Journal of Applied Mathematics and Physics, 2026, 14(1), 1-18
https://www.scirp.org/journal/jamp
ISSN Online: 2327-4379
ISSN Print: 2327-4352

Stability of Traveling Wave Solutions for Degenerate Fisher Type Equations with Fractional Laplacian

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How to cite this paper: Wang, Y.H., Zhang, Y.N. and Lu, H.Q. (2026) Stability of Traveling Wave Solutions for Degenerate Fisher Type Equations with Fractional Laplacian. *Journal of Applied Mathematics and Physics*, **14**, 1-18.
https://doi.org/10.4236/jamp.2026.141001

Received: December 2, 2025
Accepted: December 26, 2025
Published: December 29, 2025

Abstract

In this paper, we study the Lyapunov stability of traveling wave solutions for degenerating Fisher type equations with fractional Laplacian. First, we give the asymptotic behavior of the derivative of the traveling wave solutions at negative infinity by using the maximum principle. Second, we construct the subsolution and supersolution of the initial value problem in a moving coordinate frame. Finally, we use the sub-super solution method to prove the Lyapunov stability of traveling wave solutions in the polynomially weighted space.

Keywords

Lyapunov Stability, Asymptotic Behavior, Sub-Super Solution Method, Traveling Wave Solution, Fisher Type Equation

1. Introduction

It is well known that the most important mathematical model in chemical reaction, biological spreading and population genetics is the following equation

$$u_t(x,t)-\Delta u(x,t)=\overline{f}(u(x,t)),\ t>0, x\in\mathbb{R}^n, \tag{1}$$

where Δ is the traditional diffusion operator and $\overline{f}\in C(\mathbb{R})$ (see [1]-[3]).

The model (1) was proposed in 1930s (see [2] [3]). Subsequently, its traveling wave solutions have been widely investigated by many authors. In one dimension space, Kametaka [4] studied the existence of the traveling wave. Fife and McLeod [5] gave the global exponential stability of the traveling wave. The existence and the stability of V-shaped traveling fronts have been proved by Ninomiya and Taniguchi [6] in two dimension space. In three dimension space, Taniguchi [7] [8] studied the uniqueness and asymptotic stability of pyramidal traveling fronts.

Matano and Nara [9] [10] studied the larger time behavior and stability of planar waves for the Allen-Cahn equations in higher dimension space.

Recently, many researchers paid their attention to the traveling waves for reaction-diffusion equations with an anomalous diffusion such as super diffusion, which plays important roles in various physical, biological and geological processes (see [11]). From the mathematical point of view, such a super diffusion is related to Lévy process and may be modeled by a fractional Laplacian operator $(-\Delta)^s u$ with $s \in (0,1)$. The Fourier transformation of $(-\Delta)^s u$ is $(2\pi|\xi|)^{2s} \hat{u}$, which is the key to modeling the super diffusion processes (see [12]). The fractional Laplacian operator can be defined in many ways (see [13] [14]). In this paper, we use the following definition

$$\begin{aligned}(-\Delta)^s u(x) &= C_{n,s}\,(P.V.)\int_{\mathbb{R}^n} \frac{u(x)-u(y)}{|x-y|^{n+2s}}\mathrm{d}y \\ &= C_{n,s} \lim_{\epsilon\to 0}\int_{|x-y|\geq\epsilon} \frac{u(x)-u(y)}{|x-y|^{n+2s}}\mathrm{d}y,\end{aligned}$$

where P.V. denotes the Cauchy principle value and $C_{n,s} = \dfrac{2^{2s} s\Gamma\left(\frac{n+2s}{2}\right)}{\pi^{\frac{n}{2}}\Gamma(1-s)}$ is a normalized constant. The above integral definition can be used for more general functions, in particular, for $u \in C^2(\mathbb{R}^n)$.

In [14], Gui and Zhao have established that the existence and uniqueness of the traveling wave solution to the following equation

$$u_t(t,x) + (-\Delta)^s u(t,x) = \tilde{f}(u(t,x)), \ \forall t > 0, x \in \mathbb{R}, \tag{2}$$

where $s \in (0,1)$ and the bistable nonlinearity $\tilde{f}$ satisfies some conditions. Ma, Niu, and Wang [15] and Cheng and Yuan [16] have obtained that the traveling wave to (2) is asymptotic stable by using the sub-super solution method and the spectral analysis method, respectively. When $\tilde{f}$ is combustive, it is shown in [13] [17] that there exists a unique traveling wave solution to (2) if and only if $s \in \left(\frac{1}{2}, 1\right)$. Cabré and Roque in [18] have already proved that there is no traveling wave for the Fisher-KPP model corresponding to (2). But, Huan and Gui [13] have established the existence of traveling wave solution to the degenerate Fisher type model (2) for $s \in \left(\frac{p}{2(p-1)}, 1\right)$.

In 2017, He, Wu and Wu [19] prove the Lyapunov stability of traveling wave front to equation (1) with $\overline{f} = u^p(1-u)$ and $p > 1$. In this paper, we investigate the Lyapunov stability of the traveling wave of the following fractional reaction diffusion equation

$$u_t(x,t) + (-\Delta)^s u(x,t) = f(u(x,t)), \ \forall t > 0, x \in \mathbb{R}, \tag{3}$$

where $0 < s < 1$, and

$$\begin{aligned} &f(u)=u^p g(u) \text{ for } p>1;\ g(u)\in C^1(\mathbb{R});\\ &g(u)>0,\forall u\in[0,1);\ g(1)=0;\ g'(1)<0. \end{aligned} \tag{4}$$

Since $f'(0)=0$, we say that the nonlinearity f of (3) is degenerate.

We say that $u\in C^2(\mathbb{R}^2)$ is a traveling wave solution of (3) if u has the form

$$u(x,t)=\phi(x-ct),\ \forall(t,x)\in\mathbb{R}^2,$$

where c is called the speed of the traveling wave, and the function ϕ is called the profile of the traveling wave. It is easy to check that ϕ satisfies

$$\begin{cases} (-\Delta)^s\phi(z)-c\phi'(z)=f(\phi(z)),\ \forall z\in\mathbb{R},\\ \lim\limits_{z\to-\infty}\phi(z)=0,\ \lim\limits_{z\to\infty}\phi(z)=1. \end{cases} \tag{5}$$

From [13], we obtain the existence of traveling wave solution of Fisher type model (2) for $s\in\left(\frac{p}{2(p-1)},1\right)$ and the asymptotic behavior of the derivative of the traveling wave solutions to Fisher type model (2) when $z\to+\infty$. Similar results were also obtained for the degenerate Fisher type equation (3).

For convenience, we state the results on traveling wave front solutions to (3) obtained in [13].

Lemma 1.1. [13] Assume that $s\in\left[\frac{p}{2(p-1)},1\right)$ and f satisfies (4). Then (5)

has a solution (c,ϕ) with $c<0$ and $\phi'>0$. That is, $\phi(x-ct)$ is an increasing traveling wave solution to (3).

Lemma 1.2. [13] Assume that f satisfies (4), let (c,ϕ) be a solution to (5) with $c<0$. Then there exists a constant $C>0$ such that

$$\frac{1}{C|z|^{1+2s}}\le\phi'(z)\le\frac{C}{|z|^{1+2s}},\ \forall x\ge1. \tag{6}$$

In order to investigate the Lyapunov stability of the traveling wave solutions to (3), we consider the Cauchy problem for (3). Namely,

$$\begin{cases} u_t(x,t)+(-\Delta)^s u(x,t)=f(u(x,t)),\ \forall t>0,x\in\mathbb{R},\\ u(x,0)=u_0(x),\ \forall x\in\mathbb{R}. \end{cases} \tag{7}$$

Let (c,ϕ) is a solution to (5) with $c<0$ obtained in Lemma 1.1 and $z=x-ct$, then, by (7), we have the following Cauchy problem

$$\begin{cases} u_t(z,t)+(-\Delta)^s u(z,t)-cu_z(z,t)=f(u(z,t)),\\ u(z,0)=u_0(z). \end{cases} \tag{8}$$

Remark 1.1. It is easy to see that $u(x,t)$ is a solution of (7) if and only if $u(z,t)$ is a solution of (8) with $z=x-ct$.

