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# Existence of Solutions for the Schrödinger-Type Bopp-Podolsky System with Indefinite Potentials

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## Abstract

This paper investigates the existence of nontrivial solutions for the Schrödinger-Bopp-Podolsky system with an indefinite potential function $V(x)$. The indefiniteness of $V$ prevents the direct application of standard variational methods such as the mountain pass theorem, as the associated Schrödinger operator $-\Delta + V$ possesses a finite-dimensional negative space, leading to a loss of coercivity and the standard linking structure in the energy functional. By employing a reduction method to handle the coupling with the Bopp-Podolsky equation and applying Morse theory combined with critical groups at infinity, we establish the existence of at least one nontrivial solution under appropriate assumptions on the indefinite potential $V$. and the nonlinearity $g$.

## Keywords

Schrödinger-Type Bopp-Podolsky System, Morse Theory, Critical Group, Indefinite Potential

## 1. Introduction

This article investigated a Schrödinger-Bopp-Podolsky system,

$$\begin{cases} -\Delta u + V(x)u + \phi u = g(x,u), & x \in \mathbb{R}^3, \\ -\Delta \phi + \rho^2 \Delta^2 \phi = 4\pi u^2, & x \in \mathbb{R}^3. \end{cases} \tag{1}$$

where $u, \phi : \mathbb{R}^3 \to \mathbb{R}$, and $a, b, \rho > 0$. The Schrödinger-Bopp-Podolsky system couples the Schrödinger equation with the Bopp-Podolsky equation. It originates from the search for standing wave solutions of the form $\psi(t,x) = \mathrm{e}^{i\omega t} u(x)$ to the Schrödinger equation coupled with the Bopp-Podolsky Lagrangian in an electromagnetic field. This framework was first introduced by Bopp [1] and later independently developed by Podolsky *et al.* [2] as a second-order gauge theory for electromagnetism.

In recent years, the Schrödinger-Bopp-Podolsky system has garnered significant attention from researchers, spurring a series of related studies. For example, Li *et al.* [3] established the existence of nontrivial ground state solutions for a nonlinear Schrödinger-Bopp-Podolsky system with critical Sobolev exponent. Their proof employed the Pohožaev-Nehari manifold method, Brézis-Nirenberg theory, monotonicity techniques, and the principle of global compactness. Meanwhile, Wang *et al.* [4] demonstrated both the existence and multiplicity of sign-changing solutions for the same system by applying perturbation methods and a minimax-based descending flow invariant set approach.

Similarly, Kirchhoff-type problems have also attracted considerable interest. Chen *et al.* [5] proved the existence of multiple solutions for an inhomogeneous Kirchhoff equation by combining the Ekeland variational principle with the mountain pass theorem. In [6], Wu obtained the existence of nontrivial solutions and infinitely many high-energy solutions for a Schrödinger-Kirchhoff-type problem using the symmetric mountain pass theorem. Additionally, Huang *et al.* [7] applied variational methods to establish existence and nonexistence results for a Kirchhoff-type problem under specified potential conditions, and further examined the "energy doubling" property of its nodal solutions.

Recently, Tang, Wang, and Wang [8] investigated the following Kirchhoff-type modification:

$$\begin{cases} -\left(a+b\int_{\mathbb{R}^3}|\nabla u|^2 x\right)\Delta u+V(x)u+\phi u=f(x,u), & x\in\mathbb{R}^3, \\ -\Delta\phi+\rho^2\Delta^2\phi=4\pi u^2, & x\in\mathbb{R}^3, \end{cases}$$

where the nonlocal term $b\left(\int_{\mathbb{R}^3}|\nabla u|^2\,\mathrm{d}x\right)\Delta u$ substantially alters the structure of the associated functional. Assuming the potential $V$ is indefinite, they employed Morse theory to establish the existence of solutions. One of the assumptions on the nonlinearity $f$ in their work is $(f_4)$: for any $r>0$,

$$\lim_{|x|\to\infty}\sup_{0<|t|\le r}\left|\frac{f(x,t)}{t}\right|=0,$$

which describes the decay property of the nonlinear term at spatial infinity. In contrast, the primary aim of the present paper is to study the standard (non-Kirchhoff) system (1) under an indefinite potential $V$. We not only remove the Kirchhoff term to focus on the intrinsic challenges posed by the indefiniteness of $V$ and the nonlocal coupling, but also replace assumption $(f_4)$ with the following condition on the nonlinearity $g$: $(g_4)$: there exist $s\in[2,6)$, $a\in L^\infty(\mathbb{R}^3)\cap L^{3/2}(\mathbb{R}^3)$, and $b\in L^\infty(\mathbb{R}^3)\cap L^{6/(6-s)}(\mathbb{R}^3)$ such that

