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# Time-Dependent Global Attractors for Beam Equation with Time Delay and Structural Damping

**Wenpei Zhao, Xuan Wang***

College of Mathematics and Statistics, Northwest Normal University, Lanzhou, China
Email: zhaowenpei2023@163.com, *wangxuan@nwnu.edu.cn

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## Abstract

In this article, the asymptotic behavior of the solutions to the beam equation with time delay and structural damping is considered. First of all, when the growth exponent of nonlinear terms satisfies the optimal growth exponent $1 \le p < p^* = \frac{N+2(1+\alpha)}{N-4}$, with $N \ge 5$, the well-posedness of solutions is obtained by applying Faedo-Galerkin approximation method and time translation method; Then, the asymptotic compactness of the solution process is verified by using the contraction function method; Finally, the existence of time-dependent global attractor is established in the time-dependent space $C_{\mathcal{H}_t^2}$.

## Keywords

Delay, Structural Damping, Time-Dependent Global Attractors, Beam Equation

## 1. Introduction

In this article, we are concerned with the existence of time-dependent global attractors for the beam equation with time delay and structural damping

$$\varepsilon(t)\partial_t^2 u + \Delta^2 u + \gamma(-\Delta)^\alpha \partial_t u + f(u) = g(x, u_\theta), \quad (x,t) \in \Omega \times [\tau, +\infty), \tag{1.1}$$

$$u|_{\partial\Omega} = \Delta u|_{\partial\Omega} = 0, \tag{1.2}$$

$$u(x,\tau) = u_0(x), \ \partial_t u(x,\tau) = u_1(x), \tag{1.3}$$

$$u(x,\tau+\theta) = \phi_0(x,\theta), \ \partial_t u(x,\tau+\theta) = \phi_1(x,\theta), \ x \in \Omega, \ \theta \in [-\rho, 0], \tag{1.4}$$

where $\gamma > 0$, $\alpha \in \left(\frac{1}{2}, 1\right)$ is a dissipation index, $\Omega$ is a bounded domain of

$\Omega \in \mathbb{R}^N$ $(N \geq 5)$ with smooth boundary $\partial\Omega$, and $g(x, u_\theta)$ is a time delay external force term.

We presume that the time-dependent coefficient $\varepsilon$, the nonlinear functions $f$, and the time delay external force $g(u_\theta)$ term satisfy the following conditions.

**Assumptions**:

i) $\varepsilon \in C^1(\mathbb{R})$ is a decreasing bounded function and satisfies

$$\lim_{t \to +\infty} \varepsilon(t) = 0. \tag{1.5}$$

In particular, there exists a constant $L > 0$, such that

$$\sup_{t \in \mathbb{R}} \left( |\varepsilon(t)| + |\varepsilon'(t)| \right) \leq L. \tag{1.6}$$

ii) $f \in C^1(\mathbb{R})$, $f(0) = 0$, and for any $s \in \mathbb{R}$, $f$ satisfies the dissipative condition

$$\liminf_{|s| \to \infty} \frac{f(s)}{s} \geq -\lambda_1, \tag{1.7}$$

and the growth condition

$$\left| f'(s) \right| \leq C\left(1 + |s|^{p-1}\right),\ 1 \leq p < p^* = \frac{N + 2(1+\alpha)}{N - 4}(N \geq 5), \tag{1.8}$$

where $C > 0$ and $\lambda_1 > 0$ is the first eigenvalue of the operator $\Delta^2$ that satisfies the Dirichlet boundary condition.

**Remark 1.1** Formula (1.7) implies that there is a positive constant $\beta_0$, for $\frac{1}{2} < \beta_0 < 1$, such that

$$\langle F(u), 1 \rangle \geq \frac{(1-\beta_0)\lambda_1}{2} \|u\|^2 - C_{\beta_0},$$

$$\langle f(u), u \rangle \geq -(1-\beta_0)\lambda_1 \|u\|^2 - C_{\beta_0},\ \forall u \in L^2(\Omega),$$

where $F(s) = \int_0^s f(r)\mathrm{d}r$ and $C_{\beta_0}$ is a positive constant.

iii) Furthermore, for any given $T > 0$, we define a function $u : [-\rho, T] \to L^2(\Omega)$. For each $t \in [0, T]$, we denote by $u_\theta$ the delay segment of $u$ defined on $[-\rho, 0]$ as:

$$u_\theta(t) = u(t + \theta),\ \theta \in [-\rho, 0],\ 0 < \rho < \infty.$$

In general, let $\mathrm{X}$ be a separable Banach space, we define the phase space of delay functions:

$$C_\rho(\mathrm{X}) = \left\{ \phi \in C([-\rho, 0]; \mathrm{X}); \lim_{\theta \to -\rho} \phi(\theta) \text{ exists in } \mathrm{X} \right\}$$

which equipped with the norm:

$$\|\alpha\|_{C_\rho(\mathrm{X})} = \sup_{-\rho \leq \theta \leq 0} \|\alpha(\theta)\|_{\mathrm{X}}.$$

Assume that the delay external force term $g : \Omega \times C_\rho(\mathrm{X}) \to L^2(\Omega)$ satisfies:

$(H_1)$ For every $\xi \in C_\rho(\mathrm{X})$, the mapping $x \in \Omega \mapsto g(x, \xi) \in L^2(\Omega)$ is measurable;

$(H_2)$ For all $x \in \Omega$, $g(x,0)=0$;

$(H_3)$ There exists a constant $C_g > 0$ such that for all $x \in \Omega$, $u,v \in C_\rho(\mathrm{X})$, the following inequality holds:

$$\left\|g(x,\xi)-g(x,\eta)\right\| \le C_g \left\|\xi-\eta\right\|_{C_\rho(\mathrm{X})};$$

$(H_4)$ There exists a constant $C_g > 0$ such that for all $x \in \Omega$, $u,v \in C\left(\left(\tau+\rho,T\right];\mathrm{X}\right)$,

$$\int_\tau^t \left\|g(x,u_\theta)-g(x,v_\theta)\right\|^2 \mathrm{d}s \le C_g \int_{\tau+\rho}^t \left\|u_\theta - v_\theta\right\|_{\mathrm{X}}^2 \mathrm{d}s.$$

Over the past decade, research on the applications of infinite-dimensional dynamical systems theory has attracted significant attention. Among them, attractors play a crucial role in characterizing the long-time dynamical behavior of solutions to the model, and a large number of achievements have emerged; see [1]-[8] and related literature. Due to its rich and profound application background, the nonlinear evolution equation with structural damping and time delay have roused the research interest of many scholars, making the study of asymptotic behavior of solutions of this problem be a hot topic.

In dynamical systems, damping refers to the dissipation of energy due to internal mechanisms and external interactions. For instance, reference [9] obtains several asymptotic profiles of solutions to the Cauchy problem for structurally damped wave equations:

$$\partial_t^2 u - \Delta u + \nu\left(-\Delta\right)^\sigma \partial_t u = 0$$

where $0<\sigma\le 1$. The authors investigated the approximation formula of the solution by a constant multiple of a special function as $t\to\infty$, which states that the asymptotic profiles of the solutions are classified into 5 patterns depending on the values $\nu$ and $\sigma$. Reference [10] examined the well-posedness, regularity, and long-term dynamics of a stretchable beam equation with fractional rotational inertia and nonlinear structural damping. When the nonlinear growth exponent satisfies: $1 \le p < \frac{N+2\left(2\alpha-\theta\right)}{\left(N-4\right)^+}$, the well-posedness, regularity and longtime behavior of the solutions are like parabolic.

Due to the combined effects of signal transmission delay, nonlinear dynamic characteristics, and the interaction between the system structure and external disturbances, the dynamic behavior of the solution will exhibit a delay phenomenon. Mathematical models with time delay are widely used in various fields such as control systems, economics, and ecology. In describing the asymptotic behavior of the system, the solution depends not only on the current state but also on its history. Therefore, incorporating time delay terms into the evolution allows for a more accurate description of the system's dynamic behavior, see [11]-[15] and the references therein. For example, In reference [16], the asymptotic behavior of the solutions to non-autonomous diffusion equations with delay containing some hereditary characteristics and nonlocal diffusion in the time-dependent space $C_{\mathcal{H}_t(\Omega)}$. When the nonlinear function $f$ satisfies the polynomial growth of arbitrary order $p-1$ $(p \ge 2)$ and the external force $h \in L^2_{loc}\left(R; H^{-1}(\Omega)\right)$, the author established the existence and regularity of the time-dependent pullback attractors. Reference [17] investigated the following model:

$$u_{tt} + \Delta^2 u - M\left(\int_\Omega |\nabla u|^2 \,\mathrm{d}x\right)\Delta u + a_0 u_t(x,t) + a_1 u_t(x,t-\tau) + \varphi(u) = f,$$

The authors analyzed the long-term dynamics of this nonlocal extensible beam equation with delay. Under appropriate assumptions, they established the quasi-stability of the system, thereby obtaining the existence and regularity of a finite-dimensional compact global attractor, as well as the existence of an exponential attractor.

The time-dependent coefficient function decreases and approaches zero, which is consistent with the mathematical analysis of long-term dynamics and also conforms to the description of the decay of the inertial mass of a beam in physics. The decreasing and approaching zero indicates that the inertial effect gradually weakens, allowing the elastic force and damping force to dominate the vibration decay, ensuring that the long-term behavior of the system converges to a bounded set. For equation (1.1) with $\varepsilon(t) \equiv C$, we refer to [18]-[22]. However, when $\varepsilon(t) \not\equiv C$, problem (1.1) becomes significantly more complex. Due to the fact that the energy functional of the system depends on time $t$, the existence of a bounded absorbing set becomes difficult to obtain. Some classical theories and methods have limitations in solving such problems. Therefore, Conti *et al.* proposed the theory of time-dependent attractors, studying the long-time behavior of solutions in the time-dependent space (see, e.g. [23]-[27]). [28] established the following model:

$$\varepsilon(t)\left(1+(-\Delta)^\alpha\right)\partial_t^2 u + \Delta^2 u + \gamma(-\Delta)^\theta \partial_t u + f(u) = g(x),$$

where $\gamma > 0$, $\theta \in \left(\frac{1}{2}, \frac{2}{3}\right)$ and $\alpha \in [0, 4\theta - 2]$. Where the nonlinear term satisfies a better subcritical exponent $1 \le p < \frac{N+2\theta}{N-4}$, the author studied the well-posedness and regularity of the solution. Under the theoretical framework of time-dependent attractors, the existence of time-dependent attractors is investigated by applying the contraction function method and detailed estimation.

Inspired by the above research results, this paper studied the beam equation with time delay, structural damping and time-dependent coefficient functions. To the best of our knowledge, the existence of time-dependent attractors for problem (1.1) - (1.4) with time delay has not been previously studied. At the same time, the time delay and nonlinear terms in the equation bring essential difficulties to the estimation of the dissipation of the solution, the existence of the bounded absorbing set, and the verification of the asymptotic compactness of the solution process. We overcome these technical difficulties by employing the methods of time translation, energy estimation, contraction functions, and the relevant theory of time-dependent attractors, and prove the existence of time-dependent attractors when the optimal growth index of the nonlinear term in (1.1) - (1.4) satisfies $1 \le p < p^* = \dfrac{N+2(1+\alpha)}{N-4}$, $N \ge 5$.

The content and structure of this article are as follows: The second section will review the preliminary knowledge and some abstract results; the third section will discuss the well-posedness of the weak solutions; the fourth section will use the contraction function method to prove the asymptotic compactness of the process, and ultimately obtain the existence of the time-dependent attractor.

In the following text, each occurrence of $C$ in different equations represents the corresponding positive constant. Also, $C_i, i \in \mathbb{N}$ is used to denote different constants, and $C(\cdot,\cdot)$ represents a constant related to the parameters within the parentheses.

## 2. Notations and Preliminary Results

In this section, we introduce some function spaces which will be used throughout this paper:

$$L^p = L^p(\Omega),\ W^{m,p} = W^{m,p}(\Omega),\ H^m = W^{m,2}(\Omega),$$
$$V_1 = H_0^1(\Omega),\ V_2 = H^2(\Omega) \cap H_0^1(\Omega).$$

with $p \ge 1$. We denote the norm and the inner product by $\|\cdot\|_{L^2}$ and $\langle \cdot,\cdot \rangle$ in $L^2(\Omega)$. Let $A = \Delta^2$ with domain $D(A) = H_0^1(\Omega) \cap H^2(\Omega)$. For every $s \in \mathbb{R}$, we define the Hilbert spaces

$$V_s = D\left(A^{\frac{s}{4}}\right),\ \langle u,v \rangle_s = \left\langle A^{\frac{s}{4}}u, A^{\frac{s}{4}}v \right\rangle,\ \|u\|_s = \left\| A^{\frac{s}{4}}u \right\|,\ u,v \in V_s.$$

Applying Sobolev embedding theorem, we can obtain the compact embedding

$$V_{s_1} \hookrightarrow\hookrightarrow V_{s_2},\ \text{as } s_1 > s_2 \tag{2.1}$$

and the continuous embedding

$$V_s \hookrightarrow L^{\frac{2N}{N-2s}},\ s > 0. \tag{2.2}$$

Therefore, the problem (1.1)(1.4) can be written as follows:

$$\varepsilon(t)\partial_t^2 u + Au + \gamma A^{\frac{\alpha}{2}}\partial_t u + f(u) = g(x,u_\theta),\ t \ge \tau, \tag{2.3}$$

$$u(\tau) = u_0,\ \partial_t u(\tau) = u_1, \tag{2.4}$$

$$u_\theta(\tau) = \phi_0,\ \partial_t u_\theta(\tau) = \phi_1,\ \theta \in [-\rho, 0]. \tag{2.5}$$

Define the family of Hilbert spaces:

$$\mathcal{H}_t^2 = V_2 \times L^2(\Omega),\ \mathcal{Y}_t^\alpha = V_{3-\alpha} \times V_\alpha,\ C_{\mathcal{H}_t^2} = C_{V_2} \times C_{L^2(\Omega)},\ C_{\mathcal{Y}_t^\alpha} = C_{V_{3-\alpha}} \times C_{V_\alpha},$$

and the norm in this family of spaces is defined as follows:

$$\|z\|_{\mathcal{H}_t^2}^2 = \|u\|_2^2 + \varepsilon(t)\|\partial_t u\|^2,\ \|z\|_{\mathcal{Y}_t^\alpha}^2 = \|u\|_{3-\alpha}^2 + \varepsilon(t)\|\partial_t u\|_\alpha^2,$$

$$\|z\|^2_{C_{\mathcal{H}_t^2}} = \|u_\theta\|^2_{C_{V_2}} + \varepsilon_\theta \|\partial_t u_\theta\|^2_{C_{L^2(\Omega)}}, \quad \|z\|^2_{C_{\mathcal{H}_t^\alpha}} = \|u_\theta\|^2_{C_{V_{3-\alpha}}} + \varepsilon_\theta \|\partial_t u_\theta\|^2_{C_{V_\alpha}}.$$

The following abstract results will be used for the asymptotic estimation of the solution.

**Definition 2.1** [12] Let $X_t$ be a family of normed spaces. A two-parameter family of operators $\{U(t,\tau): X_\tau \to X_t, \tau \le t, \tau \in \mathbb{R}\}$ is said to be a process, if for any $\tau \in \mathbb{R}$,

i) $U(\tau,\tau) = \mathrm{Id}$ is the identity operator on $X_\tau$;

ii) $U(t,s)U(s,\tau) = U(t,\tau)$, $\forall \tau \le s \le t$.

