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Measuring Similarity between Generalized Fuzzy Numbers Based on Standard Deviation for Fuzzy Recommendation Problems

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Abstract

This study proposes a new method to measure the similarity among generalized fuzzy numbers based on a standard deviation. The properties of the proposed similarity measure are demonstrated with 44 sets of generalized fuzzy numbers applied to compare the proposed method with existing similarity measures. Comparative results indicate the proposed method is better than existing methods. Solving fuzzy recommendation problems uses the proposed similarity measure.

Keywords

Generalized Fuzzy Number, Standard Deviation, Similarity Measure, Fuzzy Recommendation Problems

1. Introduction

Measuring the similarity between fuzzy numbers is important in fuzzy decision-making [1]-[3] and fuzzy risk analysis [4]-[7]. Researchers have presented methods for determining the degree of similarity between fuzzy numbers [4] [5], and [8]-[13]. However, those similarity measures have limitations. These could not consistently or accurately yield the degree of similarity between two generalized fuzzy numbers. Therefore, this study proposes a new method to measure the degree of similarity among generalized fuzzy numbers based on a standard deviation to overcome these shortcomings. The proposed similarity measure is compared to eight existing methods [4] [5], and [8]-[13] using 44 sets of generalized fuzzy numbers. Comparative results reveal that the proposed method of measuring similarity overcomes the limitations of the existing methods. Furthermore, we applied the proposed similarity measure to solve fuzzy recommendation problems, specifically exemplified by evaluating the quality of healthcare at a hospital.

The remainder of the paper is in five sections. Section 2 briefly reviews the concept of standard deviation [14], the definitions of generalized fuzzy numbers [15] [16], and existing measures of similarity among fuzzy numbers [4] [5], and [8]-[13]. Section 3 analyses and describes the shortcomings of the existing similarity measure. Section 4 presents a new method based on standard deviation to calculate the degree of similarity between generalized fuzzy numbers, and demonstrates some properties of the proposed similarity measure. Section 5 compares the proposed similarity measure with existing measures using 44 sets of generalized fuzzy numbers. Section 6 applies the proposed similarity measure to solve a fuzzy recommendation problem. Section 7 presents the conclusions.

2. Preliminaries

This section reviews the concept of standard deviation [14], the definitions of generalized fuzzy numbers [15] [16], and existing measures of similarity among fuzzy numbers [4] [5], and [8]-[13].

2.1. Standard Deviation

The positive square root of the variance is the standard deviation. The standard deviation gives a measure of the dispersion of the data from the sample mean. The standard deviation is:

$$S_A = \sqrt{\frac{\sum_{i=1}^{n}\left(a_i - \bar{a}\right)^2}{n-1}}, \tag{1}$$

where $i = 1, 2, \cdots, n$ [14].

2.2. Generalized Fuzzy Numbers

Chen [15] [16] denoted a generalized trapezoidal fuzzy number as $\tilde{A} = \left(a, b, c, d; w\right)$, where $0 \le w \le 1$, and a, b, c, and d are real numbers. If $w = 1$, then the generalized fuzzy number $\tilde{A}$ is called a normal trapezoidal fuzzy number, denoted as $\tilde{A} = \left(a, b, c, d\right)$. If $b = c$, then $\tilde{A}$ is called a generalized triangular fuzzy number. If $a = b$, $c = d$ and $w = 1$, then $\tilde{A}$ is called a crisp interval. If $a = b = c = d$ and $w = 1$, then $\tilde{A}$ is a real number. **Figure 1** displays a generalized trapezoidal fuzzy number $\tilde{A}$, while $\tilde{A}$ denotes the opinion of a decision-maker. The value of w represents the degree of confidence of the decision maker's opinion.

2.3. The Existing Similarity Measures between Fuzzy Numbers

Chen [4] proposed a distance-based similarity measure of trapezoidal fuzzy numbers. Consider two trapezoidal fuzzy numbers $\tilde{A}$ and $\tilde{B}$, where $\tilde{A} = \left(a_1, a_2, a_3, a_4\right)$ and $\tilde{B} = \left(b_1, b_2, b_3, b_4\right)$. The degree of similarity $S\left(\tilde{A}, \tilde{B}\right)$, between $\tilde{A}$ and $\tilde{B}$ can be calculated as follows:

$$S\left(\tilde{A},\tilde{B}\right)=1-\frac{\sum_{i=1}^{4}\left|a_i-b_i\right|^2}{4}, \tag{2}$$

where $S\left(\tilde{A},\tilde{B}\right)\in[0,1]$. If $\tilde{A}$ and $\tilde{B}$ represent triangular fuzzy numbers, where $\tilde{A}=\left(a_1,a_2,a_3\right)$ and $\tilde{B}=\left(b_1,b_2,b_3\right)$, then the degree similarity $S\left(\tilde{A},\tilde{B}\right)$ between $\tilde{A}$ and $\tilde{B}$ can be calculated as follows [4]:

$$S\left(\tilde{A},\tilde{B}\right)=1-\frac{\sum_{i=1}^{3}\left|a_i-b_i\right|^2}{3}. \tag{3}$$

A larger value of $S\left(\tilde{A},\tilde{B}\right)$ represents a greater similarity between the generalized fuzzy numbers $\tilde{A}$ and $\tilde{B}$.

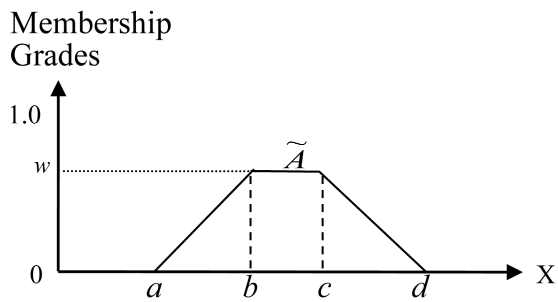

Figure 1. Generalized trapezoidal fuzzy number $\tilde{A}$.

Lee [11] proposed a similarity measure for trapezoidal fuzzy numbers and applied the similarity measure to handle fuzzy opinions for group decision-making. If $\tilde{A}$ and $\tilde{B}$ are trapezoidal fuzzy numbers $\tilde{A}=\left(a_1,a_2,a_3,a_4\right)$ and $\tilde{B}=\left(b_1,b_2,b_3,b_4\right)$, then the degree of similarity $S\left(\tilde{A},\tilde{B}\right)$ between $\tilde{A}$ and $\tilde{B}$ can be calculated as follows:

$$S\left(\tilde{A},\tilde{B}\right)=1-\frac{\left\|\tilde{A}-\tilde{B}\right\|_{lP}}{\left\|U\right\|}\times 4^{\frac{-1}{p}}, \tag{4}$$

where U is the universe of discourse,

$$\left\|U\right\|=\max\left(U\right)-\min\left(U\right), \tag{5}$$

And

$$\left\|\tilde{A}-\tilde{B}\right\|_{lP}=\left(\sum_{i=1}^{4}\left|a_i-b_i\right|^p\right)^{\frac{-1}{p}}. \tag{6}$$

A larger value of $S\left(\tilde{A},\tilde{B}\right)$ corresponds to greater similarity between the trapezoidal fuzzy numbers $\tilde{A}$ and $\tilde{B}$.

Hsieh and Chen [10] proposed a similarity measure based on the "graded mean integration representation distance", where the degree of similarity $S\left(\tilde{A},\tilde{B}\right)$ between fuzzy numbers $\tilde{A}$ and $\tilde{B}$ is calculated as follows:

$$S\left(\tilde{A},\tilde{B}\right)=\frac{1}{1+d\left(\tilde{A},\tilde{B}\right)}, \tag{7}$$

where

$$d\left(\tilde{A},\tilde{B}\right)=\left|p\left(\tilde{A}\right)-p\left(\tilde{B}\right)\right|, \tag{8}$$

$p\left(\tilde{A}\right)$ and $p\left(\tilde{B}\right)$ are the "graded mean integration representation" of $\tilde{A}$ and $\tilde{B}$, respectively. If $\tilde{A}$ and $\tilde{B}$ represent triangular fuzzy numbers, then the "graded mean integration representation" $p\left(\tilde{A}\right)$ and $p\left(\tilde{B}\right)$ of $\tilde{A}$ and $\tilde{B}$ are defined as:

$$p\left(\tilde{A}\right)=\frac{a_1+4a_2+a_3}{6}, \tag{9}$$

$$p\left(\tilde{B}\right)=\frac{b_1+4b_2+b_3}{6}. \tag{10}$$

If $\tilde{A}$ and $\tilde{B}$ denote trapezoidal fuzzy numbers, then the "graded mean integration representation" $p\left(\tilde{A}\right)$ and $p\left(\tilde{B}\right)$ of $\tilde{A}$ and $\tilde{B}$ are defined as follows:

$$p\left(\tilde{A}\right)=\frac{a_1+2a_2+2a_3+a_4}{6}, \tag{11}$$

$$p\left(\tilde{B}\right)=\frac{b_1+2b_2+2b_3+b_4}{6}. \tag{12}$$

A larger value of $S\left(\tilde{A},\tilde{B}\right)$ corresponds to a greater similarity between the generalized fuzzy numbers $\tilde{A}$ and $\tilde{B}$.

