Scientific Research Publishing

Journal of Applied Mathematics and Physics, 2025, 13(12), 4444-4469
https://www.scirp.org/journal/jamp
ISSN Online: 2327-4379
ISSN Print: 2327-4352

Normalized Solutions for the Kirchhoff-Schrödinger Systems with Weakly Attractive Potentials

Wenhao Tu, Li Wang*, Li Chen

College of Science, East China Jiaotong University, Nanchang, China
Email: 15717005896@163.com, *wangli.423@163.com, chenlimath0318@163.com

How to cite this paper: Tu, W.H., Wang, L. and Chen, L. (2025) Normalized Solutions for the Kirchhoff-Schrödinger Systems with Weakly Attractive Potentials. *Journal of Applied Mathematics and Physics*, **13**, 4444-4469.
https://doi.org/10.4236/jamp.2025.1312244

Received: November 27, 2025
Accepted: December 20, 2025
Published: December 23, 2025

Abstract

In this paper, we consider the following Kirchhoff-Schrödinger system with weakly attractive potentials:

$$\begin{cases} -\left(a+b\int_{\mathbb{R}^N}|\nabla u_1|^2\,\mathrm{d}x\right)\Delta u_1 + V_1u_1 = \lambda_1 u_1 + \nu_1|u_1|^{p_1-2}u_1 + \alpha q_1|u_1|^{q_1-2}u_1|u_2|^{q_2}, \\ -\left(a+b\int_{\mathbb{R}^N}|\nabla u_2|^2\,\mathrm{d}x\right)\Delta u_2 + V_2u_2 = \lambda_2 u_2 + \nu_2|u_2|^{p_2-2}u_2 + \alpha q_2|u_1|^{q_1}|u_2|^{q_2-2}u_2, \end{cases}$$

having prescribed mass $\int_{\mathbb{R}^N}|u_i|^2\,\mathrm{d}x = m_i$, where $a>0$, $b,\alpha,\nu_i>0$, $q_i>1$, $N=2,3$, $\lambda_i\in\mathbb{R}$ are Lagrange multiplier and $V_i\in C^1(\mathbb{R}^N)$ are potential functions for $i=1,2$. When $2+\frac{8}{N}<q_1+q_2<2^*$ and $(p_1,p_2)\in\mathbb{R}^2$, we prove the existence of multiple solutions, which are positive radial vectors. $2^*=2N/(N-2)$ is the Sobolev critical exponent. The proof is based on variational techniques and constrained minimization arguments.

Keywords

Schrödinger Systems, Normalized Solutions, Weakly Attractive Potentials, Variational Method, Pohožaev Manifold

1. Introduction and Main Results

The focus of this paper is on the nonlocal Schrödinger system with weakly attractive potentials in $H^1(\mathbb{R}^N)\times H^1(\mathbb{R}^N)$:

$$\begin{cases} -\left(a+b\int_{\mathbb{R}^N}|\nabla u_1|^2\,\mathrm{d}x\right)\Delta u_1 + V_1u_1 = \lambda_1 u_1 + \nu_1|u_1|^{p_1-2}u_1 + \alpha q_1|u_1|^{q_1-2}u_1|u_2|^{q_2}, \\ -\left(a+b\int_{\mathbb{R}^N}|\nabla u_2|^2\,\mathrm{d}x\right)\Delta u_2 + V_2u_2 = \lambda_2 u_2 + \nu_2|u_2|^{p_2-2}u_2 + \alpha q_2|u_1|^{q_1}|u_2|^{q_2-2}u_2, \end{cases} \tag{1.1}$$

with the normalization constraint

$$\int_{\mathbb{R}^N} |u_1|^2 \,\mathrm{d}x = m_1 > 0 \text{ and } \int_{\mathbb{R}^N} |u_2|^2 \,\mathrm{d}x = m_2 > 0, \tag{1.2}$$

where $a,b,\alpha,\nu_i > 0$ for $i = 1,2$. Assume $q_1, q_2 > 1$ and $2 < p_1, p_2, q_1 + q_2 < 2^*$. $\lambda_1, \lambda_2 \in \mathbb{R}$ are Lagrange multipliers. $V_1, V_2 \in C^1(\mathbb{R}^N)$ are potential functions satisfying

(V_1) $\lim_{|x|\to\infty} V_i(x) = \sup_{x\in\mathbb{R}^N} V_i(x) = 0 \ge V_i(x)$ and there exists $\omega_i \in [0,1/2)$ such that $|V_i|_{N/2} \le \omega_i S$, where

$$S = \inf_{u\in D^{1,2}(\mathbb{R}^N)\setminus\{0\}} \frac{\int_{\mathbb{R}^N} |\nabla u|^2 \,\mathrm{d}x}{\left(\int_{\mathbb{R}^N} |u|^{2^*} \,\mathrm{d}x\right)^{2/2^*}};$$

(V_2) set $W_i(x) := (\nabla V_i(x)\cdot x)/2, W_i \in C^1(\mathbb{R}^N), \lim_{|x|\to\infty} W_i(x) = 0$ and there exists $\rho_i \in [0,1)$ such that $|W_i|_{N/2} \le \rho_i S$.

(V_3) set $Y_i(x) := \gamma_{q_i} q_i W_i(x) + Z_i(x)$, where $Z_i(x) := \nabla W_i(x)\cdot x$ and $Z_i \in L^s(\mathbb{R}^N)$ for some $s \in [N/2, \infty]$, there exists σ_i such that $|Y_{i,+}|_{N/2} \le \sigma_i S$, where $Y_{i,+} = \max\{Y_i, 0\}$.

An example satisfying the conditions (V_1)-(V_3) is $V_i(x) = -\dfrac{c}{|x|^d + 1}, x \in \mathbb{R}^N$ with constant $d > 2$ and suitable small constant c. Obviously, $V = 0$ also satisfies the conditions (V_1) (V_3).