The organization of this paper as follows. In section 2, we construct the polynomially weighted space and present some properties for the inverse of the weight function. In section 3, we show the asymptotic behavior of the derivative of the traveling wave solutions as $z \to -\infty$ which is very important for the proof of Lyapunov stability of the traveling wave solution. In section 4, we first establish the subsolution and supersolution for (8), then state and prove the Lyapunov stability result for equation (3) in the polynomially weighted space by applying sub-super solution method. Conclusions are provided in section 5.

2. Preliminaries

In this section, we construct the polynomially weighted space and present some properties for the inverse of the weight function.

Define a space

$$X = \{u : \mathbb{R} \to \mathbb{R} \text{ is uniformly continuous and bounded}\}.$$

It is easy to see that X is a Banach space with the norm

$$\|u\|_X = \operatorname*{esssup}_{x\in\mathbb{R}} |u(x)|.$$

Define the weighted space

$$X_0 = \{u \in X \mid uw_0 \in X\},$$

with the norm

$$\|u\|_{X_0} = \|uw_0\|_X ,$$

where the weight function $w_0(z)$ is defined by

$$w_0(z) = \begin{cases} 2|z|^{2s}, & z \le -1, \\ 1+|z|^{4s}, & -1 < z < 0, \\ 1, & z \ge 0. \end{cases}$$

We can check that X_0 is a Banach space with the X_0 norm.

From Lemma 1.2 and Lemma 3.1, we know that the derivative of the traveling wave solutions to (3) decay algebraically at infinity. So, we choose the algebraic weight function which matches the decay rate of the traveling wave solutions to (3).

Defining the inverse function as $w(z) = w_0^{-1}(z)$, we find that $w(z)$ satisfies

$$w(z) = \begin{cases} \dfrac{1}{2|z|^{2s}}, & z \le -1, \\ \dfrac{1}{1+|z|^{4s}}, & -1 < z < 0, \\ 1, & z \ge 0. \end{cases}$$

By direct computation, we obtain the derivative of $w(z)$

$$w'(z) = \begin{cases} \dfrac{s}{|z|^{2s+1}}, & z \le -1, \\ \dfrac{4s|z|^{4s-1}}{\left(1+|z|^{4s}\right)^2}, & -1 < z < 0, \\ 0, & z \ge 0. \end{cases}$$

It is easy to be computed that $w'(z) \le 4sw(z)$.

Lemma 2.1. [13] Let $\beta > 1$, we consider the function

$$\psi_\beta(z) = \begin{cases} \dfrac{1}{|z|^\beta}, & \forall z < -1, \\ 0, & \forall z \ge -1. \end{cases}$$

Then we have the following estimate

$$(-\Delta)^s \psi_\beta(z) = -\frac{C_{1,s}}{(\beta-1)|z|^{1+2s}} + o\left(\frac{1}{|z|^{1+2s}}\right), \text{ as } z \to -\infty.$$

Lemma 2.2. [13] Consider the function

$$\varphi(z) = \begin{cases} 1, & \forall z \le -1, \\ 0, & \forall z > -1. \end{cases}$$

Then we have the following estimate

$$(-\Delta)^s \varphi(z) = \frac{C_{1,s}}{2s|z|^{2s}} + o\left(\frac{1}{|z|^{2s}}\right), \text{ as } z \to -\infty.$$

Lemma 2.3. For $w(z)$, we have

i) $(-\Delta)^s w(z) \ge -C_1 |z|^{-(2s+1)}$, $\forall z \in [0, +\infty)$;

ii) $(-\Delta)^s w(z) \ge -C_2 |z|^{-2s}$, $\forall z \in (-\infty, -1]$;

ii) For any $M > 1$, $(-\Delta)^s w(z) \ge -C_3$, $\forall z \in [-M, M]$.

Where C_1, C_2, C_3 are positive constants.

Proof. i) For $z \ge 0$, we have $w(z) = 1$. Consider the fractional Laplacian

$$\begin{aligned} (-\Delta)^s w(z) &= C_{1,s} P.V. \int_{\mathbb{R}} \frac{w(z) - w(y)}{|z-y|^{1+2s}} \mathrm{d}y \\ &= C_{1,s} P.V. \int_{\mathbb{R}} \frac{1 - w(y)}{|z-y|^{1+2s}} \mathrm{d}y \\ &= C_{1,s} \left(\int_{-\infty}^{-1} \frac{1 - \dfrac{1}{2|y|^{2s}}}{|z-y|^{1+2s}} \mathrm{d}y + \int_{-1}^{0} \frac{1 - \dfrac{1}{1+|y|^{4s}}}{|z-y|^{1+2s}} \mathrm{d}y \right) \\ &> 0. \end{aligned}$$

Therefore, there exists $C_1 > 0$ such that

$$(-\Delta)^s w(z) > -C_1 |z|^{-(2s+1)}.$$

ii) For $z \le -1$, we have $w(z) = \dfrac{1}{2|z|^{2s}}$. Consider the fractional Laplacian

$$\left|(-\Delta)^s w(z)\right| = \left|C_{1,s}P.V.\int_{\mathbb{R}}\frac{w(z)-w(z+y)}{|y|^{1+2s}}\mathrm{d}y\right| = \left|C_{1,s}P.V.\int_{\mathbb{R}}\frac{\frac{1}{2|z|^{2s}}-\frac{1}{2|z+y|^{2s}}}{|y|^{1+2s}}\mathrm{d}y\right|$$

$$= C_{1,s}\left|\int_{-\infty}^{-z-1}\frac{\frac{1}{2|z|^{2s}}-\frac{1}{2|z+y|^{2s}}}{|y|^{1+2s}}\mathrm{d}y + \int_{-z-1}^{-z}\frac{\frac{1}{2|z|^{2s}}-\frac{1}{1+|z+y|^{4s}}}{|y|^{1+2s}}\mathrm{d}y + \int_{-z}^{+\infty}\frac{\frac{1}{2|z|^{2s}}-1}{|y|^{1+2s}}\mathrm{d}y\right|$$

$$\le \frac{C_{1,s}}{2}\left(\left|\int_{-\infty}^{-z-1}\frac{\frac{1}{|z|^{2s}}-\frac{1}{|z+y|^{2s}}}{|y|^{1+2s}}\mathrm{d}y\right| + \left|\int_{-z-1}^{-z}\frac{\frac{1}{|z|^{2s}}-\frac{2}{1+|z+y|^{4s}}}{|y|^{1+2s}}\mathrm{d}y\right| + \left|\int_{-z}^{+\infty}\frac{\frac{1}{|z|^{2s}}-2}{|y|^{1+2s}}\mathrm{d}y\right|\right)$$

$$= \frac{C_{1,s}}{2}\left(|I_1|+|I_2|+|I_3|\right).$$

Let us begin by approximating I_1. By changing of variables $y = zx$, we know that

$$|I_1| = \left|\int_{-\infty}^{-\frac{1}{z}-1}\frac{1}{|z|^{4s}}\left(\frac{1-\frac{1}{|1+x|^{2s}}}{|x|^{1+2s}}\right)\mathrm{d}x\right|$$

$$= \frac{1}{|z|^{4s}}\left(\left|\int_{-\infty}^{-\frac{1}{z}-1}\frac{1}{|x|^{1+2s}}\mathrm{d}x\right| + \left|\int_{-\infty}^{-\frac{1}{z}-1}\frac{1}{|1+x|^{2s}|x|^{1+2s}}\mathrm{d}x\right|\right)$$

$$\le \frac{1}{2s|z|^{4s}}\frac{|z|^{2s}}{|z+1|^{2s}} + \frac{1}{|z|^{4s}}\left|\int_{-\infty}^{-\frac{1}{z}-1}\frac{1}{\left|\frac{1}{z}\right|^{2s}|x|^{1+2s}}\mathrm{d}x\right|$$

$$= \frac{1}{2s}\left(\frac{1}{|z|^{2s}}\frac{1}{|z+1|^{2s}} + \frac{1}{|z+1|^{2s}}\right).$$

