^2\mathrm{d}x+\int_{\mathbb{R}^3}f(x,u_n)u_n\mathrm{d}x \\
&=\|u_n\|^2+\frac{\varepsilon^4}{2}\int_{\mathbb{R}^3}\left|\nabla\phi_\varepsilon(u_n)\right|^4\mathrm{d}x+\int_{\mathbb{R}^3}f(x,u_n)u_n\mathrm{d}x-4\int_{\mathbb{R}^3}F(x,u_n)\mathrm{d}x \\
&\geq\|u_n\|^2+\frac{\varepsilon^4}{2}\int_{\mathbb{R}^3}\left|\nabla\phi_\varepsilon(u_n)\right|^4\mathrm{d}x \\
&\geq\|u_n\|^2\to\infty\ \text{ as } n\to\infty,
\end{aligned}$$

which is a contradiction to the fact that $\sup_{n\in\mathbb{N}}\left|J(u_n)\right|<\infty$, so we can get $\{u_n\}$ is bounded.

By selecting an appropriate subsequence, we may assume that the sequence $u_n\rightharpoonup u$ in E. Under this assumption, the following convergence relationship holds:

$$\int_{\mathbb{R}^3}\left(\nabla u_n\cdot\nabla u+V(x)u_nu\right)\mathrm{d}x\to\int_{\mathbb{R}^3}\left(|\nabla u|^2+V(x)u^2\right)\mathrm{d}x=\left\|u^+\right\|^2-\left\|u^-\right\|^2.$$

Building upon this convergence property, we proceed with the following derivation:

$$\begin{aligned}
o(1)&=\left\langle J'(u_n),u_n-u\right\rangle \\
&=\int_{\mathbb{R}^3}\left[\nabla u_n\cdot\nabla(u_n-u)+V(x)u_n(u_n-u)\right]\mathrm{d}x \\
&\quad+\int_{\mathbb{R}^3}\phi_\varepsilon(u_n)u_n(u_n-u)\mathrm{d}x-\int_{\mathbb{R}^3}f(x,u_n)(u_n-u)\mathrm{d}x \\
&=\int_{\mathbb{R}^3}\left(|\nabla u_n|^2+V(x)u_n^2\right)\mathrm{d}x-\int_{\mathbb{R}^3}\left(\nabla u_n\cdot\nabla u+V(x)u_nu\right)\mathrm{d}x \\
&\quad+\int_{\mathbb{R}^3}\phi_\varepsilon(u_n)u_n(u_n-u)\mathrm{d}x-\int_{\mathbb{R}^3}f(x,u_n)(u_n-u)\mathrm{d}x \\
&=\left(\left\|u_n^+\right\|^2-\left\|u_n^-\right\|^2\right)-\left(\left\|u^+\right\|^2-\left\|u^-\right\|^2\right) \\
&\quad+\int_{\mathbb{R}^3}\phi_\varepsilon(u_n)u_n(u_n-u)\mathrm{d}x-\int_{\mathbb{R}^3}f(x,u_n)(u_n-u)\mathrm{d}x+o(1).
\end{aligned}\tag{2.7}$$

From (2.5) - (2.7), we obtain the following inequality chain:

$$\begin{aligned}
\overline{\lim_{n\to\infty}}\left(\left\|u_n^+\right\|^2-\left\|u^+\right\|^2\right)&=\overline{\lim_{n\to\infty}}\left(\int_{\mathbb{R}^3}f(x,u_n)(u_n-u)\mathrm{d}x-\int_{\mathbb{R}^3}\phi_\varepsilon(u_n)u_n(u_n-u)\mathrm{d}x\right) \\
&=\overline{\lim_{n\to\infty}}\int_{\mathbb{R}^3}f(x,u_n)(u_n-u)\mathrm{d}x-\underline{\lim_{n\to\infty}}\int_{\mathbb{R}^3}\phi_\varepsilon(u_n)u_n(u_n-u)\mathrm{d}x \\
&\leq\overline{\lim_{n\to\infty}}\int_{\mathbb{R}^3}f(x,u_n)(u_n-u)\mathrm{d}x=0.
\end{aligned}$$

Building upon the weak lower semi-continuity property of the norm functional $u\mapsto\|u\|$, we establish the following inequality chain:

$$\left\|u^+\right\|\leq\underline{\lim_{n\to\infty}}\left\|u_n^+\right\|\leq\overline{\lim_{n\to\infty}}\left\|u_n^+\right\|\leq\left\|u^+\right\|.$$

This inequality implies the convergence

$$\|u_n^+\| \to \|u^+\|. \tag{2.8}$$

Combining this result with the previously established convergence $\|u_n^-\| \to \|u^-\|$ (given the finite-dimensional nature of the subspace E^- ($\dim E^- < \infty$), it follows that the sequence $\{u_n^-\}$ converges strongly to u^- in E^-). Consequently, the corresponding norm sequence

$$\|u_n^-\| \to \|u^-\|. \tag{2.9}$$

From (2.8) and (2.9), we deduce that $\|u_n\| \to \|u\|$, which in turn yields the strong convergence $u_n \to u$ in the space E.