Chen and Chen [5] proposed a method to measure the degree of similarity among generalized fuzzy numbers using the Simple Center of Gravity Method (SCGM) that calculates the center of gravity (COG) of generalized fuzzy numbers, followed by the degree of similarity between those numbers. The degree of similarity $S\left(\tilde{A},\tilde{B}\right)$ between the generalized trapezoidal fuzzy numbers $\tilde{A}$ and $\tilde{B}$, where $\tilde{A}=\left(a_1,a_2,a_3,a_4;w_{\tilde{A}}\right)$ and $\tilde{B}=\left(b_1,b_2,b_3,b_4;w_{\tilde{B}}\right)$ is calculated as follows:

$$S\left(\tilde{A},\tilde{B}\right)=\left(1-\frac{\sum_{i=1}^{4}\left|a_i-b_i\right|}{4}\right)\times\left(1-\left|x_{\tilde{A}^*}-x_{\tilde{B}^*}\right|\right)^{B\left(S_{\tilde{A}},S_{\tilde{B}}\right)}\times\frac{\min\left(y_{\tilde{A}}^*,y_{\tilde{B}}^*\right)}{\max\left(y_{\tilde{A}}^*,y_{\tilde{B}}^*\right)}, \tag{13}$$

where $S\left(\tilde{A},\tilde{B}\right)\in[0,1]$, and $B\left(S_{\tilde{A}},S_{\tilde{B}}\right)$ is defined as follows.

$$B\left(S_{\tilde{A}},S_{\tilde{B}}\right)=\begin{cases}1, & \text{if } S_{\tilde{A}}+S_{\tilde{B}}>0,\\ 0, & \text{if } S_{\tilde{A}}+S_{\tilde{B}}=0,\end{cases} \tag{14}$$

where $S_{\tilde{A}}$ and $S_{\tilde{B}}$ are the length of the bases of the generalized trapezoidal fuzzy numbers $\tilde{A}$ and $\tilde{B}$, respectively, and defined as:

$$S_{\tilde{A}}=a_4-a_1 \tag{15}$$

$$S_{\tilde{B}}=b_4-b_1 \tag{16}$$

The two values $y_{\tilde{A}}^*$ and $x_{\tilde{A}^*}$, are calculated as follows:

$$y_{\tilde{A}}^*=\begin{cases}\dfrac{w_{\tilde{A}}\times\left(\dfrac{a_3-a_2}{a_4-a_1}+2\right)}{6}, & \text{if } a_1\neq a_4 \text{ and } 0<w_{\tilde{A}}\leq 1,\\[2ex] \dfrac{w_{\tilde{A}}}{2}, & \text{if } a_1=a_4 \text{ and } 0<w_{\tilde{A}}\leq 1,\end{cases} \tag{17}$$

$$x_{\tilde{A}^*} = \frac{y^*_{\tilde{A}}(a_3+a_2)+(a_4+a_1)(w_{\tilde{A}}-y^*_{\tilde{A}})}{2w_{\tilde{A}}}. \quad (18)$$

Similarly, the two values $y^*_{\tilde{B}}$ and $x_{\tilde{B}^*}$ are calculated by (17) and (18). A larger $S(\tilde{A},\tilde{B})$ corresponds to a greater similarity between the generalized fuzzy numbers $\tilde{A}$ and $\tilde{B}$.

Yong *et al.* [13] proposed a method to measure the degree of similarity among generalized fuzzy numbers based on the radius of gyration (ROG) points. In their method, the degree of similarity $S(\tilde{A},\tilde{B})$ between the generalized trapezoidal fuzzy numbers $\tilde{A}=(a_1,a_2,a_3,a_4;w_{\tilde{A}})$ and $\tilde{B}=(b_1,b_2,b_3,b_4;w_{\tilde{B}})$ are calculated as follows:

$$S(\tilde{A},\tilde{B}) = \left[1-\frac{\sum_{i=1}^{4}|a_i-b_i|}{4}\right]\times\left[1-\left|r_y^{\tilde{A}}-r_y^{\tilde{B}}\right|^{B(S_{\tilde{A}},S_{\tilde{B}})}\right]\times\frac{\min(r_x^{\tilde{A}},r_x^{\tilde{B}})}{\max(r_x^{\tilde{A}},r_x^{\tilde{B}})}, \quad (19)$$

where $S(\tilde{A},\tilde{B})\in[0,1]$. The value $B(S_{\tilde{A}},S_{\tilde{B}})$ is from (14). The two values $r_x^{\tilde{A}}$ and $r_y^{\tilde{A}}$ are the ROG of the generalized fuzzy number $\tilde{A}$, and are calculated as follows:

$$r_x^{\tilde{A}} = \sqrt{\frac{(I_x)_1+(I_x)_2+(I_x)_3}{\frac{((a_3-a_2)(a_4-a_1))\times w_{\tilde{A}}}{2}}}, \quad (20)$$

$$r_y^{\tilde{A}} = \sqrt{\frac{(I_y)_1+(I_y)_2+(I_y)_3}{\frac{((a_3-a_2)(a_4-a_1))\times w_{\tilde{A}}}{2}}}. \quad (21)$$

where I_x is the moment of inertia of $\tilde{A}$ on the x-axis

$$(I_x)_1 = \frac{(a_2-a_1)w^3_{\tilde{A}}}{12}, \quad (22)$$

$$(I_x)_2 = \frac{(a_3-a_2)w^3_{\tilde{A}}}{3}, \quad (23)$$

$$(I_x)_3 = \frac{(a_4-a_3)w^3_{\tilde{A}}}{12}, \quad (24)$$

$$(I_y)_1 = \frac{(a_2-a_1)^3 w^3_{\tilde{A}}}{4}+\frac{(a_2-a_1)a_1^2 w_{\tilde{A}}}{2}+\frac{2(a_2-a_1)a_1 w_{\tilde{A}}}{3}, \quad (25)$$

$$(I_y)_2 = \frac{(a_3-a_2)^3 w_{\tilde{A}}}{3}+(a_3-a_2)a_2^2 w_{\tilde{A}}+(a_3-a_2)^2 a_2 w_{\tilde{A}}, \quad (26)$$

$$(I_y)_3 = \frac{(a_4-a_3)^3 w_{\tilde{A}}}{4}+\frac{(a_4-a_3)a_3^2 w_{\tilde{A}}}{2}+\frac{(a_4-a_3)^2 a_3 w_{\tilde{A}}}{3}. \quad (27)$$

In a special case, if the generalized fuzzy number $\tilde{A}=(a,b,c,d;w)$ and $a=b=c=d$ and $0\le w\le 1$, such that $\tilde{A}$ is a real number, then the two values $r_x^{\tilde{A}}$ and $r_y^{\tilde{A}}$ are obtained using the following:

$$r_x^{\tilde{A}} = \frac{\sqrt{3}}{3}w, \quad (28)$$

$$r_y^{\tilde{A}} = a. \tag{29}$$

A larger $S(\tilde{A},\tilde{B})$ corresponds to greater similarity between the generalized fuzzy numbers $\tilde{A}$ and $\tilde{B}$.