For the study of Kirchhoff, Li, Luo and Yang [1] studied a Kirchhoff equation with a combined nonlinearity given by:

$$\begin{cases} -\left(a + b\int_{\mathbb{R}^N} |\nabla u|^2 \,\mathrm{d}x\right)\Delta u + \lambda u = |u|^{p-2} u + \nu |u|^{q-2} u, \ x \in \mathbb{R}^N, \\ \int_{\mathbb{R}^N} |u|^2 \,\mathrm{d}x = c, \end{cases} \tag{1.3}$$

where $a,b,c,\nu > 0$. They demonstrated the existence of multiple solutions when $2 < q < \frac{10}{3}$ and $\frac{14}{3} < p < 6$, as well as ground states for the cases $2 < q < \frac{10}{3} < p = 6$ and $\frac{14}{3} < q < p \le 6$. Additionally, their work provided insights into the asymptotic behavior of the obtained solutions. In contrast, when $\nu \le 0$, Carrião, Miyagaki and Vicente [2] investigated the scenario. They established the existence of ground states for the equation when $2 < q < 2^* = p$ or $2 < q \le \bar{p} = \frac{14}{3} < p < 2^*$, offering complementary results to those of Li *et al.*

Ye [3] studied the nonautonomous Kirchhoff-type problem:

$$\begin{cases} -\left(a + b\int_{\mathbb{R}^3} |\nabla u|^2 \,\mathrm{d}x\right)\Delta u + \lambda u = |u|^{p-2} u + V(x)|u|^{q-2} u, \ x \in \mathbb{R}^N, \\ \int_{\mathbb{R}^N} |u|^2 \,\mathrm{d}x = c, \end{cases} \tag{1.4}$$

where $p=\frac{14}{3}, q=4$, and $V\in L^{\infty}_{\mathrm{loc}}\left(\mathbb{R}^3\right)$ satisfies

$$V(x)\geq 0,\ \lim_{|x|\to\infty}V(x)=0.$$

Using the concentration compactness principle, Ye demonstrated the existence of a minimizer under certain conditions. Specifically, the author established that there exist constants $a_0,b_0,c_0>0$ such that for $a<a_0$, $b<b_0$, and $c<c_0$, the problem (1.4) admits a minimizer. This result highlights the role of the compactness argument in proving the existence of solutions in the presence of nonautonomous potentials.

Over the past two decades, systems, which similar to (1.1), have garnered significant attention due to their important physical background. When $b>0$, the $\left(b\int_{\mathbb{R}^N}|\nabla u|^2\,\mathrm{d}x\right)\Delta u$, leading to its classification as a Kirchhoff-Schrödinger system, is nonlocal term of system (1.1). This nonlocal nature is associated with the following equation

$$\tau\frac{\partial^2 u}{\partial t^2}-\left(\frac{P_0}{h}+\frac{E}{2L}\int_0^L\left|\frac{\partial u}{\partial x}\right|^2\mathrm{d}x\right)\frac{\partial^2 u}{\partial x^2}=0$$

which was proposed by Kirchhoff in 1883 [4]. This equation not only serves as an extension of the classical D'Alembert's wave equation but also relevant in biological contexts. Over the last few decades, Kirchhoff-type problems have drawn considerable attention, beginning with Lions' foundational work [5], which established abstract framework for such problems. Since then, numerous important results have been developed, evidenced by works in [6]-[10]. For a more detailed discussion on the physical implications of systems such as (1.1), as well as additional physical and mathematical interpretations, we refer the reader to [11]-[13], along with the references cited in these works.

Building on this foundation, there has been growing interest in exploring the existence of normalized solutions. For example, the existence and multiplicity of normalized solutions for Schrödinger systems have garnered significant attention in recent years, as explored in studies such as [14]-[16] and related references. A notable example is the system

$$\begin{cases}-\Delta u+\lambda_1 u=\nu_1|u|^{p-2}u+\mu_1|u|^{p_1-2}u+\alpha q_1|u|^{q_1-2}u|v|^{q_2} & \text{in } \mathbb{R}^N,\\ -\Delta v+\lambda_2 v=\nu_2|v|^{p-2}v+\mu_2|v|^{p_2-2}v+\alpha q_2|u|^{q_1}|v|^{q_2-2}v & \text{in } \mathbb{R}^N,\end{cases}\tag{1.5}$$

which has been analyzed under various parameter settings. For $N\geq 3$, when $\mu_1=\mu_2=0$, $p=4$, and $q_1=q_2=2$, the existence and multiplicity of normalized solutions were established in [17]-[19]. For $N=3,4$, when $\nu_1=\nu_2=0$, $q_1,q_2>1$, $p_1,q_1+q_2\in\left(2,2^*\right)$ and $p_2\in\left(2,2^*\right]$, Li and Zou [20] investigated the associated Pohožaev manifold and proved the existence of a normalized solution. When $N=4$, $p=3$, $p_1,p_2\in(2,4)$ and $q_1=q_2=2$, Luo *et al.* [21] analyzed the existence, nonexistence, and asymptotic properties of normalized solutions, particularly in the Sobolev critical case. More recently, Liu and Fang [15] studied the system for $N = 3,4$, $q_1, q_2 > 1$ and $p_1, p_2, q_1 + q_2 \in \left(2 + 4/N, 2^*\right]$, proving both the existence and nonexistence of normalized solutions. These studies illustrate the breadth of research on Schrödinger systems, addressing various nonlinearities, couplings, dimensions, and parameter regimes.

Furthermore, Hu and Mao [22] considered the following Kirchhoff-Schrödinger system

$$\begin{cases} -\left(a + b\int_{\mathbb{R}^N} \left|\nabla v_1\right|^2 \mathrm{d}x\right)\Delta v_1 = \lambda_1 v_1 + \mu_1 \left|v_1\right|^{p_1-2} v_1 + \alpha q_1 \left|v_1\right|^{q_1-2} v_1 \left|u_2\right|^{q_2}, \\ -\left(a + b\int_{\mathbb{R}^N} \left|\nabla v_2\right|^2 \mathrm{d}x\right)\Delta v_2 = \lambda_2 v_2 + \mu_2 \left|u_2\right|^{p_2-2} v_2 + \alpha q_2 \left|v_1\right|^{q_1} \left|v_2\right|^{q_2-2} v_2, \end{cases} \tag{1.6}$$

having prescribed mass $\int_{\mathbb{R}^N} \left|v_i\right|^2 \mathrm{d}x = c_i > 0$, where $q_i > 1$, $a, b, \alpha, \mu_i > 0$ for $i = 1,2$, $2 < p_1, p_2, q_1 + q_2 < 2^*$ and $N = 2,3$. They established multiple positive radial vector solutions using variational methods combined with a constrained minimization approach.

Building on the results discussed earlier, a natural question emerges: does system (1.1) admit multiple normalized solutions? In this paper, we confirm this with a positive answer. Notably, when $V \neq 0$, the existing literature offers limited insights into system (1.1). Our objective is to address this gap by extending the findings of [22] to the Kirchhoff-Schrödinger system with potentials, thereby broadening the understanding of such systems.