$$|g(x,t)|\le a(x)|t|+b(x)|t|^{s-1}.$$

Condition $(g_4)$ represents a broader growth condition. Unlike $(f_4)$, it does not require decay of $g$ as $|x|\to\infty$. Instead, it controls the global behavior of the nonlinearity through the functions $a(x)$ and $b(x)$ belonging to specific Lebesgue spaces. This marks a substantive difference from $(f_4)$ and allows us to handle a wider class of nonlinearities.

Recently, significant attention has been devoted to the study of coupled elliptic systems arising from quantum electrodynamics, particularly those modeling the interaction between a charged particle and an electromagnetic field. A notable advancement beyond classical Maxwell theory is the Bopp-Podolsky electrodynacs, which accounts for short-range interactions more accurately. In the pioneering work by d'Avenia and Siciliano [9], the stationary Schrödinger-Bopp-Podolsky system was introduced and studied via variational methods, laying the foundation for subsequent investigations.

Motivated by the existence results for semiclassical solutions of the Schrödinger equation with critical nonlinearity, as examined by Ding and Lin [10], this paper focuses on the following critical Schrödinger-Bopp-Podolsky system in the semiassical regime:

$$\begin{cases} -\varepsilon^2\Delta u + V(x)u + Q(x)\phi u = h(x,u) + K(x)|u|^4 u & \text{in } \mathbb{R}^3, \\ -\Delta\phi + a^2\Delta^2\phi = 4\pi Q(x)u^2 & \text{in } \mathbb{R}^3, \end{cases}$$

where $u,\phi:\mathbb{R}^3\to\mathbb{R}$ are unknown functions, and $\varepsilon, a > 0$ are parameters. The functions $V, K, Q$ satisfy appropriate conditions, and the nonlinearity $h$ is subcritical. In the equivalent formulation with $\lambda = 1/\varepsilon^2$, the system becomes

$$\begin{cases} -\Delta u + \lambda V(x)u + \lambda Q(x)\phi u = \lambda h(x,u) + \lambda K(x)|u|^4 u & \text{in } \mathbb{R}^3, \\ -\Delta\phi + a^2\Delta^2\phi = 4\pi Q(x)u^2 & \text{in } \mathbb{R}^3, \end{cases}$$

and the aim is to study the behavior as $\lambda\to+\infty$.

In recent studies, Xueqing Peng [11] focused on the existence and multiplicity of solutions in three dimensions, considering scenarios of general potentials, constant potentials, coercive potentials, and steep potentials. The specific content is as follows:

$$\begin{cases} -\Delta u + V(x)u + K(x)\phi u = f(u), \\ -\Delta\phi + a^2\Delta^2\phi = 4\pi u^2. \end{cases}$$

where $a,\lambda \ge 0$, $K(x) > 0$, $V(x)$ is the potential, and $f$ is the nonlinear term. When $K(x)\equiv 1$ and $\lambda = 1$, they obtained three theorems concerning the multiplicity of solutions: i) Under some weaker conditions on $f$, they obtained infinitely many high-energy solutions using the symmetric mountain pass theorem; ii) They proved the existence of ground state solutions under a natural constraint and obtained infinitely many radial solutions using Krasnoselskii genus theory; iii) By using the invariant set of descending flow method, they studied the existence and multiplicity of sign-changing solutions.

The paper is organized as follows. At the end of Section 1, we introduce the variational framework and the required conditions by Theorem 1. In Section 2, we list the preliminaries required by the proof of Theorem 1. In Section 3, we focus on verifying the (PS) condition of the functional and computing the critical groups at infinity. In Section 4, we finish the proof of Theorem 1. In Section 5, we summarize the article.