Chen [8] proposed a method to measure the degree of similarity among generalized fuzzy numbers using the geometric-mean operator to eliminate the shortcomings of the similarity measure of Yong *et al.* [13]. The degree of similarity $S(\tilde{A},\tilde{B})$ between the generalized fuzzy numbers $\tilde{A}$ and $\tilde{B}$ is calculated as follows:

$$S(\tilde{A},\tilde{B}) = \left[\sqrt[4]{\prod_{i=1}^{4}\left(2-|a_i-b_i|\right)}-1\right] \times \frac{\min\left(y_{\tilde{A}}^{*}, y_{\tilde{B}}^{*}\right)}{\max\left(y_{\tilde{A}}^{*}, y_{\tilde{B}}^{*}\right)}, \tag{30}$$

where $y_{\tilde{A}}^{*}$ and $y_{\tilde{B}}^{*}$ are given by (17). A larger $S(\tilde{A},\tilde{B})$ corresponds to a greater similarity between the generalized fuzzy numbers $\tilde{A}$ and $\tilde{B}$.

Wei and Chen [12] presented a method using the arithmetic-mean averaging operator and the perimeter of the generalized trapezoidal fuzzy number. The degree of similarity of $S(\tilde{A},\tilde{B})$ between $\tilde{A}$ and $\tilde{B}$ is calculated as follows:

$$S(\tilde{A},\tilde{B}) = \left[1-\frac{\sum_{i=1}^{4}|a_i-b_i|}{4}\right] \times \frac{\min\left(\left(p(\tilde{A}),p(\tilde{B})\right)+\min\left(w_{\tilde{A}},w_{\tilde{B}}\right)\right)}{\max\left(\left(p(\tilde{A}),p(\tilde{B})\right)+\max\left(w_{\tilde{A}},w_{\tilde{B}}\right)\right)}, \tag{31}$$

where $p(\tilde{A})$ and $p(\tilde{B})$ are the perimeters of generalized trapezoidal fuzzy numbers $\tilde{A}$ and $\tilde{B}$, respectively. These two values are calculated as follows:

$$P(\tilde{A}) = \sqrt{(a_1-a_2)^2+w_{\tilde{A}}^2}+\sqrt{(a_3-a_4)^2+w_{\tilde{A}}^2}+(a_3-a_2)+(a_4-a_1), \tag{32}$$

$$P(\tilde{B}) = \sqrt{(b_1-b_2)^2+w_{\tilde{B}}^2}+\sqrt{(b_3-b_4)^2+w_{\tilde{B}}^2}+(b_3-b_2)+(b_4-b_1). \tag{33}$$

A larger value of $S(\tilde{A},\tilde{B})$ corresponds to greater similarity between the generalized fuzzy numbers $\tilde{A}$ and $\tilde{B}$.

Chen [9] presented a method to measure the degree of similarity among generalized fuzzy numbers based on the quadratic-mean. The degree of similarity of $S(\tilde{A},\tilde{B})$ between the generalized fuzzy numbers $\tilde{A}$ and $\tilde{B}$ is calculated as follows:

$$S(\tilde{A},\tilde{B}) = \left[1-\sqrt{\frac{\sum_{i=1}^{4}(a_i-b_i)^2}{4}}\right] \times \frac{\min\left(w_{\tilde{A}},w_{\tilde{B}}\right)}{\max\left(w_{\tilde{A}},w_{\tilde{B}}\right)}, \tag{34}$$

where $S(\tilde{A},\tilde{B}) \in [0,1]$. A larger value of $S(\tilde{A},\tilde{B})$ corresponds to a greater similarity between the generalized fuzzy numbers $\tilde{A}$ and $\tilde{B}$.

3. Analysis of the Existing Similarity Measure

This section analyses the weaknesses of the existing similarity measure. In 2003, Chen and Chen [5] described the three properties of the similarity measure between generalized fuzzy numbers $\tilde{A}$ and $\tilde{B}$ as:

Property 1: Two generalized fuzzy numbers $\tilde{A}$ and $\tilde{B}$ are identical if and only if $S(\tilde{A},\tilde{B})=1$.

Property 2: $S(\tilde{A},\tilde{B})=S(\tilde{B},\tilde{A})$.

Property 3: If $\tilde{A}=(a,a,a,a;1.0)$ and $\tilde{B}=(b,b,b,b;1.0)$ are two real numbers, then $S(\tilde{A},\tilde{B})=1-|a-b|$.

Chen and Chen [5] also proposed a method for measuring the degree of similarity among generalized fuzzy numbers using the SCGM. Yong *et al.* [13] pointed out that the method of Chen and Chen [5] cannot handle two generalized fuzzy numbers with the same COG points, and presented a method for measuring the degree of similarity among generalized fuzzy numbers based on the ROG to overcome the weaknesses of Chen and Chen's [5] method. Chen [8] proposed a method using the geometric-mean averaging operator to eliminate the drawbacks of the similarity measure of Yong *et al.* [13]. Wei and Chen [12] noted that Chen's method [8] cannot correctly measure the similarity between two generalized fuzzy numbers in some cases, and proposed a method that uses the arithmetic-mean averaging operator and the perimeter of the generalized trapezoidal fuzzy number. In 2009, Chen indicated that Wei and Chen's method [12] has similar weaknesses identified in Chen's method [8]. Therefore, Chen proposed a method that uses the quadratic-mean operator to eliminate these shortcomings [9].

However, this study indicates that the similarity measure based on the quadratic-mean [9] still has drawbacks:

1) Consider the following two sets of generalized fuzzy numbers (**Figure 2**):

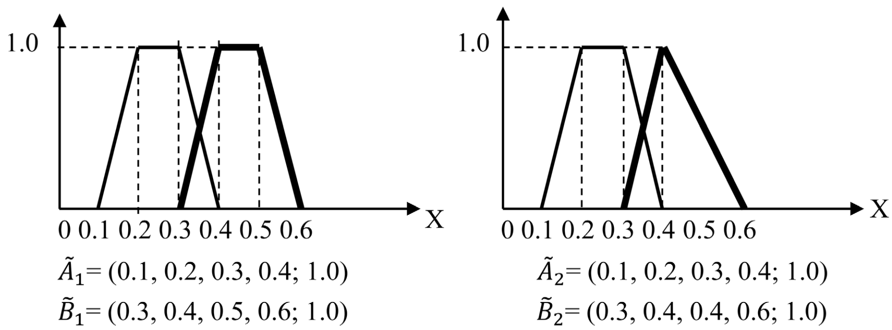

Figure 2. Two sets of generalized fuzzy numbers.

The degrees of similarity $S(\tilde{A}_1,\tilde{B}_1)=0.8$ and $S(\tilde{A}_2,\tilde{B}_2)=0.819722$ can be calculated from (34). However, $S(\tilde{A}_1,\tilde{B}_1)$ should not exceed $S(\tilde{A}_2,\tilde{B}_2)$ because the shapes of $\tilde{A}_1$ and $\tilde{B}_1$ are more similar than the shapes of $\tilde{A}_2$ and $\tilde{B}_2$.

2) Consider the following two sets of generalized fuzzy numbers:

According to Chen's method [9], the degrees of similarity are $S(\tilde{A}_1,\tilde{B}_1)=0.9293$ and $S(\tilde{A}_2,\tilde{B}_2)=0.9293$. However, $S(\tilde{A}_1,\tilde{B}_1)$ cannot reasonably equal the degree of similarity $S(\tilde{A}_2,\tilde{B}_2)$ because again the shapes of $\tilde{A}_2$ and $\tilde{B}_2$ are more similar to each other than those of $\tilde{A}_1$ and $\tilde{B}_1$, and the relative distance between $\tilde{A}_1$ and $\tilde{B}_1$ is the same as the relative distance between $\tilde{A}_2$ and $\tilde{B}_2$ (**Figure 3**).

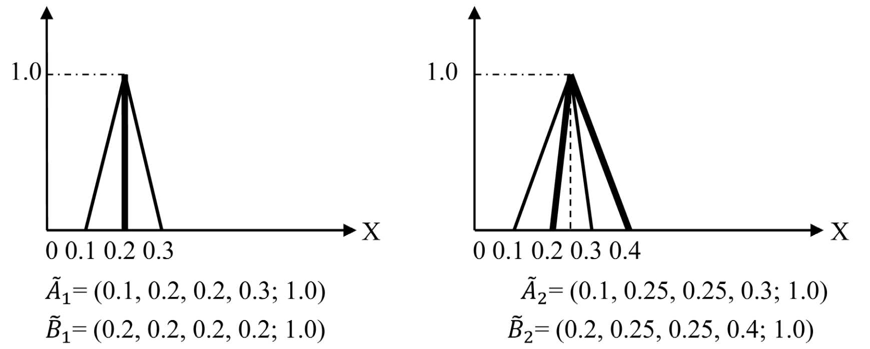

Figure 3. Two sets of generalized fuzzy numbers.