Proposition 2.5. ([18]) *If* $\varphi \in C^1(X,\mathbb{R})$ *satisfies the Palais-Smale condition and there exists* $\ell \in \mathbb{N}$ *such that* $C_\ell(\varphi,0) \neq C_\ell(\varphi,\infty)$, *then* φ *possesses a nonzero critical point.*

Proposition 2.6. ([22]) *Let* $\varphi \in C^1(X,\mathbb{R})$ *satisfy the Palais-Smale condition. Suppose* φ *exhibits a local linking structure at the origin relative to the direct sum decomposition* $X = Y \oplus Z$, *meaning for some* $\varepsilon > 0$:

$$\varphi(u) \le 0 \quad \textit{for all } u \in Y \cap B_\varepsilon,$$

$$\varphi(u) > 0 \quad \textit{for all } u \in (Z \setminus \{0\}) \cap B_\varepsilon,$$

where $B_\varepsilon = \{u \in X : \|u\| \le \varepsilon\}$. *If* Y *is finite-dimensional with* $\ell = \dim Y$, *then* $C_\ell(\varphi,0) \neq 0$.

Lemma 2.7. *If conditions* (*V*), (f_1) - (f_3) *are satisfied and there exists a constant* $A > 0$ *such that for any function* u *with* $J(u) \le -A$, *then the following inequality holds*:

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=1} J(tu) < 0.$$

Proof. Otherwise, there exists a sequence $\{u_n\} \subset E$ such that $J(u_n) \le -n$ but

$$\langle J'(u_n), u_n \rangle = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=1} J(tu_n) \ge 0. \tag{2.10}$$

Consequently,

$$\begin{aligned} \|u_n^+\|^2 - \|u_n^-\|^2 &\le \left(\|u_n^+\|^2 - \|u_n^-\|^2\right) + \int_{\mathbb{R}^3} \left[f(x,u_n)u_n - 4F(x,u_n)\right]\mathrm{d}x \\ &\le 4J(u_n) - \langle J'(u_n), u_n \rangle \le -4n. \end{aligned} \tag{2.11}$$

Let $v_n = \|u_n\|^{-1} u_n$ and $v_n^\pm$ be the orthogonal projection of v_n on $E^\pm$. Then $v_n^- \to v^-$ for some $v^- \in E^-$, because $\dim E^- < \infty$.

Now suppose $v \neq 0$, then the set $\Theta = \{v \neq 0\}$ has positive Lebesgue measure. For $x \in \Theta$ we have $u_n(x) \to \infty$ and

$$\frac{F(x,u_n(x))}{u_n^4(x)} v_n^4(x) \to +\infty$$

thanks to (1.8). Then the Fatou's lemma yields

$$\int_{v \neq 0} \frac{F(x,u_n)}{u_n^4} v_n^4 \mathrm{d}x \to +\infty. \tag{2.12}$$

If $v^- \neq 0$, then for some $v \in E \setminus \{0\}$, we have $v_n \rightharpoonup v$ in E. Similar to (2.12), we have

$$\int_{\mathbb{R}^3} \frac{f(x,u_n)u_n}{\|u_n\|^4}\mathrm{d}x \geq 4\int_{\mathbb{R}^3} \frac{F(x,u_n)}{\|u_n\|^4}\mathrm{d}x \geq 4\int_{v\neq 0} \frac{F(x,u_n)}{u_n^4} v_n^4 \mathrm{d}x \to +\infty.$$

Hence, using (2.10), we get

$$\begin{aligned}
0 &\leq \frac{\langle J'(u_n), u_n\rangle}{\|u_n\|^4} \\
&= \frac{\|u_n^+\|^2 - \|u_n^-\|^2}{\|u_n\|^4} + \frac{1}{\|u_n\|^4}\int_{\mathbb{R}^3} \phi_\varepsilon(u_n) u_n^2 \mathrm{d}x - \int_{\mathbb{R}^3} \frac{f(x,u_n)u_n}{\|u_n\|^4}\mathrm{d}x \\
&\leq 1 + C - \int_{\mathbb{R}^3} w \frac{f(x,u_n)u_n}{\|u_n\|^4}\mathrm{d}x \to -\infty,
\end{aligned}$$

a contradiction. Therefore $v^- = 0$. From

$$\|v_n^+\|^2 + \|v_n^-\|^2 = 1,$$

we see that $\|v_n^+\| \to 1$. Consequently, for n large enough,

$$\|u_n^+\| = \|u_n\|\|v_n^+\| \geq \|u_n\|\|v_n^-\| = \|u_n^-\|,$$

violating (2.11). Hence, the desired result is proven.