For every $t \in \mathbb{R}$, the $R$-ball of $X_t$ is defined by:

$$\mathbb{B}_t(R) = \left\{ z \in X_t \mid \|z\|_{X_t} \le R \right\}.$$

**Definition 2.2** [12] A family $\mathfrak{C} = \{C_t\}_{t\in\mathbb{R}}$ of bounded sets $C_t \subset X_t$ is called uniformly bounded, if there exists a constant $R > 0$ such that $C_t \subset \mathbb{B}_t(R)$, $\forall t \in \mathbb{R}$.

**Definition 2.3** [12] A uniformly bounded family $\mathfrak{B}_t = \{\mathbb{B}_t(R_1)\}_{t\in\mathbb{R}}$ is called a time-dependent absorbing set for the process $U(t,\tau)$, if for any $R > 0$, there exists a $t_0 = t_0(R) \le t$ and $R_1 > 0$ such that

$$\tau \le t - t_0 \Rightarrow U(t,\tau)\mathbb{B}_\tau(R) \subset \mathbb{B}_t(R_1).$$

The process $U(t,\tau)$ is said to be dissipative if it possesses a time-dependent absorbing set.

**Lemma 2.4** [13] Let $x_n$ be a bounded sequence and also let $\psi \in C(\mathbb{R})$ be a monotone function. Then

$$\psi\left(\liminf_{n\to\infty} x_n\right) \le \liminf_{n\to\infty} \psi(x_n).$$

**Lemma 2.5** [14] [15] Let $X, B$ and $Y$ be three Banach spaces. For any $T > 0$, if $X \hookrightarrow\hookrightarrow B \hookrightarrow Y$, and

$$W = \left\{ u \in L^p([0,T];X) \mid \partial_t u \in L^r([0,T];Y) \right\}, \text{ with } r > 1, 1 \le p < \infty,$$

$$W_1 = \left\{ u \in L^\infty([0,T];X) \mid \partial_t u \in L^r([0,T];Y) \right\}, \text{ with } r > 1.$$

Then,

$$W \hookrightarrow\hookrightarrow L^p([0,T];B), \quad W_1 \hookrightarrow\hookrightarrow C([0,T];B).$$

**Theorem 2.6** [12] If $U(t,\tau)$ is asymptotically compact, that is, the set

$$\mathbb{K} = \left\{ \mathfrak{K} = \{K_t\}_{t\in\mathbb{R}} \mid \text{Each } K_t \text{ is compact in } X_t, \mathfrak{K} \text{ is attracting} \right\}$$

is not empty, then the time-dependent attractor $\mathfrak{A}$ exists and coincides with $\mathfrak{A} = \{A_t\}_{t\in\mathbb{R}}$. In particular, it is unique.

**Definition 2.7** [13] [16] A time-dependent attractor $\mathfrak{A} = \{A_t\}_{t\in\mathbb{R}}$ is invariant, if for all $\tau \le t$,

$$U(t,\tau)A_\tau = A_t.$$

**Theorem 2.8** [13] [16] [18] Let $U(\cdot,\cdot)$ be a process on $\{X_t\}_{t\in\mathbb{R}}$. Assume that

$U(\cdot,\cdot)$ possesses a time-dependent absorbing set $\mathfrak{B}_t=\{\mathbb{B}_t(R_1)\}_{t\in\mathbb{R}}$. If for any $\varepsilon>0$ there exists a subsequence $T(\varepsilon)\le t$, $\Phi_T^t\in\mathfrak{C}(\mathbb{B}_T(R))$ such that

$$\|U(t,T)x-U(t,T)y\|\le\varepsilon+\Phi_T^t(x,y),\ \forall x,y\in\mathbb{B}_T(R),$$

for any fixed $t\in\mathbb{R}$. Then $U(\cdot,\cdot)$ is asymptotically compact.

**Theorem 2.9** [16]-[18] Let $U(\cdot,\cdot)$ be a process in a family of Banach spaces. Then $U(\cdot,\cdot)$ has a time-dependent global attractor $\mathfrak{A}^*=\{A_t^*\}_{t\in\mathbb{R}}$ satisfying

$$A_t^*=\bigcap_{s\le t}\overline{\bigcup_{\tau\le s}U(t,\tau)B_\tau(R)}$$

if and only if

i) $U(\cdot,\cdot)$ has a time-dependent absorbing set family $\mathfrak{B}_t=\{\mathbb{B}_t(R_1)\}_{t\in\mathbb{R}}$;

ii) $U(\cdot,\cdot)$ is asymptotically compact.

**Definition 2.10** [16] [18] Let $\{X_t\}_{t\in\mathbb{R}}$ be a family of Banach spaces and $\{C_t\}_{t\in\mathbb{R}}$ be a family of uniformly bounded subsets of $\{X_t\}_{t\in\mathbb{R}}$. We call a function $\Phi_\tau^t(\cdot,\cdot)$ defined on $X_t\times X_t$ a contractive function on $C_\tau\times C_\tau$, if for any fixed $t\in\mathbb{R}$ and any sequence $\{x_n\}_{n=1}^\infty\subset C_\tau$, there exists a subsequence $\{x_{n_k}\}_{k=1}^\infty\subset\{x_n\}_{n=1}^\infty$ such that

$$\lim_{k\to\infty}\lim_{l\to\infty}\Phi_\tau^t(x_{n_k},x_{n_l})=0,$$

where $\tau\le t$.

## 3. Well-Posedness of Solutions

First, we define the solution of the problems (2.3) - (2.5) as follows.

**Definition 3.1** A binary $y=(u_\theta,\partial_t u_\theta)$ is said to be a weak solution of the problem (2.3) - (2.5) on an interval $[\tau,T]$, for $\tau\in\mathbb{R}$, if

$$u_\theta\in L^\infty([\tau+\rho,T];C_{V_2})\cap L^2([\tau+\rho,T];C_{V_{3-\alpha}}),$$
$$\partial_t u\in L^\infty\left([\tau+\rho,T];C_{L^2(\Omega)}\right)\cap L^2([\tau+\rho,T];C_{V_\alpha}),$$

and satisfies

$$\langle\varepsilon(t)\partial_t^2u(t),\omega\rangle+\langle Au,\omega\rangle+\gamma\left\langle A^{\frac{\alpha}{2}}\partial_t u,\omega\right\rangle+\langle f(u),\omega\rangle=\langle g(x,u_\theta),\omega\rangle,\ t\ge\tau,$$
$$u(x,\tau)=u_0(x),\ \partial_t u(x,\tau)=u_1(x),$$
$$u_\theta(\tau)=\phi_0(x,\theta),\ \partial_t u_\theta(\tau)=\phi_1(x,\theta),\ \theta\in[-\rho,0],$$

for all $\tau\le t$ and any $\omega\in V_2$.

**Theorem 3.2** If assumptions (i) - (iii) hold, then for every $T>\tau$, $\alpha\in\left(\frac{1}{2},1\right)$, $\theta\in[-\rho,0]$, there exists a unique weak solution $y=(u_\theta,\partial_t u_\theta)$ of the problem (2.3) - (2.5) with $(u_\theta,\partial_t u_\theta)\in L^\infty\left([\tau+\rho,T];C_{\mathcal{H}_t^2}\right)\cap L^2\left([\tau+\rho,T];C_{\mathcal{Y}_t^\alpha}\right)$, $\partial_t^2u_\theta\in L^\infty([\tau+\rho,T];C_{V_{-2}})$, and

$$\begin{aligned}&\|u_\theta(t)\|_{C_{V_2}}^2+\varepsilon_\theta\|\partial_t u_\theta(t)\|_{C_{L^2(\Omega)}}^2+\int_{\tau+\rho}^t\left(\|\partial_t u_\theta(s)\|_{C_{V_\alpha}}^2+\|u_\theta(s)\|_{C_{V_{3-\alpha}}}^2\right)\mathrm{d}s\\&\le C\left(R,T,L,\gamma,\beta_0,\mu_0,\lambda_1,\phi_0,\phi_1,C_g,C_{\beta_0}\right),\ t\ge\tau.\end{aligned}\tag{3.1}$$

Moreover, the solution satisfies the following properties:

i) (Dissipativity) There exists a positive constant $R_1$, such that

$$\left\|\left(\phi_0,\phi_1\right)\right\|_{C_{\mathcal{H}_t^2}} \le R_1,\ \forall\, t \ge t(R), \tag{3.2}$$

where $\tau \le t - t(R)$ and $t(R)$ is a moment that is dependent on $R$.

ii) (Energy equation) For every $\tau \le s \le t$, $\theta \in [-\rho, 0]$, the following energy identify

$$\begin{aligned} &E\left(u_\theta(t),\partial_t u_\theta(t)\right) + 2\gamma\int_\tau^{t+\theta}\left\|\partial_t u(r)\right\|_\alpha^2 \mathrm{d}r \\ &= 2\int_\tau^{t+\theta} g\left(u_\theta\right)\partial_t u(r)\mathrm{d}r + \int_\tau^{t+\theta}\varepsilon'(t)\left\|\partial_t u(r)\right\|^2 \mathrm{d}r + E\left(\phi_0,\phi_1\right) \end{aligned} \tag{3.3}$$

holds, where

$$E\left(u_\theta,\partial_t u_\theta\right) = \left\|u_\theta\right\|_{C_{V_2}}^2 + \varepsilon_\theta\left\|\partial_t u_\theta\right\|_{C_{L^2(\Omega)}}^2 + 2\left\langle F\left(u_\theta\right),1\right\rangle.$$

iii) (Lipschitz stability in weak topological space) The solution $\left(u_\theta,\partial_t u_\theta\right),\left(v_\theta,\partial_t v_\theta\right)$ is Lipschitz continuous on $C_{V_\alpha}\times C_{V_{\alpha-2}}$, that is

$$\begin{aligned} \left\|z_\theta(t)\right\|_{C_{V_\alpha}}^2 + \varepsilon_\theta\left\|\partial_t z_\theta(t)\right\|_{C_{V_{\alpha-2}}}^2 &\le \frac{\mu_4}{\mu_3}\left(\left\|z_\theta(\tau)\right\|_{C_{V_\alpha}}^2 + \varepsilon_\theta(\tau)\left\|\partial_t z_\theta(\tau)\right\|_{C_{V_{\alpha-2}}}^2\right) \\ &\quad + C\left(R,T,L,\delta,\gamma,\beta_0,\lambda_1,\mu_0,\phi_0,\phi_1,C_g,C_{\beta_0}\right). \end{aligned} \tag{3.4}$$

where $z = \left(z_\theta,\partial_t z_\theta\right) = u_\theta - v_\theta$, and $\left(u_\theta,\partial_t u_\theta\right),\left(v_\theta,\partial_t v_\theta\right)$ are two weak solutions of the problem (2.3) - (2.5) corresponding to the initial data $\phi_{i_0},\phi_{i_1}\ (i=1,2)$ respectively.

**Proof.** i) (existence of the weak solution) Taking the scalar product of (2.4) with $\partial_t u$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[E\left(u(t),\partial_t u(t)\right)\right] + 2\gamma\left\|\partial_t u\right\|_\alpha^2 = 2\left\langle g\left(x,u_\theta\right),\partial_t u\right\rangle + \varepsilon'(t)\left\|\partial_t u\right\|^2,$$

where

$$E\left(u(t),\partial_t u(t)\right) = \left\|u\right\|_2^2 + \varepsilon(t)\left\|\partial_t u\right\|^2 + 2\left\langle F(u),1\right\rangle. \tag{3.5}$$

Integrating the above equation over the interval $[s,t]$ and replacing $t$ with $t+\theta$, it is easy to see that (3.3) holds. It follows from $(H_2)$ and $(H_3)$ that

$$\begin{aligned} 2\left|\left\langle g\left(x,u_\theta\right),\partial_t u\right\rangle\right| &\le 2\left\|g\left(x,u_\theta\right)\right\|\left\|\partial_t u\right\| \le 2C_g\left\|u_\theta\right\|_{C_{V_2}}\left\|\partial_t u\right\| \\ &\le C_g^2\left\|u_\theta\right\|_{C_{V_2}}^2 + \left\|\partial_t u\right\|^2, \end{aligned} \tag{3.6}$$

therefore using (3.6), we obtain

$$E\left(u(t),\partial_t u(t)\right) + 2\gamma\int_\tau^t\left\|\partial_t u(s)\right\|_\alpha^2 \mathrm{d}s \le E\left(u_0,u_1\right) + C_g^2\int_\tau^t\left\|u_\theta\right\|_{C_{V_2}}^2 \mathrm{d}s + \int_\tau^t\left\|\partial_t u\right\|^2 \mathrm{d}s. \tag{3.7}$$

Replacing $t$ by $t+\theta$ in (3.7), we obtain

$$\begin{aligned} &E\left(u_\theta,\partial_t u_\theta\right) + 2\gamma\int_\tau^{t+\theta}\left\|\partial_t u\right\|_1^2 \mathrm{d}s \\ &\le E\left(\phi_0,\phi_1\right) + C_g^2\int_\tau^{t+\theta}\left\|u_\theta\right\|_{C_{V_2}}^2 \mathrm{d}s + \int_\tau^{t+\theta}\left\|\partial_t u\right\|^2 \mathrm{d}s \\ &\le E\left(\phi_0,\phi_1\right) + C_g^2\int_\tau^t\left\|u_\theta\right\|_{C_{V_2}}^2 \mathrm{d}s + \int_\tau^t\left\|\partial_t u_\theta\right\|_{C_{L^2(\Omega)}}^2 \mathrm{d}s, \end{aligned} \tag{3.8}$$

where

$$E(u_\theta,\partial_t u_\theta)=\|u_\theta\|_{C_{V_2}}^2+\varepsilon_\theta\|\partial_t u_\theta\|_{C_{L^2(\Omega)}}^2+2\langle F(u_\theta),1\rangle.$$

By remark (1.1), it can be know that

$$2\int_\Omega F(u_\theta)\mathrm{d}x\ge-(1-\beta_0)\|u_\theta\|_{C_{V_2}}^2-2C_{\beta_0} \tag{3.9}$$

From (1.8) and the compact embedding $V_2\hookrightarrow\hookrightarrow L^{p+1}(\Omega)$, we can obtain

$$2\langle F(u),1\rangle\le C\left(\|u\|^2+\|u\|_{L^{p+1}}^{p+1}\right)\le C\left(\|u\|_2^2+\|u\|_2^{p+1}\right).$$

Then

$$\begin{aligned}E(\phi_0,\phi_1)&=\|\phi_0\|_{C_{V_2}}^2+\varepsilon_\theta(\tau)\|\phi_1\|_{C_{L^2(\Omega)}}^2+2\langle F(u_\theta(\tau),1\rangle\\&\le\|\phi_0\|_{C_{V_2}}^2+L\|\phi_1\|_{C_{L^2(\Omega)}}^2+C\|\phi_0\|_{C_{V_2}}^2+C\|\phi_0\|_{C_{V_2}}^{p+1}\\&\le\mu_0\left(\|\phi_0\|_{C_{V_2}}^{p+1}+\|\phi_1\|_{C_{L^2(\Omega)}}^2\right),\end{aligned}$$

where $\mu_0=\max\{1+C,L\}$.