System (1) has a variational structure. Its corresponding energy functional $\mathcal{J}:H^1(\mathbb{R}^3)\times D^{1,2}(\mathbb{R}^3)\to\mathbb{R}$ is

$$\mathcal{J}(u,\phi)=\frac{1}{2}\int_{\mathbb{R}^3}\left(|\nabla u|^2+V(x)u^2\right)x-\frac{1}{16\pi}\int_{\mathbb{R}^3}|\nabla\phi|^2 x$$
$$-\frac{\rho^2}{16\pi}\int_{\mathbb{R}^3}|\Delta\phi|^2 x+\frac{1}{2}\int_{\mathbb{R}^3}\phi u^2 x-\int_{\mathbb{R}^3}G(x,u)x$$

where $G(x,t)=\int_0^t g(x,s)s$. It can be shown that $(u,\phi)$ is a solution of system (1) if and only if it is a critical point of the functional $\mathcal{J}$. However, due to the indefiniteness of the potential function, although there is a local linking at the origin, all critical point theorems require the functional to satisfy global compactness conditions. For specific details, refer to [12], which presents essential difficulties for the study of its critical points. We employ the reduction method proposed by Benci *et al.* in [13], with appropriate modifications. Since $\partial_\phi$ is a $C^1$ functional. When $H_h$ is the graph of the map $h:u\in H^1(\mathbb{R}^3)\mapsto\phi_u\in D^{1,2}$, by the implicit function theorem, we have

$$H_h=\left\{(u,\phi)\in H^1(\mathbb{R}^3)\times D^{1,2}:\partial_\phi\partial(u,\phi)=0\right\},\quad \Phi\in C^1\left(H^1(\mathbb{R}^3);D^{1,2}\right).$$

Therefore, if $u$ is a critical point of $\Phi:H^1(\mathbb{R}^3)\to\mathbb{R}$

$$\Phi(u)=\frac{1}{2}\int_{\mathbb{R}^3}\left(|\nabla u|^2+V(x)u^2\right)x+\frac{1}{4}\int_{\mathbb{R}^3}\phi_u u^2 x-\int_{\mathbb{R}^3}G(x,u)x,$$

then $(u,\phi)$ is a solution of (1). More details can be found in [14] [15].

Now we give some conditions on $V$ and $g$:

(V) $V\in C(\mathbb{R}^3)$ is a bounded function such that the quadratic form

$$Q(u)=\frac{1}{2}\int_{\mathbb{R}^3}\left(|\nabla u|^2+V(x)u^2\right)x$$

is nondegenerate and the negative space of $Q$ is finite-dimensional.

(g1) $g\in C(\mathbb{R}^3\times\mathbb{R})$, and there exist $C>0$ and $p\in(4,6)$ such that

$$|g(x,t)|\le C\left(1+|t|^{p-1}\right).$$

(g2) $g$ is superlinear at zero, i.e., $g(x,t)=o(t)$ as $t\to 0$, uniformly in $x\in\mathbb{R}^3$.

(g3) For $(x,t)\in\mathbb{R}^3\times\mathbb{R}\setminus\{0\}$, $0<4G(x,t)\le tg(x,t)$. Furthermore, for almost all $x\in\mathbb{R}^3$,

$$\lim_{|t|\to\infty}\frac{G(x,t)}{t^4}=+\infty.$$

(g4) For $s\in[2,6)$, there exist $a\in L^\infty(\mathbb{R}^3)\cap L^{3/2}(\mathbb{R}^3)$, $b\in L^\infty(\mathbb{R}^3)\cap L^{6/(6-s)}(\mathbb{R}^3)$ such that

$$|g(x,t)|\le a(x)|t|+b(x)|t|^{s-1}.$$

## 2. Preliminaries

**Theorem 1** If conditions (V) and (g1)-(g4) are satisfied, then system (1) has a nontrivial solution.

To prove Theorem 1, we recall concepts and conclusions related to infinite-dimensional Morse theory [16], [17], and introduce several important propositions.

**Definition 1** Let $X$ be a Banach space, $\varphi : X \to \mathbb{R}$ a $C^1$ functional, and $u$ an isolated critical point of $\varphi$ with $\varphi(u) = c$. Let $\varphi_c = \varphi^{-1}(-\infty, c)$. Then the $q$-th critical group of $\varphi$ at $u$ is defined as:

$$C_q(\varphi, u) := H_q(\varphi_c, \varphi_c \setminus \{u\}),\ q \in \mathbb{N} = 0, 1, 2, \cdots$$

where $H_q$ denotes singular homology with coefficients in $\mathbb{Z}$.