For I_2, we can obtain that

$$|I_2| = \left|\int_{-z-1}^{-z}\frac{1}{|y|^{1+2s}}\frac{1}{|z|^{2s}}\mathrm{d}y - \int_{-z-1}^{-z}\frac{1}{|y|^{1+2s}}\frac{2}{1+|z+y|^{4s}}\mathrm{d}y\right|$$

$$\le \left|\frac{1}{2s|z|^{2s}}\left(\frac{1}{|z|^{2s}}-\frac{1}{|z+1|^{2s}}\right)\right| + 2\left|\int_{-z-1}^{-z}\frac{1}{|y|^{1+2s}}\frac{1}{1+|z+y|^{4s}}\mathrm{d}y\right|$$

$$\le \left|\frac{1}{2s|z|^{2s}}\left(\frac{1}{|z|^{2s}}-\frac{1}{|z+1|^{2s}}\right)\right| + 2\left|\int_{-z-1}^{-z}\frac{1}{|y|^{1+2s}}\mathrm{d}y\right|$$

$$= \left(\frac{1}{2s|z|^{2s}}+\frac{1}{s}\right)\left(\frac{1}{|z|^{2s}}-\frac{1}{|z+1|^{2s}}\right).$$

For I_3, we have

$$
\begin{aligned}
|I_3| &= \left|\int_{-z}^{+\infty}\frac{1}{|z|^{2s}|y|^{1+2s}}\mathrm{d}y - 2\int_{-z}^{+\infty}\frac{1}{|y|^{1+2s}}\mathrm{d}y\right| \\
&= \frac{1}{|z|^{2s}}\left|\frac{1}{s}-\frac{1}{2s|z|^{2s}}\right|.
\end{aligned}
$$

Finally, from the estimates of I_1, I_2 and I_3, we obtain

$$
\begin{aligned}
\left|(-\Delta)^s w(z)\right| &\le \frac{C_{1,s}}{2}\left(|I_1|+|I_2|+|I_3|\right) \\
&\le \frac{C_{1,s}}{4s}\left(\frac{1}{|z|^{2s}|z+1|^{2s}}+\frac{1}{|z+1|^{2s}}+\frac{1}{|z|^{4s}}-\frac{1}{|z|^{2s}|z+1|^{2s}}\right. \\
&\qquad \left.+\frac{2}{|z|^{2s}}-\frac{2}{|z+1|^{2s}}+\frac{2}{|z|^{2s}}-\frac{1}{|z|^{4s}}\right) \\
&= \frac{C_{1,s}}{4s}\left(\frac{4}{|z|^{2s}}-\frac{1}{|z+1|^{2s}}\right) \\
&\le \frac{C_{1,s}}{|z|^{2s}}.
\end{aligned}
$$

It follows that there exists $C_2 > 0$ such that

$$(-\Delta)^s w(z) \ge -C_2|z|^{-2s}.$$

iii) For $x \in (-1,0)$, we have $w(z)=\frac{1}{1+|z|^{4s}}$. Consider the fractional Laplacian

$$
\begin{aligned}
\left|(-\Delta)^s w(z)\right| &= \left|C_{1,s}P.V.\int_{\mathbb{R}}\frac{w(z)-w(z+y)}{|y|^{1+2s}}\mathrm{d}y\right| = \left|C_{1,s}P.V.\int_{\mathbb{R}}\frac{\frac{1}{1+|z|^{4s}}-w(z+y)}{|y|^{1+2s}}\mathrm{d}y\right| \\
&= C_{1,s}\left|\int_{-\infty}^{-z-1}\frac{\frac{1}{1+|z|^{4s}}-\frac{1}{2|z+y|^{2s}}}{|y|^{1+2s}}\mathrm{d}y+\int_{-z-1}^{-z}\frac{\frac{1}{1+|z|^{4s}}-\frac{1}{1+|z+y|^{4s}}}{|y|^{1+2s}}\mathrm{d}y+\int_{-z}^{+\infty}\frac{\frac{1}{1+|z|^{4s}}-1}{|y|^{1+2s}}\mathrm{d}y\right| \\
&\le C_{1,s}\left(\left|\int_{-\infty}^{-z-1}\frac{\frac{1}{1+|z|^{4s}}-\frac{1}{2|z+y|^{2s}}}{|y|^{1+2s}}\mathrm{d}y\right|+\left|\int_{-z-1}^{-z}\frac{\frac{1}{1+|z|^{4s}}-\frac{1}{1+|z+y|^{4s}}}{|y|^{1+2s}}\mathrm{d}y\right|+\left|\int_{-z}^{+\infty}\frac{\frac{1}{1+|z|^{4s}}-1}{|y|^{1+2s}}\mathrm{d}y\right|\right) \\
&= C_{1,s}\left(|I_1|+|I_2|+|I_3|\right).
\end{aligned}
$$

For L_1, by changing of variables $y = zx$, we can get

$$\begin{aligned}
\left|L_1\right| &= \left|\int_{-\infty}^{-z-1} \frac{1}{1+|z|^{4s}} \cdot \frac{1}{|y|^{1+2s}} \mathrm{d}y - \int_{-\infty}^{-z-1} \frac{1}{2|z+y|^{2s}} \cdot \frac{1}{|y|^{1+2s}} \mathrm{d}y\right| \\
&= \frac{1}{1+|z|^{4s}} \left|\int_{-\infty}^{-z-1} \frac{1}{|y|^{1+2s}} \mathrm{d}y\right| + \frac{1}{2}\left|\int_{-\infty}^{-\frac{1}{z}-1} \frac{1}{|z|^{4s}} \cdot \frac{1}{|1+x|^{2s}|x|^{1+2s}} \mathrm{d}x\right| \\
&\le \frac{1}{2s|z+1|^{2s}\left(|z|^{4s}+1\right)} + \frac{1}{2|z|^{4s}}\left|\int_{-\infty}^{-\frac{1}{z}-1} \frac{1}{\left|\frac{1}{z}\right|^{2s}|x|^{1+2s}} \mathrm{d}x\right| \\
&= \frac{1}{2s|z+1|^{2s}\left(|z|^{4s}+1\right)} + \frac{1}{4s|z+1|^{2s}}.
\end{aligned}$$

For L_2, we have

$$\begin{aligned}
\left|L_2\right| &= \left|\int_{-z-1}^{-z} \frac{1}{1+|z|^{4s}} \cdot \frac{1}{|y|^{1+2s}} \mathrm{d}y\right| + \left|\int_{-z-1}^{-z} \frac{1}{1+|z+y|^{4s}} \cdot \frac{1}{|y|^{1+2s}} \mathrm{d}y\right| \\
&\le \left(\frac{1}{2s\left(1+|z|^{4s}\right)} + \frac{1}{2s}\right)\left(\frac{1}{|z|^{2s}} - \frac{1}{|z+1|^{2s}}\right).
\end{aligned}$$

For L_3, we can directly compute to obtain the estimate

$$\begin{aligned}
\left|L_3\right| &= \left|\int_{-z}^{+\infty} \frac{1}{1+|z|^{4s}} \cdot \frac{1}{|y|^{1+2s}} \mathrm{d}y + \int_{-z}^{+\infty} \frac{1}{|y|^{1+2s}} \mathrm{d}y\right| \\
&= \frac{1}{2s|z|^{2s}}\left(1+\frac{1}{1+|z|^{4s}}\right).
\end{aligned}$$