3) Consider the following two sets of generalized fuzzy numbers (**Figure 4**):

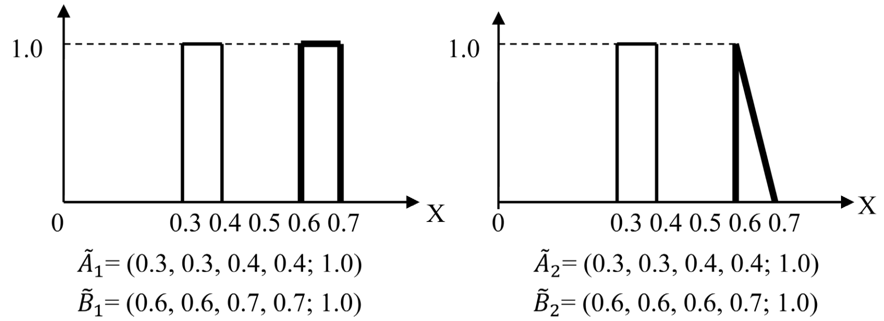

Figure 4. Two sets of generalized fuzzy numbers.

According to Chen's method [9], the degrees of similarity are $S(\tilde{A}_1,\tilde{B}_1)=0.7$ and $S(\tilde{A}_2,\tilde{B}_2)=0.7216$. However, $S(\tilde{A}_2,\tilde{B}_2)$ cannot reasonably exceed $S(\tilde{A}_1,\tilde{B}_1)$ because the shapes of $\tilde{A}_1$ and $\tilde{B}_1$ are more similar than those of $\tilde{A}_2$ and $\tilde{B}_2$.

4) Consider the following two sets of generalized fuzzy numbers shown in **Figure 5**:

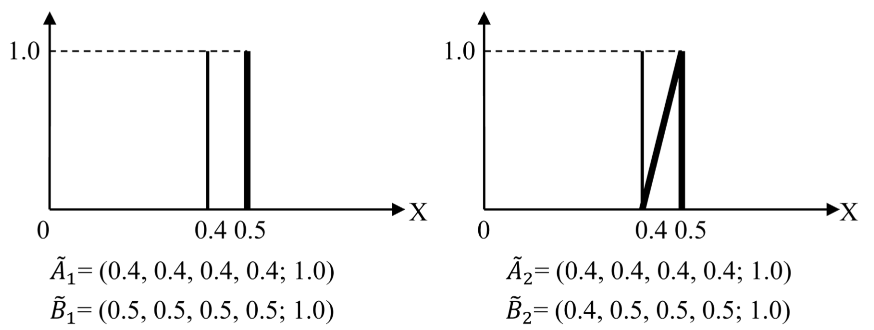

Figure 5. Two sets of generalized fuzzy numbers.

According to Chen's method [9], the degrees of similarity are $S(\tilde{A}_1,\tilde{B}_1)=0.9$ and $S(\tilde{A}_2,\tilde{B}_2)=0.9134$. However, $S(\tilde{A}_2,\tilde{B}_2)$ should not exceed $S(\tilde{A}_1,\tilde{B}_1)$ because the shapes of $\tilde{A}_1$ and $\tilde{B}_1$ are more similar than those of $\tilde{A}_2$ and $\tilde{B}_2$.

This discussion demonstrates that Chen's method [9] still has some shortcomings in some situations. Hence, to measure the similarity among generalized fuzzy numbers requires a new method.

4. New Similarity Measure among Generalized Fuzzy Numbers Based on Standard Deviation

This section presents a new similarity measure based on the standard deviation operator to calculate the degree of similarity among generalized fuzzy numbers, as well as elucidating some properties of the proposed similarity measure. The choice of standard deviation as the basis for the new measure is justified by its strength as a statistical measure of dispersion, which effectively captures the distribution and spread of the generalized fuzzy number $\tilde{A}$'s key control points $(a_1, a_2, a_3, a_4; w_{\tilde{A}})$, thereby quantifying the fuzzy number's shape and spread in the single parameter $S_{\tilde{A}}$. Consider two generalized trapezoidal fuzzy numbers $\tilde{A}$ and $\tilde{B}$, where $\tilde{A} = (a_1, a_2, a_3, a_4; w_{\tilde{A}})$ and $\tilde{B} = (b_1, b_2, b_3, b_4; w_{\tilde{B}})$, $0 < w_{\tilde{A}} < 1$, $0 < w_{\tilde{B}} < 1$, $0 \le a_1 \le a_2 \le a_3 \le a_4$, and $0 \le b_1 \le b_2 \le b_3 \le b_4 \le 1$. The degree of similarity of $S(\tilde{A}, \tilde{B})$ between the generalized trapezoidal fuzzy numbers $\tilde{A}$ and $\tilde{B}$ is calculated as:

$$S(\tilde{A}, \tilde{B}) = \left[1 - \sqrt{\frac{\sum_{i=1}^{4}(a_i - b_i)^2}{2}}\right] \times \left[1 - \sqrt{\frac{|S_{\tilde{A}} - S_{\tilde{B}}|}{2}}\right] \times \frac{\min(w_{\tilde{A}}, w_{\tilde{B}})}{\max(w_{\tilde{A}}, w_{\tilde{B}})}, \tag{35}$$

where $S_{\tilde{A}}$ and $S_{\tilde{B}}$ are the standard deviations

$$S_{\tilde{A}} = \sqrt{\frac{\sum_{i=1}^{4}(a_i - \bar{a})^2}{n-1}}, \tag{36}$$

$$S_{\tilde{B}} = \sqrt{\frac{\sum_{i=1}^{4}(b_i - \bar{b})^2}{n-1}}, \tag{37}$$

and $S(\tilde{A}, \tilde{B}) \in [0,1]$. A higher value of $S(\tilde{A}, \tilde{B})$ corresponds to a greater similarity between the fuzzy numbers $\tilde{A}$ and $\tilde{B}$. The proposed similarity measure based on the standard deviation has the following properties:

Property 1: Two generalized trapezoidal fuzzy numbers $\tilde{A}$ and $\tilde{B}$ are identical if and only if $S(\tilde{A}, \tilde{B}) = 1$.

Proof: 1) If $\tilde{A}$ and $\tilde{B}$ are identical, then $a_1 = b_1$, $a_2 = b_2$, $a_3 = b_3$, $a_4 = b_4$ and $w_{\tilde{A}} = w_{\tilde{B}}$. The degree of similarity between $\tilde{A}$ and $\tilde{B}$ can be calculated as follows:

$$\begin{aligned} S(\tilde{A}, \tilde{B}) &= \left[1 - \sqrt{\frac{\sum_{i=1}^{4}(a_i - b_i)^2}{2}}\right] \times \left[1 - \sqrt{\frac{|S_{\tilde{A}} - S_{\tilde{B}}|}{2}}\right] \times \frac{\min(w_{\tilde{A}}, w_{\tilde{B}})}{\max(w_{\tilde{A}}, w_{\tilde{B}})} \\ &= \left[1 - \sqrt{\frac{0}{2}}\right] \times \left[1 - \sqrt{\frac{|0|}{2}}\right] \times \frac{1}{1} \\ &= 1. \end{aligned}$$

2) If $S(\tilde{A}, \tilde{B}) = 1$, then

$$S(\tilde{A},\tilde{B}) = \left[1-\sqrt{\frac{\sum_{i=1}^{4}(a_i-b_i)^2}{2}}\right]\times\left[1-\sqrt{\frac{|S_{\tilde{A}}-S_{\tilde{B}}|}{2}}\right]\times\frac{\min(w_{\tilde{A}},w_{\tilde{B}})}{\max(w_{\tilde{A}},w_{\tilde{B}})}$$
$$=1.$$

This result indicates that $\sum_{i=1}^{4}(a_i-b_i)^2$ must be 0, and $\frac{\min(w_{\tilde{A}},w_{\tilde{B}})}{\max(w_{\tilde{A}},w_{\tilde{B}})}$ must be 1. So, $a_1=b_1$, $a_2=b_2$, $a_3=b_3$, $a_4=b_4$ and $w_{\tilde{A}}=w_{\tilde{B}}$. In this situation, $S_{\tilde{A}}$ will be the same as $S_{\tilde{B}}$ by (36) and (37). Hence, the generalized trapezoidal fuzzy numbers $\tilde{A}$ and $\tilde{B}$ are identical.