**Definition 2** [18] If $\varphi$ satisfies the (PS) condition and the critical values of $\varphi$ are bounded below by $a$, then the $q$-th critical group of $\varphi$ at infinity is defined as:

$$C_q(\varphi, \infty) := H_q(X, \varphi_a),\ q \in \mathbb{N}.$$

**Remark:** By the deformation lemma, the homology on the right-hand side is independent of the choice of $a$.

**Proposition 1** [18] Let $\varphi \in C^1(X, \mathbb{R})$ satisfy the (PS) condition, and suppose there exists $l \in \mathbb{N}$ such that $C_l(\varphi, 0) \neq C_l(\varphi, \infty)$. Then $\varphi$ has a nonzero critical point.

**Proposition 2** [16] If $\varphi \in C^1(X, \mathbb{R})$ satisfies the (PS) condition, let $B_\varepsilon = \{u \in X \mid \|u\| \le \varepsilon\}$. If $\varphi$ has a local linking at 0 with respect to the decomposition $X = X^- \oplus X^+$, then there exists $\varepsilon > 0$ such that

$$\begin{cases} \varphi(u) \le 0, & u \in X^- \cap B_\varepsilon, \\ \varphi(u) > 0, & u \in (X^+ \setminus \{0\}) \cap B_\varepsilon. \end{cases}$$

If $l = \dim X^- < \infty$, then $C_l(\varphi, 0) \neq 0$.

**Proposition 3** [12] There exists a constant $k > 0$ such that for all $u \in H^1(\mathbb{R}^3)$,

$$0 \le \frac{1}{4}\int_{\mathbb{R}^3} \phi_u u^2 x \le k\|u\|^4.$$

Now we give the proof of Theorem 1.

## 3. Proof of the Main Lemmas

In this paper, for convenience, we always denote $X = H^1(\mathbb{R}^3)$. As mentioned earlier, to solve system (1), it suffices to find a critical point of the $C^1$ functional $\Phi : X \to \mathbb{R}$, where

$$\Phi(u) = \frac{1}{2}\int_{\mathbb{R}^3} \left(|\nabla u|^2 + V(x)u^2\right)x + \frac{1}{4}\int_{\mathbb{R}^3} \phi_u u^2 x - \int_{\mathbb{R}^3} G(x, u)x.$$

Due to condition (V), there exists an equivalent norm $\|\cdot\|$ on $X$ such that $u^\pm$ are the orthogonal projections of $u$ onto $X^\pm$, where $X^\pm$ are the positive/negative subspaces of the quadratic form $Q$. Since our working space $X \hookrightarrow L^s(\mathbb{R}^3)$ is non-compact, we need assumption (g4) to recover the (PS) condition.

Therefore, we have the following lemma. Due to our assumption on (V), $V$ may be negative somewhere, and the Sobolev embedding $H^1(\mathbb{R}^3) \hookrightarrow L^2(\mathbb{R}^3)$ is non-compact, we need condition (g4) to recover the (PS) condition.

**Lemma 1** Assume $g: \mathbb{R}^3 \times \mathbb{R}$ is continuous and satisfies (g4). Then the functional $S: X \to \mathbb{R}$ defined by

$$S(u) = \int_{\mathbb{R}^3} G(x,u)x$$

is well-defined and belongs to $C^1$, with

$$\langle S'(u), v \rangle = \int_{\mathbb{R}^3} g(x,u)vx, \ \forall v \in X.$$

Moreover, $S'$ is compact.

**Proof:** From (g4) we have

$$|g(x,t)| \le |a|_\infty |t| + |b|_\infty |t|^{s-1},$$

so $S$ is well-defined and $C^1$. The compactness of $S'$ follows from the fact proved in [19] that $S': D^{1,2}(\mathbb{R}^3) \to \mathbb{R}$ is compact, where $S: D^{1,2}(\mathbb{R}^3) \to \mathbb{R}$, $S(u) = \int_{\mathbb{R}^3} G(x,u)x$. By the continuity of the embedding $i: X \to D^{1,2}(\mathbb{R}^3)$, if $u_n \rightharpoonup u$ in $X$, then $u_n \rightharpoonup u$ in $D^{1,2}(\mathbb{R}^3)$ (for brevity, we denote $i(u_n)$ as $u_n$). Without loss of generality, after taking a subsequence, $S'(u_n) \rightharpoonup S'(u)$ in $\left(D^{1,2}(\mathbb{R}^3)\right)^*$. Since

$$\begin{aligned}\|S'(u_n) - S'(u)\| &= \sup_{v \in X \setminus \{0\}} \frac{1}{\|v\|} \left| \int_{\mathbb{R}^3} \left(g(x,u_n) - g(x,u)\right) vx \right| \\ &\le \sup_{v \in D^{1,2}(\mathbb{R}^3) \setminus \{0\}} \frac{1}{\|v\|_{D^{1,2}}} \left| \int_{\mathbb{R}^3} \left(g(x,u_n) - g(x,u)\right) vx \right| \\ &= \|S'(u_n) - S'(u)\|_{D^{1,2}} \to 0,\end{aligned}$$

since $S'(u_n) \to S'(u)$.