Finally, using the above inequality L_1, L_2 and L_3, we obtain

$$\begin{aligned}
\left|(-\Delta)^s w(z)\right| &\le C_{1,s}\left(\left|I_1\right|+\left|I_2\right|+\left|I_3\right|\right) \\
&\le C_{1,s}\left(\frac{1}{2s|z+1|^{2s}} \cdot \frac{1}{1+|z|^{4s}} + \frac{1}{4s|z+1|^{2s}} + \frac{1}{2s|z|^{2s}} \cdot \frac{1}{1+|z|^{4s}}\right. \\
&\quad - \frac{1}{2s|z+1|^{2s}} \cdot \frac{1}{1+|z|^{4s}} + \frac{1}{2s|z|^{2s}} - \frac{1}{2s|z+1|^{2s}} \\
&\quad \left. + \frac{1}{2s|z|^{2s}} \cdot \frac{1}{1+|z|^{4s}} + \frac{1}{2s|z|^{2s}}\right) \\
&= C_{1,s}\left(\frac{1}{s|z|^{2s}} \cdot \frac{1}{1+|z|^{4s}} + \frac{1}{s|z|^{2s}} + \frac{1}{4|z+1|^{2s}}\right) \\
&\le \frac{C_{1,s}}{|z|^{2s}}\left(\frac{2}{s}+\frac{1}{4}\right).
\end{aligned}$$

For any $M > 1$, the function $f(z) = \dfrac{1}{|z|^{2s}}$ is bounded on the closed interval $[-M, M]$. By combining (ii) and (iii), we know that there exists $C_3 > 0$ such that

$$(-\Delta)^s w(z) \ge -C_3.$$

3. Asymptotic Rate at $-\infty$

In this section, we give the asymptotic behavior of the derivative of the traveling wave solutions to (3) when $z \to -\infty$.

Lemma 3.1. Let $\frac{p-1}{2(p-2)} < s < 1$, let (c,ϕ) be a solution to (5) with $c<0$. Then there exists some constant $C>0$ such that

$$\frac{1}{C|z|^{2s}} \le \phi'(z) \le \frac{C}{|z|^{2s}}, \ \forall z \le -1.$$

Proof. From Theorem 1.3 in [13], we have

$$\frac{1}{C|z|^{2s}} \le \phi'(z), \ \forall z \le -1. \tag{9}$$

Consider the function $\Psi(z) = \varphi(-z-2) + \psi_{2s}(z)$ for all $z \in \mathbb{R}$, we know that

$$\Psi(z) = \frac{1}{|z|^{2s}}, \ \Psi'(z) = \frac{2s}{|z|^{2s+1}}, \ \forall z < -1. \tag{10}$$

By Lemma 2.1 and Lemma 2.2, we have

$$\begin{aligned}(-\Delta)^s \Psi(z) &= \frac{C_{1,s}}{2s|z|^{2s}} + o\left(\frac{1}{|z|^{2s}}\right) - \frac{C_{1,s}}{(2s-1)|z|^{2s+1}} + o\left(\frac{1}{|z|^{2s+1}}\right) \\ &= \frac{C_{1,s}}{2s|z|^{2s}} + o\left(\frac{1}{|z|^{2s}}\right), \ \text{as } z \to -\infty.\end{aligned}$$

Hence we get

$$\begin{aligned}(-\Delta)^s \Psi(z) + c\Psi'(z) &= \frac{C_{1,s}}{2s|z|^{2s}} + o\left(\frac{1}{|z|^{2s}}\right) + \frac{2cs}{|z|^{2s+1}} \\ &= \frac{C_{1,s}}{2s|z|^{2s}} + o\left(\frac{1}{|z|^{2s}}\right) \\ &\ge \frac{C_{1,s}}{4s|z|^{2s}}, \ \text{as } z \to -\infty.\end{aligned}$$

Therefore there exists some large $R>0$ such that

$$(-\Delta)^s \Psi(z) + c\Psi'(z) \ge 0, \ z \le -R.$$

Let $v(z) = \phi'(z) > 0$ for all $z \in \mathbb{R}$, then it satisfies

$$(-\Delta)^s v(z) + cv'(z) = f'(\phi(z)) \cdot \phi'(z).$$

From (4), it is easy to obtain that there exists $\theta \in (0,1)$ such that $f'(\phi) \ge 0$ for all $\phi \in [0,\theta]$. So we get

$$(-\Delta)^s v(z)+cv'(z)\geq 0,\ \ z\leq -R.$$

Since $v(z)=\phi'(z)$ is bounded, there exists some $C_1>0$ such that $|v(z)|<C_1$ for all $z\in\mathbb{R}$. Since $f(\phi)=\phi^p g(\phi)$, there exist some $C_2>0$, $M_1>0$ and $M_2>0$ such that

$$\begin{aligned} f'(\phi(x)) &= p\phi^{p-1}g(\phi)+\phi^p g'(\phi) \\ &\leq pM_1\phi^{p-1}+M_2\phi^p \\ &\leq C_2\left(\frac{1}{|z|^{(2s-1)(p-1)}}+\frac{1}{|z|^{(2s-1)p}}\right). \end{aligned}$$

Since $\frac{p-1}{2(p-2)}<s<1$, we have

$$f'(\phi(z))v(z)\leq \frac{\gamma}{|z|^{\alpha}},\ \ \alpha\geq 2s,$$

where γ is a positive constant.

Let $\omega(z)=C\Psi(z)-v(z)$, we can obtain

$$\begin{aligned} (-\Delta)^s\omega(z)+c\omega'(z) &\geq \frac{C_{1,s}}{4s|z|^{2s}}-f'(\phi(z))\cdot\phi'(z) \\ &\geq \frac{C_{1,s}}{4s|z|^{2s}}-\frac{\gamma}{|z|^{\alpha}} \\ &\geq 0,\ \ \forall z\leq -R. \end{aligned}$$

For all $z\in\mathbb{R}$, we have $\Psi(z)>0$. So there exists some constant $C>0$ such that

$$\|v(x)\|_{C(\mathbb{R})}<C \ \text{ and } \ \|v(z)\|_{C(\mathbb{R})}<C\inf_{z\in[-R,-1]}\Psi(z).$$

Since $\Psi(z)=\varphi(-z-2)=1$ for all $z\geq -1$, we know that $C\Psi(z)=C\geq\|v(z)\|_{C(\mathbb{R})}$ for all $z\geq -1$. In summary, we obtain

$$C\Psi(z)\geq v(z),\ \ \forall z\geq -R.$$

Then we have

$$\begin{cases} (-\Delta)^s\omega(z)+c\omega'(z)\geq 0, & z\leq -R, \\ \lim\limits_{z\to-\infty}\omega(z)=0, & \\ \omega(z)\geq 0, & \forall z\geq -R. \end{cases}$$

By the maximum principle (see Lemma 4.4 in [13]) and (10), we know that $\omega(z)\geq 0$ in $\mathbb{R}$, which implies

$$\phi'(z)=v(z)\leq\frac{C}{|z|^{2s}},\ \ \forall z\leq -1. \tag{11}$$

By (9) and (11), we have

$$\frac{1}{C|z|^{2s}} \le \phi'(z) \le \frac{C}{|z|^{2s}}, \ \forall z \le -1.$$

4. Lyapunov Stability of Traveling Wave Solutions

In this section, we first construct the subsolution and supersolution for (8), then study the Lyapunov stability of traveling wave solutions with $f(u) = u^p g(u)$.

Theorem 4.1. Assume f satisfies (4). Let $\phi(z)$ be the traveling wave solution of (3) with wave speed $c<0$, there exists $\tilde{\epsilon}>0$ such that for all $\epsilon \in (0,\tilde{\epsilon})$, if the initial condition $u_0(z) \in [0,1]$ with $u_0(z) \in X_0$ and $\|u_0(z) - \phi(z)\|_{X_0} < \epsilon$, then the global solution $u(z,t)$ of (8) satisfies

$$\underline{u}(z,t) \le u(z,t) \le \overline{u}(z,t), \ \text{for all } t \ge 0, z \in \mathbb{R}. \tag{12}$$

where

$$\underline{u}(z,t) = \phi(z) - \epsilon A_1 \left(1 - \mathrm{e}^{-\gamma t}\right) - \epsilon \mathrm{e}^{-\gamma t} w_0^{-1}(z), \tag{13}$$

$$\overline{u}(z,t) = \phi(z) + \epsilon A_2 \left(1 - \mathrm{e}^{-\gamma t}\right) + \epsilon \mathrm{e}^{-\gamma t} w_0^{-1}(z), \tag{14}$$

where $A_i \, (i=1,2)$ and γ are positive constants independent of ϵ.