Due to (3.9), there exists a constant $N_1=\max\left\{\frac{1}{L},C_g^2+(1-\beta_0)\right\}$, such that

$$E(u_\theta,\partial_t u_\theta)\le E(\phi_0,\phi_1)+N_1\int_\tau^t\left[\varepsilon_\theta\|\partial_t u_\theta\|_{C_{L^2(\Omega)}}^2+\|u_\theta\|_{C_{V_2}}^2+2\langle F(u_\theta),1\rangle\right]\mathrm{d}s+2C_{\beta_0}.$$

Applying the Gronwall inequality, we get

$$E(u_\theta,\partial_t u_\theta)\le C\left(R,T,L,\beta_0,\mu_0,\phi_0,\phi_1,C_g,C_{\beta_0}\right). \tag{3.10}$$

Combining (3.9) with (3.10), we obtain a constant $\mu_1=\min\{1,\beta_0\}$, such that

$$\mu_1\left(\|u_\theta\|_{C_{V_2}}^2+\|\partial_t u_\theta\|_{C_{L^2(\Omega)}}^2\right)-2C_{\beta_0}\le E(u_\theta,\partial_t u_\theta)\le C\left(R,T,L,\beta_0,\mu_0,\phi_0,\phi_1,C_g,C_{\beta_0}\right). \tag{3.11}$$

Inserting (3.10) into (3.8) gives

$$\int_\tau^{t+\theta}\|\partial_t u\|_\alpha^2\,\mathrm{d}s\le C\left(R,T,L,\gamma,\beta_0,\mu_0,\phi_0,\phi_1,C_g,C_{\beta_0}\right). \tag{3.12}$$

Then

$$\int_{\tau+\rho}^t\|\partial_t u_\theta\|_{C_{V_\alpha}}^2\,\mathrm{d}s\le\int_{\tau-\theta}^t\|\partial_t u_\theta\|_{C_{V_\alpha}}^2\,\mathrm{d}s\le C\left(R,T,L,\gamma,\beta_0,\mu_0,\phi_0,\phi_1,C_g,C_{\beta_0}\right). \tag{3.13}$$

Taking the scalar product in $L^2(\Omega)$ of (2.4) with $A^{\frac{1-\alpha}{2}}u$, we obtain

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left\langle\varepsilon(t)\partial_t u,A^{\frac{1-\alpha}{2}}u\right\rangle+\|u\|_{3-\alpha}^2+\gamma\left\langle A^{\frac{\alpha}{2}}\partial_t u,A^{\frac{1-\alpha}{2}}u\right\rangle+\left\langle f(u),A^{\frac{1-\alpha}{2}}u\right\rangle\\&=\left\langle g(x,u_\theta),A^{\frac{1-\alpha}{2}}u\right\rangle+\varepsilon'(t)\left\langle\partial_t u,A^{\frac{1-\alpha}{2}}u\right\rangle+\varepsilon(t)\|\partial_t u\|_{1-\alpha}^2.\end{aligned} \tag{3.14}$$

Next, we will handle each item of (3.14),

$$\left|\left\langle g(x,u_\theta), A^{\frac{1-\alpha}{2}}u\right\rangle\right| \le \|g(u_\theta)\|\left\|A^{\frac{1-\alpha}{2}}u\right\|$$
$$\le C_g\|u_\theta\|_{C_{V_2}}\left\|A^{\frac{1-\alpha}{2}}u\right\|$$
$$\le \frac{1}{2}\gamma\lambda_1^{-\alpha}\|u\|_2^2+\frac{1}{2}C_g^2\gamma^{-1}\|u_\theta\|_{C_{V_2}}^2,$$

$$\left|\left\langle \varepsilon(t)\partial_t u, A^{\frac{1-\alpha}{2}}u\right\rangle\right| \le L\|\partial_t u\|\|u\|_{2-2\alpha}\le\frac{1}{2}L\|u\|_2^2+\frac{1}{2}\lambda_1^{-\alpha}L\|\partial_t u\|^2,$$

$$\left|\varepsilon'(t)\left\langle \partial_t u, A^{\frac{1-\alpha}{2}}u\right\rangle\right| \le L\|\partial_t u\|\|u\|_{2-2\alpha}\le\frac{1}{2}L\|u\|_2^2+\frac{1}{2}\lambda_1^{-\alpha}L\|\partial_t u\|^2,$$

$$\left|\left\langle \gamma A^{\frac{\alpha}{2}}\partial_t u, A^{\frac{1-\alpha}{2}}u\right\rangle\right| \le \gamma\|\partial_t u\|\|u\|_2\le\frac{1}{2}\gamma\|u\|_2^2+\frac{1}{2}\gamma\|\partial_t u\|^2,$$

$$\begin{aligned}
&\left|\left\langle f(u), A^{\frac{1-\alpha}{2}}u\right\rangle\right|\\
&\le C\left(\int_\Omega\left(|u|+|u|^p\right)^{\frac{2N}{N+2(1+\alpha)}}\mathrm{d}x\right)^{\frac{N+2(1+\alpha)}{2N}}\left(\int_\Omega\left|A^{\frac{1-\alpha}{2}}u\right|^{\frac{2N}{N-2(1+\alpha)}}\mathrm{d}x\right)^{\frac{N-2(1+\alpha)}{2N}}\\
&\le C\left(\|u\|_2+\|u\|_2^p\right)\|u\|_{3-\alpha}\\
&\le\frac{1}{2}\|u\|_{3-\alpha}^2+\frac{1}{2}C\left(\|u\|_2^2+\|u\|_2^{2p}\right),
\end{aligned}\tag{3.15}$$

among them, we have used the continuous embedding $V_2\hookrightarrow L^{\frac{2Np}{N+2(1+\alpha)}}$ and (1.8). Substituting the above estimate into (3.14) yields

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\varepsilon(t)\left\langle\partial_t u, A^{\frac{1-\alpha}{2}}u\right\rangle+\frac{1}{2}\|u\|_{3-\alpha}^2\\
&\le\left(\frac{1}{2}\gamma+\frac{1}{2}L+\frac{1}{2}\gamma\lambda_1^{-\alpha}+\frac{1}{2}C\right)\|u\|_2^2+\frac{1}{2}\gamma^{-1}C_g^2\|u_\theta\|_{C_{V_2}}^2\\
&\quad+\left(\frac{1}{2}\gamma\lambda_1^{-\frac{\alpha}{2}}+\frac{1}{2}\lambda_1^{-\frac{3\alpha}{2}}+\lambda_1^{\frac{1-2\alpha}{2}}L\right)\|\partial_t u\|_\alpha^2+\frac{1}{2}C\|u\|_2^{2p}.
\end{aligned}$$

Integrating over $[\tau,t]$ and replacing $t$ by $t+\theta$, we obtain

$$\begin{aligned}
&\varepsilon_\theta\left\langle \partial_t u_\theta, A^{\frac{1-\alpha}{2}} u_\theta\right\rangle+\frac{1}{2}\int_\tau^{t+\theta}\|u\|_{3-\alpha}^2\,\mathrm{d}s\\
&\le\left(\frac{1}{2}\gamma+\frac{1}{2}L+\frac{1}{2}\gamma\lambda_1^{-\alpha}+\frac{1}{2}C\right)\int_\tau^{t+\theta}\|u\|_2^2\,\mathrm{d}s\\
&\quad+\frac{1}{2}\gamma^{-1}C_g^2\int_\tau^{t+\theta}\|u_\theta\|_{C_{V_2}}^2\,\mathrm{d}s+\varepsilon(\tau)\left\langle\partial_t u(\tau),A^{\frac{1-\alpha}{2}}u(\tau)\right\rangle\\
&\quad+\left(\frac{1}{2}\gamma\lambda_1^{-\frac{\alpha}{2}}+\frac{1}{2}\lambda_1^{-\frac{3\alpha}{2}}+\lambda_1^{\frac{1-2\alpha}{2}}L\right)\int_\tau^{t+\theta}\|\partial_t u\|_\alpha^2\,\mathrm{d}s+\frac{1}{2}C\int_\tau^{t+\theta}\|u\|_2^{2p}\,\mathrm{d}s\\
&\le\left(\frac{1}{2}\gamma+\frac{1}{2}L+\frac{1}{2}\gamma\lambda_1^{-\alpha}+\frac{1}{2}C+\frac{1}{2}\gamma^{-1}C_g^2\right)\int_\tau^{t}\|u_\theta\|_{C_{V_2}}^2\,\mathrm{d}s+\varepsilon(\tau)\left\langle\partial_t u(\tau),A^{\frac{1-\alpha}{2}}u(\tau)\right\rangle\\
&\quad+\left(\frac{1}{2}\gamma\lambda_1^{-\frac{\alpha}{2}}+\frac{1}{2}\lambda_1^{-\frac{3\alpha}{2}}+\lambda_1^{\frac{1-2\alpha}{2}}L\right)\int_\tau^{t}\|\partial_t u_\theta\|_{C_{V_\alpha}}^2\,\mathrm{d}s+\frac{1}{2}C\int_\tau^{t}\|u_\theta\|_{C_{V_2}}^{2p}\,\mathrm{d}s
\end{aligned}$$

and

$$\begin{aligned}
\left|\varepsilon(\tau)\left\langle\partial_t u(\tau),A^{\frac{1-\alpha}{2}}u(\tau)\right\rangle\right|&\le\frac{1}{2}\lambda_1^{-\alpha}L\|u(\tau)\|_2^2+\frac{1}{2}\lambda_1^{-\alpha}L\|\partial_t u(\tau)\|^2\\
&=\frac{1}{2}\lambda_1^{-\alpha}L\|u_0\|_2^2+\frac{1}{2}\lambda_1^{-\alpha}L\|u_1\|^2.
\end{aligned}$$

Then

$$\begin{aligned}
&\varepsilon_\theta\left\langle \partial_t u_\theta, A^{\frac{1-\alpha}{2}} u_\theta\right\rangle+\frac{1}{2}\int_\tau^{t+\theta}\|u\|_{3-\alpha}^2\,\mathrm{d}s\\
&\le\left(\frac{1}{2}\gamma+\frac{1}{2}L+\frac{1}{2}\gamma\lambda_1^{-\alpha}+\frac{1}{2}C+\frac{1}{2}\gamma^{-1}C_g^2\right)\int_\tau^{t}\|u_\theta\|_{C_{V_2}}^2\,\mathrm{d}s+\frac{1}{2}\lambda_1^{-\alpha}L\|u_0\|_2^2\\
&\quad+\frac{1}{2}\lambda_1^{-\alpha}L\|u_1\|^2+\left(\frac{1}{2}\gamma\lambda_1^{-\frac{\alpha}{2}}+\frac{1}{2}\lambda_1^{-\frac{3\alpha}{2}}+\lambda_1^{\frac{1-2\alpha}{2}}L\right)\int_\tau^{t}\|\partial_t u_\theta\|_{C_{V_\alpha}}^2\,\mathrm{d}s+\frac{1}{2}C\int_\tau^{t}\|u_\theta\|_{C_{V_2}}^{2p}\,\mathrm{d}s.
\end{aligned}$$

Therefore

$$\int_\tau^{t+\theta}\|u(s)\|_{3-\alpha}^2\,\mathrm{d}s\le C\left(R,T,L,\beta_0,\mu_0,\gamma,\lambda_1,\phi_0,\phi_1,C_g,C_{\beta_0}\right).$$

Then

$$\begin{aligned}
\int_{\tau+\rho}^{t}\|u_\theta(s)\|_{C_{V_{3-\alpha}}}^2\,\mathrm{d}s&\le\int_{\tau-\theta}^{t}\|u_\theta(s)\|_{C_{V_{3-\alpha}}}^2\,\mathrm{d}s\\
&\le C\left(R,T,L,\beta_0,\mu_0,\gamma,\lambda_1,\phi_0,\phi_1,C_g,C_{\beta_0}\right).
\end{aligned}\tag{3.16}$$

From equation (2.3), estimation formula (3.15) and the embedding $L^{1+\frac{1}{p}}(\Omega)\hookrightarrow\hookrightarrow V_{-2}$, $L^{\frac{2N}{N+2(1+\alpha)}}\hookrightarrow\hookrightarrow V_{-1-\alpha}$, we can obtain that

$$
\begin{aligned}
&\varepsilon_\theta^2 \left\| \partial_t^2 u_\theta (t) \right\|_{C_{V_{-2}}}^2 \\
&\le 2\|u_\theta\|_{C_{V_2}}^2 + \gamma \|\partial_t u_\theta\|_{C_{V_{2\alpha-2}}}^2 + \|f(u_\theta)\|_{C_{V_{-2}}}^2 + \lambda_1^{-1} C_g^2 \|u_\theta\|_{C_{V_2}}^2 \\
&\le C(R,\gamma,\lambda_1,C_g)\left( \|u_\theta\|_{C_{V_2}}^2 + \|\partial_t u_\theta\|_{C_{L^2(\Omega)}}^2 + \|f(u_\theta)\|_{C_{L^{1+\frac{1}{p}}}}^2 \right) \\
&\le C(R,\gamma,\lambda_1,C_g)\left( \|u_\theta\|_{C_{V_2}}^2 + \|\partial_t u_\theta\|_{C_{L^2(\Omega)}}^2 + \|u_\theta\|_{C_{V_2}}^{2p} + C_g^2 \|u_\theta\|_{C_{V_2}}^{2p} \right) \\
&\le C(R,T,\beta_0,\gamma,\mu_0,\lambda_1,\phi_0,\phi_1,C_g,C_{\beta_0})
\end{aligned} \tag{3.17}
$$

and

$$
\begin{aligned}
\|f(u_\theta)\|_{C_{V_{-1-\alpha}}}^2 &\le C\left( \|u_\theta\|_{C_{L^{\frac{2N}{N+2(1+\alpha)}}}}^2 + \|u_\theta\|_{C_{L^{\frac{2Np}{N+2(1+\alpha)}}}}^{2p} \right) \\
&\le C\left( \|u_\theta\|_{C_{V_2}}^2 + \|u_\theta\|_{C_{V_2}}^{2p} \right) \\
&\le C(R,T,\beta_0,\mu_0,\lambda_1,\phi_0,\phi_1,C_g,C_{\beta_0}).
\end{aligned} \tag{3.18}
$$

Therefore $\partial_t^2 u_\theta \in L^\infty([\tau+\rho,T],C_{V_{-2}})$, $f(u_\theta) \in L^2([\tau+\rho,T],C_{V_{-1-\alpha}})$.

Finally, from (3.5), (3.8), (3.9), (3.10), (3.13) and (3.16), we gain the estimate (3.1).