**Lemma 2** Under assumptions (V) and (g1) - (g4), the functional $\Phi$ satisfies the (PS) condition.

**Proof:** Let $\{u_n\}$ be a (PS) sequence of $\Phi$, i.e., $\sup_n |\Phi(u_n)| < \infty$, $\Phi'(u_n) \to 0$. We aim to prove that $\{u_n\}$ is bounded in $X$. We argue by contradiction. Suppose there exists a subsequence of $\{u_n\}$, still denoted by $\{u_n\}$, such that $\|u_n\| \to \infty$ as $n \to \infty$. Let $v_n = \frac{u_n}{\|u_n\|}$. Perform an orthogonal decomposition of the Hilbert space $X = X^- \oplus X^+$, then $v_n = v_n^+ + v_n^-$, where $v_n^+ \in X^+$, $v_n^- \in X^-$. We discuss two cases.

If $v = 0$, since $\dim X^- < +\infty$, we have $v_n^- \to v^-$. Hence $\|v_n^-\| \to 0$. When $n$ is sufficiently large, $\|v_n^-\|^2 < \frac{1}{4}$. Combined with $\|v_n^+\|^2 + \|v_n^-\|^2 = 1$, we also have $\|v_n^+\|^2 - \|v_n^-\|^2 = \left(1 - \|v_n^-\|^2\right) - \|v_n^-\|^2 \ge \frac{1}{2}$. Since $\{u_n\}$ is a (PS) sequence, $\{\Phi(u_n)\}$ is bounded and $\Phi'(u_n) \to 0$, so

$$\left|\langle \Phi'(u_n), u_n \rangle\right| \le \|\Phi'(u_n)\| \|u_n\| \le 4\|u_n\|.$$

Furthermore, we have

$$\begin{aligned}
1+\sup_{n}\left|\Phi\left(u_n\right)\right|+\left\|u_n\right\| &\geq \Phi\left(u_n\right)-\frac{1}{4}\left\langle\Phi'\left(u_n\right),u_n\right\rangle \\
&=\frac{1}{4}\int_{\mathbb{R}^3}\left(\left|\nabla u_n\right|^2+V\left(x\right)u_n^2\right)x+\int_{\mathbb{R}^3}\left(\frac{1}{4}g\left(x,u_n\right)u_n-G\left(x,u_n\right)\right)x \\
&=\frac{1}{4}\left(\left\|u_n^+\right\|^2-\left\|u_n^-\right\|^2\right)+\int_{\mathbb{R}^3}\left(\frac{1}{4}g\left(x,u_n\right)u_n-G\left(x,u_n\right)\right)x \\
&=\frac{1}{4}\left\|u_n\right\|^2\left(\left\|v_n^+\right\|^2-\left\|v_n^-\right\|^2\right)+\int_{\mathbb{R}^3}\left(\frac{1}{4}g\left(x,u_n\right)u_n-G\left(x,u_n\right)\right)x\geq\frac{1}{8}\left\|u_n\right\|^2,
\end{aligned}$$

which clearly contradicts $\left\|u_n\right\|\to\infty$.

If $v\neq 0$, then the set $\mathcal{N}=\left\{x\mid v\left(x\right)\neq 0\right\}$ has positive Lebesgue measure. For $x\in\mathcal{N}$, since $v_n=\frac{u_n}{\left\|u_n\right\|}$, as $n\to\infty$, $\left|u_n\left(x\right)\right|\to\infty$, and

$$\frac{G\left(x,u_n\left(x\right)\right)}{u_n^4\left(x\right)}v_n^4\left(x\right)\to+\infty, \tag{3.1}$$