Proof. For all $t \ge 0$ and $z \in \mathbb{R}$, we define the functions
$\underline{u}(z,t) = \phi(z) - \epsilon A_1 \left(1 - \mathrm{e}^{-\gamma t}\right) - \epsilon \mathrm{e}^{-\gamma t} w_0^{-1}(z)$,
$\overline{u}(z,t) = \phi(z) + \epsilon A_2 \left(1 - \mathrm{e}^{-\gamma t}\right) + \epsilon \mathrm{e}^{-\gamma t} w_0^{-1}(z)$,
where $A_1 > 0$, $A_2 > 0$ and $\gamma > 0$.

By the initial condition, we have

$$\phi(z) - \epsilon w_0^{-1}(z) < u_0(z) < \phi(z) + \epsilon w_0^{-1}(z). \tag{15}$$

Setting $t = 0$ in (13) and (14), we obtain

$$\underline{u}(z,0) = \phi(z) - \epsilon w_0^{-1}(z), \ \overline{u}(z,0) = \phi(z) + \epsilon w_0^{-1}(z). \tag{16}$$

By (15) and (16), we have $\underline{u}(z,0) \le u_0(z) \le \overline{u}(z,0)$.

We next prove (12) holds.

1) For all $t > 0$ and $z \in \mathbb{R}$, we prove $u(z,t) \ge \underline{u}(z,t)$. Let $\zeta_1 = z - \epsilon A_1 \left(1 - \mathrm{e}^{-\gamma t}\right)$. By (5), we obtain

$$\begin{aligned} N\underline{u}(z,t) &= \underline{u}_t + (-\Delta)^s \underline{u} - c\underline{u}_z - f(\underline{u}) \\ &= -\gamma \epsilon A_1 \mathrm{e}^{-\gamma t} \phi'(\zeta_1) + \epsilon \gamma \mathrm{e}^{-\gamma t} w(z) + (-\Delta)^s \phi(\zeta_1) \\ &\quad - \epsilon \mathrm{e}^{-\gamma t} (-\Delta)^s w(z) - c\phi'(\zeta_1) + c\epsilon \mathrm{e}^{-\gamma t} w'(z) - f(\underline{u}) \\ &= -\gamma \epsilon A_1 \mathrm{e}^{-\gamma t} \phi'(\zeta_1) + \epsilon \gamma \mathrm{e}^{-\gamma t} w(z) - \epsilon \mathrm{e}^{-\gamma t} (-\Delta)^s w(z) \\ &\quad + c\epsilon \mathrm{e}^{-\gamma t} w'(z) + f(\phi(\zeta_1)) - f\left(\phi(\zeta_1) - \epsilon \mathrm{e}^{-\gamma t} w(z)\right). \end{aligned}$$

By $\phi(-\infty) = 0$, $f'(0) = 0$ and Lemma 3.1, we know that there exist some constants $\gamma_1 > 0$, $M_1 > 0$, $\beta_1 > 0$ and $\epsilon > 0$ such that

$$\begin{cases} f\left(\phi(z)\right)-f\left(\phi(z)-\epsilon \mathrm{e}^{-\gamma t}w(z)\right)\le \gamma_1\epsilon \mathrm{e}^{-\gamma t}w(z)=\gamma_1\epsilon \mathrm{e}^{-\gamma t}\dfrac{1}{2|z|^{2s}}, & z<-M_1,\\ \phi'(z)\ge \dfrac{\beta_1}{|z|^{2s}}. \end{cases} \tag{17}$$

By $\phi(+\infty)=1$, $f'(1)<0$ and Lemma 1.2, we know that there exist some constants $\gamma\in(0,\gamma_1)$, $M_2>0$, $\beta_2>0$ and $\epsilon>0$ such that

$$\begin{cases} f\left(\phi(z)\right)-f\left(\phi(z)-\epsilon \mathrm{e}^{-\gamma t}w(z)\right)\le -\gamma\epsilon \mathrm{e}^{-\gamma t}w(z)=-\gamma\epsilon \mathrm{e}^{-\gamma t}, & z>M_2,\\ \phi'(z)\ge \dfrac{\beta_2}{|z|^{1+2s}}. \end{cases} \tag{18}$$

Let $\tilde{M}=\max\{M_1,M_2+1,2\}$. Since $\phi'(z)>0$, $z\in\mathbb{R}$, we know that there exists $\delta_0>0$ such that

$$\min_{z\in\left[-\tilde{M}-1,\tilde{M}+1\right]}\phi'(z)\ge\delta_0>0. \tag{19}$$

Let

$$\begin{aligned} &K_1=\|f'\|_{L^\infty(0,2)},\\ &A_1=\max\left\{\frac{\gamma_1+2C_2+2cs+\gamma}{2\gamma\beta_1},\frac{C_1}{\beta_2\gamma},\frac{\gamma+C_3+4cs+K_1}{\gamma\delta_0}\right\}, \end{aligned} \tag{20}$$

and $\epsilon\in(0,\epsilon_1)$ with $\epsilon_1=\dfrac{1}{A_1}$. There are three cases to consider.

Case 1. $z<-\tilde{M}$.

In case 1, we have $\zeta_1=z-\epsilon A_1\left(1-\mathrm{e}^{-\gamma t}\right)<-\tilde{M}-\epsilon A_1\left(1-\mathrm{e}^{-\gamma t}\right)<-\tilde{M}$ for any $t>0$. By the definition of $w(z)$, we know that $w(z)=\dfrac{1}{2|z|^{2s}}$, $w'(z)=\dfrac{s}{|z|^{2s+1}}$. Using (17) and Lemma 2.3, we have

$$\begin{aligned} N\underline{u}(z,t)&=-\gamma\epsilon A_1\mathrm{e}^{-\gamma t}\phi'(\zeta_1)+\epsilon\gamma \mathrm{e}^{-\gamma t}w(z)-\epsilon \mathrm{e}^{-\gamma t}(-\Delta)^s w(z)+c\epsilon \mathrm{e}^{-\gamma t}w'(z)\\ &\quad+f\left(\phi(\zeta_1)\right)-f\left(\phi(\zeta_1)-\epsilon \mathrm{e}^{-\gamma t}w(z)\right)\\ &\le -\gamma A_1\epsilon\beta_1\mathrm{e}^{-\gamma t}|z|^{-2s}+\frac{1}{2}\epsilon\gamma \mathrm{e}^{-\gamma t}|z|^{-2s}+C_2\epsilon \mathrm{e}^{-\gamma t}|z|^{-2s}+cs\epsilon \mathrm{e}^{-\gamma t}|z|^{-(2s+1)}\\ &\quad+\frac{1}{2}\gamma_1\epsilon \mathrm{e}^{-\gamma t}|z|^{-2s}\\ &\le -\gamma A_1\epsilon\beta_1\mathrm{e}^{-\gamma t}|z|^{-2s}+\frac{1}{2}\epsilon\gamma \mathrm{e}^{-\gamma t}|z|^{-2s}+C_2\epsilon \mathrm{e}^{-\gamma t}|z|^{-2s}+cs\epsilon \mathrm{e}^{-\gamma t}|z|^{-2s}\\ &\quad+\frac{1}{2}\gamma_1\epsilon \mathrm{e}^{-\gamma t}|z|^{-2s}\\ &=\epsilon \mathrm{e}^{-\gamma t}|z|^{-2s}\left(-\gamma A_1\beta_1+\frac{1}{2}\gamma+C_2+cs+\frac{1}{2}\gamma_1\right)\\ &\le 0. \end{aligned}$$

Case 2. $z>\tilde{M}$.