Recently, growing attention has been directed toward the Kirchhoff-Schrödinger system, which is exemplified by (1.1). Similar to the case of the equation, where λ is considered an unknown Lagrange multiplier, the problem (1.1) can be viewed as a characteristic value issue. Within this framework, solving (1.1)-(1.2) involves analyzing constrained variational problems. Normalized solutions are then obtained by identifying the critical points of the energy functional $I : H^1\left(\mathbb{R}^N\right) \times H^1\left(\mathbb{R}^N\right) \to \mathbb{R}$, defined by

$$\begin{aligned} I\left(u_1, u_2\right) = {} & \frac{a}{2}\sum_{i=1}^{2}\left|\nabla u_i\right|_2^2 + \frac{b}{4}\sum_{i=1}^{2}\left|\nabla u_i\right|_2^4 + \frac{1}{2}\sum_{i=1}^{2}\int_{\mathbb{R}^N} V_i u_i^2 \mathrm{d}x \\ & - \sum_{i=1}^{2}\frac{v_i}{p_i}\int_{\mathbb{R}^N}\left|u_i\right|^{p_i} \mathrm{d}x - \alpha\int_{\mathbb{R}^N}\left|u_1\right|^{q_1}\left|u_2\right|^{q_2} \mathrm{d}x \end{aligned} \tag{1.7}$$

on $S\left(m_1\right) \times S\left(m_2\right)$, where $S(m) := \left\{u \in H^1\left(\mathbb{R}^N\right) : |u|_2^2 = m\right\}$ for all positive constant m.

To address compactness issues, we restrict our analysis to a radial framework. Specifically, we seek critical points of the functional I restricted on $S_r\left(m_1\right) \times S_r\left(m_2\right)$, where

$$S_r(m) := \left\{u \in H^1_{\mathrm{rad}}\left(\mathbb{R}^N\right) : \int_{\mathbb{R}^N} u^2 = m, \int_{\mathbb{R}^N} V u^2 < +\infty\right\}$$

and $H^1_{\mathrm{rad}}\left(\mathbb{R}^N\right)$ represents the space of radial H^1-functions on $\mathbb{R}^N$. The radial restriction allows us to invoke compact embedding theorems, which are essential for the variational arguments. By invoking the principle of symmetric criticality, we ensure that critical points of I constrained to $S_r\left(m_1\right) \times S_r\left(m_2\right)$ are also critical points of I when constrained on the broader set $S(m_1)\times S(m_2)$. Set

$$\gamma_p := \frac{N(p-2)}{2p}.$$

Clearly, critical points of $I|_{S_r(m_1)\times S_r(m_2)}$ lie within the Pohožaev manifold

$$\mathcal{M} := \left\{(u_1,u_2)\in S_r(m_1)\times S_r(m_2) : J(u_1,u_2)=0\right\},$$

where

$$J(u_1,u_2) := a\sum_{i=1}^{2}|\nabla u_i|_2^2 + b\sum_{i=1}^{2}|\nabla u_i|_2^4 - \frac{1}{2}\sum_{i=1}^{2}\int_{\mathbb{R}^N} W_i u_i^2 \mathrm{d}x - \sum_{i=1}^{2}\nu_i\gamma_{p_i}|u_i|_{p_i}^{p_i} \\ -\alpha\gamma_{q_1+q_2}(q_1+q_2)\int_{\mathbb{R}^N}|u_1|^{q_1}|u_2|^{q_2}\,\mathrm{d}x,$$

which is the Pohožaev identity.

To present our results, we define the set $\mathcal{B}(\tau) := \left\{(u_1,u_2)\in S_r(m_1)\times S_r(m_2) : |\nabla u_1|_2^2 + |\nabla u_2|_2^2 < \tau\right\}$ for all $\tau > 0$ and assume the following condition:

(M$_1$) $2 < p_1, p_2 < 2+\frac{4}{N}$, $2+\frac{8}{N} < q_1+q_2 < 2^*$.

The mathematical role of these constraints, such as ensuring the nonlinearities are well-behaved and avoiding compactness loss.

Now, the main result can be stated as follows.

Theorem 1.1. *Given that condition* (M$_1$) *is satisfied. Subsequently, there exist three positive constants*

$$a^* := \max\left\{\left(\frac{2\max\{\rho_1,\rho_2\}+\max\{\sigma_1,\sigma_2\}}{2-\max\{\gamma_{p_1}p_1,\gamma_{p_2}p_2\}}\right)+\max\{\rho_1,\rho_2\},\right. \\ \left.\frac{\max\{\rho_1,\rho_2\}\left(\sum_{i=1}^{2}\nu_i+\alpha q\right)}{\sum_{i=1}^{2}\nu_i\left(1-\gamma_{p_i}\right)},\max\{\omega_1,\omega_2\}\right\},\quad \tau_0=\tau_0(m_1,m_2)>0,$$

$\alpha_0=\alpha_0(m_1,m_2)$ *ensuring that, for* $\alpha\in(0,\alpha_0)$ *and* $a>a^*$,

(i) *System* (1.1)-(1.2) *admits a solution* (u_1,u_2), *which is positive radial vector, for some* $\lambda_1,\lambda_2<0$. *Additionally,* $I(u_1,u_2)>0$;

(ii) *System* (1.1)-(1.2) *admits a solution* (v_1,v_2), *which is positive radial vector, for some* $\lambda_1,\lambda_2<0$. *Additionally,* $(v_1,v_2)\in\mathcal{B}(\tau_0)$ *and* $I(v_1,v_2)<0$.

We now proceed to present the proof of Theorem 1.1. To prove Theorem1.1 (i), we begin by demonstrating that under the condition (M$_1$), the following inequality holds:

$$\inf_{\mathcal{B}(\tau)} I(u_1,u_2) < 0 \text{ for any } \tau>0. \tag{1.8}$$

Additionally, we show the existence of constants $\tau_0=\tau_0(m_1,m_2)>0$, $\alpha_0=\alpha_0(m_1,m_2)>0$ such that for any $0<\alpha\le\alpha_0$, we have

$$\inf_{\partial\mathcal{B}(\tau_0)} I(u_1,u_2) > 0. \tag{1.9}$$

With these results in hand, a min-max structure of the mountain pass type can be introduced. Specifically, there exists $\overline{\tau} \in (0, \tau_0)$, ensuing that for all $0 < \alpha \le \alpha_0$, the desired conclusions follow that

$$\gamma(m_1, m_2) := \inf_{\eta \in \Gamma} \max_{t \in [0,1]} E(\eta(t)) > \max\{E(\eta(0)), E(\eta(1))\}, \tag{1.10}$$

where

$$\Gamma := \left\{\eta \in (S_r(m_1) \times S_r(m_2), C[0,1]) : \eta(0) \in \mathcal{B}(\overline{\tau}), \eta(1) \notin \overline{\mathcal{B}(\tau_0)}, E(\eta(1) < 0)\right\}.$$