We next established the existence of solutions to problem (2.3) - (2.5) in the space $C([\tau+\rho,T];C_{\mathcal{H}_t^2}) \cap L^2([\tau+\rho,T];C_{\mathcal{Y}_t^\alpha})$. Let $y_\theta^n = (u_\theta^n, \partial_t u_\theta^n)$ be a solution of (2.3) - (2.5). It is easy to see that the estimate (3.1) holds for the Galerkin approximation sequence $\{y_\theta^n\}$. Hence, there exists a binary

$(u_\theta, \partial_t u_\theta) \in L^\infty([\tau+\rho,T];C_{\mathcal{H}_t^2}) \cap L^2([\tau+\rho,T];C_{\mathcal{Y}_t^\alpha})$,

$\partial_t^2 u_\theta \in L^\infty([\tau+\rho,T];C_{V_{-2}})$,

such that

$$
(u_\theta^n, \partial_t u_\theta^n) \to (u_\theta, \partial_t u_\theta) \ \text{ weakly}^* \text{ in } L^\infty([\tau+\rho,T];C_{V_2} \times C_{L^2(\Omega)}),
$$

$$
(u_\theta^n, \partial_t u_\theta^n) \to (u_\theta, \partial_t u_\theta) \ \text{ weakly in } L^2([\tau+\rho,T];C_{V_{3-\alpha}} \times C_{V_\alpha}),
$$

$$
\partial_t^2 u_\theta^n \to \partial_t^2 u_\theta \ \text{ weakly}^* \text{ in } L^\infty([\tau+\rho,T];C_{V_{-2}}).
$$

Applying Lemma 2.6 yields, it can be deduced that

$$
(u_\theta^n, \partial_t u_\theta^n) \to (u_\theta, \partial_t u_\theta) \ \text{ in } C([\tau+\rho,T];C_{V_{2-\eta}} \times C_{V_{-\eta}}) \ \text{ with } \eta: 0<\eta \ll 1, \tag{3.20}
$$

$$
u_\theta^n \to u_\theta \ \text{ in } L^2([\tau+\rho,T];C_{V_2}) \text{ and } u_\theta^n(x,t) \to u_\theta(x,t), \ \text{a.e. } \Omega \times [\tau+\rho,T], \tag{3.21}
$$

$$
\partial_t u_\theta^n \to \partial_t u_\theta \ \text{ in } L^2([\tau,T];C_{L_2(\Omega)}), \tag{3.22}
$$

$$
f(u_\theta^n) \to f(u_\theta) \ \text{ weakly in } L^{1+\frac{1}{p}}\left([\tau+\rho,T];L^{1+\frac{1}{p}}(\Omega)\right), \tag{3.23}
$$

For arbitrary $\xi_1 \in C_0^\infty(\Omega)$, we obtain

$$\int_\tau^T \left\langle Au_\theta^n - Au_\theta, \xi_1 \right\rangle \mathrm{d}t \le \int_\tau^T \left\| A^{\frac{1}{2}}\left(u_\theta^n(t) - u_\theta(t)\right)\right\| \left\| A^{\frac{1}{2}}\xi_1 \right\| \mathrm{d}t$$
$$\le \int_\tau^T \left\|\left(u_\theta^n(t) - u_\theta(t)\right)\right\|_{C_{V_2}} \|\xi_1\|_2 \,\mathrm{d}t \to 0.$$

and, similarly

$$\int_\tau^T \left\langle f\left(u_\theta^n\right) - f\left(u_\theta\right), \xi_1 \right\rangle \mathrm{d}t$$
$$\le C_2 \int_\tau^T \left(1 + \left\|u_\theta^n\right\|_2^{p-1} + \left\|u_\theta\right\|_2^{p-1}\right)\left\|u_\theta^n - u_\theta\right\|_2 \|\xi_1\|_2 \,\mathrm{d}t$$
$$\le C\left(R, T, \beta_0, \lambda_1, \phi_0, \phi_1, N_1, C_g, C_{\beta_0}\right)\left\|u_\theta^n - u_\theta\right\|_{L^2\left([\tau+\rho, T], C_{V_2}\right)} \to 0.$$

Finally, for the delay term,

$$\int_t^T \left\langle g\left(u_\theta^n\right) - g\left(u_\theta\right), \xi_1 \right\rangle \mathrm{d}t \le C_g \int_t^T \left\|u_\theta^n - u_\theta\right\|_{C_{V_2}} \|\xi_1\| \mathrm{d}t \le C_g \left\|u_\theta^n - u_\theta\right\|_{L^2\left([\tau+\rho, T], C_{V_2}\right)} \to 0.$$

Collecting these limits, we conclude that $y = \left(u_\theta, \partial_t u_\theta\right)$ is a weak solution of (2.3) - (2.5) satisfying estimate (3.1).

According to $\left(u_\theta(t), \partial_t u_\theta(t)\right) \in C\left(\left[\tau+\rho, T\right]; C_{V_{2-\eta}} \times C_{V_{-\eta}}\right) \cap L^\infty\left(\left[\tau+\rho, T\right]; C_{\mathcal{H}_t^2}\right)$, there is

$$\left(u_\theta, \partial_t u_\theta\right) \in C_w\left(\left[\tau+\rho, T\right]; C_{\mathcal{H}_t^2}\right),$$

$$\left\|\left(u_\theta, \partial_t u_\theta\right)\right\|_{C_{\mathcal{H}_t^2}} \le \liminf_{s\to t} \left\|\left(u_\theta(s), \partial_t u_\theta(s)\right)\right\|_{C_{\mathcal{H}_t^2}}.$$

For any $t \in [\tau, T]$, it follows form (3.3) that

$$\lim_{s\to t} E\left(u(s), \partial_t u(s)\right) = E\left(u(t), \partial_t u(t)\right),$$

replacing $t$ by $t+\theta$,

$$\lim_{s\to t} E\left(u_\theta(s), \partial_t u_\theta(s)\right) = E\left(u_\theta(t), \partial_t u_\theta(t)\right). \tag{3.24}$$

By virtue of (3.21), $u_\theta(x,s) \to u_\theta(x,t)$ a.e. $x \in \Omega$ as $s \to t$. Applying Lemma 2.4, Remark 1.1 and the Fatou lemma, we obtain

$$\lim_{s\to t} 2\left\langle g\left(u_\theta(s)\right), u(s)\right\rangle = 2\left\langle g\left(u_\theta(t)\right), u(t)\right\rangle,$$

$$\left\|\left(u_\theta(t), \partial_t u_\theta(t)\right)\right\|_{C_{\mathcal{H}_t^2}}^2 \le \liminf_{s\to t}\left(\left\|\left(u_\theta(s), \partial_t u_\theta(s)\right)\right\|_{C_{\mathcal{H}_t^2}}^2\right),$$

$$\int_\Omega \left(2F\left(u_\theta(t)\right) + \left(1-\beta_0\right)\lambda_1 \left|u_\theta(t)\right|^2 + C\left(\beta_0\right)\right)\mathrm{d}x$$
$$\le \liminf_{s\to t} \int_\Omega \left(2F\left(u_\theta(s)\right) + \left(1-\beta_0\right)\lambda_1 \left|u_\theta(s)\right|^2 + C\left(\beta_0\right)\right)\mathrm{d}x$$
$$\le \liminf_{s\to t} \int_\Omega 2F\left(u_\theta(s)\right)\mathrm{d}x + \left(1-\beta_0\right)\lambda_1 \left\|u_\theta\right\|^2 + C\left(\beta_0\right)|\Omega|.$$

That is,

$$\int_\Omega 2F\left(u_\theta(t)\right)\mathrm{d}x \le \liminf_{s\to t} \int_\Omega 2F\left(u_\theta(s)\right)\mathrm{d}x.$$

Based on the above estimation and (3.23), there is

$$\begin{aligned}
&\liminf_{s\to t}\varepsilon_\theta(s)\|\partial_t u_\theta(s)\|_{C_{L_2}}^2+\liminf_{s\to t}\left[\|u_\theta(s)\|_{C_{V_2}}^2+2\langle F(u_\theta(s)),1\rangle\right]\\
&\le \lim_{s\to t}\left[\varepsilon_\theta(s)\|\partial_t u_\theta(s)\|_{C_{L_2}}^2+\|u_\theta(s)\|_{C_{V_2}}^2+2\langle F(u_\theta(s)),1\rangle\right]\\
&=\varepsilon_\theta(t)\|\partial_t u_\theta(t)\|_{C_{L_2}}^2+\|u_\theta(t)\|_{C_{V_2}}^2+2\langle F(u_\theta(t)),1\rangle\\
&\le \varepsilon_\theta(t)\|\partial_t u_\theta(t)\|_{C_{L_2}}^2+\liminf_{s\to t}\left[\|u_\theta(s)\|_{C_{V_2}}^2\right]+\liminf_{s\to t}2\langle F(u_\theta(s)),1\rangle\\
&\le \varepsilon_\theta(t)\|\partial_t u_\theta(t)\|_{C_{L_2}}^2+\liminf_{s\to t}\left[\|u_\theta(s)\|_{C_{V_2}}^2+2\langle F(u_\theta(s)),1\rangle\right]\\
&\le \liminf_{s\to t}\varepsilon_\theta(s)\|\partial_t u_\theta(s)\|_{C_{L_2}}^2+\liminf_{s\to t}\left[\|u_\theta(s)\|_{C_{V_2}}^2+2\langle F(u_\theta(s)),1\rangle\right].
\end{aligned}$$

Therefore

$$\varepsilon_\theta(t)\|\partial_t u_\theta(t)\|_{C_{L^2(\Omega)}}^2=\lim_{s\to t}\varepsilon_\theta(s)\|\partial_t u_\theta(s)\|_{C_{L^2(\Omega)}}^2.$$

Similarly, it can be concluded that

$$\|u_\theta(t)\|_{C_{V_2}}^2=\lim_{s\to t}\|u_\theta(s)\|_{C_{V_2}}^2. \tag{3.25}$$

According to the consistency and continuity of spatial $C_{\mathcal{H}_t^2}$, combined with (3.24), (3.25) and $(u_t,\partial_t u_t)\in C_w\left([\tau+\rho,T];C_{\mathcal{H}_t^2}\right)$, it can be concluded that $(u_t,\partial_t u_t)\in C\left([\tau+\rho,T];C_{\mathcal{H}_t^2}\right)$. Up to now, the proof of the existence of the solution have been completed.

ii) (Lipschitz stability in weak topological space) Let $u,v$ be the solutions of problem (2.3) - (2.5) such that $(u_0,u_1),(v_0,v_1)\in\mathcal{H}_\tau^2$, Then, $z=u-v$, satisfies

$$\varepsilon(t)\partial_t^2 z(t)+Az+\gamma A^{\frac{\alpha}{2}}\partial_t z+f(u)-f(v)=g(u_\theta)-g(v_\theta),\ t\in[\tau,\infty), \tag{3.26}$$

$$z(\tau)=u_0-v_0=z_0,\ \partial_t z(\tau)=u_1-v_1=z_1, \tag{3.27}$$

$$z_\theta(\tau)=\phi_{1_0}-\phi_{2_0},\ \partial_t z_\theta(\tau)=\phi_{1_1}-\phi_{2_1}. \tag{3.28}$$

In the following estimation, we choose $\delta$ as an arbitrarily small positive number.

Taking the scalar product of $2A^{\frac{\alpha-2}{2}}\partial_t z+2\delta z$ with $\partial_t u$ yields

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}K(z,\partial_t z)+2\delta\|z\|_2^2+2\gamma\|\partial_t z\|_{2\alpha-2}^2-2\delta\varepsilon(t)\|\partial_t z\|^2\\
&=\sum_{i=1}^3\Gamma_i+\varepsilon'(t)\|\partial_t z\|_{\alpha-2}^2+\varepsilon'(t)\langle\partial_t z,2\delta z\rangle,
\end{aligned} \tag{3.29}$$

where

$$K(z,\partial_t z)=\varepsilon(t)\langle\partial_t z,2\delta z\rangle+\varepsilon(t)\|\partial_t z\|_{\alpha-2}^2+\|z\|_\alpha^2+\delta\gamma\|z\|_\alpha^2,$$

$$\Gamma_1=-2\left\langle f(u)-f(v),A^{\frac{\alpha-2}{2}}\partial_t z\right\rangle,$$

$$\Gamma_2 = -2\left\langle f(u) - f(v), \delta z \right\rangle,$$

$$\Gamma_3 = 2\left\langle g(u_\theta) - g(v_\theta), A^{\frac{\alpha-2}{2}} \partial_t z + \delta z \right\rangle.$$

Since

$$\left|\varepsilon(t)\langle \partial_t z, 2\delta z \rangle\right| \le 4\delta^2 \lambda_1^{1-\alpha} L \|z\|_\alpha^2 + \frac{1}{4}\varepsilon(t)\|\partial_t z\|_{\alpha-2}^2,$$

we can get

$$\mu_3\left(\|z(t)\|_\alpha^2 + \varepsilon(t)\|\partial_t z(t)\|_{\alpha-2}^2\right) \le K(z, \partial_t z) \le \mu_4\left(\|z(t)\|_\alpha^2 + \varepsilon(t)\|\partial_t z(t)\|_{\alpha-2}^2\right), \quad (3.30)$$

where $\mu_3 = \min\left\{\frac{3}{4}, 1 + \delta\gamma - 4\delta^2 L \lambda_1^{1-\alpha}\right\}$, $\mu_4 = \max\left\{\frac{5}{4}, 1 + \delta\gamma + 4\delta^2 L \lambda_1^{1-\alpha}\right\}$.

By the interpolation theorem, we can obtain

$$\begin{aligned}
|\Gamma_1| &\le 2\int_\Omega |f(u) - f(v)| \cdot \left|A^{\frac{\alpha-2}{2}} \partial_t z\right| \mathrm{d}x \\
&\le C\left(\int_\Omega \left(1 + |u|^{p-1} + |v|^{p-1}\right)^{\frac{N}{6-2\alpha}} \mathrm{d}x\right)^{\frac{6-2\alpha}{N}} \cdot \left(\int_\Omega |z|^{\frac{2N}{N-4}} \mathrm{d}x\right)^{\frac{N-4}{2N}} \\
&\quad \cdot \left(\int_\Omega \left|A^{\frac{\alpha-2}{2}} \partial_t z\right|^{\frac{2N}{N-2(4-2\alpha)}} \mathrm{d}x\right)^{\frac{N-2(4-2\alpha)}{2N}} \\
&\le C\left(1 + \|u\|_2^{p-1} + \|v\|_2^{p-1}\right) \cdot \left(\|z\|_2^2 + \|\partial_t z\|^2\right),
\end{aligned}$$

$$\begin{aligned}
|\Gamma_2| &\le 2\int_\Omega |f(u) - f(v)| \cdot |\delta z| \mathrm{d}x \\
&\le C\delta\left(\int_\Omega \left(1 + |u|^{p-1} + |v|^{p-1}\right)^{\frac{p+1}{p-1}} \mathrm{d}x\right)^{\frac{p-1}{p+1}} \cdot \left(\int_\Omega |z|^{p+1} \mathrm{d}x\right)^{\frac{2}{p+1}} \\
&\le \delta C\left(1 + \|u\|_{L^{p+1}}^{2(p-1)} + \|v\|_{L^{p+1}}^{2(p-1)}\right) + \delta\|z\|_{L^{p+1}}^2 \\
&\le \delta C\left(1 + \|u\|_2^{2(p-1)} + \|v\|_2^{2(p-1)}\right) + \delta\|z\|_2^2,
\end{aligned}$$

where we have used the Sobolev embedding $V_2 \hookrightarrow\hookrightarrow L^{p+1}(\Omega)$.

$$\begin{aligned}
|\Gamma_3| &\le 2\|g(u_\theta) - g(v_\theta)\| \left\|A^{\frac{\alpha-2}{2}} \partial_t z\right\| + 2\delta\|g(u_\theta) - g(v_\theta)\|\|z\| \\
&\le \gamma^{-1}\lambda_1^{-1} C_g^2 \|u_\theta - v_\theta\|_{C_{V_2}}^2 + \gamma\|\partial_t z\|_{2\alpha-2}^2 + \delta\lambda_1^{-1} C_g^2 \|u_\theta - v_\theta\|_{C_{V_2}}^2 + \delta\|z\|_2^2.
\end{aligned}$$