In case 2, we have $\zeta_1=z-\epsilon A_1\left(1-\mathrm{e}^{-\gamma t}\right)>\tilde{M}-\epsilon A_1\left(1-\mathrm{e}^{-\gamma t}\right)>\tilde{M}-1$ for any $t>0$. By $w(z)=1$, $w'(z)=0$, (18) and Lemma 2.3, we have

$$
\begin{aligned}
N\underline{u}(z,t) &= -\gamma\epsilon A_1 \mathrm{e}^{-\gamma t}\phi'(\zeta_1) + \epsilon\gamma \mathrm{e}^{-\gamma t} w(z) - \epsilon \mathrm{e}^{-\gamma t}(-\Delta)^s w(z) + c\epsilon \mathrm{e}^{-\gamma t} w'(z) \\
&\quad + f(\phi(\zeta_1)) - f(\phi(\zeta_1) - \epsilon \mathrm{e}^{-\gamma t} w(z)) \\
&\le -\gamma\epsilon A_1\beta_2 \mathrm{e}^{-\gamma t}|z|^{-(2s+1)} + \epsilon\gamma \mathrm{e}^{-\gamma t} + C_1\epsilon e^{-\gamma t}|z|^{-(2s+1)} - \gamma\epsilon \mathrm{e}^{-\gamma t} \\
&= |z|^{-(2s+1)}\epsilon \mathrm{e}^{-\gamma t}(C_1 - \beta_2\gamma A_1) \\
&\le 0.
\end{aligned}
$$

Case 3. $z \in [-\tilde{M}, \tilde{M}]$.

In case 3, we have $w(z) \le 1$, $w'(z) \le 4sw(z) \le 4s$ and $\zeta_1 = z - \epsilon A_1(1-\mathrm{e}^{-\gamma t}) \in [-\tilde{M}-1, \tilde{M}]$ for any $t > 0$. From (19), (20) and Lemma 2.3, we have

$$
\begin{aligned}
N\underline{u}(z,t) &= -\gamma\epsilon A_1 e^{-\gamma t}\phi'(\zeta_1) + \epsilon\gamma \mathrm{e}^{-\gamma t} w(z) - \epsilon \mathrm{e}^{-\gamma t}(-\Delta)^s w(z) + c\epsilon \mathrm{e}^{-\gamma t} w'(z) \\
&\quad + f(\phi(\zeta_1)) - f(\phi(\zeta_1) - \epsilon \mathrm{e}^{-\gamma t} w(z)) \\
&\le -\delta_0\gamma\epsilon A_1 \mathrm{e}^{-\gamma t} + \epsilon\gamma \mathrm{e}^{-\gamma t} + \epsilon \mathrm{e}^{-\gamma t} C_3 + 4sc\epsilon \mathrm{e}^{-\gamma t} + K_1\epsilon \mathrm{e}^{-\gamma t} w(z) \\
&\le \epsilon \mathrm{e}^{-\gamma t}(-\delta_0\gamma A_1 + \gamma + C_3 + 4sc + K_1) \\
&\le 0.
\end{aligned}
$$

Therefore, we have proved that for all $\epsilon \in (0, \epsilon_1)$, $\underline{u}(z,t)$ is a subsolution of (8). By the comparison principle, we obtain

$$
u(z,t) \ge \phi(z) - \epsilon A_1(1-\mathrm{e}^{-\gamma t}) - \epsilon e^{-\gamma t} w(z), t \ge 0, z \in \mathbb{R}. \tag{21}
$$

2) In this part, it suffices to prove that $N\bar{u}(z,t) \ge 0$. Let $\zeta_2 = z + \epsilon A_2(1-\mathrm{e}^{-\gamma t})$. By (5), we obtain

$$
\begin{aligned}
N\bar{u}(z,t) &= \bar{u}_t + (-\Delta)^s \bar{u} - c\bar{u}_z - f(\bar{u}) \\
&= \gamma\epsilon A_2 \mathrm{e}^{-\gamma t}\phi'(\zeta_1) - \epsilon\gamma \mathrm{e}^{-\gamma t} w(z) + (-\Delta)^s \phi(\zeta_2) \\
&\quad + \epsilon \mathrm{e}^{-\gamma t}(-\Delta)^s w(z) - c\phi'(\zeta_2) - c\epsilon \mathrm{e}^{-\gamma t} w'(z) - f(\bar{u}) \\
&= \gamma\epsilon A_2 \mathrm{e}^{-\gamma t}\phi'(\zeta_2) - \epsilon\gamma \mathrm{e}^{-\gamma t} w(z) + \epsilon \mathrm{e}^{-\gamma t}(-\Delta)^s w(z) \\
&\quad - c\epsilon \mathrm{e}^{-\gamma t} w'(z) + f(\phi(\zeta_2)) - f(\phi(\zeta_2) + \epsilon \mathrm{e}^{-\gamma t} w(z)).
\end{aligned}
$$

For all $\epsilon > 0$, there exist γ_1, $\beta_1 > 0$, $M_3 > 0$ such that

$$
\begin{cases}
f(\phi(z)) - f(\phi(z) + \epsilon \mathrm{e}^{-\gamma t} w(z)) \ge -\gamma_1\epsilon \mathrm{e}^{-\gamma t} w(z) = -\dfrac{\gamma_1\epsilon \mathrm{e}^{-\gamma t}}{2|z|^{2s}}, \quad z < -M_3, \\
\phi'(z) \ge \dfrac{\beta_1}{|z|^{2s}}.
\end{cases} \tag{22}
$$

For all $\epsilon > 0$, there exist $\gamma \in (0, \gamma_1)$, $\beta_2 > 0$ and $M_4 > 0$ such that

$$
\begin{cases}
f(\phi(z)) - f(\phi(z) + \epsilon \mathrm{e}^{-\gamma t} w(z)) \ge \gamma\epsilon \mathrm{e}^{-\gamma t} w(z) = \gamma\epsilon \mathrm{e}^{-\gamma t}, \quad z > M_4, \\
\phi'(z) \ge \dfrac{\beta_2}{|z|^{1+2s}}.
\end{cases} \tag{23}
$$

Let $M = \max\{M_4, M_3 + 1, 2\}$. Since $\phi'(z) > 0$ for all $z \in \mathbb{R}$, there exists $\delta_1 > 0$ such that

$$\min_{z\in[-M-1,M+1]}\phi'(z)\geq\delta_1>0. \tag{24}$$

Let

$$A_2=\max\left\{\frac{\gamma_1+2C_2+2cs+\gamma}{2\gamma\beta_1},\frac{C_1}{\beta_2\gamma},\frac{\gamma+C_3+4cs+K_1}{\gamma\delta_0}\right\}, \tag{25}$$

and $\epsilon\in(0,\epsilon_2)$ with $\epsilon_2=\frac{1}{A_2}$.

Similarly, we consider the following three cases.

Case 1. $z<-M$.

By the definition of $w(z)$, we know that $w(z)=\frac{1}{2|z|^{2s}}$ and $w'(z)=\frac{s}{|z|^{2s+1}}$. we have $\zeta_2=z+\epsilon A_2\left(1-\mathrm{e}^{-\gamma t}\right)<-M+\epsilon A_2\left(1-\mathrm{e}^{-\gamma t}\right)<-M_3$ for all $t>0$. By (22) and Lemma 2.3, we have

$$\begin{aligned}N\overline{u}(z,t)&=\gamma\epsilon A_2\mathrm{e}^{-\gamma t}\phi'(\zeta_2)-\epsilon\gamma\mathrm{e}^{-\gamma t}w(z)+\epsilon\mathrm{e}^{-\gamma t}(-\Delta)^s w(z)-c\epsilon\mathrm{e}^{-\gamma t}w'(z)\\&\quad+f(\phi(\zeta_2))-f\left(\phi(\zeta_2)+\epsilon\mathrm{e}^{-\gamma t}w(z)\right)\\&\geq\gamma\epsilon A_2\beta_1\mathrm{e}^{-\gamma t}|z|^{-2s}-\frac{1}{2}\epsilon\gamma\mathrm{e}^{-\gamma t}|z|^{-2s}-C_2\epsilon\mathrm{e}^{-\gamma t}\|z|^{-2s}-cs\epsilon\mathrm{e}^{-\gamma t}\|z|^{-(2s+1)}\\&\quad-\frac{1}{2}\gamma_1\epsilon\mathrm{e}^{-\gamma t}|z|^{-2s}\\&\geq\gamma\epsilon A_2\beta_1\mathrm{e}^{-\gamma t}|z|^{-2s}-\frac{1}{2}\epsilon\gamma e^{-\gamma t}|z|^{-2s}-C_2\epsilon\mathrm{e}^{-\gamma t}|z|^{-2s}-cs\epsilon\mathrm{e}^{-\gamma t}|z|^{-2s}\\&\quad-\frac{1}{2}\gamma_1\epsilon\mathrm{e}^{-\gamma t}|z|^{-2s}\\&=\epsilon\mathrm{e}^{-\gamma t}|z|^{-2s}\left(\gamma A_2\beta_1-\frac{1}{2}\gamma-C_2-cs-\frac{1}{2}\gamma_1\right)\geq0.\end{aligned}$$

Case 2. $z>M$.