This approach allows us to search for a mountain pass solution. There are two primary challenges in the proof. One is to show the boundedness of $(PS)_{\gamma(m_1,m_2)}$ sequence $\{(u_1^n, u_2^n)\}$ in $S_r(m_1) \times S_r(m_2)$. By employing the method described in [23], we get a $(PS)_{\gamma(m_1,m_2)}$ sequence, along with the property that $J(u_1^n, u_2^n) \to 0$. Utilizing this property, we can show that the functional is coercive, leading to the (u_1^n, u_2^n) converges weakly to (u_1, u_2) in $H^1_{\text{rad}}(\mathbb{R}^N) \times H^1_{\text{rad}}(\mathbb{R}^N)$. The other is proving the compactness of $(PS)_{\gamma(m_1,m_2)}$ sequence. Establishing $(u_1, u_2) \in S_r(m_1) \times S_r(m_2)$ is the crucial step. To address this requirement, we must excluding both the semi-trivial solutions and the trivial solutions of the problem (1.1), which introduces a complication not present in the case of Kirchhoff equation. In order to tackle this challenge, we employ the technique outlined in [[24], Lemma A.2] and integrating the uniqueness of positive solutions to Equation (2.1) with energy estimations.

In order to prove Theorem (1.1) (ii), we naturally introduce the following minimization problem for all $0 < \alpha \le \alpha_0$, derived from Equation (1.8) and Equation (1.9),

$$c(m_1, m_2) := \inf_{\mathcal{B}(\tau_0)} I(u_1, u_2) < 0. \tag{1.11}$$

Moreover, we define

$$\begin{aligned} \Upsilon_{(u_1,u_2)}(\theta) &:= I(\theta * u_1, \theta * u_2) \\ &= \frac{a}{2} \mathrm{e}^{2\theta} \sum_{i=1}^{2} |\nabla u_i|_2^2 + \frac{b}{4} \mathrm{e}^{4\theta} \sum_{i=1}^{2} |\nabla u_i|_2^4 + \frac{1}{2} \sum_{i=1}^{2} \int_{\mathbb{R}^N} V_i(\mathrm{e}^{-\theta} x) u_i^2 \mathrm{d}x \\ &\quad - \sum_{i=1}^{2} \frac{\nu_i}{p_i} \mathrm{e}^{\gamma_{p_i} p_i \theta} |u_i|_{p_i}^{p_i} - \alpha \mathrm{e}^{\gamma_q q \theta} \int_{\mathbb{R}^N} |u_1|^{q_1} |u_2|^{q_2} \, \mathrm{d}x, \end{aligned} \tag{1.12}$$

where $\theta * u_i := \mathrm{e}^{\frac{N}{2}\theta} u_i(\mathrm{e}^{\theta} x)$ for $i = 1, 2$, $q = q_1 + q_2$. It follows from $(\Upsilon_{(u_1,u_2)})'(0) = J(u_1, u_2)$ that we divide $\mathcal{M}$ into

$$\mathcal{M}_+ := \left\{(u_1, u_2) \in \mathcal{M} : \left(\Upsilon_{(u_1,u_2)}\right)(0) > 0\right\},$$

$$\mathcal{M}_0 := \left\{(u_1, u_2) \in \mathcal{M} : \left(\Upsilon_{(u_1,u_2)}\right)(0) = 0\right\},$$

$$\mathcal{M}_- := \left\{(u_1, u_2) \in \mathcal{M} : \left(\Upsilon_{(u_1,u_2)}\right)(0) < 0\right\},$$

where

$$\left(\Upsilon_{(u_1,u_2)}\right)(0)=2a\sum_{i=1}^{2}\left|\nabla u_i\right|_2^2+4b\sum_{i=1}^{2}\left|\nabla u_i\right|_2^4+\frac{1}{2}\sum_{i=1}^{2}\int_{\mathbb{R}^N}Z_iu_i^2\mathrm{d}x$$
$$-\sum_{i=1}^{2}\nu_i\gamma_{p_i}^2p_i\left|u_i\right|_{p_i}^{p_i}-\alpha\left(\gamma_q q\right)^2\int_{\mathbb{R}^N}\left|u_1\right|^{q_1}\left|u_2\right|^{q_2}\mathrm{d}x.$$

In this paper, we define the L^p-norm as $\left|u\right|_p:=\left(\int_{\mathbb{R}^N}\left|u\right|^p\mathrm{d}x\right)^{\frac{1}{p}}$ and the H^1-norm as $\left\|u\right\|:=\left(\int_{\mathbb{R}^N}\left|\nabla u\right|^2\mathrm{d}x+\int_{\mathbb{R}^N}\left|u\right|^2\mathrm{d}x\right)^{\frac{1}{2}}$. The symbols $\rightharpoonup$ and $\rightarrow$ represent weak and strong convergence in the corresponding function spaces, respectively. The notation $:=$ and $=:$ is used to indicate definitions, and $C,C_1,C_2,K_1,K_2,\cdots$ denote positive constants.

The structure of the paper is as follows: Section 2 introduces preliminary results. In Section 3, we prove the existence of mountain pass solutions, specifically addressing Theorem 1.1 (i). Section 4 explores the connection between the functional's structure and the Pohožaev manifold, concluding the proof of Theorem 1.1 (ii).

2. Preliminarie Results

In this segment, we introduce foundational outcomes that are slated for recurrent application in the subsequent sections of the manuscript. Initially, we summarize some key inequalities.

Lemma 2.1. ([25] *Gagliardo-Nirenberg inequality*): *For any* $v\in H^1\left(\mathbb{R}^N\right)$, *there exists a constant* $C_{N,p}>0$ *such that*

$$\left|v\right|_p\le C_{N,p}\left|\nabla v\right|_2^{\gamma_p}\left|v\right|_2^{1-\gamma_p},$$

where $\gamma_p=\dfrac{N\left(p-2\right)}{2p}$.