Then by Fatou's lemma and Proposition 4, we have

$$\begin{aligned}
\int_{v\neq 0}\frac{G\left(x,u_n\right)}{u_n^4}v_n^4x&=\frac{1}{\left\|u_n\right\|^4}\int_{v\neq 0}G\left(x,u_n\right)x\leq\frac{1}{\left\|u_n\right\|^4}\int_{\mathbb{R}^3}G\left(x,u_n\right)x \\
&=\frac{1}{\left\|u_n\right\|^4}\left(\frac{1}{2}\int_{\mathbb{R}^3}\left(\left|\nabla u_n\right|^2+V\left(x\right)u_n^2\right)x+\frac{1}{4}\int_{\mathbb{R}^3}\phi_{u_n}u_n^2x-\Phi\left(u_n\right)\right) \\
&=\frac{1}{\left\|u_n\right\|^4}\left(\frac{1}{2}\left(\left\|u_n^+\right\|^2-\left\|u_n^-\right\|^2\right)+\frac{1}{4}\int_{\mathbb{R}^3}\phi_{u_n}u_n^2x-\Phi\left(u_n\right)\right) \\
&\leq\frac{1}{\left\|u_n\right\|^4}\left(\frac{1}{2}\left(\left\|u_n^+\right\|^2-\left\|u_n^-\right\|^2\right)+k\left\|u_n\right\|^4-\Phi\left(u_n\right)\right) \\
&=\frac{\left\|u_n\right\|^2}{2\left\|u_n\right\|^4}+k-\frac{\Phi\left(u_n\right)}{\left\|u_n\right\|^4}\leq 1+k.
\end{aligned} \tag{3.2}$$

It is easy to see that (3.1) contradicts (3.2). Hence the (PS) sequence $\left\{u_n\right\}$ is bounded.

Next, a property concerning $N\left(u\right)$ is inspired by [12]. Define the $C^1$ functional $N:X\to\mathbb{R}$ by $N\left(u\right)=\frac{1}{4}\int_{\mathbb{R}^3}\phi_u u^2\mathrm{d}x$, then for all $u,v\in X$, $\left\langle N'\left(u\right),v\right\rangle=\int_{\mathbb{R}^3}\phi_u uvx$. Since the (PS) sequence $\left\{u_n\right\}$ is bounded in $X$, we can select a subsequence $\left\{u_n\right\}$ such that $u_n\rightharpoonup u$ in $X$. Moreover, similar to the discussions in [20, 21], $N:X\to\mathbb{R}$ is weakly lower semicontinuous, and $N':X\to X^*$ is weakly sequentially continuous, where $X^*=H^{-1}\left(\mathbb{R}^3\right)$ is the dual space of $X=H^1\left(\mathbb{R}^3\right)$. Thus,

$$\liminf_{n\to\infty}N\left(u_n\right)\geq N\left(u\right),\ \lim_{n\to\infty}\left\langle N'\left(u_n\right),u\right\rangle=\left\langle N'\left(u\right),u\right\rangle.$$

Furthermore,

$$\begin{aligned}
\liminf_{n\to\infty}\int_{\mathbb{R}^3}\phi_{u_n}u_n\left(u_n-u\right)x&=\liminf_{n\to\infty}\left(4N\left(u_n\right)-\left\langle N'\left(u_n\right),u\right\rangle\right) \\
&\geq 4N\left(u\right)-\left\langle N'\left(u\right),u\right\rangle=0.
\end{aligned}$$

Thus,

$$\liminf_{n\to\infty}\int_{\mathbb{R}^3}\phi_{u_n}u_n\left(u_n-u\right)x\geq 0. \tag{3.3}$$

Therefore, it suffices to prove that $\{u_n\}$ has a convergent subsequence. Similar to [12], we can prove

$$\left\|u_n^+\right\|^2-\left\|u^+\right\|^2=o(1)+\int_{\mathbb{R}^3}g\left(x,u_n\right)\left(u_n-u\right)x-\int_{\mathbb{R}^3}\phi_{u_n}u_n\left(u_n-u\right)x. \tag{3.4}$$

Since $u_n\rightharpoonup u$ in $X$, by Lemma 1, $S'\left(u_n\right)\to S'(u)$ and

$$\begin{aligned}\left|\int_{\mathbb{R}^3}g\left(x,u_n\right)\left(u_n-u\right)\right|x&=\left|\left\langle S'\left(u_n\right),u_n-u\right\rangle\right|\\&\leq\left|\left\langle S'\left(u_n\right)-S'(u),u_n-u\right\rangle\right|+\left|\left\langle S'(u),u_n-u\right\rangle\right|\\&\leq\left\|S'\left(u_n\right)-S'(u)\right\|\left\|u_n-u\right\|+o(1).\end{aligned}$$