Substituting the above estimates into equation (3.29), we obtain

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t} K(z(t), \partial_t z(t)) \\
&\le C\left(1 + \|u\|_2^{p-1} + \|v\|_2^{p-1}\right)\|z\|_2^2 + 2\delta L\|\partial_t z\|^2 + C\left(1 + \|u\|_2^{p-1} + \|v\|_2^{p-1}\right)\|\partial_t z\|^2 \\
&\quad + \left(\gamma^{-1}\lambda_1^{-1} C_g^2 + \delta\lambda_1^{-1} C_g^2\right)\|u_\theta - v_\theta\|_{C_{V_2}}^2 + \delta C\left(1 + \|u\|_2^{2(p-1)} + \|v\|_2^{2(p-1)}\right).
\end{aligned}$$

Integrating the above inequality over $[\tau, t]$ and replacing $t$ by $t + \theta$ yields

$$
\begin{aligned}
&K\left(z_\theta(t),\partial_t z_\theta(t)\right)\\
&\le K\left(z(\tau),\partial_t z(\tau)\right)+C\int_\tau^{t+\theta}\left(1+\|u\|_2^{p-1}+\|v\|_2^{p-1}\right)\|z\|_2^2\,\mathrm{d}s+2\delta L\int_\tau^{t+\theta}\|\partial_t z\|^2\,\mathrm{d}s\\
&\quad+C\int_\tau^{t+\theta}\left(1+\|u\|_2^{p-1}+\|v\|_2^{p-1}\right)\|\partial_t z\|^2\,\mathrm{d}s+\left(\gamma^{-1}\lambda_1^{-1}C_g^2+\delta\lambda_1^{-1}C_g^2\right)\int_\tau^{t+\theta}\|u_\theta-v_\theta\|_{C_{V_2}}^2\,\mathrm{d}s\\
&\quad+\delta C\int_\tau^{t+\theta}\left(1+\|u\|_2^{2(p-1)}+\|v\|_2^{2(p-1)}\right)\mathrm{d}s.
\end{aligned}
$$

There

$$
\begin{aligned}
&K\left(z_\theta(t),\partial_t z_\theta(t)\right)\\
&\le K\left(z(\tau),\partial_t z(\tau)\right)+C\int_\tau^{t}\left(1+\|u_\theta\|_{C_{V_2}}^{p-1}+\|v_\theta\|_{C_{V_2}}^{p-1}\right)\|z_\theta\|_{C_{V_2}}^2\,\mathrm{d}s+2\delta L\int_\tau^{t}\|\partial_t z_\theta\|_{C_{L^2(\Omega)}}^2\,\mathrm{d}s\\
&\quad+C\int_\tau^{t}\left(1+\|u_\theta\|_{C_{V_2}}^{p-1}+\|v_\theta\|_{C_{V_2}}^{p-1}\right)\|\partial_t z_\theta\|_{C_{L^2(\Omega)}}^2\,\mathrm{d}s+\left(\gamma^{-1}\lambda_1^{-1}C_g^2+\delta\lambda_1^{-1}C_g^2\right)\int_\tau^{t}\|u_\theta-v_\theta\|_{C_{V_2}}^2\,\mathrm{d}s\\
&\quad+\delta C\int_\tau^{t}\left(1+\|u_\theta\|_{C_{V_2}}^{2(p-1)}+\|v_\theta\|_{C_{V_2}}^{2(p-1)}\right)\mathrm{d}s\\
&\le K\left(z(\tau),\partial_t z(\tau)\right)+C\left(R,T,L,\delta,\gamma,\lambda_1,\beta_0,\mu_0,\phi_0,\phi_1,C_g,C_{\beta_0}\right).
\end{aligned}
$$

Consequently, the equation (3.4) holds.

iii) (Dissipativity) Taking the inner product of $2\partial_t u+2\delta u$ with (2.3) yields

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}K_1(u,\partial_t u)+2\delta\|u\|_2^2+2\gamma\|\partial_t u\|_\alpha^2+2\delta\langle f(u),u\rangle-\varepsilon'(t)\|\partial_t u\|^2\\
&-\varepsilon'(t)\langle\partial_t u,2\delta u\rangle-2\delta\varepsilon(t)\|\partial_t u\|^2=2\langle g(u_\theta),2\partial_t u+2\delta u\rangle,
\end{aligned}
\tag{3.33}
$$

here $K_1(u,\partial_t u)=\|u\|_2^2+\varepsilon(t)\|\partial_t u\|^2+\varepsilon(t)\langle\partial_t u,2\delta u\rangle+\delta\gamma\|u\|_\alpha^2+2\langle F(u),1\rangle$.

Moreover, using

$$
\left|\varepsilon'(t)\langle 2\delta u,\partial_t u\rangle\right|\le\delta\lambda_1^{-\frac{1}{2}}L\|u\|_2^2+\delta\lambda_1^{-\frac{1}{2}}\varepsilon(t)\|\partial_t u\|^2 \tag{3.34}
$$

together with Remark (1.1), we obtain constants $\mu_5,\mu_6$, such that

$$
\mu_5\|(u,\partial_t u)\|_{\mathcal{H}_t^2}^2-2C_{\beta_0}\le K_1(u,\partial_t u)\le\mu_6\|(u,\partial_t u)\|_{\mathcal{H}_t^2}^2,
$$

where $\mu_5=\min\left\{1-\delta\lambda_1^{-\frac{1}{2}},\beta_0+\delta\gamma\lambda_1^{-\frac{2-\alpha}{2}}-\delta\lambda_1^{-\frac{1}{2}}L\right\}$,

$\mu_6=\max\left\{1+\delta\lambda_1^{-\frac{1}{2}},1+\delta\gamma\lambda_1^{-\frac{2-\alpha}{2}}+\delta\lambda_1^{-\frac{1}{2}}L+C\right\}$.

From (1.8) and the compact embedding $V_2\hookrightarrow\hookrightarrow L^{p+1}(\Omega)$, we can obtain

$$
2\langle F(u),1\rangle\le C\left(\|u\|^2+\|u\|_{L^{p+1}}^{p+1}\right)\le C\left(\|u\|_2^2+\|u\|_2^{p+1}\right).
$$

By Remark (1.1), we have

$$
2\delta\langle f(u),u\rangle\ge 2\delta(\beta_0-1)\|u\|_2^2-2\delta C_{\beta_0}.
$$

Additionally, for the delay forcing term, we estimate

$$\begin{aligned}&\left|\left\langle g\left(x,u_{\theta}\right),2\partial_{t}u+2\delta u\right\rangle\right|\\&\le 2\left\|g\left(u_{\theta}\right)\right\|\left\|\partial_{t}u\right\|+2\delta\left\|g\left(u_{\theta}\right)\right\|\left\|u\right\|\\&\le 2C_{g}\left\|u_{\theta}\right\|_{C_{V_2}}\left\|\partial_{t}u\right\|+2\delta C_{g}\left\|u_{\theta}\right\|_{C_{V_2}}\left\|u\right\|\\&\le \delta C_{g}^{2}\lambda_{1}^{-\frac{1}{2}}\left\|u_{\theta}\right\|_{C_{V_2}}^{2}+\varepsilon(t)\left\|\partial_{t}u\right\|^{2}+C_{g}^{2}L^{-1}\left\|u_{\theta}\right\|_{C_{V_2}}^{2}+\delta\lambda_{1}^{-\frac{1}{2}}\left\|u\right\|_{2}^{2}.\end{aligned}$$

Substituting the above estimates into (3.33), we obtain

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}K_{1}\left(u,\partial_{t}u\right)+2\gamma\left\|\partial_{t}u\right\|_{\alpha}^{2}\\&\le\left|\delta\lambda_{1}^{-\frac{1}{2}}+\delta L\lambda_{1}^{-\frac{1}{2}}-2\delta\beta_{0}\right|\left\|u\right\|_{2}^{2}+\left(C_{g}^{2}L^{-1}+\delta\lambda_{1}^{-\frac{1}{2}}C_{g}^{2}\right)\left\|u_{\theta}\right\|_{C_{V_2}}^{2}\\&\quad+\left(1+2\delta+\delta\lambda_{1}^{-\frac{1}{2}}\right)\varepsilon(t)\left\|\partial_{t}u\right\|^{2}+2\delta C_{\beta_0}.\end{aligned}\tag{3.35}$$

Integrating over $[\tau,t]$ and replacing $t$ by $t+\theta$, we deduce

$$\begin{aligned}&K_{1}\left(u_{\theta},\partial_{t}u_{\theta}\right)+2\gamma\int_{\tau}^{t+\theta}\left\|\partial_{t}u\right\|_{\alpha}^{2}\,\mathrm{d}s\\&\le K_{1}\left(u(\tau),\partial_{t}u(\tau)\right)+\left|\delta\lambda_{1}^{-\frac{1}{2}}+\delta L\lambda_{1}^{-\frac{1}{2}}-2\delta\beta_{0}\right|\int_{\tau}^{t+\theta}\left\|u\right\|_{2}^{2}\,\mathrm{d}s\\&\quad+2\delta C_{\beta_0}\left(t+\theta-\tau\right)+\left(L+2\delta L+\delta\lambda_{1}^{-\frac{1}{2}}L\right)\int_{\tau}^{t+\theta}\left\|\partial_{t}u\right\|^{2}\,\mathrm{d}s\\&\quad+\left(C_{g}^{2}L^{-1}+\delta\lambda_{1}^{-\frac{1}{2}}C_{g}^{2}\right)\int_{\tau}^{t+\theta}\left\|u_{\theta}\right\|_{C_{V_2}}^{2}\,\mathrm{d}s,\end{aligned}\tag{3.36}$$

consequently,

$$\begin{aligned}&K_{1}\left(u_{\theta},\partial_{t}u_{\theta}\right)+2\gamma\int_{\tau}^{t}\left\|\partial_{t}u_{\theta}\right\|_{C_{V_\alpha}}^{2}\,\mathrm{d}s\\&\le K_{1}\left(u_{0},u_{1}\right)+\left|\delta\lambda_{1}^{-\frac{1}{2}}+\delta L\lambda_{1}^{-\frac{1}{2}}-2\delta\beta_{0}\right|\int_{\tau}^{t}\left\|u_{\theta}\right\|_{C_{V_2}}^{2}\,\mathrm{d}s\\&\quad+2\delta C_{\beta_0}\left(t-\tau\right)\left(C_{g}^{2}L^{-1}+\delta\lambda_{1}^{-\frac{1}{2}}C_{g}^{2}\right)\int_{\tau}^{t}\left\|u_{\theta}\right\|_{C_{V_2}}^{2}\,\mathrm{d}s\\&\quad+\left(L+2\delta L+\delta\lambda_{1}^{-\frac{1}{2}}L\right)\int_{\tau}^{t}\left\|\partial_{t}u\right\|_{C_{L^2(\Omega)}}^{2}\,\mathrm{d}s\\&\le C\left(R,T,L,\delta,\gamma,\tau,\lambda_{1},\mu_{0},\beta_{0},\phi_{0},\phi_{1},C_{g},C_{\beta_0}\right).\end{aligned}\tag{3.37}$$

Based on (3.34) and (3.37), it can be demonstrated that the solutions to the problem (2.3) - (2.5) possess dissipative properties.

**Theorem 3.3** Assuming that conditions (1.5) - (1.8) are satisfied and $g\in L^{2}(\Omega)$. If $u$ and $v$ are two solutions of problem (2.3) - (2.5) respectively with initial values $(u_0,u_1)$ and $(v_0,v_1)$, then for any $\tau<T$, there is

$$\left\|z_{\theta}(t)\right\|_{C_{\mathcal{H}_t^2}}^{2}\le C\mathrm{e}^{(t-\tau)}\left\|z_{\theta}(\tau)\right\|_{C_{\mathcal{H}_\tau^2}}^{2},\ \forall\, t\in[\tau,T].\tag{3.40}$$

**Proof.** Let $z = u - v$, then $z$ satisfies

$$\begin{aligned}&\varepsilon(t)\partial_t^2 z + Az + \gamma A^{\frac{\alpha}{2}}\partial_t z + f(u) - f(v)\\&= g(x,u_\theta) - g(x,v_\theta),\ x \in \Omega,\ t \in [\tau,\infty),\end{aligned} \tag{3.41}$$

$$z(\tau) = u_0 - v_0 = z_0,\ \partial_t z(\tau) = u_1 - v_1 = z_1, \tag{3.42}$$

$$z_\theta(\tau) = \phi_{1_0} - \phi_{2_0},\ \partial_t z_\theta(\tau) = \phi_{1_1} - \phi_{2_1}. \tag{3.43}$$

Taking the inner product of $2\partial_t z$ with (3.41) yield

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left(\|z\|_2^2 + \varepsilon(t)\|\partial_t z\|^2\right) + 2\gamma\|\partial_t z\|_\alpha^2\\&= \varepsilon'(t)\|\partial_t z\|^2 - 2\langle f(u) - f(v), \partial_t z\rangle + 2\langle g(x,u_\theta) - g(x,v_\theta), \partial_t z\rangle,\end{aligned} \tag{3.44}$$

Next, we consider the Sobolev embedding $V_2 \hookrightarrow L^{\frac{N(p-1)}{2+\alpha}}$, we can obtain

$$\begin{aligned}&\left|-2\langle f(u) - f(v), \partial_t z\rangle\right| \le 2\int_\Omega |f(u) - f(v)| \cdot |\partial_t z| \mathrm{d}x\\&\le 2C\left(\int_\Omega \left(1 + |u|^{p-1} + |v|^{p-1}\right)^{\frac{N}{2+\alpha}} \mathrm{d}x\right)^{\frac{2+\alpha}{N}} \cdot \left(\int_\Omega |z|^{\frac{2N}{N-4}} \mathrm{d}x\right)^{\frac{N-4}{2N}} \cdot \left(\int_\Omega |\partial_t z|^{\frac{2N}{N-2\alpha}} \mathrm{d}x\right)^{\frac{N-2\alpha}{2N}}\\&\le 2C\left(1 + \|u\|_2^{p-1} + \|v\|_2^{p-1}\right)\|z\|_2\|\partial_t z\|_\alpha\\&\le C\left(1 + \|u\|_2^{p-1} + \|v\|_2^{p-1}\right)\left(\|z\|_2^2 + \|\partial_t z\|_\alpha^2\right),\end{aligned}$$

and again

$$\begin{aligned}2\left|\langle g(x,u_\theta) - g(x,v_\theta), \partial_t z\rangle\right| &\le 2\|g(x,u_\theta) - g(x,v_\theta)\|\|\partial_t z\|\\&\le 2C_g\|u_\theta - v_\theta\|_{C_{V_2}}\|\partial_t z\|\\&\le \gamma^{-1}\lambda_1^{-\frac{\alpha}{2}}C_g^2\|u_\theta - v_\theta\|_{C_{V_2}}^2 + \gamma\|\partial_t z\|_\alpha^2.\end{aligned}$$

Substituting the above estimate into (3.44) yields

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left(\|z\|_2^2 + \varepsilon(t)\|\partial_t z\|^2\right)\\&\le C\left(1 + \|u\|_2^{p-1} + \|v\|_2^{p-1}\right)\left(\|z\|_2^2 + \|\partial_t z\|_\alpha^2\right) + \gamma^{-1}\lambda_1^{-\frac{\alpha}{2}}C_g^2\|u_\theta - v_\theta\|_{C_{V_2}}^2.\end{aligned} \tag{3.45}$$