By Lemma 2.1, we get

$$\begin{aligned}\int_{\mathbb{R}^N}\left|u_1\right|^{q_1}\left|u_2\right|^{q_2}\mathrm{d}x&\le\left|u_1\right|_r^{q_1}\left|u_2\right|_r^{q_2}\\&\le C_{N,r}m_1^{\frac{\left(1-\gamma_q\right)q_1}{2}}m_2^{\frac{\left(1-\gamma_q\right)q_2}{2}}\left|\nabla u_1\right|_2^{\gamma_q q_1}\left|\nabla u_2\right|_2^{\gamma_q q_2}\\&\le C\left(\left|\nabla u_1\right|_2^2+\left|\nabla u_2\right|_2^2\right)^{\frac{\gamma_q q}{2}},\end{aligned}$$

where $q:=q_1+q_2$.

We shall require certain findings about the Kirchhoff equation in the following proof:

$$\begin{cases}-\left(a+b\int_{\mathbb{R}^N}\left|\nabla u\right|^2\mathrm{d}x\right)\Delta u+Vu=\lambda u+\nu\left|u\right|^{p-2}u & \text{in }\mathbb{R}^N,\\ \int_{\mathbb{R}^N}\left|u\right|^2\mathrm{d}x=m & \text{in }\mathbb{R}^N,\end{cases}\tag{2.1}$$

where $N=2,3$ and $a,b>0,2<p<2^*$. We define the energy functional of Equation (2.1) as

$$E_\nu(u)=\frac{a}{2}|\nabla u|_2^2+\frac{b}{4}|\nabla u|_2^4+\frac{1}{2}\int_{\mathbb{R}^N}Vu^2\mathrm{d}x-\frac{\nu}{p}|u|_p^p.$$

The corresponding minimization energy is

$$l(m,\nu):=\inf_{u\in S(m)}E_\nu(u),\tag{2.2}$$

Lemma 2.2. [26]-[28] *If* $2<p<2+\frac{4}{N}$ *and* $m,\nu>0$. *Subsequently, Equation* (2.1) *possesses an exclusive positive radial solution* u_0 (*up to translations*) *for certain values of* $\lambda_0<0$, *and* $E_\nu(u_0)=l(m,\nu)<0$.

Lemma 2.3. *Assume* V_i, W_i *are defined in* (V₁), (V₂) *and* (V₃). *If* $\{(u_1^n,u_2^n)\}$ *is bounded in* $H_{\text{rad}}^1(\mathbb{R}^N)\times H_{\text{rad}}^1(\mathbb{R}^N)$, *then we have*

(i) $\int_{\mathbb{R}^N}V_i(u_i^n)^2\mathrm{d}x\to\int_{\mathbb{R}^N}V_iu_i^2\mathrm{d}x$, $(i=1,2)$,

(ii) $\int_{\mathbb{R}^N}W_i(u_i^n)^2\mathrm{d}x\to\int_{\mathbb{R}^N}W_iu_i^2\mathrm{d}x$, $(i=1,2)$.

Proof. Since $\{(u_1^n,u_2^n)\}$ is bounded in $H_{\text{rad}}^1(\mathbb{R}^N)\times H_{\text{rad}}^1(\mathbb{R}^N)$, we may assume that

$$(u_1^n,u_2^n)\rightharpoonup(u_1,u_2)\text{ in }H_{\text{rad}}^1(\mathbb{R}^N)\times H_{\text{rad}}^1(\mathbb{R}^N),$$
$$(u_1^n,u_2^n)\to(u_1,u_2)\text{ in }L^s(\mathbb{R}^N)\times L_{\text{loc}}^s(\mathbb{R}^N)\text{ for }s\in[1,2^*).$$

By $\lim_{|x|\to\infty}V_i(x)=0$, there exists two constant $M>0$ and $R>0$ such that $|V_i|<M$, $\int_{B_R}\left[(u_i^n)^2-(u_i)^2\right]\mathrm{d}x<\frac{\varepsilon}{M}$ and $|V_i|<\frac{\varepsilon}{M}$ where $|x|>R$, then

$$\begin{aligned}&\int_{\mathbb{R}^N}V_i\left[(u_i^n)^2-(u_i)^2\right]\mathrm{d}x\\&=\int_{B_R}V_i\left[(u_i^n)^2-(u_i)^2\right]\mathrm{d}x+\int_{B_R^c}V_i\left[(u_i^n)^2-(u_i)^2\right]\mathrm{d}x\\&\le M\int_{B_R}\left[(u_i^n)^2-(u_i)^2\right]\mathrm{d}x+\frac{\varepsilon}{M}\int_{B_R^c}\left[(u_i^n)^2-(u_i)^2\right]\mathrm{d}x\\&\le 2\varepsilon\to 0,\end{aligned}$$

where n is large enough. Similarly, the result of $\int_{\mathbb{R}^N}W_i(u_i^n)^2\mathrm{d}x\to\int_{\mathbb{R}^N}W_iu_i^2\mathrm{d}x$ can be obtained by the above proof.

Lemma 2.4. [[29], *Lemma* 2.4] *Assume that* $q_1,q_2>1, 2<q_1+q_2<2^*$. *If* $(u_1^n,u_2^n)\rightharpoonup(u_1,u_2)$ *in* $H^1(\mathbb{R}^N)\times H^1(\mathbb{R}^N)$,

then up to a subsequence

$$\int_{\mathbb{R}^N}|u_1|^{q_1}|u_2|^{q_2}\mathrm{d}x+\int_{\mathbb{R}^N}|u_1^n-u_1|^{q_1}|u_2^n-u_2|^{q_2}\mathrm{d}x=\int_{\mathbb{R}^N}|u_1^n|^{q_1}|u_2^n|^{q_2}\mathrm{d}x+o(1).$$

Lemma 2.5. *Suppose that the condition* (M₁) *is satisfied and* $\tau>0$, *then* $\inf_{\mathcal{B}(\tau)}I(u_1,u_2)<0$.