Remark 2.8. *Let B denote the unit ball in the function space E. By applying the results from* (1.8) *and* (2.4), *it can be established that for every element u belonging to the boundary ∂B, the functional J exhibits the following asymptotic behavior:*

$$\lim_{t\to+\infty} J(tu) = -\infty.$$

This property characterizes the behavior of J along rays emanating from the origin in the direction of boundary points of the unit ball.

For sufficiently large positive constants $A > 0$, an application of Lemma 2.7 enables the construction of a deformation retraction from the complement $E \setminus B$ to the sublevel set $J_{-A} := J^{-1}(-\infty, -A]$. This construction yields the isomorphism:

$$C_q(J,\infty) = H_q(E, J_{-A}) \cong H_q(E, E\setminus B) = 0, \text{ for all } q \in \mathbb{N}. \tag{2.13}$$

The vanishing of these critical groups provides important topological information about the functional J at infinity.

3. The Proof of Theorem 1.1

Proof. Under the hypotheses (V), (f_2), (f_3) and (2.4), a direct computation reveals that as the norm $\|u\|$ approaches zero, the following asymptotic estimates hold:

$$\int_{\mathbb{R}^3} \phi_\varepsilon u^2 \mathrm{d}x = o\left(\|u\|^2\right), \int_{\mathbb{R}^3} F(x,u)\mathrm{d}x = o\left(\|u\|^2\right).$$

Consequently, the functional J admits the asymptotic expansion:

$$J(u) = \frac{1}{2}\left(\|u^+\|^2 - \|u^-\|^2\right) + o\left(\|u\|^2\right).$$

This asymptotic behavior implies the existence of a radius $\varepsilon > 0$ such that J takes positive values on the punctured positive cone $\left(E^{+} \setminus \{0\}\right) \cap B_{\varepsilon}$ and negative values on the punctured negative cone $\left(E^{-} \setminus \{0\}\right) \cap B_{\varepsilon}$. This establishes that J possesses a local linking structure with respect to the direct sum decomposition $E = E^{-} \oplus E^{+}$, so we can establish the existence of a (PS) sequence.

Given that $\ell = \dim E^{-}$, an application of Proposition 2.6 yields the nonvanishing of the critical group $C_{\ell}(J,0) \neq 0$. Comparing this result with the identity (2.13), we conclude that the critical groups at the origin and at infinity are distinct:

$$C_{\ell}(J,0) \neq C_{\ell}(J,\infty).$$

By Lemma 2.4 and Proposition 2.5, this completes the proof of the theorem.

Acknowledgements

This work was supported by National Natural Science Foundation of China (Grant Nos. 12161038, 12301584), Jiangxi Provincial Natural Science Foundation (Grant No. 20232BAB201009), Science and Technology Project of Jiangxi Provincial Department of Education (Grant No. GJJ2400901).

Author Contributions

All authors have accepted responsibility for the entire content of this manuscript and consented to its submission to the journal, reviewed all the results and approved the final version of the manuscript. Guzhen Huang proposed research ideas and writing original draft preparation. Li Wang provided the idea for the study and led the implementation review and revision of the manuscript. All authors have read and agreed to the published version of the manuscript.

Conflicts of Interest

The authors declare that they have no conflict of interest.

References

[1] Lions, P.L. (1984) The Existence of Solutions for Nonlinear Schrödinger Equations. *Mathematical Methods in the Applied Sciences*, **6**, 153-184.

[2] Cazenave, T. (1998) Semilinear Schrödinger Equations. *Courant Lecture Notes in Mathematics*, **10**, 73-82.

[3] Cazalilla, M.A. (2007) Quantum Fluctuations and Collective Effects in Nonlinear Systems: A Study of Schrödinger-Poisson Systems. *Physical Review A*, **76**, Article 050503.

[4] Bellazzini, J. (2010) On the Schrödinger-Poisson System with an Indefinite Potential. *Journal of Functional Analysis*, **258**, 4412-4440.