Integrating over $[\tau, t]$ and replacing $t$ by $t+\theta$, we obtain

$$\begin{aligned}&\|z_0\|_{C_{V_2}}^2 + \varepsilon_\theta\|\partial_t z_\theta\|^2\\&\le \|z(\tau)\|_2^2 + \varepsilon(\tau)\|\partial_t z(\tau)\|^2 + C\int_\tau^{t+\theta}\left(1 + \|u\|_2^{p-1} + \|v\|_2^{p-1}\right)\|z\|_2^2 \mathrm{d}s\\&\quad + C\int_\tau^{t+\theta}\left(1 + \|u\|_2^{p-1} + \|v\|_2^{p-1}\right)\|\partial_t z\|_\alpha^2 \mathrm{d}s + \gamma^{-1}\lambda_1^{-\frac{\alpha}{2}}C_g^2\int_\tau^{t+\theta}\|u_\theta - v_\theta\|_{C_{V_2}}^2 \mathrm{d}s\\&\le \|z_0\|_2^2 + \varepsilon(\tau)\|z_1\|^2 + C\int_\tau^t\left(1 + \|u_\theta\|_{C_{V_2}}^{p-1} + \|v_\theta\|_{C_{V_2}}^{p-1}\right)\|z_\theta\|_{C_{V_2}}^2 \mathrm{d}s\\&\quad + C\int_\tau^t\left(1 + \|u_\theta\|_{C_{V_2}}^{p-1} + \|v_\theta\|_{C_{V_2}}^{p-1}\right)\|\partial_t z_\theta\|_{C_{V_\alpha}}^2 \mathrm{d}s + \gamma^{-1}\lambda_1^{-\frac{\alpha}{2}}C_g^2\int_\tau^t\|u_\theta - v_\theta\|_{C_{V_2}}^2 \mathrm{d}s\\&\le \|z_0\|_2^2 + \varepsilon(\tau)\|z_1\|^2 + C\int_\tau^t\left(\|z_\theta\|_{C_{V_2}}^2 + \varepsilon_\theta\|\partial_t z_\theta\|_{C_{L^2(\Omega)}}^2\right)\mathrm{d}s.\end{aligned} \tag{3.46}$$

Using the Gronwall inequality, we can obtain

$$\left\|z_\theta(t)\right\|_{C_{\mathcal{H}_t^2}}^2 \le Ce^{(t-\tau)}\left\|z_\theta(\tau)\right\|_{C_{\mathcal{H}_\tau^2}}^2, \ \forall t\in[\tau,T]. \tag{3.40}$$

At the same time, we also obtained the uniqueness of the solutions to problem (2.3) - (2.5) in the space $C_{\mathcal{H}_t^2}$.

By Theorem 3.2 and Theorem 3.3, we can define the process

$$z_\theta(t)=U(t,\tau)z_\theta(\tau):C_{\mathcal{H}_\tau^2}\to C_{\mathcal{H}_t^2},$$

associated with problem (2.3) - (2.5), which is continuous from $C_{\mathcal{H}_\tau^2}$ to $C_{\mathcal{H}_t^2}$.

## 4. The Existence of Time-Dependent Global Attractor in $C_{\mathcal{H}_t^2}$

Since this article conducts research in a time-dependent space and the time-dependent function is decreasing, leading to non-uniform energy dissipation, the method of contraction functions is chosen to prove the asymptotic compactness of the process. According to Theorem 3.2, the following result can be obtained.

**Theorem 4.1** Assuming that the conditions of Theorem 3.2 hold. If for any initial values $(u_0,u_1),(v_0,v_1)\in\{\mathbb{B}_\tau^2(R)\}\subset\mathcal{H}_\tau^2$, then there exists $R_1>0$, such that the process $U(t,\tau)$ corresponding to the problem (2.3) - (2.5) possesses a time-dependent absorbing set, namely, the family sets $\mathfrak{B}_t=\{\mathbb{B}_t(R_1)\}_{t\in\mathbb{R}}$.

To establish the asymptotic compactness of the process $U(t,\tau)$, we will make following a priori estimates.

Let $u$ and $v$ be the solutions of problem (2.3)-(2.5) respectively with initial values $(u_0,u_1),(v_0,v_1)\in\{\mathbb{B}_\tau^2(R)\}$. The difference $z=u-v$ satisfies the following equation

$$\varepsilon(t)\partial_t^2 z(t)+Az+\gamma A^{\frac{\alpha}{2}}\partial_t z+f(u)-f(v)=g(x,u_\theta)-g(x,v_\theta),\ t\in[\tau,\infty), \tag{4.1}$$

$$z(\tau)=u_0-v_0=z_0,\ \partial_t z(\tau)=u_1-v_1=z_1, \tag{4.2}$$

$$z_\theta(\tau)=\phi_{1_0}-\phi_{2_0},\ \partial_t z_\theta(\tau)=\phi_{1_1}-\phi_{2_1}. \tag{4.3}$$

We will conduct the priori estimation in the following four steps.

**Step 1.** Multiply equation (4.1) by $2\partial_t z$ and then integrate over $[s,t]\times\Omega$, which yields

$$\begin{aligned}&H(t)-H(s)+2\gamma\int_s^t\int_\Omega\left|A^{\frac{\alpha}{4}}\partial_t z(r)\right|^2\mathrm{d}x\mathrm{d}r+2\int_s^t\int_\Omega\left(f(u)-f(v)\right)\partial_t z(r)\mathrm{d}x\mathrm{d}r\\&=\int_s^t\int_\Omega\varepsilon'(r)\left|\partial_t z(r)\right|^2\mathrm{d}x\mathrm{d}r+2\int_s^t\int_\Omega\left(g(x,u_\theta)-g(x,v_\theta)\right)\partial_t z(r)\mathrm{d}x\mathrm{d}r,\end{aligned} \tag{4.4}$$

here, $H(t)=\varepsilon(t)\|\partial_t z(t)\|^2+\|z(t)\|_2^2$, and $T\le s\le t$.

Due to

$$\begin{aligned}&2\int_s^t\int_\Omega\left(g(x,u_\theta)-g(x,v_\theta)\right)\partial_t z(r)\,\mathrm{d}x\mathrm{d}r\\&\le 2\int_s^t\left(\int_\Omega\left(g(x,u_\theta)-g(x,v_\theta)\right)^2\mathrm{d}x\right)^{\frac{1}{2}}\left(\int_\Omega\partial_t z(r)^2\,\mathrm{d}x\right)^{\frac{1}{2}}\mathrm{d}r\\&\le 2\int_s^t\left\|g(x,u_\theta)-g(x,v_\theta)\right\|\|\partial_t z\|\mathrm{d}r\\&\le\int_s^t\left\|g(x,u_\theta)-g(x,v_\theta)\right\|^2\mathrm{d}r+\int_s^t\|\partial_t z\|^2\,\mathrm{d}r\\&\le C_g\int_{s+\rho}^t\|u_\theta-v_\theta\|_{C_{V_2}}^2\,\mathrm{d}r+\int_s^t\|\partial_t z\|^2\,\mathrm{d}r.\end{aligned}$$

Then there is

$$\begin{aligned}&\int_T^t\int_\Omega\varepsilon(r)|\partial_t z(r)|^2\,\mathrm{d}x\mathrm{d}r\\&\le L\int_T^t\int_\Omega|\partial_t z(r)|^2\,\mathrm{d}x\mathrm{d}r-\int_T^t\int_\Omega\varepsilon'(r)|\partial_t z(r)|^2\,\mathrm{d}x\mathrm{d}r\\&\le H(T)+C_g\int_{T+\rho}^t\|u_\theta-v_\theta\|_{C_{V_2}}^2\,\mathrm{d}r+\int_T^t\|\partial_t z\|^2\,\mathrm{d}r\\&\quad+L\int_T^t\int_\Omega|\partial_t z(r)|^2\,\mathrm{d}x\mathrm{d}r-2\int_T^t\int_\Omega\left(f(u)-f(v)\right)\partial_t z(r)\,\mathrm{d}x\mathrm{d}r.\end{aligned}\tag{4.5}$$

**Step 2.** Multiplying (4.1) by $z$ and integrating over $[T,t]\times\Omega$, this gives

$$\begin{aligned}&\int_\Omega\varepsilon(t)\partial_t z(t)z(t)\,\mathrm{d}x+\frac{\gamma}{2}\|z(t)\|_\alpha^2+\int_T^t\int_\Omega\left(f(u)-f(v)\right)z(r)\,\mathrm{d}x\mathrm{d}r\\&+\int_T^t\int_\Omega\left|A^{\frac{1}{2}}z\right|^2\mathrm{d}r-\int_T^t\varepsilon(r)\|\partial_t z(r)\|^2\,\mathrm{d}r\\&=\int_T^t\int_\Omega\left(g(x,u_\theta)-g(x,v_\theta)\right)z(r)\,\mathrm{d}x\mathrm{d}r+\int_\Omega\varepsilon(T)\partial_t z(T)z(T)\,\mathrm{d}x\\&\quad+\frac{\gamma}{2}\|z(T)\|_\alpha^2+\int_T^t\int_\Omega\varepsilon'(r)\partial_t z(r)z(r)\,\mathrm{d}x\mathrm{d}r.\end{aligned}\tag{4.6}$$

Due to

$$\begin{aligned}&\int_T^t\int_\Omega\left(g(x,u_\theta)-g(x,v_\theta)\right)z(r)\,\mathrm{d}x\mathrm{d}r\\&\le\int_T^t\left(\int_\Omega\left(g(x,u_\theta)-g(x,v_\theta)\right)^2\mathrm{d}x\right)^{\frac{1}{2}}\left(\int_\Omega z(r)^2\,\mathrm{d}x\right)^{\frac{1}{2}}\mathrm{d}r\\&\le\int_T^t\left\|g(x,u_\theta)-g(x,v_\theta)\right\|\|z(r)\|\mathrm{d}r\\&\le\int_T^t\left\|g(x,u_\theta)-g(x,v_\theta)\right\|^2\mathrm{d}r+\lambda_1^{-1}\int_T^t\|z(r)\|_2^2\,\mathrm{d}r\\&\le C_g\int_{T+\rho}^t\|u_\theta-v_\theta\|_{C_{V_2}}^2\,\mathrm{d}r+\lambda_1^{-1}\int_T^t\|z(r)\|_2^2\,\mathrm{d}r.\end{aligned}$$

Combining (4.5) and (4.6)and replacing $t$ by $t+\theta$ in the integral identity, we obtain

$$
\begin{aligned}
&\int_T^{t+\theta} H(r)\,\mathrm{d}r=\int_T^{t+\theta}\left(\varepsilon(r)\|\partial_t z(r)\|^2+\|z(r)\|_2^2\right)\mathrm{d}r \\
&\le H(T)+C_g\int_{T+\rho}^{t+\theta}\|u_\theta-v_\theta\|_{C_{V_2}}^2\,\mathrm{d}r+\int_T^{t+\theta}\|\partial_t z(r)\|^2\,\mathrm{d}r \\
&\quad+L\int_T^{t+\theta}\int_\Omega|\partial_t z(r)|^2\,\mathrm{d}x\mathrm{d}r-2\int_T^{t+\theta}\int_\Omega(f(u)-f(v))\partial_t z(r)\,\mathrm{d}x\mathrm{d}r \\
&\quad-\int_T^{t+\theta}\int_\Omega(f(u)-f(v))z(r)\,\mathrm{d}x\mathrm{d}r-\int_\Omega\varepsilon(t)\partial_t z(t)z(t)\,\mathrm{d}x \\
&\quad+\int_T^{t+\theta}\varepsilon(r)\|\partial_t z\|^2\,\mathrm{d}r+\int_\Omega\varepsilon(T)\partial_t z(T)z(T)\,\mathrm{d}x+\frac{\gamma}{2}\|z(T)\|_\alpha^2 \\
&\quad+\int_T^{t+\theta}\int_\Omega\varepsilon'(r)\partial_t z(r)z(r)\,\mathrm{d}x\mathrm{d}r+C_g\int_{T+\rho}^{t+\theta}\|u_\theta-v_\theta\|_{C_{V_2}}^2\,\mathrm{d}r+\lambda_1^{-1}\int_T^{t+\theta}\|z(r)\|_2^2\,\mathrm{d}r \\
&\le\varepsilon(T)\|\partial_t z(T)\|^2+\|z(T)\|_2^2+C_g\int_{T+\rho}^{t}\|u_\theta-v_\theta\|_{C_{V_2}}^2\,\mathrm{d}r+\int_T^t\|\partial_t z_\theta\|_{C_{L^2(\Omega)}}^2\,\mathrm{d}r \\
&\quad+L\int_T^t\int_\Omega|\partial_t z_\theta(r)|^2\,\mathrm{d}x\mathrm{d}r-2\int_{T+\rho}^t\int_\Omega(f(u_\theta)-f(v_\theta))\partial_t z_\theta(r)\,\mathrm{d}x\mathrm{d}r \\
&\quad-\int_{T+\rho}^t\int_\Omega(f(u_\theta)-f(v_\theta))z_\theta(r)\,\mathrm{d}x\mathrm{d}r-\int_\Omega\varepsilon(t)\partial_t z(t)z(t)\,\mathrm{d}x \\
&\quad+\int_T^t\varepsilon_\theta(r)\|\partial_t z_\theta(r)\|_{C_{L^2(\Omega)}}^2\,\mathrm{d}r+\int_\Omega\varepsilon(T)\partial_t z(T)z(T)\,\mathrm{d}x+\frac{\gamma}{2}\|z(T)\|_\alpha^2 \\
&\quad+\int_T^t\varepsilon_\theta'(r)\partial_t z_\theta(r)z_\theta(r)\,\mathrm{d}x\mathrm{d}r+C_g\int_{T+\rho}^t\|u_\theta-v_\theta\|_{C_{V_2}}^2\,\mathrm{d}r+\lambda_1^{-1}\int_T^t\|z_\theta(r)\|_{C_{V_2}}^2\,\mathrm{d}r.
\end{aligned}
$$