Property 2: $S(\tilde{A},\tilde{B}) = S(\tilde{B},\tilde{A})$.

Proof: Since

$$S(\tilde{A},\tilde{B}) = \left[1-\sqrt{\frac{\sum_{i=1}^{4}(a_i-b_i)^2}{2}}\right]\times\left[1-\sqrt{\frac{|S_{\tilde{A}}-S_{\tilde{B}}|}{2}}\right]\times\frac{\min(w_{\tilde{A}},w_{\tilde{B}})}{\max(w_{\tilde{A}},w_{\tilde{B}})},$$

$$S(\tilde{B},\tilde{A}) = \left[1-\sqrt{\frac{\sum_{i=1}^{4}(b_i-a_i)^2}{2}}\right]\times\left[1-\sqrt{\frac{|S_{\tilde{B}}-S_{\tilde{A}}|}{2}}\right]\times\frac{\min(w_{\tilde{B}},w_{\tilde{A}})}{\max(w_{\tilde{B}},w_{\tilde{A}})},$$

where $\sum_{i=1}^{4}(a_i-b_i)^2 = \sum_{i=1}^{4}(b_i-a_i)^2$, $|S_{\tilde{A}}-S_{\tilde{B}}| = |S_{\tilde{B}}-S_{\tilde{A}}|$ and $\frac{\min(w_{\tilde{A}},w_{\tilde{B}})}{\max(w_{\tilde{A}},w_{\tilde{B}})} = \frac{\min(w_{\tilde{B}},w_{\tilde{A}})}{\max(w_{\tilde{B}},w_{\tilde{A}})}$, thus, $S(\tilde{A},\tilde{B}) = S(\tilde{B},\tilde{A})$.

Property 3: If $\tilde{A}=(a,a,a,a;1.0)$ and $\tilde{B}=(b,b,b,b;1.0)$ are two real numbers, then $S(\tilde{A},\tilde{B}) = 1-|a-b|$ where $a\in[0,1]$, $b\in[0,1]$.

Proof: The following is derived from (35):

$$S(\tilde{A},\tilde{B}) = \left[1-\sqrt{\frac{\sum_{i=1}^{4}(a_i-b_i)^2}{2}}\right]\times\left[1-\sqrt{\frac{|S_{\tilde{A}}-S_{\tilde{B}}|}{2}}\right]\times\frac{\min(w_{\tilde{A}},w_{\tilde{B}})}{\max(w_{\tilde{A}},w_{\tilde{B}})}$$
$$=\left[1-\sqrt{\frac{4(a-b)^2}{2}}\right]\times\left[1-\sqrt{\frac{|0|}{2}}\right]\times\frac{\min(1,1)}{\max(1,1)}$$
$$=\left[1-\sqrt{(a-b)^2}\right]\times1\times1 = 1-|a-b|$$

Consider the two generalized fuzzy numbers $\tilde{A}=(0.1,0.2,0.3,0.4;1.0)$ and $\tilde{B}=(0.3,0.4,0.6;1.0)$. The generalized trapezoidal fuzzy number $\tilde{B}=(0.3,0.4,0.4,0.6;1.0)$ represents the generalized triangular fuzzy number $\tilde{B}$. Based on (36) and (37), the standard deviations of $\tilde{A}$ and $\tilde{B}$ are calculated as follows:

$$S_{\tilde{A}} = \sqrt{\frac{\sum_{i=1}^{4}(a_i-\bar{a})^2}{n-1}}$$
$$=\sqrt{\frac{(-0.15)^2+(-0.05)^2+(0.05)^2+(0.15)^2}{4-1}}$$
$$=0.129099,$$

$$
\begin{aligned}
S_{\tilde{B}} &= \sqrt{\frac{\sum_{i=1}^{4}\left(b_i-\bar{b}\right)^2}{n-1}} \\
&= \sqrt{\frac{(-0.125)^2+(-0.025)^2+(0.025)^2+(0.125)^2}{4-1}} \\
&= 0.125831.
\end{aligned}
$$

The degree of similarity $S\left(\tilde{A},\tilde{B}\right)$ between the two generalized fuzzy numbers is calculated as:

$$
\begin{aligned}
S\left(\tilde{A},\tilde{B}\right) &= \left[1-\sqrt{\frac{\sum_{i=1}^{4}\left(a_i-b_i\right)^2}{2}}\right]\times\left[1-\sqrt{\frac{\left|S_{\tilde{A}}-S_{\tilde{B}}\right|}{2}}\right]\times\frac{\min\left(w_{\tilde{A}},w_{\tilde{B}}\right)}{\max\left(w_{\tilde{A}},w_{\tilde{B}}\right)} \\
&= \left[1-\sqrt{\frac{(-0.2)^2+(-0.2)^2+(0.1)^2+(0.2)^2}{2}}\right] \\
&\quad \times\left[1-\sqrt{\frac{(0.129099-0.125831)}{2}}\right]\times\frac{\min(1,1)}{\max(1,1)} \\
&= 0.819722\times 0.959572\times 1 = 0.786583.
\end{aligned}
$$

5. Comparing Existing Similarity Measures with Proposed Similarity Measure

This section compares the proposed similarity measure with eight existing similarity measures [4] [5], and [8]-[13] by using 44 sets of generalized fuzzy numbers, which are shown in **Figure 6**. **Table 1** compares the calculations of the nine similarity measures. Generally, desirable similarity measure must exhibit sufficient sensitivity to the differences in **shape**, **spread**, and **relative distance** between two generalized fuzzy numbers $\tilde{A}$ and $\tilde{B}$. A person also uses the criteria for fuzzy number ranking problems [17]. The parts of 44 sets of generalized fuzzy numbers are from [4] [5], and [8]-[13], and some are extended in **Figures 1-4** in section 3. The aforementioned criteria can determine these sets of generalized fuzzy numbers.

Table 1 and **Figure 6** show that the existing similarity measures [4] [5], and [8]-[13] have some shortcomings described below.

1) Set 1 in **Figure 6** indicates that the two generalized fuzzy numbers $\tilde{A}$ and $\tilde{B}$ are unequal because their shapes differ. However, based on **Table 1**, Hsieh and Chen's [10] similarity measure yields $S\left(\tilde{A},\tilde{B}\right)=1$.

2) Sets 3 and 4 are different sets of generalized fuzzy numbers because the shapes and spreads of generalized fuzzy numbers $\tilde{A}$ and $\tilde{B}$ in Set 4 are more similar than in Set 3, and the relative distance between the generalized fuzzy numbers $\tilde{A}$ and $\tilde{B}$ in Set 3 is the same as that in Set 4. However, the methods of Chen [4], Hsieh and Chen [10] and Lee [11] yield the same degree of similarity for Sets 3 and 4.

3) In Set 5, $\tilde{A}$ and $\tilde{B}$ differ in their shapes; however, the methods of Chen [4], Hsieh and Chen [10] and Lee [11] yield $S\left(\tilde{A},\tilde{B}\right)=1$.

4) Set 6 indicates that Lee's method cannot correctly calculate the degree of similarity between two identical real values because the denominator $\|U\|$ of (4) would become zero yielding $S\left(\tilde{A},\tilde{B}\right)=\infty$. Additionally, in Set 7 the degree of similarity

of $S(\tilde{A},\tilde{B})$ is not zero. However, Lee's method [11] yields $S(\tilde{A},\tilde{B})=0$.