Proof. Set $u^t(x)=t^{\frac{N}{2}}u(tx)$, through simple calculations, we obtain

$$\int_{\mathbb{R}^N}(u^t)^2\mathrm{d}x=\int_{\mathbb{R}^N}u^2\mathrm{d}x,\int_{\mathbb{R}^N}(\nabla u^t)^2\mathrm{d}x=t^2\int_{\mathbb{R}^N}(\nabla u)^2\mathrm{d}x,$$
$$\int_{\mathbb{R}^N}(u^t)^p\mathrm{d}x=t^{\gamma_pp}\int_{\mathbb{R}^N}u^p\mathrm{d}x,$$

where $p \in \left(2, 2^*\right)$. Assume that t is sufficiently small, then $\left(u_1^t, u_2^t\right) \in \mathcal{B}(\tau)$, where $\left(u_1, u_2\right) \in S_r\left(m_1\right) \times S_r\left(m_2\right)$. Since

$$\begin{aligned} I\left(u_1^t, u_2^t\right) = &\frac{a}{2} t^2 \sum_{i=1}^{2} \left|\nabla u_i\right|_2^2 + \frac{b}{4} t^4 \sum_{i=1}^{2} \left|\nabla u_i\right|_2^4 + \frac{1}{2} \sum_{i=1}^{2} \int_{\mathbb{R}^N} V_i\left(t^{-1} x\right) u_i^2 \mathrm{d}x \\ &- \sum_{i=1}^{2} \frac{\nu_i}{p_i} t^{\gamma_{p_i} p_i} \left|u_i\right|_{p_i}^{p_i} - \alpha t^{\gamma_q q} \int_{\mathbb{R}^N} \left|u_1\right|^{q_1} \left|u_2\right|^{q_2} \mathrm{d}x, \end{aligned}$$

it can be readily verified that $I\left(u_1^t, u_2^t\right) < 0$ when t is sufficiently small if (M$_1$) holds.

Lemma 2.6. *Suppose the condition* (M$_1$) *is satisfied, then there exist* $\tau_0 = \tau_0\left(m_1, m_2\right) > 0$, $\alpha_0 = \alpha_0\left(m_1, m_2\right) > 0$ *such that for any* $0 < \alpha \le \alpha_0$,

$$\inf_{\mathcal{B}(2\tau_0) \setminus \mathcal{B}(\tau_0)} I\left(u_1, u_2\right) > 0.$$

Furthermore, there exists $\varepsilon_0 > 0$, ensuring that

$$c\left(m_1, m_2\right) < \inf_{\mathcal{B}(\tau_0) \setminus \mathcal{B}(\tau_0 - \varepsilon_0)} I\left(u_1, u_2\right).$$

Proof. Set $\tau = \left|\nabla u_1\right|_2^2 + \left|\nabla u_2\right|_2^2$, thus, for all $\left(u_1, u_2\right) \in S_r\left(m_1\right) \times S_r\left(m_2\right)$, we have

$$\begin{aligned} I\left(u_1, u_2\right) &= \frac{a}{2} \tau + \frac{b}{4} \sum_{i=1}^{2} \left|\nabla u_i\right|_2^4 + \frac{1}{2} \sum_{i=1}^{2} \int_{\mathbb{R}^N} V_i u_i^2 \mathrm{d}x - \sum_{i=1}^{2} \frac{\nu_i}{p_i} \left|u_i\right|_{p_i}^{p_i} - \alpha \int_{\mathbb{R}^N} \left|u_1\right|^{q_1} \left|u_2\right|^{q_2} \mathrm{d}x \\ &\ge \frac{a - \max\left(\omega_1, \omega_2\right)}{2} \tau + \frac{b}{8} \tau^2 - \sum_{i=1}^{2} K_i \left|\nabla u_i\right|_2^{\gamma_{p_i} p_i} - \alpha K_3 \tau^{\frac{\gamma_q q}{2}} \\ &\ge \frac{b}{8} \tau^2 - \sum_{i=1}^{2} K_i \tau^{\frac{\gamma_{p_i} p_i}{2}} - \alpha K_3 \tau^{\frac{\gamma_q q}{2}} := f(\tau), \end{aligned}$$

where $a > a^*$, $K_i = \frac{\nu_i}{p_i} C_{N, p_i} m_i^{\frac{\left(1 - \gamma_{p_i}\right) p_i}{2}} \left(i = 1, 2\right)$ and $K_3 = C_{N, r} m_1^{\frac{\left(1 - \gamma_q\right) q_1}{2}} m_2^{\frac{\left(1 - \gamma_q\right) q_2}{2}}$. According to (M$_1$), we know that $0 < \gamma_{p_i} p_i < 2$, $4 < \gamma_q q < 2^*$. Select a sufficiently large value for $\tau_0 > 0$ ensuring that

$$\sum_{i=1}^{2} K_i \left(\tau_0\right)^{\frac{\gamma_{p_i} p_i - 4}{2}} \le \frac{b}{32}.$$

Moreover, we can take $\alpha_0 > 0$ sufficiently small ensuring that

$$\alpha_0 K_3 \left(2\tau_0\right)^{\frac{\gamma_q q - 4}{2}} \le \frac{b}{32}.$$

Hence, for any $0 < \alpha \le \alpha_0$ and $\left(u_1, u_2\right) \in \mathcal{B}\left(2\tau_0\right) \setminus \mathcal{B}\left(\tau_0\right)$, *i.e.*, $\tau_0 \le \tau < 2\tau_0$, we have

$$\begin{aligned} I\left(u_1, u_2\right) &\ge \frac{b}{8} \tau^2 - \sum_{i=1}^{2} K_i \tau^{\frac{\gamma_{p_i} p_i}{2}} - \alpha K_3 \tau^{\frac{\gamma_q q}{2}} \\ &= \tau^2 \left(\frac{b}{8} - \sum_{i=1}^{2} K_i \tau^{\frac{\gamma_{p_i} p_i - 4}{2}} - \alpha K_3 \tau^{\frac{\gamma_q q - 4}{2}} \right) \\ &\ge b \tau_0^2 \left(\frac{1}{8} - \frac{1}{32} - \frac{1}{32} \right) = \frac{b}{16} \tau_0^2. \end{aligned}$$

According to the continuity of $f(\tau)$ and the fact that $f(\tau_0) > 0$, we can find an extremely small number $\varepsilon_0 > 0$ such that $f(\tau) \geq 0$ when $\tau \in [\tau_0 - \varepsilon_0, \tau_0]$. Thus

$$I(u_1, u_2) \geq f(\tau) \geq 0 > c(m_1, m_2)$$

for any $(u_1, u_2) \in \overline{\mathcal{B}(\tau_0)} \setminus \mathcal{B}(\tau_0 - \varepsilon_0)$.