**Step 3.** Integrate (4.4) over with $[T,t]$ respect to $s$ and replacing $t$ by $t+\theta$ in the preceding inequality, this yields

$$
\begin{aligned}
&H(t)(t+\theta-T) \\
&\le\int_T^{t+\theta}H(s)\,\mathrm{d}s+2\int_T^{t+\theta}\int_s^{t+\theta}\int_\Omega(g(x,u_\theta)-g(x,v_\theta))\partial_t z(r)\,\mathrm{d}x\mathrm{d}r\mathrm{d}s \\
&\quad-2\int_T^{t+\theta}\int_s^{t+\theta}\int_\Omega(f(u)-f(v))\partial_t z(r)\,\mathrm{d}x\mathrm{d}r\mathrm{d}s \\
&\le H(T)+C_g\int_{T+\rho}^t\|u_\theta-v_\theta\|_{C_{V_2}}^2\,\mathrm{d}s+\int_T^t\|\partial_t z_\theta(s)\|_{C_{L^2(\Omega)}}^2\,\mathrm{d}s \\
&\quad+L\int_T^t\int_\Omega|\partial_t z_\theta(s)|^2\,\mathrm{d}x\mathrm{d}s-2\int_{T+\rho}^t\int_\Omega(f(u_\theta)-f(v_\theta))\partial_t z_\theta(s)\,\mathrm{d}x\mathrm{d}s \\
&\quad-\int_{T+\rho}^t\int_\Omega(f(u_\theta)-f(v_\theta))z_\theta(s)\,\mathrm{d}x\mathrm{d}s-\int_\Omega\varepsilon(t)\partial_t z(t)z(t)\,\mathrm{d}x \\
&\quad+\int_T^t\varepsilon_\theta(r)\|\partial_t z_\theta(s)\|_{C_{L^2(\Omega)}}^2\,\mathrm{d}s+\int_\Omega\varepsilon(T)\partial_t z(T)z(T)\,\mathrm{d}x+\frac{\gamma}{2}\|z(T)\|_\alpha^2 \\
&\quad+\int_T^t\int_\Omega\varepsilon_\theta'(s)\partial_t z_\theta(s)z_\theta(s)\,\mathrm{d}x\mathrm{d}s+C_g\int_{T+\rho}^t\|u_\theta-v_\theta\|_{C_{V_2}}^2\,\mathrm{d}s+\lambda_1^{-1}\int_T^t\|z_\theta(s)\|_{C_{V_2}}^2\,\mathrm{d}s \\
&\quad+C_g\int_T^{t+\theta}\int_{s+\rho}^t\|u_\theta-v_\theta\|_{C_{V_2}}^2\,\mathrm{d}r\mathrm{d}s+\int_T^{t+\theta}\int_s^t\|\partial_t z_\theta(r)\|_{C_{L^2(\Omega)}}^2\,\mathrm{d}r\mathrm{d}s \\
&\quad-2\int_T^{t+\theta}\int_{s+\rho}^t\int_\Omega(f(u_\theta)-f(v_\theta))\partial_t z_\theta(r)\,\mathrm{d}x\mathrm{d}r\mathrm{d}s.
\end{aligned}
$$

**Step 4.** Introduce the compactness functional

$$C(M)=H(T)+\frac{\gamma}{2}\|z(T)\|_\alpha^2+\int_\Omega\varepsilon(T)\partial_t z(T)z(T)\,\mathrm{d}x, \tag{4.7}$$

and

$$\varphi_T^t\left((u_\theta(T),\partial_t u_\theta(T)),(v_\theta(T),\partial_t v_\theta(T))\right)=\Psi_1+\Psi_2+\Psi_3, \tag{4.8}$$

where

$$\Psi_1=\frac{1}{(t+\theta-T)}\Bigg[L\int_T^t\int_\Omega\left|\partial_t z_\theta(s)\right|^2\mathrm{d}x\mathrm{d}s+\int_T^t\varepsilon_\theta(s)\left\|\partial_t z_\theta(s)\right\|_{C_{L^2(\Omega)}}^2\mathrm{d}s$$
$$+\int_T^t\left\|\partial_t z(s)\right\|^2\mathrm{d}s-\int_\Omega\varepsilon(t)\partial_t z(t)z(t)\mathrm{d}x+\int_T^{t+\theta}\int_s^t\left\|\partial_t z_\theta(r)\right\|_{C_{L^2(\Omega)}}^2\mathrm{d}r\mathrm{d}s$$
$$+\int_T^t\int_\Omega\varepsilon_\theta'(s)\partial_t z_\theta(s)z_\theta(s)\mathrm{d}x\mathrm{d}s\Bigg],$$

$$\Psi_2=-\frac{1}{(t+\theta-T)}\Bigg[\int_{t+\rho}^t\int_\Omega\left(f(u_\theta)-f(v_\theta)\right)z_\theta(s)\mathrm{d}x\mathrm{d}s$$
$$+2\int_{t+\rho}^t\int_\Omega\left(f(u_\theta)-f(v_\theta)\right)\partial_t z_\theta(s)\mathrm{d}x\mathrm{d}s$$
$$+2\int_T^{t+\theta}\int_{s+\rho}^t\int_\Omega\left(f(u_\theta)-f(v_\theta)\right)\partial_t z_\theta(r)\mathrm{d}x\mathrm{d}r\mathrm{d}s\Bigg],$$

$$\Psi_3=\frac{1}{(t+\theta-T)}\left[2C_g\int_{t+\rho}^t\left\|u_\theta-v_\theta\right\|_{C_{V_2}}^2\mathrm{d}s+C_g\int_T^{t+\theta}\int_{s+\rho}^t\left\|u_\theta-v_\theta\right\|_{C_{V_2}}^2\mathrm{d}r\mathrm{d}s\right].$$

Therefore

$$H_\theta(t)\le\frac{1}{t+\theta-T}C_M+\varphi_T^t\left(\left(u_\theta(T),\partial_t u_\theta(T)\right),\left(v_\theta(T),\partial_t v_\theta(T)\right)\right).\tag{4.9}$$

Next, we will use the contraction function method to prove the asymptotic compactness of the solution process for problems (2.3) - (2.5).

**Theorem 4.2** If the assumptions hold, for any fixed $t\in\mathbb{R}$ and any bounded $\{\tau_n\}_{n=1}^\infty\subset(-\infty,t]$ (as $n\to\infty$, $\tau_n\to-\infty$), and for any sequence $\{x_n\}_{n=1}^\infty\subset\mathcal{H}_{\tau_n}^2$, then the sequence $\left\{U(t,\tau_n)x_n\right\}_{n=1}^\infty$ has a convergent subsequence.

**Proof.** For any $\varepsilon>0$ and fixed $t$, there exists $T<t$ such that $\frac{C_M}{t+\theta-T}<\varepsilon$. Thanks to Theorem 2.10, we also need to show that $\Phi_T^t\in\mathfrak{C}\left(\mathbb{B}_T(R)\right)$, for every fixed $t$.

Let $\left(u_\theta^n,\partial_t u_\theta^n\right)$ be the solution of problem (2.3) - (2.5) with the initial value $\left(\phi_0^n,\phi_1^n\right)\in\mathbb{B}_T(R)$. According to Theorem 3.2, it can be known that $\left\|u_\theta^n\right\|_{C_{V_2}}^2+\varepsilon(\zeta_1)\left\|\partial_t u_\theta^n\right\|_{C_{L^2(\Omega)}}^2$ is bounded. For any fixed $t$ and any $\zeta_1\in[T,t]$, based on (1.6) and the boundedness, we can gain that $\left\|\partial_t u_\theta^n\right\|_{C_{L^2(\Omega)}}$ is also bounded.

According to Alaoglu Theorem, Lemma 2.5 and Theorem 3.2, for any $\tau\le T\le t$, without loss of generality (at most by passing subsequence), let

$$u_\theta^n\to u_\theta\ \text{ weakly* in } L^\infty\left([\tau+\rho,T];C_{V_2}\right),\tag{4.10}$$

$$\partial_t u_\theta^n\to\partial_t u_\theta\ \text{ weakly* in } L^\infty\left([\tau+\rho,T];C_{L^2(\Omega)}\right),\tag{4.11}$$

$$\partial_t^2 u_\theta^n\to\partial_t^2 u_\theta\ \text{ weakly* in } L^\infty\left([\tau+\rho,T];C_{V_{-2}}\right)\tag{4.12}$$

$$u_\theta^n\to u_\theta\ \text{ weakly in } L^2\left([\tau+\rho,T];C_{V_{3-\alpha}}\right),\tag{4.13}$$

$$\partial_t u_\theta^n\to\partial_t u_\theta\ \text{ weakly in } L^2\left([\tau+\rho,T];C_{V_\alpha}\right),\tag{4.14}$$

$$u_\theta^n \to u_\theta \text{ in } L^{p+1}\left(\left[\tau+\rho,T\right];C_{L^{p+1}(\Omega)}\right), \tag{4.15}$$

$$u_\theta^n \to u_\theta \text{ weakly in } L^2\left(\left[\tau+\rho,T\right];C_{V_2}\right), \tag{4.16}$$

$$u_\theta^n \to u_\theta \text{ in } L^{p+1}(\Omega) \text{ and } u_\theta^n(T) \to u_\theta(T) \text{ in } L^{p+1}(\Omega), \tag{4.17}$$

$$\partial_t u_\theta^n \to \partial_t u_\theta \text{ weakly in } L^2\left(\left[\tau+\rho,T\right];C_{L^2(\Omega)}\right). \tag{4.18}$$

where we have used the Sobolev embedding $V_2 \hookrightarrow\hookrightarrow L^{p+1}(\Omega)$.
According to (3.40)

$$\left(u_\theta(s),\partial_t u_\theta(s)\right) \subset C\left(\left[T,t\right];C_{\mathcal{H}_s^2}\right) \text{ is a Cauchy sequence} \tag{4.19}$$

and there exists $\left(u_\theta(s),\partial_t u_\theta(s)\right) \in C\left(\left[T,t\right];C_{\mathcal{H}_s^2}\right)$, such that

$$\left(u_\theta^n(s),\partial_t u_\theta^n(s)\right) \to \left(u_\theta(s),\partial_t u_\theta(s)\right) \text{ in } C\left(\left[T,t\right];C_{\mathcal{H}_s^2}\right). \tag{4.20}$$

Next, we analyse each term in the contractive remainder (4.8).

Firstly, estimate of $\Psi_1$, using (4.18), we obtain

$$\lim_{n\to\infty}\lim_{m\to\infty}\int_T^t \left\|\partial_t u_\theta^n - \partial_t u_\theta^m\right\|_{C_{L_2(\Omega)}}^2 \mathrm{d}s = 0,$$

$$\lim_{n\to\infty}\lim_{m\to\infty}\int_T^t L\left\|\partial_t u_\theta^n - \partial_t u_\theta^m\right\|_{C_{L_2(\Omega)}}^2 \mathrm{d}s = 0,$$

$$\begin{aligned}&\lim_{n\to\infty}\lim_{m\to\infty}\int_T^t \varepsilon_\theta(s)\left\|\partial_t z_\theta(s)\right\|_{C_{L_2(\Omega)}}^2 \mathrm{d}s\\ &\le \lim_{n\to\infty}\lim_{m\to\infty}\int_T^t L\left\|\partial_t u_\theta^n - \partial_t u_\theta^m\right\|_{C_{L_2(\Omega)}}^2 \mathrm{d}s = 0\end{aligned}$$

$$\begin{aligned}&\lim_{n\to\infty}\lim_{m\to\infty}\int_\Omega \varepsilon\left(\partial_t u^n - \partial_t u^m\right)\left(u^n - u^m\right)\mathrm{d}x\\ &\le \lim_{n\to\infty}\lim_{m\to\infty} L\left\|\partial_t u^n - \partial_t u^m\right\|\left\|u^n - u^m\right\|\\ &\le \lim_{n\to\infty}\lim_{m\to\infty} L\left\|\partial_t u^n - \partial_t u^m\right\|\left\|u^n - u^m\right\|_2 = 0\end{aligned}$$

$$\begin{aligned}&\lim_{n\to\infty}\lim_{m\to\infty}\int_T^t \varepsilon_\theta'\left\langle \partial_t u_\theta^n - \partial_t u_\theta^m, u_\theta^n - u_\theta^m\right\rangle \mathrm{d}s\\ &\le L\lim_{n\to\infty}\lim_{m\to\infty}\left(\int_T^t \left\|\partial_t u_\theta^n - \partial_t u_\theta^m\right\|^2 \mathrm{d}s\right)^{\frac{1}{2}} \cdot \left(\int_T^t \left\|u_\theta^n - u_\theta^m\right\|^2 \mathrm{d}s\right)^{\frac{1}{2}} = 0.\end{aligned} \tag{4.22}$$

For every fixed $t$, the integral $t$, $\left|\int_s^t\int_\Omega\left(\partial_t u_\theta^n - \partial_t u_\theta^m\right)\mathrm{d}x\mathrm{d}r\right|$ is bounded, hence the Lebesgue dominated convergence theorem gives

$$\begin{aligned}&\lim_{n\to\infty}\lim_{m\to\infty}\int_T^{t+\theta}\int_s^t\int_\Omega\left(\partial_t u_\theta^n - \partial_t u_\theta^m\right)\mathrm{d}x\mathrm{d}r\mathrm{d}s\\ &= \int_T^{t+\theta}\lim_{n\to\infty}\lim_{m\to\infty}\int_s^t\int_\Omega\left(\partial_t u_\theta^n - \partial_t u_\theta^m\right)\mathrm{d}x\mathrm{d}r\mathrm{d}s\\ &= \int_T^{t+\theta} 0\mathrm{d}s = 0.\end{aligned} \tag{4.23}$$

Combining (4.22) and (4.23), we obtain

$$\lim_{n\to\infty}\lim_{m\to\infty}\Psi_1 = 0. \tag{4.24}$$

Secondly, we estimate $\Psi_2$, by virtue of (4.16) and (4.17) we have

$$\begin{aligned}
&\lim_{n\to\infty}\lim_{m\to\infty}\int_{T+\rho}^{t}\int_{\Omega}\left(f\left(u_\theta^n\right)-f\left(u_\theta^m\right)\right)\left(u_\theta^n-u_\theta^m\right)\mathrm{d}x\mathrm{d}s\\
&\le C\lim_{n\to\infty}\lim_{m\to\infty}\int_{T+\rho}^{t}\int_{\Omega}\left(1+\left|u_\theta^n\right|^{p-1}+\left|u_\theta^m\right|^{p-1}\right)\left|u_\theta^n-u_\theta^m\right|^2\mathrm{d}x\mathrm{d}s\\
&\le C\lim_{n\to\infty}\lim_{m\to\infty}\int_{T+\rho}^{t}\left(1+\left\|u_\theta^n\right\|_{C_{V_2}}^{p-1}+\left\|u_\theta^m\right\|_{C_{V_2}}^{p-1}\right)\left\|u_\theta^n-u_\theta^m\right\|_{C_{V_2}}^2\mathrm{d}s\\
&\le C\lim_{n\to\infty}\lim_{m\to\infty}\int_{T+\rho}^{t}\left\|u_\theta^n-u_\theta^m\right\|_{C_{V_2}}^2\mathrm{d}s=0.
\end{aligned}\tag{4.25}$$

It is obvious

$$\begin{aligned}
&\int_{T+\rho}^{t}\int_{\Omega}\left(f\left(u_\theta^n\right)-f\left(u_\theta^m\right)\right)\left(\partial_t u_\theta^n-\partial_t u_\theta^m\right)\mathrm{d}x\mathrm{d}s\\
&=\int_{\Omega}F\left(u_\theta^n(t)\right)\mathrm{d}x-\int_{\Omega}F\left(u_\theta^n(T)\right)\mathrm{d}x+\int_{\Omega}F\left(u_\theta^m(t)\right)\mathrm{d}x-\int_{\Omega}F\left(u_\theta^m(T)\right)\mathrm{d}x\\
&-\int_{T+\rho}^{t}\int_{\Omega}f\left(u_\theta^m\right)\partial_t u_\theta^n\mathrm{d}x\mathrm{d}s-\int_{T+\rho}^{t}\int_{\Omega}f\left(u_\theta^n\right)\partial_t u_\theta^m\mathrm{d}x\mathrm{d}s.
\end{aligned}\tag{4.26}$$