We know that $\zeta_2=z+\epsilon A_2\left(1-\mathrm{e}^{-\gamma t}\right)>M+\epsilon A_2\left(1-\mathrm{e}^{-\gamma t}\right)>M_4$ for all $t>0$, $w(z)=1$ and $w'(z)=0$. Useing (23) and Lemma 2.3, we obtain

$$\begin{aligned}N\overline{u}(z,t)&=\gamma\epsilon A_2\mathrm{e}^{-\gamma t}\phi'(\zeta_2)-\epsilon\gamma\mathrm{e}^{-\gamma t}w(z)+\epsilon\mathrm{e}^{-\gamma t}(-\Delta)^s w(z)-c\epsilon\mathrm{e}^{-\gamma t}w'(z)\\&\quad+f(\phi(\zeta_2))-f\left(\phi(\zeta_2)+\epsilon\mathrm{e}^{-\gamma t}w(z)\right)\\&\geq\gamma\epsilon A_2\beta_2\mathrm{e}^{-\gamma t}|z|^{-(2s+1)}-\epsilon\gamma\mathrm{e}^{-\gamma t}-C_1\epsilon\mathrm{e}^{-\gamma t}|z|^{-(2s+1)}+\gamma\epsilon\mathrm{e}^{-\gamma t}\\&=|z|^{-(2s+1)}\epsilon\mathrm{e}^{-\gamma t}\left(\beta_2\gamma A_2-C_1\right)\\&\geq0.\end{aligned}$$

Case 3. $z\in[-M,M]$.

In this case, we have $w(z)\leq1$, $w'(z)\leq4sw(z)\leq4s$ and $\zeta_2=z+\epsilon A_2\left(1-\mathrm{e}^{-\gamma t}\right)\in[-M,M+1]$ for all $t>0$. Using (24), (25) and Lemma 2.3, we have

$$\begin{aligned}N\overline{u}(z,t) &= \gamma\epsilon A_2 \mathrm{e}^{-\gamma t}\phi'(\zeta_2) - \epsilon\gamma \mathrm{e}^{-\gamma t} w(z) + \epsilon \mathrm{e}^{-\gamma t}(-\Delta)^s w(z) - c\epsilon \mathrm{e}^{-\gamma t} w'(z) \\ &\quad + f(\phi(\zeta_2)) - f(\phi(\zeta_2) + \epsilon \mathrm{e}^{-\gamma t} w(z)) \\ &\geq \delta_1 \gamma \epsilon A_2 \mathrm{e}^{-\gamma t} - \epsilon\gamma \mathrm{e}^{-\gamma t} - \epsilon \mathrm{e}^{-\gamma t} C_3 - 4sc\epsilon \mathrm{e}^{-\gamma t} - K_1 \epsilon \mathrm{e}^{-\gamma t} w(z) \\ &\geq \epsilon \mathrm{e}^{-\gamma t}(\delta_1 \gamma A_2 - \gamma - C_3 - 4sc - K_1) \\ &\geq 0.\end{aligned}$$

Therefore, we have proved that for all $\epsilon \in (0, \epsilon_2)$, $\overline{u}(z,t)$ is a supersolution of (8). By the comparison principle, we obtain

$$u(z,t) \leq \phi(z) + \epsilon A_2 (1 - \mathrm{e}^{-\gamma t}) + \epsilon \mathrm{e}^{-\gamma t} w(z), t \geq 0, z \in \mathbb{R}. \tag{26}$$

In summary, let $\tilde{\epsilon} = \min\{\epsilon_1, \epsilon_2\}$. Then, for all $\epsilon \in (0, \tilde{\epsilon})$, we have

$$\underline{u}(z,t) \leq u(z,t) \leq \overline{u}(z,t), \text{ for all } z \in \mathbb{R}, t \geq 0.$$

Theorem 4.2. (Lyapunov stability of traveling wave solution) Assume that $\frac{p}{2(p-2)} < s < 1$ and f satisfies (4). Then the traveling wave solution $\phi(x - ct)$ established in Lemma 1.1 is Lyapunov stable. That is, there exists a positive constant $\tilde{\epsilon}$ such that for all $\epsilon \in (0, \tilde{\epsilon})$, if the initial condition $u_0(x) \in [0,1]$ with $u_0(x) \in X_0$ and $\|u_0(x) - \phi(x)\|_{X_0} \leq \epsilon$, then the solution $u(x,t)$ of (7) exists uniquely for all $t > 0$ and satisfies

$$\|u(x,t) - \phi(x - ct)\|_{X_0} \leq \tilde{K}\epsilon, \ \forall t > 0,$$

where $\tilde{K} > 0$ is independent on t and $u_0(x)$.

Proof. From [18], Cabré and Roque have proved that (7) admits a unique global solution $u(x,t)$ by using semigroup theory. From Remark 1.1, we know that $u(z,t)$ is a unique global solution of (8) with $z = x - ct$.

Let $u(z,t)$ be the unique global solution for (8). By Theorem 4.1, for all $\epsilon \in (0, \tilde{\epsilon})$, we have

$$\underline{u}(z,t) \leq u(z,t) \leq \overline{u}(z,t), \text{ for all } z \in \mathbb{R}, t \geq 0.$$

Subtracting $\phi(z)$ on both sides in (21) and using the mean value theorem, we can get

$$\begin{aligned}u(z,t) - \phi(z) &\geq \phi(z - \epsilon A_1(1 - \mathrm{e}^{-\gamma t})) - \epsilon \mathrm{e}^{-\gamma t} w(z) - \phi(z) \\ &= -\phi'(\theta_1)\epsilon A_1(1 - \mathrm{e}^{-\gamma t}) - \epsilon \mathrm{e}^{-\gamma t} w(z) \\ &\geq -\phi'(\theta_1)\epsilon A_1(1 - \mathrm{e}^{-\gamma t}) - \epsilon w(z) \\ &\geq -\phi'(\theta_1) A_1 - \epsilon w(z) \\ &= -\epsilon[\phi'(\theta_1) A_1 + w(z)],\end{aligned}$$

where $\epsilon A_1(1 - \mathrm{e}^{-\gamma t}) \leq \epsilon A_1$ and $\theta_1 \in (z - \epsilon A_1(1 - \mathrm{e}^{-\gamma t}), z)$.

For (26), by the same method, we obtain

$$\begin{aligned} u(z,t)-\phi(z) &\le \phi\left(z+\epsilon A_2\left(1-\mathrm{e}^{-\gamma t}\right)\right)+\epsilon \mathrm{e}^{-\gamma t}w(z)-\phi(z) \\ &= \phi'(\theta_2)\epsilon A_2\left(1-\mathrm{e}^{-\gamma t}\right)+\epsilon \mathrm{e}^{-\gamma t}w(z) \\ &\le \phi'(\theta_2)A_2+\epsilon w(z) \\ &= \epsilon\left[\phi'(\theta_2)A_2+w(z)\right], \end{aligned}$$

where $\epsilon A_2\left(1-\mathrm{e}^{-\gamma t}\right)\le \epsilon A_2$ and $\theta_2\in\left(z,z+\epsilon A_2\left(1-\mathrm{e}^{-\gamma t}\right)\right)$.