Lemma 2.7. *Assume that* $2 < p_1, p_2, q_1 + q_2 < 2^*$. *Suppose that the* (PS) *sequence* $\{(u_1^n, u_2^n)\}$ *restricted on* $S_r(m_1) \times S_r(m_2)$ *is bounded, then we can find a* $(u_1, u_2) \in H^1_{\mathrm{rad}}(\mathbb{R}^N) \times H^1_{\mathrm{rad}}(\mathbb{R}^N)$ *and a sequence* $\{(\lambda_1^n, \lambda_2^n)\} \subset \mathbb{R}^2$ *such that up to a sub-sequence*

(i) $(u_1^n, u_2^n) \rightharpoonup (u_1, u_2)$ in $H^1_{\mathrm{rad}}(\mathbb{R}^N) \times H^1_{\mathrm{rad}}(\mathbb{R}^N)$, $(u_1^n, u_2^n) \to (u_1, u_2)$, in $L^p(\mathbb{R}^N) \times L^p(\mathbb{R}^N)$ for $p \in (2, 2^*)$.

(ii) $(\lambda_1^n, \lambda_2^n) \to (\lambda_1, \lambda_2)$ in $\mathbb{R}^2$.

(iii) $I'(u_1^n, u_2^n) - \lambda_1^n (u_1^n, 0) - \lambda_2^n (0, u_2^n) \to 0$ in $H^{-1}_{\mathrm{rad}}(\mathbb{R}^N) \times H^{-1}_{\mathrm{rad}}(\mathbb{R}^N)$.

(iv) problem (1.1) admits a solution (u_1, u_2) for some $\lambda_1, \lambda_2 \leq 0$ if (u_1, u_2) satisfies the additional property $J(u_1^n, u_2^n) \to 0$, where (λ_1, λ_2) is defined by (ii).

Proof. (i) is clear. By the fact of $\left(I|_{S_r(c_1) \times S_r(c_2)}\right)'(u_1^n, u_2^n) \to 0$ and Proposition 5.12 in [30], we can take two sequences of real numbers $\{\lambda_1^n\}, \{\lambda_2^n\}$ so that

$$\begin{aligned} & a\sum_{i=1}^{2}\int_{\mathbb{R}^N} \nabla u_i^n \nabla \varphi_i \mathrm{d}x + b\sum_{i=1}^{2}\int_{\mathbb{R}^N} \left|\nabla u_i^n\right|^2 \mathrm{d}x \int_{\mathbb{R}^N} \nabla u_i^n \nabla \varphi_i \mathrm{d}x + \sum_{i=1}^{2}\int_{\mathbb{R}^N} V_i u_i^n \varphi_i \mathrm{d}x \\ & - \sum_{i=1}^{2} \nu_i \int_{\mathbb{R}^N} \left|u_i^n\right|^{p_i - 2} u_i^n \varphi_i \mathrm{d}x - \alpha q_1 \int_{\mathbb{R}^N} \left|u_1^n\right|^{q_1 - 2} u_1^n \left|u_2^n\right|^{q_2} \varphi_1 \mathrm{d}x \\ & - \alpha q_2 \int_{\mathbb{R}^N} \left|u_1^n\right|^{q_1} \left|u_2^n\right|^{q_2 - 2} u_2^n \varphi_2 \mathrm{d}x - \sum_{i=1}^{2}\int_{\mathbb{R}^N} \lambda_i^n u_i^n \varphi_i \mathrm{d}x = o(1)\left\|(\varphi_1, \varphi_2)\right\|, \end{aligned} \tag{2.3}$$

where $o(1) \to 0$ as $n \to +\infty$. For further information, it is advisable to consult [[23], Lemma 3.2]. Testing Equation (2.3) with $(u_1^n, 0)$ and $(0, u_2^n)$, we get

$$a\left|\nabla u_1^n\right|_2^2 + b\left|\nabla u_1^n\right|_2^4 + \int_{\mathbb{R}^N} V_1 \left(u_1^n\right)^2 \mathrm{d}x - \nu_1 \left|u_1^n\right|_{p_1}^{p_1} - \alpha q_1 \int_{\mathbb{R}^N} \left|u_1^n\right|^{q_1} \left|u_2^n\right|^{q_2} \mathrm{d}x - o(1) = \lambda_1^n m_1,$$

$$a\left|\nabla u_2^n\right|_2^2 + b\left|\nabla u_2^n\right|_2^4 + \int_{\mathbb{R}^N} V_2 \left(u_2^n\right)^2 \mathrm{d}x - \nu_2 \left|u_2^n\right|_{p_2}^{p_2} - \alpha q_2 \int_{\mathbb{R}^N} \left|u_1^n\right|^{q_1} \left|u_2^n\right|^{q_2} \mathrm{d}x - o(1) = \lambda_2^n m_2.$$

By the boundedness of u_1^n, u_2^n in $L^p(\mathbb{R}^N)$ and $H^1_{\mathrm{rad}}(\mathbb{R}^N)$ for $p \in (2, 2^*)$, it can be deduced that the sequences $\{\lambda_1^n\}, \{\lambda_2^n\}$ are bounded. Consequently, it is reasonable to presume that λ_i^n converges strongly to $\lambda_i (i = 1, 2)$. Based on (ii) and (iii), the proof of (iv) is achievable through the method described in [[31], Proposition 2.10]. As the verification process is identical, we do not proof it here. □

Lemma 2.8. *Suppose that the Lemma 2.6 are satisfied, it follows that* u_1^n *converges strongly to* u_1 *in* $H^1_{\mathrm{rad}}(\mathbb{R}^N)$ *when* $\lambda_1 < 0$. *In this similar way, we can get the sequence* u_2^n *converges strongly to* u_2 *in* $H^1_{\mathrm{rad}}(\mathbb{R}^N)$ *when* $\lambda_2 < 0$.

Proof. Set

$$\lim_{n\to+\infty}\left|\nabla u_i^n\right|_2^2 = B_i \quad (i=1,2). \tag{2.4}$$

By Lemmas 2.3 and 2.6, we can get

$$\int_{\mathbb{R}^N} V_1\left(u_1^n\right)^2 \mathrm{d}x \to \int_{\mathbb{R}^N} V_1 u_1^2 \mathrm{d}x, \left|u_1^n\right|_{p_1}^{p_1} \to \left|u_1\right|_{p_1}^{p_1}, \int_{\mathbb{R}^N} \left|u_1^n\right|^{q_1}\left|u_2^n\right|^{q_2} \mathrm{d}x \to \int_{\mathbb{R}^N} \left|u_1\right|^{q_1}\left|u_2\right|^{q_2} \mathrm{d}x$$

and

$$\begin{aligned}&\left\langle I'\left(u_1^n,u_2^n\right)-\lambda_1^n\left(u_1^n,0\right),\left(u_1^n,0\right)\right\rangle\\ &\to 0=\left(a+bB_1\right)\left|\nabla u_1\right|_2^2+\int_{\mathbb{R}^N} V_1 u_1^2 \mathrm{d}x-\nu_1\left|u_1\right|_{p_1}^{p_1}-\alpha r\int_{\mathbb{R}^N}\left|u_1\right|^{q_1}\left|u_2\right|^{q_2}\mathrm{d}x-\lambda_1\left|u_1\right|_2^2.\end{aligned}$$