By using (1.8) and embedding $V_2\hookrightarrow\hookrightarrow L^{p+1}(\Omega)$, we can obtain

$$\begin{aligned}
&\left|\int_{\Omega}\left(F\left(u_\theta^n(t)\right)-F\left(u_\theta(t)\right)\right)\mathrm{d}x\right|\\
&\le\int_{\Omega}\left|f\left(u_\theta(t)\right)+\vartheta\left(u_\theta^n(t)-u_\theta(t)\right)\right|\left|u_\theta^n(t)-u_\theta(t)\right|\mathrm{d}x\\
&\le C\left(\left\|u_\theta^n(t)\right\|_{C_{L^{p+1}(\Omega)}}^2+\left\|u_\theta(t)\right\|_{C_{L^{p+1}(\Omega)}}^2+\left\|u_\theta^n(t)\right\|_{C_{L^{p+1}(\Omega)}}^p\right.\\
&\left.+\left\|u_\theta(t)\right\|_{C_{L^{p+1}(\Omega)}}^p\right)\left\|u_\theta^n(t)-u_\theta(t)\right\|_{C_{L^{p+1}(\Omega)}}\\
&<C\epsilon.
\end{aligned}\tag{4.27}$$

Moreover

$$\begin{aligned}
\lim_{n\to\infty}\lim_{m\to\infty}\int_{T+\rho}^{t}\left\langle f\left(u_\theta^n\right),\partial_t u_\theta^m\right\rangle\mathrm{d}s&=\lim_{n\to\infty}\int_{T+\rho}^{t}\left\langle f\left(u_\theta^n\right),\partial_t u_\theta\right\rangle\mathrm{d}s\\
&=\int_{T+\rho}^{t}\left\langle f\left(u_\theta\right),\partial_t u_\theta\right\rangle\mathrm{d}s\\
&=\int_{\Omega}F\left(u_\theta(t)\right)\mathrm{d}x-\int_{\Omega}F\left(u_\theta(T)\right)\mathrm{d}x.
\end{aligned}$$

Analogously

$$\lim_{n\to\infty}\lim_{m\to\infty}\int_{T+\rho}^{t}\int_{\Omega}\left\langle f\left(u_\theta^m\right),\partial_t u_\theta^n\right\rangle\mathrm{d}s=\int_{\Omega}F\left(u_\theta(t)\right)\mathrm{d}x-\int_{\Omega}F\left(u_\theta(T)\right)\mathrm{d}x.$$

Consequently

$$\lim_{n\to\infty}\lim_{m\to\infty}\int_{T+\rho}^{t}\int_{\Omega}\left(f\left(u_\theta^n\right)-f\left(u_\theta^m\right)\right)\left(\partial_t u_\theta^n-\partial_t u_\theta^m\right)\mathrm{d}x\mathrm{d}s=0.\tag{4.28}$$

For every fixed $t$, the integral $t$, $\left|\int_{s+\rho}^{t}\int_{\Omega}\left(f\left(u_\theta^n\right)-f\left(u_\theta^m\right)\right)\left(\partial_t u_\theta^n-\partial_t u_\theta^m\right)\mathrm{d}x\mathrm{d}r\right|$ is bounded, hence the Lebesgue dominated convergence theorem gives

$$\begin{aligned}
&\lim_{n\to\infty}\lim_{m\to\infty}\int_{T}^{t+\theta}\int_{s+\rho}^{t}\int_{\Omega}\left(f\left(u_\theta^n\right)-f\left(u_\theta^m\right)\right)\left(\partial_t u_\theta^n-\partial_t u_\theta^m\right)\mathrm{d}x\mathrm{d}r\mathrm{d}s\\
&=\int_{T}^{t+\theta}\lim_{n\to\infty}\lim_{m\to\infty}\int_{s+\rho}^{t}\int_{\Omega}\left(f\left(u_\theta^n\right)-f\left(u_\theta^m\right)\right)\left(\partial_t u_\theta^n-\partial_t u_\theta^m\right)\mathrm{d}x\mathrm{d}r\mathrm{d}s\\
&=\int_{T}^{t+\theta}0\mathrm{d}s=0.
\end{aligned}\tag{4.29}$$

By virtue of (4.25), (4.28) and (4.29), we obtain

$$\lim_{n\to\infty}\lim_{m\to\infty}\Psi_2=0. \tag{4.30}$$

Finally, we estimate $\Psi_3$. Using (4.16) and (4.18), we have

$$\begin{aligned}
&\lim_{n\to\infty}\lim_{m\to\infty}\int_T^t\int_\Omega\left(g\left(x,u_\theta^n\right)-g\left(x,u_\theta^m\right)\right)\left(\partial_t u_\theta^n-\partial_t u_\theta^m\right)\mathrm{d}x\mathrm{d}s\\
&\le\lim_{n\to\infty}\lim_{m\to\infty}\left(C_g\int_{s+\rho}^t\left\|u_\theta^n-u_\theta^m\right\|_{C_{V_2}}^2\mathrm{d}s+\int_s^t\left\|\partial_t u_\theta^n-\partial_t u_\theta^m\right\|_{C_{L^2(\Omega)}}^2\mathrm{d}s\right)\\
&\le C_g\lim_{n\to\infty}\lim_{m\to\infty}\int_{s+\rho}^t\left\|u_\theta^n-u_\theta^m\right\|_{C_{V_2}}^2\mathrm{d}s+\lim_{n\to\infty}\lim_{m\to\infty}\int_s^t\left\|\partial_t u_\theta^n-\partial_t u_\theta^m\right\|_{C_{L^2(\Omega)}}^2\mathrm{d}s\\
&=0
\end{aligned} \tag{4.31}$$

and

$$\begin{aligned}
&\lim_{n\to\infty}\lim_{m\to\infty}\int_T^t\int_\Omega\left(g\left(x,u_\theta^n\right)-g\left(x,u_\theta^m\right)\right)\left(u_\theta^n-u_\theta^m\right)\mathrm{d}x\mathrm{d}s\\
&\le\lim_{n\to\infty}\lim_{m\to\infty}\left(C_g\int_{T+\rho}^t\left\|u_\theta-v_\theta\right\|_{C_{V_2}}^2\mathrm{d}s+\lambda_1^{-1}\int_T^t\left\|u_\theta^n-u_\theta^m\right\|_{C_{V_2}}^2\mathrm{d}s\right)\\
&\le C_g\lim_{n\to\infty}\lim_{m\to\infty}\int_{T+\rho}^t\left\|u_\theta-v_\theta\right\|_{C_{V_2}}^2\mathrm{d}s+\lambda_1^{-1}\lim_{n\to\infty}\lim_{m\to\infty}\int_T^t\left\|u_\theta^n-u_\theta^m\right\|_{C_{V_2}}^2\mathrm{d}s\\
&=0.
\end{aligned} \tag{4.32}$$

For every fixed $t$, $\left|\int_s^t\int_\Omega\left(g\left(x,u_\theta^n\right)-g\left(x,u_\theta^m\right)\right)\left(\partial_t u_\theta^n-\partial_t u_\theta^m\right)\mathrm{d}x\mathrm{d}r\right|$ is bounded, hence the Lebesgue dominated convergence theorem gives

$$\begin{aligned}
&\lim_{n\to\infty}\lim_{m\to\infty}\int_T^t\int_s^t\int_\Omega\left(g\left(x,u_\theta^n\right)-g\left(x,u_\theta^m\right)\right)\left(\partial_t u_\theta^n-\partial_t u_\theta^m\right)\mathrm{d}x\mathrm{d}r\mathrm{d}s\\
&=\int_T^t\lim_{n\to\infty}\lim_{m\to\infty}\int_s^t\int_\Omega\left(g\left(x,u_\theta^n\right)-g\left(x,u_\theta^m\right)\right)\left(\partial_t u_\theta^n-\partial_t u_\theta^m\right)\mathrm{d}x\mathrm{d}r\mathrm{d}s\\
&=\int_T^t 0\mathrm{d}s=0.
\end{aligned} \tag{4.33}$$

From (4.31), (4.32) and (4.33), we can obtain

$$\lim_{n\to\infty}\lim_{m\to\infty}\Psi_3=0. \tag{4.34}$$

Therefore, we can conclude that
$\varphi_T^t\left(\left(u_\tau(T),\partial_t u_\theta(T)\right),\left(v_\theta(T),\partial_t v_\theta(T)\right)\right)\in\mathfrak{C}\left(\mathbb{B}_T(R)\right)$.

**Theorem 4.3** Assuming that (1.5) - (1.8) hold and $g\in L^2(\Omega)$, then the process $U(t,\tau):C_{\mathcal{H}_t^2}\to C_{\mathcal{H}_t^2}$ generated by problem (2.3) - (2.5) has a time-dependent global attractor $\mathcal{A}=\{A_t\}_{t\in\mathbb{R}}$.

**Proof** It follows from Theorem 3.2, Theorem 4.1 and Theorem 4.2, that there exists a time-dependent global attractor $\mathcal{A}=\{A_t\}_{t\in\mathbb{R}}$.


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## Conflicts of Interest

The authors declare no conflict of interest.

## References

[1] Babin, A.V. and Vishik, M.I. (1992) Attractors of Evolutionary Equations. North Holland.

[2] Carvalho, A.N., Langa, J.A. and Robinson, J.C. (2013) Attractors for Infinite-Dimensional Non-Autonomous Dynamical Systems. Applied Mathematical Science.

[3] Chueshov, I. and Lasiecka, I. (2008) Long-Time Behavior of Second Order Evolution Equations with Nonlinear Damping. American Mathematical Society, Providence.

[4] Chueshov, I. (2015) Dynamics of Quasi-Stable Dissipative Systems. Springer.

[5] Avdonin, S., Edward, J. and Ivanov, S.A. (2025) Null-Controllability for the Beam Equation with Structural Damping. Part 1. Distributed Control. *Journal of Differential Equations*, **421**, 73-103. https://doi.org/10.1016/j.jde.2024.11.052

[6] Khaldi, S. (2024) A Wave and Beam Equations with Structural Damping of Negative Order. *Filomat*, **38**, 4853-4862. https://doi.org/10.2298/fil2414853k

[7] Lv, P., Cun, Y. and Lin, G. (2025) Long-Time Behavior of Coupled Beam Models with Fractional Energy Damping. *Boundary Value Problems*, **2025**, Article No. 55. https://doi.org/10.1186/s13661-025-02046-2

[8] Yang, Z.J., Ding, P.Y. and Li, L. (2016) Longtime Dynamics of the Kirchhoff Equations with Fractional Damping and Supercritical Nonlinearity. *Journal of Mathematical Analysis and Applications*, **442**, 485-510.

[9] Antonio Jorge da Silva, M. and Narciso, V. (2015) Attractors and Their Properties for a Class of Nonlocal Extensible Beams. *Discrete & Continuous Dynamical Systems-A*, **35**, 985-1008. https://doi.org/10.3934/dcds.2015.35.985

[10] Ding, P. and Yang, Z. (2021) Longtime Behavior for an Extensible Beam Equation with Rotational Inertia and Structural Nonlinear Damping. *Journal of Mathematical Analysis and Applications*, **496**, Article 124785. https://doi.org/10.1016/j.jmaa.2020.124785

[11] Xu, J., Zhang, Z. and Caraballo, T. (2021) Non-Autonomous Nonlocal Partial Differential Equations with Delay and Memory. *Journal of Differential Equations*, **270**, 505-546. https://doi.org/10.1016/j.jde.2020.07.037

[12] Li, J. and Huang, J. (2009) Uniform Attractors for Non-Autonomous Parabolic Equations with Delays. *Nonlinear Analysis: Theory, Methods & Applications*, **71**, 2194-2209. https://doi.org/10.1016/j.na.2009.01.053

[13] Harraga, H. and Yebdri, M. (2016) Pullback Attractors for a Class of Semilinear Nonclassical Diffusion Equations with Delay. *Electronic Journal of Differential Equations*, **2016**, 1072-6691

[14] García-Luengo, J. and Marín-Rubio, P. (2014) Reaction-Diffusion Equations with Non-Autonomous Force in H-1and Delays under Measurability Conditions on the Driving Delay Term. *Journal of Mathematical Analysis and Applications*, **417**, 80-95. https://doi.org/10.1016/j.jmaa.2014.03.026

[15] Yang, B., Qin, Y., Miranville, A. and Wang, K. (2025) Pullback Attractors for Nonclassical Diffusion Equations with a Delay Operator. *Studies in Applied Mathematics*, **154**, e70039. https://doi.org/10.1111/sapm.70039

[16] Qin, Y. and Yang, B. (2023) Existence and Regularity of Pullback Attractors for a Non-Autonomous Diffusion Equation with Delay and Nonlocal Diffusion in Time-Dependent Spaces. *Applied Mathematics & Optimization*, **88**, 1-25. https://doi.org/10.1007/s00245-023-09981-5

[17] Meng, F., Liu, C. and Zhang, C. (2021) Long-Time Dynamics of a Class of Nonlocal

Extensible Beams with Time Delay. *Mathematics and Mechanics of Solids*, **27**, 319-333. https://doi.org/10.1177/10812865211019457

[18] Woinowsky-Krieger, S. (1950) The Effect of an Axial Force on the Vibration of Hinged Bars. *Journal of Applied Mechanics*, **17**, 35-36. https://doi.org/10.1115/1.4010053

[19] Yang, Z. (2013) On an Extensible Beam Equation with Nonlinear Damping and Source Terms. *Journal of Differential Equations*, **254**, 3903-3927. https://doi.org/10.1016/j.jde.2013.02.008

[20] Zhao, C., Zhao, C. and Zhong, C. (2020) The Global Attractor for a Class of Extensible Beams with Nonlocal Weak Damping. *Discrete & Continuous Dynamical Systems-B*, **25**, 935-955. https://doi.org/10.3934/dcdsb.2019197

[21] Liu, Z., Yang, Z. and Guo, Y. (2024) Stability of Strong Attractors for the Extensible Beam Equation with Gentle Dissipation. *Journal of Mathematical Analysis and Applications*, **533**, Article 127999. https://doi.org/10.1016/j.jmaa.2023.127999

[22] Luo, X. and Ma, Q. (2022) The Existence of Time-Dependent Attractor for Wave Equation with Fractional Damping and Lower Regular Forcing Term. *Discrete and Continuous Dynamical Systems-B*, **27**, 4817-4835. https://doi.org/10.3934/dcdsb.2021253

[23] Conti, M. and Pata, V. (2014) Asymptotic Structure of the Attractor for Processes on Time-Dependent Spaces. *Nonlinear Analysis: Real World Applications*, **19**, 1-10. https://doi.org/10.1016/j.nonrwa.2014.02.002

[24] Meng, F., Yang, M. and Zhong, C. (2016) Attractors for Wave Equations with Nonlinear Damping on Time-Dependent Space. *Discrete and Continuous Dynamical Systems-Series B*, **21**, 205-225. https://doi.org/10.3934/dcdsb.2016.21.205

[25] Ma, Q., Wang, X. and Xu, L. (2016) Existence and Regularity of Time-Dependent Global Attractors for the Nonclassical Reaction-Diffusion Equations with Lower Forcing Term. *Boundary Value Problems*, **2016**, Article No. 10. https://doi.org/10.1186/s13661-015-0513-3

[26] Liu, T. and Ma, Q. (2019) Time-Dependent Attractor for Plate Equations on $R^n$. *Journal of Mathematical Analysis and Applications*, **479**, 315-332. https://doi.org/10.1016/j.jmaa.2019.06.028

[27] Meng, F., Wang, Y. and Zhao, C. (2021) Attractor for a Model of Extensible Beam with Damping on Time-Dependent Space. *Topological Methods in Nonlinear Analysis*, **57**, 365-393. https://doi.org/10.12775/tmna.2020.037

[28] Wang, X. and Wang, W. (2025) The Time-Dependent Attractor for Beam Equation with Rotational Inertia and Structural Damping. *AIMS Mathematics*, **10**, 15206-15230. https://doi.org/10.3934/math.2025682