Thus,

$$\limsup_{n\to\infty}\int_{\mathbb{R}^3}g\left(x,u_n\right)\left(u_n-u\right)x=0. \tag{3.5}$$

Using (3.3), (3.4), and (3.5), we have

$$\begin{aligned}&\limsup_{n\to\infty}\left(\left\|u_n^+\right\|^2-\left\|u^+\right\|^2\right)\\&=\limsup_{n\to\infty}\left(\int_{\mathbb{R}^3}g\left(x,u_n\right)\left(u_n-u\right)x-\int_{\mathbb{R}^3}\phi_{u_n}u_n\left(u_n-u\right)x\right)\\&\leq\limsup_{n\to\infty}\int_{\mathbb{R}^3}g\left(x,u_n\right)\left(u_n-u\right)x-\liminf_{n\to\infty}\int_{\mathbb{R}^3}\phi_{u_n}u_n\left(u_n-u\right)x\\&\leq\limsup_{n\to\infty}\int_{\mathbb{R}^3}g\left(x,u_n\right)\left(u_n-u\right)x=0.\end{aligned}$$

Combined with the weak lower semicontinuity of the norm functional $u\mapsto\|u\|$, we have

$$\left\|u^+\right\|\leq\liminf_{n\to\infty}\left\|u_n^+\right\|\leq\limsup_{n\to\infty}\left\|u_n^+\right\|\leq\left\|u^+\right\|.$$

Therefore, $\left\|u_n^+\right\|\to\left\|u^+\right\|$. On the other hand, since $\dim X^-<+\infty$, we have $\left\|u_n^-\right\|\to\left\|u^-\right\|$. Furthermore,

$$\left\|u_n^-\right\|^2\to\left\|u^-\right\|^2,\ \left\|u_n^+\right\|^2\to\left\|u^+\right\|^2.$$

Combined with $\left\|u_n\right\|^2=\left\|u_n^+\right\|^2+\left\|u_n^-\right\|^2$, we obtain $\left\|u_n\right\|\to\|u\|$, hence $u_n\to u$ in $X$.

**Lemma 3** If (V), (g1), (g2), and (g3) hold, then there exists $A>0$ such that when $\Phi(u)\leq -A$, $\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=1}\Phi(tu)<0$.

**Proof:** This proof follows a standard procedure and is omitted here.

**Lemma 4** $C_q(\Phi,\infty)=0$ for all $q=0,1,\cdots,n$.

**Proof:** Let $A>0$ be as defined in Lemma 3. Since the critical group at infinity is defined via $\Phi_{-A}:=\Phi^{-1}(-\infty,-A)$, directly computing $H_q\left(X,\Phi_{-A}\right)$ is difficult. Therefore, we need to find a space with simple topological structure. Consider $X\setminus B$, where $B=\left\{u\in X\mid\|u\|<1\right\}$. Denote $S=\partial B$ as the unit sphere. By a strong deformation retraction, $\Phi_{-A}$ is topologically equivalent to $X\setminus B$, thus transforming the computation of $H_q\left(X,\Phi_{-A}\right)$ into computing $H_q\left(X,X\setminus B\right)$.

Consider when $\|u\|\le 2$, we have $-A<\inf\Phi\left(u\right)$. Since $g$ satisfies (g3), for any $v\in S$, as $s\to+\infty$,

$$\begin{aligned}\Phi\left(sv\right)&=\frac{1}{2}\int_{\mathbb{R}^3}\left(\left|\nabla\left(sv\right)\right|^2+V\left(x\right)\left(sv\right)^2\right)x+\frac{1}{4}\int_{\mathbb{R}^3}\phi_{sv}\left(sv\right)^2x-\int_{\mathbb{R}^3}G\left(x,sv\right)x\\&=\frac{1}{2}\left(\left\|\left(sv\right)^+\right\|^2-\left\|\left(sv\right)^-\right\|^2\right)+\frac{1}{4}\int_{\mathbb{R}^3}s^4\phi_v v^2x-\int_{\mathbb{R}^3}G\left(x,sv\right)x\\&=s^4\left(\frac{\left\|v^+\right\|^2-\left\|v^-\right\|^2}{2s^2}+\frac{1}{4}\int_{\mathbb{R}^3}\phi_v v^2x-\frac{1}{s^4}\int_{\mathbb{R}^3}G\left(x,sv\right)x\right)\to-\infty,\end{aligned}$$