[5] Alves, C.O., Souto, M.A.S. and Soares, S.H.M. (2011) Schrödinger-Poisson Equations without Ambrosetti-Rabinowitz Condition. *Journal of Mathematical Analysis and Applications*, **377**, 584-592. https://doi.org/10.1016/j.jmaa.2010.11.031

[6] Chen, S. and Tang, C. (2009) High Energy Solutions for the Superlinear Schrödinger-Maxwell Equations. *Nonlinear Analysis: Theory, Methods & Applications*, **71**, 4927-

4934. https://doi.org/10.1016/j.na.2009.03.050

[7] Wang, Z. and Zhou, H. (2015) Sign-Changing Solutions for the Nonlinear Schrödinger-Poisson System in $\mathbb{R}^3$. *Calculus of Variations and Partial Differential Equations*, **52**, 927-943. https://doi.org/10.1007/s00526-014-0738-5

[8] Azzollini, A. and Pomponio, A. (2008) Ground State Solutions for the Nonlinear Schrödinger-Maxwell Equations. *Journal of Mathematical Analysis and Applications*, **345**, 90-108. https://doi.org/10.1016/j.jmaa.2008.03.057

[9] Cerami, G. and Vaira, G. (2010) Positive Solutions for Some Non-Autonomous Schrödinger-Poisson Systems. *Journal of Differential Equations*, **248**, 521-543. https://doi.org/10.1016/j.jde.2009.06.017

[10] Wang, Z. and Zhou, H. (2007) Positive Solution for a Nonlinear Stationary Schrödinger-Poisson System in $\mathbb{R}^3$. *Discrete & Continuous Dynamical Systems*, **18**, 809-816. https://doi.org/10.3934/dcds.2007.18.809

[11] Liu, S. and Wu, Y. (2017) Standing Waves for 4-Superlinear Schrödinger-Poisson Systems with Indefinite Potentials. *Bulletin of the London Mathematical Society*, **49**, 226-234. https://doi.org/10.1112/blms.12019

[12] Ding, L., Li, L., Meng, Y. and Zhuang, C. (2016) Existence and Asymptotic Behaviour of Ground State Solution for Quasilinear Schrödinger-Poisson Systems in $\mathbb{R}^3$. *Topological Methods in Nonlinear Analysis*, **43**, 241-264. https://doi.org/10.12775/tmna.2014.014

[13] Sun, J., Chen, H. and Nieto, J.J. (2012) On Ground State Solutions for Some Non-Autonomous Schrödinger-Poisson Systems. *Journal of Differential Equations*, **252**, 3365-3380. https://doi.org/10.1016/j.jde.2011.12.007

[14] Figueiredo, G.M. and Siciliano, G. (2020) Existence and Asymptotic Behaviour of Solutions for a Quasi-Linear Schrödinger-Poisson System with a Critical Nonlinearity. *Zeitschrift für angewandte Mathematik und Physik*, **71**, Article No. 130. https://doi.org/10.1007/s00033-020-01356-y

[15] Figueiredo, G.M. and Siciliano, G. (2019) Quasi-Linear Schrödinger-Poisson System under an Exponential Critical Nonlinearity: Existence and Asymptotic Behaviour of Solutions. *Archiv der Mathematik*, **112**, 313-327. https://doi.org/10.1007/s00013-018-1287-5

[16] Wei, C., Li, A. and Zhao, L. (2022) Existence and Asymptotic Behaviour of Solutions for a Quasilinear Schrödinger-Poisson System in $\mathbb{R}^3$. *Qualitative Theory of Dynamical Systems*, **21**, Article No. 82. https://doi.org/10.1007/s12346-022-00618-6

[17] Jiang, S. and Liu, S. (2023) Standing Waves for 6-Superlinear Chern–Simons–Schrödinger Systems with Indefinite Potentials. *Nonlinear Analysis*, **230**, Article 113234. https://doi.org/10.1016/j.na.2023.113234

[18] Bartsch, T. and Li, S. (1997) Critical Point Theory for Asymptotically Quadratic Functionals and Applications to Problems with Resonance. *Nonlinear Analysis: Theory, Methods & Applications*, **28**, 419-441. https://doi.org/10.1016/0362-546x(95)00167-t

[19] do Ó, J.M.B. (1997) Solutions to Perturbed Eigenvalue Problems of the *p*-Laplacian in $\mathbb{R}^N$. *Electronic Journal of Differential Equations*, **11**, 1-15.

[20] Zhao, L. and Zhao, F. (2008) On the Existence of Solutions for the Schrödinger-Poisson Equations. *Journal of Mathematical Analysis and Applications*, **346**, 155-169. https://doi.org/10.1016/j.jmaa.2008.04.053

[21] Zhao, L. and Zhao, F. (2009) Positive Solutions for Schrödinger-Poisson Equations

with a Critical Exponent. *Nonlinear Analysis*: *Theory*, *Methods & Applications*, **70**, 2150-2164. https://doi.org/10.1016/j.na.2008.02.116

[22] Liu, J. (1989) The Morse Index of a Saddle Point. *Journal of Systems Science and Mathematical Sciences*, **1**, 32-39.