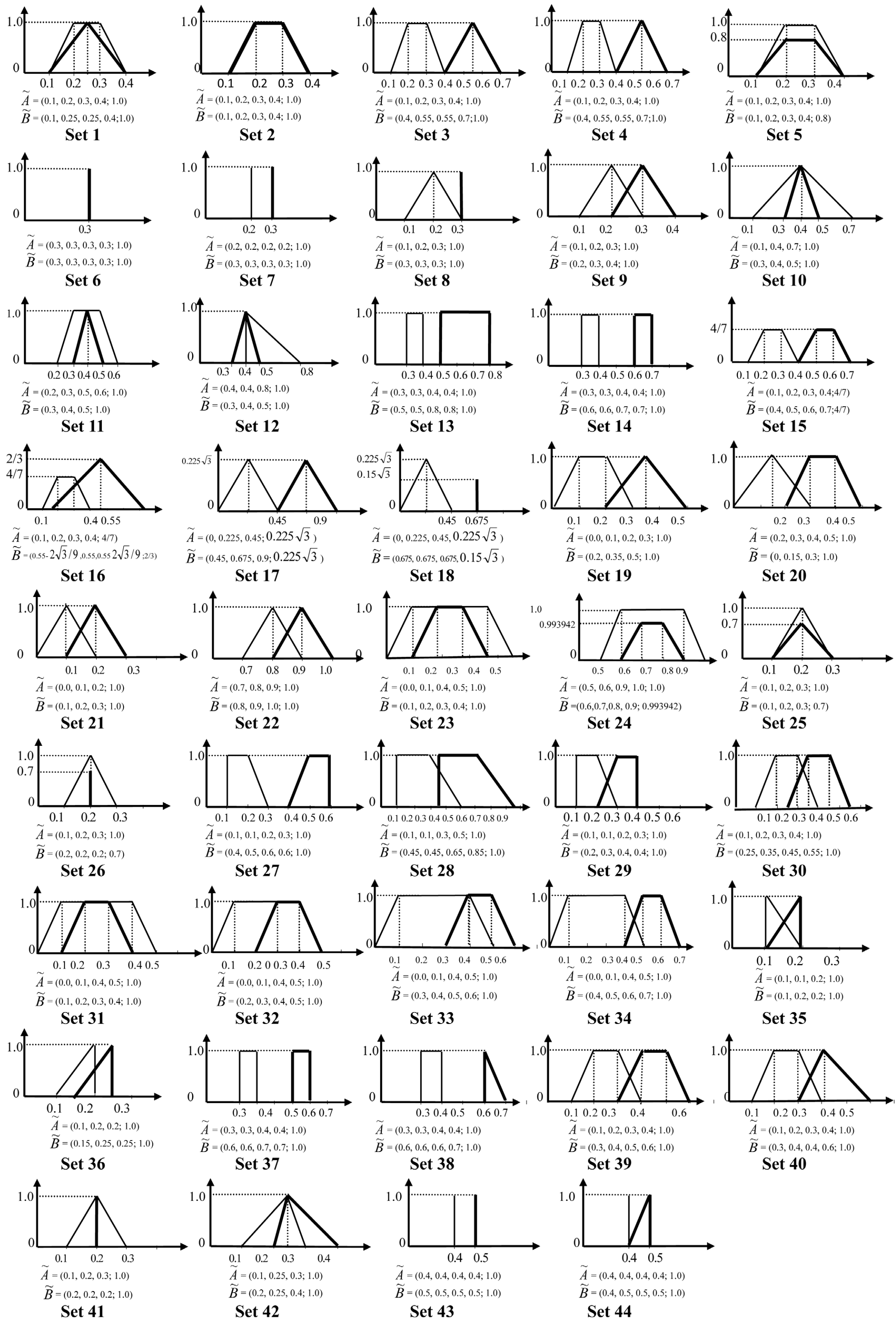

Figure 6. The 44 sets of generalized fuzzy numbers [9].

5) Sets 7 and 8 are different sets of generalized fuzzy numbers. However, the methods of Hsieh and Chen [10], and Chen [4] yield the same degree of similarity for Sets 7 and 8.

Table 1. Comparison of the calculation results made using the proposed similarity measure and the existing methods.

	Lee's Method [11]	Hsieh-and-Chen's Method [10]	Chen's Method [4]	Chen-and-Chen's Method [5]	Yong *et al.*'s Method [13]	Chen's Method [8]	Wei and Chen's Method [12]	Chen's Method [9]	The Proposed Method
Set 1	0.9617	1	0.975	0.8357	0.7954	0.8356	0.95	0.9646	0.9091
Set 2	1	1	1	1	1	1	1	1	1
Set 3	0.5	0.7692	0.7	0.42	0.4028	0.5997	0.68	0.6979	0.6578
Set 4	0.5	0.7692	0.7	0.49	0.4931	0.7	0.7	0.7	0.7
Set 5	1	1	1	0.8	0.8	0.8	0.8248	0.8	0.8
Set 6	*	1	1	1	1	1	1	1	1
Set 7	0	0.909	0.9	0.9	0.9	0.9	0.9	0.9	0.9
Set 8	0.5	0.909	0.9	0.54	0.5754	0.5991	0.8411	0.8775	0.7002
Set 9	0.6667	0.909	0.9	0.81	0.8112	0.9	0.9	0.9	0.9
Set 10	0.8333	1	0.9	0.9	0.8854	0.8974	0.7833	0.8586	0.6132
Set 11	0.75	1	0.9	0.72	0.6914	0.72	0.8003	0.9	0.6978
Set 12	0.8	0.9375	0.9	0.78	0.7744	0.8959	0.8289	0.8419	0.6371
Set 13	0.4	0.7692	0.7	0.49	0.4868	0.6971	0.6222	0.6838	0.5195
Set 14	0.25	0.7692	0.7	0.49	0.4904	0.7	0.7	0.7	0.7
Set 15	0.5	0.7692	0.7	0.49	0.4931	0.7	0.7	0.7	0.7
Set 16	0.6407	0.7692	0.7	0.49	0.4576	0.6915	0.5014	0.5617	0.3907
Set 17	0.5	0.6897	0.55	0.3025	0.309	0.55	0.55	0.55	0.55
Set 18	0.3333	0.6897	0.55	0.3025	0.2945	0.5418	0.2464	0.3485	0.2429
Set 19	0.6	0.8333	0.8	0.5486	0.5278	0.6854	0.7794	0.7969	0.7510
Set 20	0.6	0.8333	0.8	0.5486	0.5266	0.6854	0.7794	0.7969	0.7510
Set 21	0.9	0.9091	0.9	0.81	0.8135	0.9	0.9	0.9	0.9
Set 22	0.9	0.9091	0.9	0.81	0.8101	0.9	0.9	0.9	0.9
Set 23	0.9	1	0.9	0.8077	0.8177	0.8077	0.8055	0.9	0.6899
Set 24	0.9	1	0.9	0.8028	0.8227	0.8028	0.8012	0.8945	0.6858
Set 25	1	1	1	0.7	0.7	0.7	0.7209	0.7	0.7
Set 26	0.75	1	0.95	0.9048	0.9366	0.9042	0.6215	0.6505	0.5191

Continued

Set 27	0.3	0.7317	0.65	0.4279	0.4288	0.6492	0.65	0.6464	0.6464
Set 28	0.5333	0.7407	0.65	0.4225	0.4289	0.65	0.65	0.65	0.65
Set 29	0.5	0.8571	0.85	0.7296	0.7296	0.8493	0.85	0.8419	0.8419
Set 30	0.6667	0.87	0.85	0.7225	0.7251	0.85	0.85	0.85	0.85
Set 31	0.8571	1	0.9	0.8077	0.8177	0.8077	0.8055	0.9	0.6899
Set 32	0.8571	0.909	0.9	0.7269	0.7674	0.8053	0.8055	0.8586	0.6582
Set 33	0.7143	0.8333	0.8	0.5744	0.609	0.7155	0.716	0.7764	0.5952
Set 34	0.5714	0.7692	0.7	0.4397	0.4686	0.6256	0.6265	0.6838	0.5242
Set 35	0.5	0.9375	0.95	0.9183	0.9187	0.9494	0.95	0.9293	0.9293
Set 36	0.6667	0.9524	0.95	0.9025	0.9029	0.95	0.95	0.95	0.95
Set 37	0.25	0.7692	0.7	0.49	0.4904	0.7	0.7	0.7	0.7
Set 38	0.3125	0.7894	0.725	0.3464	0.3678	0.483	0.7035	0.7216	0.6767
Set 39	0.6	0.8333	0.8	0.64	0.6429	0.8	0.8	0.8	0.8
Set 40	0.65	0.8571	0.825	0.5775	0.5526	0.7067	0.8044	0.8197	0.7866
Set 41	0.75	1	0.95	0.6333	0.669	0.6329	0.8879	0.9293	0.7415
Set 42	0.8333	0.9677	0.95	0.8867	0.8876	0.9494	0.95	0.9293	0.9293
Set 43	0	0.9091	0.9	0.9	0.9	0.9	0.9	0.9	0.9
Set 44	0.25	0.9231	0.925	0.5756	0.6101	0.6163	0.8937	0.9134	0.7690

Note: "*" means that the similarity measure cannot calculate the degree of similarity between two generalized fuzzy numbers. ""means incorrect results.