since $\|v\|=1<2$, we have $\Phi\left(v\right)>-A$. Combined with $\Phi\left(sv\right)\to-\infty$, by the intermediate value theorem, for the above $v\in S$, there exists $s_v>0$ such that $\Phi\left(s_vv\right)=-A$. Let $u=s_vv$. By Lemma 3,

$$\left.\frac{\mathrm{d}}{\mathrm{d}s}\Phi\left(sv\right)\right|_{s=s_v}=\frac{1}{s_v}\left.\frac{\mathrm{d}}{\mathrm{d}t}\Phi\left(tu\right)\right|_{t=1}<0.$$

Treat $\Phi\left(sv\right)=-A$ as an equation in $s$, and let $F\left(v,s\right):=\Phi\left(sv\right)+A=0$. By the implicit function theorem, at $\left(v,s_v\right)$, since $\frac{\partial F}{\partial s}=\left.\frac{\mathrm{d}}{\mathrm{d}s}\Phi\left(sv\right)\right|_{s=s_v}<0$, there exists a continuous function $Y:S\to\left(0,+\infty\right)$ such that $Y\left(v\right)=s_v$. Thus, for each $v\in S$, there corresponds a unique $s_v$. Referring to [22], construct a strong deformation retraction $\eta:X\setminus B\to\Phi_{-A}$, specifically defined as: when $\Phi\left(u\right)\le-A$, $\eta\left(u\right)=Y\left(\frac{u}{\|u\|}\right)\frac{u}{\|u\|}=s_vv$, where $v=\frac{u}{\|u\|}$. Since $\Phi_{-A}$ is a strong deformation retract of $X\setminus B$, the topological pairs $\left(X,\Phi_{-A}\right)$ and $\left(X,X\setminus B\right)$ are homotopy equivalent. A strong deformation retraction is a homotopy equivalence, then their relative homology groups satisfy
$C_q\left(\Phi,\infty\right)=H_q\left(X,\Phi_{-A}\right)\cong H_q\left(X,X\setminus B\right)=0$, $q=0,1,\cdots,n$. Lemma 4 is proved.

## 4. Proof of Theorem 1

From conditions (V), (g2), and (g3) in Theorem 1, it is easy to see that as $\|u\|\to0$,

$$\int_{\mathbb{R}^3}\phi_uu^2x=O\left(\|u\|^2\right),\ \int_{\mathbb{R}^3}G\left(x,u\right)x=o\left(\|u\|^2\right),$$

thus,

$$\Phi\left(u\right)=\frac{1}{2}\left(\left\|u^+\right\|^2-\left\|u^-\right\|^2\right)+o\left(\|u\|^2\right).$$

Therefore, there exists $\varepsilon>0$ such that $\Phi$ is positive on $\left(X^+\setminus\{0\}\right)\cap B_\varepsilon$ and negative on $\left(X^-\setminus\{0\}\right)\cap B_\varepsilon$. That is, $\Phi$ has a local linking structure with respect to the decomposition $X=X^-\oplus X^+$. Since $l=\dim X^-$, by Proposition 2, $C_l\left(\Phi,0\right)\ne0$. By Lemma 4, $C_l\left(\Phi,0\right)\ne C_l\left(\Phi,\infty\right)$. Applying Proposition 1, $\Phi$ has a nonzero critical point $u$, and thus equation (1) has a nontrivial solution.

## 5. Conclusions

This paper has established the existence of nontrivial solutions for the standard Schrödinger-Bopp-Podolsky system (1) with an indefinite potential. The main contribution lies in extending the analysis of indefinite potential problems from the Kirchhoff-type framework studied by Tang *et al.* [8] back to the Schrödinger-Bopp-Podolsky system, highlighting that the indefiniteness of the potential itself, combined with the Bopp-Podolsky nonlocal term, creates a sufficiently rich struure for the existence of solutions via Morse theory.

The result adds to the understanding of coupled Schrödinger-Bopp-Podolsky systems under sign-changing potentials, a scenario relevant in physical models where attractive and repulsive regions coexist. The methodology, combining the reduction technique with Morse theory and critical groups at infinity, provides a robust framework that could be applicable to other nonlocal systems with indefite linear parts.

## Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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