Therefore, we have

$$-\epsilon\left[\phi'(\theta_1)A_1+w(z)\right]\le u(z,t)-\phi(z)\le \epsilon\left[\phi'(\theta_2)A_2+w(z)\right]. \tag{27}$$

Since $w_0(z)>0$ for all $z\in\mathbb{R}$, multiplying $w_0(z)$ on both sides in (27), we obtain

$$\begin{cases} \left(u(z,t)-\phi(z)\right)w_0(z)\ge -\epsilon\left[\phi'(\theta_1)A_1w_0(z)+1\right], \\ \left(u(z,t)-\phi(z)\right)w_0(z)\le \epsilon\left[\phi'(\theta_2)A_2w_0(z)+1\right]. \end{cases} \tag{28}$$

By Lemma 1.2 and Lemma 3.1, we have

$$\phi'(z)\le \frac{\beta_1}{|z|^{2s}},\quad \forall z\le -1,$$

$$\frac{\beta_2}{|z|^{1+2s}}\le \phi'(z)\le \frac{\beta_2}{|z|^{1+2s}},\quad z\ge 1.$$

Next, using the definition of $w_0(z)$, it follows that

i) *For all* $z\ge 0$, $w_0(z)=1$; *as* $z\to+\infty$, $\phi'(z)\to 0$, *hence* $\phi'(z)w_0(z)\to 0$.

ii) *For all* $z\le -1$, $w_0(z)=2|z|^{2s}$, $\phi'(z)\le \frac{\beta_1}{|z|^{2s}}$, *hence* $0<\phi'(z)w_0(z)\le 2\beta_1$.

iii) *For any* $M>0$, *it is evident that* $w_0(z)$ *and* $\phi'(z)$ *are bounded for all* $z\in[-M,M]$, *so there exists* $B>0$ *such that* $\left|\phi'(z)w_0(z)\right|\le B$.

Let $\tilde{K}=\max\left\{B,2A_1\beta_1+1,2A_2\beta_1+1\right\}$. By (28), we have

$$\left\|u(z,t)-\phi(z)\right\|_{X_0}\le \tilde{K}\epsilon,\ \text{for all } t>0.$$

Let $z=x-ct$, by Remark 1.1, we have

$$\left\|u(x,t)-\phi(x-ct)\right\|_{X_0}\le \tilde{K}\epsilon,\ \text{for all } t>0.$$

The proof is complete.

5. Conclusion

In this study, we investigate the Lyapunov stability of traveling wave solutions for degenerate Fisher type equations with fractional Laplacian. Firstly, we use the maximum principle to prove the asymptotic behavior of the derivative of the traveling wave solutions to (3) when $x\to-\infty$, which provides an important foundation for the stability analysis of traveling wave solutions. Secondly, we construct the subsolution and supersolution of the Cauchy problem (8). Finally, the Lyapunov stability of traveling wave solutions in polynomially weighted spaces is proved by sub-super solution method. This research develops the stability theory of traveling wave solutions for degenerate fractional reaction-diffusion equations. We can further investigate the Lyapunov stability of traveling wave solutions for degenerate bistable equations with fractional Laplacian.

Acknowledgements

This research is supported in part by the Natural Science Foundation of China (Grant No.61803236).

Conflicts of Interest

The authors declare there is no conflict of interest.

References

[1] Bebernes, J. and Eberly, D. (1989) Mathematical Problems from Combustion Theory (Applied Mathematical Sciences, 83). Springer.

[2] Fisher, R.A. (1937) The Wave of Advance of Advantageous Genes. *Annals of Eugenics*, **7**, 335-369.

[3] Kolmogorov, A., Petrovskii, I. and Piskunov, N. (1937) A Study of the Equation of Diffusion with Increase in the Quantity of Matter, and Its Application to a Biological Problem. *Moscow University Mathematics Bulletin*, **1**, 1-25.

[4] Kametaka, Y. (1962) On the Stabilization of Solutions of the Cauchy Problem for Equations Arising in the Theory of Combustion. *Matematicheskii Sbornik*, **59**, 245-288.

[5] Fife, P.C. and McLeod, J.B. (1977) The Approach of Solutions of Nonlinear Diffusion Equations to Travelling Front Solutions. *Archive for Rational Mechanics and Analysis*, **65**, 335-361. https://doi.org/10.1007/bf00250432

[6] Ninomiya, H. and Taniguchi, M. (2005) Existence and Global Stability of Traveling Curved Fronts in the Allen-Cahn Equations. *Journal of Differential Equations*, **213**, 204-233. https://doi.org/10.1016/j.jde.2004.06.011

[7] Taniguchi, M. (2007) Traveling Fronts of Pyramidal Shapes in the Allen-Cahn Equations. *SIAM Journal on Mathematical Analysis*, **39**, 319-344. https://doi.org/10.1137/060661788

[8] Taniguchi, M. (2009) The Uniqueness and Asymptotic Stability of Pyramidal Traveling Fronts in the Allen-Cahn Equations. *Journal of Differential Equations*, **246**, 2103-2130. https://doi.org/10.1016/j.jde.2008.06.037

[9] Matano, H. and Nara, M. (2011) Large Time Behavior of Disturbed Planar Fronts in the Allen-Cahn Equation. *Journal of Differential Equations*, **251**, 3522-3557. https://doi.org/10.1016/j.jde.2011.08.029

[10] Matano, H., Nara, M. and Taniguchi, M. (2009) Stability of Planar Waves in the Allen-Cahn Equation. *Communications in Partial Differential Equations*, **34**, 976-1002. https://doi.org/10.1080/03605300902963500

[11] Volpert, V.A., Nec, Y. and Nepomnyashchy, A.A. (2010) Exact Solutions in Front Propagation Problems with Superdiffusion. *Physica D*: *Nonlinear Phenomena*, **239**, 134-144. https://doi.org/10.1016/j.physd.2009.10.011

[12] Metzler, R. and Klafter, J. (2004) The Restaurant at the End of the Random Walk: Recent Developments in the Description of Anomalous Transport by Fractional Dynamics. *Journal of Physics A*: *Mathematical and General*, **37**, R161-R208.

https://doi.org/10.1088/0305-4470/37/31/r01

[13] Gui, C. and Huan, T. (2015) Traveling Wave Solutions to Some Reaction Diffusion Equations with Fractional Laplacians. *Calculus of Variations and Partial Differential Equations*, **54**, 251-273. https://doi.org/10.1007/s00526-014-0785-y

[14] Gui, C. and Zhao, M. (2015) Traveling Wave Solutions of Allen-Cahn Equation with a Fractional Laplacian. *Annales de l'Institut Henri Poincaré C, Analyse non linéaire*, **32**, 785-812. https://doi.org/10.1016/j.anihpc.2014.03.005

[15] Ma, L., Niu, H. and Wang, Z. (2019) Global Asymptotic Stability of Traveling Waves to the Allen-Cahn Equation with a Fractional Laplacian. *Communications on Pure & Applied Analysis*, **18**, 2457-2472. https://doi.org/10.3934/cpaa.2019111

[16] Cheng, H. and Yuan, R. (2022) The Stability of Traveling Waves for Allen-Cahn Equations with Fractional Laplacian. *Applicable Analysis*, **101**, 263-273. https://doi.org/10.1080/00036811.2020.1738399

[17] Mellet, A., Roquejoffre, J.M. and Sire, Y. (2014) Existence and Asymptotics of Fronts in Non Local Combustion Models. *Communications in Mathematical Sciences*, **12**, 1-11. https://doi.org/10.4310/cms.2014.v12.n1.a1

[18] Cabré, X. and Roquejoffre, J.M. (2013) The Influence of Fractional Diffusion in Fisher-KPP Equations. *Communications in Mathematical Physics*, **320**, 679-722. https://doi.org/10.1007/s00220-013-1682-5

[19] He, J., Wu, Y. and Wu, Y. (2017) Large Time Behavior of Solutions for Degenerate p-Degree Fisher Equation with Algebraic Decaying Initial Data. *Journal of Mathematical Analysis and Applications*, **448**, 1-21. https://doi.org/10.1016/j.jmaa.2016.10.037