6) Sets 8 and 9 are different sets of generalized fuzzy numbers because the shapes and spreads of $\tilde{A}$ and $\tilde{B}$ in Set 9 are more similar than the shapes and spreads of $\tilde{A}$ and $\tilde{B}$ in Set 8. However, the methods of Chen [4] and Hsieh and Chen [10] yield the same degree of similarity for both sets.

7) Sets 10, 11 and 12 are different sets of generalized fuzzy numbers. However, Chen's method [4] yields the same degree of similarity for each set of generalized fuzzy numbers.

8) In Set 10, $\tilde{A}$ and $\tilde{B}$ differ in shape where Hsieh and Chen's method [10] yields $S(\tilde{A},\tilde{B})=1$. Furthermore, Set 11 reveals the same problem encountered in Hsieh and Chen's method [10].

9) Set 15 has a greater similarity than Set 16 does because the shapes and spreads of the generalized fuzzy numbers in Set 15 are more similar than those in Set 16 are. However, the methods of Chen [4], Chen and Chen [5] and Hsieh and Chen [10] yield the same degrees of similarity for both sets. Moreover, the methods of Lee [11] and Yong *et al.* [13] yield the incorrect result that Set 16 has more similarities than Set 15 does.

10) Sets 17 and 18 are different sets of generalized fuzzy numbers because the shapes and spreads of the generalized fuzzy numbers in Set 17 are more similar than those in Set 18 are. However, the methods of Chen [4], Chen and Chen [5] and Hsieh and Chen [10] yield the same degrees of similarity for Sets 17 and 18.

11) Set 19 has the same degree of similarity as Set 20 because the relative distance between the generalized fuzzy numbers in Set 19 equals the relative distance of those in Set 20. The generalized fuzzy number $\tilde{A}$ in both sets has the same shape, as does the generalized fuzzy number $\tilde{B}$. However, the method of Yong *et al.* [13] yields the degree of similarity of Set 19 exceeds that of Set 20.

12) Set 21 has the same degree of similarity as Set 22 because the relative distance between the generalized fuzzy numbers in Set 21 equals the relative distance of those in Set 22. Moreover, the shapes of the four generalized fuzzy numbers in Sets 21 and 22 are identical although the method of Yong *et al.* [13] yields the degree of similarity in Set 21 exceeds that of Set 22.

13) The relative distance between the generalized fuzzy numbers in Set 23 equals the relative distance of those in Set 24. The shapes of the generalized fuzzy number $\tilde{A}$ in both sets are identical, but those of the generalized fuzzy number $\tilde{B}$ in both sets differ. Thus, the degree of similarity in Set 23 exceeds that in Set 24. However, the methods of Chen [4], Hsieh and Chen [10], Lee [11] yield the same degree of similarity for both sets. Moreover, the method of Yong *et al.* [13] yields the incorrect result the degree of similarity in Set 24 exceeds that of Set 23.

14) The generalized fuzzy numbers in Set 25 are more similar than in Set 26 because the shapes of the generalized fuzzy numbers in Set 25 are all triangular, whereas those of the generalized fuzzy numbers in Set 26 differ. However, **Table 1** indicates that the methods of Chen [4], Chen and Chen [5] and Yong *et al.* [13] yield the incorrect result that generalized fuzzy numbers in Set 26 are more similar than those in Set 25.

15) Set 25 $\tilde{A}$ and $\tilde{B}$ differ because their shapes are different due to the fact that $\tilde{A}$ is higher than $\tilde{B}$. However, **Table 1** indicates that the methods of Chen [4], Hsieh and Chen [10] and Lee [11] yield $S\left(\tilde{A},\tilde{B}\right)=1$, and Set 26 $\tilde{A}$ and $\tilde{B}$ are unequal. However, Hsieh and Chen's method [10] yields $S\left(\tilde{A},\tilde{B}\right)=1$.

16) The relative distance between the generalized fuzzy numbers in Set 28 is shorter than that between the identical generalized fuzzy numbers in Set 27. However, the methods of Chen [4] and Wei and Chen [12] yield the same degree of similarity for Sets 27 and 28. Moreover, Chen and Chen's method [5] yields Set 27 has greater similarity than Set 28 does.

17) The generalized fuzzy numbers in Set 30 are more similar than in Set 29 because the shapes and spreads of the generalized fuzzy numbers in Set 30 are more similar than those in Set 29 are. However, the methods of Chen [4], and Wei and Chen [12] yield the same degree of similarity for Sets 29 and 30, while the methods of Chen and Chen [5] and Yong *et al.*[13] yield Set 29 has a greater similarity than Set 30.

18) The generalized fuzzy numbers in Set 31 are more similar than those in Sets

32, 33 and 34 because relative distance between the two generalized fuzzy numbers $\tilde{A}$ and $\tilde{B}$ varies among the sets, and the distance in Set 31 is shorter than those in Sets 32, 33 and 34. However, the methods of Lee [11], Chen [4], and Wei and Chen [12] yield the same degree of similarity for Sets 31 and 32.

19) The generalized fuzzy numbers in Set 36 are more similar than in Set 35 because the shapes of the generalized fuzzy numbers in Set 36 are identical, whereas those in Set 35 are not. However, the methods of Chen [4], and Wei and Chen [12] yield the same degree of similarity for Sets 35 and 36 while the method of Yong *et al.* [13] yields the degree of similarity in Set 35 exceeds that in Set 36.

20) The generalized fuzzy numbers in Set 37 are more similar than in Set 38 because the generalized fuzzy numbers in Set 37 are all rectangular, whereas the generalized fuzzy numbers in set 38 have different shapes. However, **Table 1** indicates that the methods of Chen [4], Hsieh and Chen [10], Lee [11], Wei and Chen [12], and Chen [9] yield the incorrect result that Set 38 is more similar than Set 37 is.

21) The generalized fuzzy numbers in Set 39 are more similar than Set 40 are because the shapes of the generalized fuzzy numbers in Set 39 are identical, but those in Set 40 are not. However, the methods of Chen [4], Hsieh and Chen [10], Lee [11], Wei and Chen [12], and Chen [9] yield the incorrect result that Set 40 is more similar than Set 39 is.

22) The relative distance between the generalized fuzzy numbers in Set 41 is the same as between the generalized fuzzy numbers in Set 42, but the shapes and spreads of the generalized fuzzy numbers in Set 42 are more similar than the generalized fuzzy numbers in Set 41. However, the methods of Chen [4], and Chen [9] yield the same degree of similarity for Set 41 and Set 42.

23) Set 41 indicates that the two generalized fuzzy numbers $\tilde{A}$ and $\tilde{B}$ differ in shapes. However, Hsieh and Chen's method [10] yields that $S\left(\tilde{A},\tilde{B}\right)=1$.

24) The shapes and spreads of the generalized fuzzy numbers in Set 43 are more similar than the generalized fuzzy numbers in Set 44. However, the methods of Chen [4], Hsieh and Chen [10], Lee [11], and Chen [9] yield the result that Set 44 is more similar than Set 43 is. Moreover, in Set 43, the degree of similarity $S\left(\tilde{A},\tilde{B}\right)$ is not zero. However, Lee's method [11] yields $S\left(\tilde{A},\tilde{B}\right)=0$.

6. Fuzzy Recommendation Process Based on Proposed Similarity Measure for Evaluating Quality of Health Care at a Hospital

The section uses the proposed similarity measure to solve fuzzy recommendation problems. Martinez *et al.* [18] pointed out the recommendation process has the following steps: a) fusion of the human linguistic evaluating values, b) calculation of the similarity between the user profile and recommended items, and c) provision of a recommendation to the user. Assume there are n items $a_1,a_2,\cdots,a_n$ described by a set of m features $B=\{b_1,b_2,\cdots,b_m\}$, and the nine-member set of linguistic terms in **Table 2** can be adopted to describe these features.

Table 2. A nine-member linguistic term set [18].

Linguistic Term	Generalized Fuzzy Numbers
Negligible	(0.0, 0.0, 0.0, 0.0; 1.0)
Very inferior	(0.0, 0.0, 0.02, 0.07; 1.0)
Inferior	(0.04, 0.1, 0.18, 0.23; 1.0)
Fairly inferior	(0.17, 0.22, 0.36, 0.42; 1.0)
Average	(0.32, 0.41, 0.58, 0.65; 1.0)
Fairly superior	(0.58, 0.63, 0.80, 0.86; 1.0)
Superior	(0.72, 0.78, 0.92, 0.97; 1.0)
Very superior	(0.93, 0.98, 1.0, 1.0; 1.0)
Outstanding	(1.0, 1.0, 1.0, 1.0; 1.0)

This study proposes an algorithm based on Chen's [19] proposed similarity measure for the fuzzy recommendation process.

Step 1: Use the weighted mean method and the generalized fuzzy number arithmetic operations to fuse the evaluated linguistic values $\tilde{R}_{ij}$ and the real numbers w_i, that are the degree of strength and weight of feature b_i in item a_j, $0 \le w_i \le 1$, $i = 1, 2, \cdots, m$ and $j = 1, 2, \cdots, n$, and yield the evaluated value $\tilde{R}_j$ for item a_j, as follows:

$$\tilde{R}_j = \frac{\sum_{i=1}^{n} (1 + w_i) \otimes \tilde{R}_{ij}}{\sum_{i=1}^{n} (1 + w_i)}, \tag{38}$$

where

$$\begin{aligned}(1 + w_i) \otimes \tilde{R}_{ij} &= (1 + w_i) \otimes \left(r_{1ij}, r_{2ij}, r_{3ij}, r_{4ij}; w_{\tilde{R}ij}\right) \\ &= \left((1 + w_i) \times r_{1ij}, (1 + w_i) \times r_{2ij}, (1 + w_i) \times r_{3ij},\right. \\ &\quad \left.(1 + w_i) \times r_{4ij}; (1 + w_i) \times w_{\tilde{R}ij}\right),\end{aligned} \tag{39}$$

and $\tilde{R}_j$ is a generalized fuzzy number.

Step 2: Use the proposed fuzzy similarity measure (i.e., formula (35)) to calculate the degree of similarity between the generalized fuzzy number $\tilde{R}_j$ and each linguistic term in **Table 2**. We translate the generalized number $\tilde{R}_j$ into a linguistic term that has the largest degree of similarity to $\tilde{R}_j$.

Step 3: The recommended item is the closest to the requirements of the user.

This study adopted the fuzzy recommendation process [19] to suggest a hospital to patients. In 2005, Choi *et al.* [20] assessed the quality of health services on four quality dimensions: degree of physician concern, degree of staff concern, convenience of the care process, and tangibles. They adopted these dimensions to evaluate patient satisfaction in hospitals in South Korea [20]. Fletcher *et al.* [21] and Ware *et al.* [22] found that the importance of each dimension varies with the age of the patient. These dimensions [20] were adopted in the recommendation process. Consider three hospitals $A = \{a_1, a_2, a_3\}$, each described by a set of four features $B=\{b_1,b_2,b_3,b_4\}$, where b_1 is the degree of physician concern; b_2 is the degree of staff concern; b_3 is the convenience of the care process, and b_4 is tangibles. Taken from Table 2, evaluated values are represented by generalized fuzzy numbers (Table 3) where $\tilde{R}_{ij}$ denotes the degree of strength of feature b_i in hospital a_j; w_i is the weight of feature b_i in the hospital; $0 \le w_i \le 1$, $i=1,2,3,4$ and $j=1,2,3$.

Table 3. Evaluating a_1, a_2 and a_3 for different hospitals.

b_i	w_i	Hospitals		
		a_1	a_2	a_3
b_1	0.7	$\tilde{R}_{11}=(0.0,0.0,0.0,0.0;1.0)$	$\tilde{R}_{12}=(0.72,0.78,0.92,0.97;1.0)$	$\tilde{R}_{13}=(0.58,0.63,0.8,0.86;1.0)$
b_2	0.6	$\tilde{R}_{21}=(0.72,0.78,0.92,0.97;1.0)$	$\tilde{R}_{22}=(0.58,0.63,0.8,0.86;1.0)$	$\tilde{R}_{23}=(0.0,0.0,0.0,0.0;1.0)$
b_3	0.9	$\tilde{R}_{31}=(0.32,0.41,0.58,0.65;1.0)$	$\tilde{R}_{32}=(1.0,1.0,1.0,1.0;1.0)$	$\tilde{R}_{33}=(0.93,0.98,1.0,1.0;1.0)$
b_4	0.4	$\tilde{R}_{41}=(0.04,0.1,0.18,0.23;1.0)$	$\tilde{R}_{42}=(0.04,0.1,0.18,0.23;1.0)$	$\tilde{R}_{23}=(0.32,0.41,0.58,0.65;1.0)$

The algorithm for the fuzzy recommendation process is adopted to recommend a hospital for patients.

[Step 1] Based on (38), (39) and Table 3, $\tilde{R}_1$ for hospital a_1 is calculated as:

$$\tilde{R}_1=\frac{\sum_{i=1}^{n}(1+w_i)\otimes\tilde{R}_{ij}}{\sum_{i=1}^{n}(1+w_i)}=(0.2752,0.3283,0.4282,0.4711;1.0).$$

The value for hospital a_2 is $\tilde{R}_2=(0.6224,0.6627,0.757,0.795;1.0)$ and for hospital a_3 is $\tilde{R}_3=(0.485,0.5314,0.617,0.6473;1.0)$.

[Step 2] Based on (35), (36) and (37) and Table 2, the degrees of similarity between $\tilde{R}_1$ for hospital a_1 and the linguistic terms in Table 2 can be calculated as:

$$S(\tilde{R}_1,\text{Negligible})=0.4857,$$

$$S(\tilde{R}_1,\text{Very inferior})=0.5343,$$

$$S(\tilde{R}_1,\text{Inferior})=0.7213,$$

$$S(\tilde{R}_1,\text{Fairly inferior})=0.8068,$$

$$S(\tilde{R}_1,\text{Average})=0.7201,$$

$$S(\tilde{R}_1,\text{Fairly superior})=0.5588,$$

$$S(\tilde{R}_1,\text{Superior})=0.4661,$$

$$S(\tilde{R}_1,\text{Very superior})=0.3291,$$

$$S(\tilde{R}_1,\text{Outstanding})=0.2923.$$

$S(\tilde{R}_1,\text{Fairly inferior})=0.8068$ has the largest value, the value $\tilde{R}_1$ for hospital a_1 is "Fairly inferior". The degree of similarity $S(\tilde{R}_2,\text{Superior})=0.797$ exceeds the degrees of similarity between $\tilde{R}_2$ and linguistic terms in **Table 2**, $\tilde{R}_2$ for hospital a_2 is "Fairly superior". The degree of similarity $S\left(\tilde{R}_3, \text{Average}\right) = 0.7206$ exceeds the degrees of similarity between $\tilde{R}_3$ and linguistic terms in **Table 2**, $\tilde{R}_3$ for hospital a_3 is "Average".

[Step 3] Based on the results, hospital a_2 is recommended as matching the needs of the user.

7. Conclusion

This study presents a new approach for calculating the similarity between generalized fuzzy numbers. The study demonstrated the proposed similarity measure with 44 sets of generalized fuzzy numbers applied for comparison with the eight existing similarity measures. **Figure 6** and **Table 1** indicate that the proposed similarity measure overcomes the shortcomings of existing similarity measures. The proposed similarity measure is more flexible and effective than existing methods. Future research could extend the application of the proposed similarity measure to other domains, such as Fuzzy Pattern Recognition, to further validate its flexibility and effectiveness across different contexts.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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