Hence

$$a\left|\nabla u_1^n\right|_2^2+b\left|\nabla u_1^n\right|_2^4-\lambda_1^n\left|u_1^n\right|_2^2 \to a\left|\nabla u_1\right|_2^2+bB_1\left|\nabla u_1\right|_2^2-\lambda_1\left|u_1\right|_2^2.$$

Since

$$\left|\nabla u_1\right|_2^2 \le \lim_{n\to+\infty}\left|\nabla u_1^n\right|_2^2,\ \left|u_1\right|_2^2 \le \lim_{n\to+\infty}\left|u_1^n\right|_2^2,$$

we get $\left|\nabla u_1^n\right|_2^2 \to \left|\nabla u_1\right|_2^2, \left|u_1^n\right|_2^2 \to \left|u_1\right|_2^2$. □

3. Proof of Theorem 1.1 (i)

Lemma 3.1. *Suppose that* (M_1) *is satisfied, then for* $\alpha\in(0,\alpha_0)$, *we can take a* (PS) *sequence* $\left\{\left(u_1^n,u_2^n\right)\right\}$ *for* $I|_{S_r(m_1)\times S_r(m_2)}$ *at the level* $\gamma(m_1,m_2)$, *which satisfies* $\left(u_1^n\right)^- \to 0$, $\left(u_2^n\right)^- \to 0$ *and* $J\left(u_1^n,u_2^n\right)\to 0$.

Proof. Our proof approach will adhere to the methodological framework delineated in [[23], Lemma 5.5]. Let $\tilde{I}:\mathbb{R}\times\left(H_{\mathrm{rad}}^1\left(\mathbb{R}^N\right)\times H_{\mathrm{rad}}^1\left(\mathbb{R}^N\right)\right)\to\mathbb{R}$ defined by

$$\tilde{I}\left(\theta,(u_1,u_2)\right)=I\left(\theta*u_1,\theta*u_2\right),$$

where $\theta*u=\mathrm{e}^{\frac{N}{2}\theta}u\left(\mathrm{e}^{\theta}x\right)$. Set $\theta*(u_1,u_2):=(\theta*u_1,\theta*u_2)$. Thus, $\theta*(u_1,u_2)\in S_r(m_1)\times S_r(m_2)$ when $(u_1,u_2)\in S_r(m_1)\times S_r(m_2)$. We set

$$\begin{aligned}\tilde{\Gamma}:=\Big\{\tilde{\eta}\in\left(S_r(m_1)\times S_r(m_2),C[0,1]\right):\tilde{\eta}(1)=\left(0,\eta(1)\right),\tilde{\eta}(0)\in\left(0,\eta(0)\right),\\ \eta(1)\notin\overline{\mathcal{B}(\tau_0)},I\left(\eta(1)\right)<0,\eta(0)\in\mathcal{B}(\overline{\tau})\Big\}\end{aligned}$$

and

$$\tilde{\gamma}(m_1,m_2):=\inf_{\tilde{\eta}\in\tilde{\Gamma}}\max_{t\in[0,1]}\tilde{I}\left(\tilde{\eta}(t)\right).$$

Pay attention to the fact that $\tilde{\gamma}(m_1,m_2)=\gamma(m_1,m_2)$. Indeed, by the definitions of $\tilde{\gamma}(m_1,m_2)$ and $\gamma(m_1,m_2)$, this equation is directly inferred from the premise that the mapps

$$\psi:\Gamma\to\tilde{\Gamma},\ \eta\mapsto\psi(\eta):=(0,\eta)$$

and

$$\zeta:\tilde{\Gamma}\to\Gamma,\tilde{\eta}=(\vartheta,\eta)\mapsto\zeta(\tilde{\eta}):=\vartheta*\eta\ \text{ with }(\vartheta*\eta)(t)=\vartheta(t)*\eta(t)$$

satisfy

$$\tilde{I}(\psi(\eta)) = I(\eta), I(\zeta(\tilde{\eta})) = \tilde{I}(\tilde{\eta}).$$

It is observed that $I(|u_1|,|u_2|) = I(u_1,u_2)$ holds when $(u_1,u_2) \in S_r(m_1) \times S_r(m_2)$. Then, we can assume the existence of a minimization sequence $v_1^n(t), v_2^n(t) \geq 0$, where $0 \leq t \leq 1$. According to [[32], Theorem 4.1], we can find a $(PS)_{\gamma(m_1,m_2)}$ sequence $\{(\theta_n,(u_1^n,u_2^n))\}$ for $\tilde{I}|_{\mathbb{R}} \times (S_r(m_1) \times S_r(m_2))$ and $\|(u_1^n,u_2^n) - (v_1^n,v_2^n)\| \to 0$. By $(u_1^n)^- \to 0, (u_2^n)^- \to 0$ and direct calculation

$$\tilde{I}(\theta,(u_1,u_2)) = \tilde{I}(0,\theta * (u_1,u_2)), (\partial_\theta \tilde{I})(\theta,(u_1,u_2)) = (\partial_\theta \tilde{I})(0,\theta * (u_1,u_2))$$

for $u = (u_1,u_2), \varphi = (\varphi_1,\varphi_2)$, we have $(\partial_u \tilde{I})(\theta,u)[\varphi] = (\partial_u \tilde{I})(0,\theta * u)[\theta * \varphi]$. Hence, $\{(0,\theta_n * (u_1^n,u_2^n))\}$ is also a $(PS)_{\gamma(m_1,m_2)}$ sequence for $\tilde{I}|_{\mathbb{R}\times(S_r(m_1)\times S_r(m_2))}$. Let

$$(w_1^n, w_2^n) := \theta_n * (u_1^n,u_2^n),$$

then $\{(w_1^n,w_2^n)\} \subset S_r(m_1) \times S_r(m_2)$ is a $(PS)_{\gamma(m_1,m_2)}$ sequence for $I|_{S_r(m_1)\times S_r(m_2)}$ and then $(\partial_\theta \tilde{I})(0,(w_1^n,w_2^n)) \to 0$ implies $J(w_1^n,w_2^n) \to 0$. ☐

According to Equation (2.4), we rewrite Equation (1.1) as

$$\begin{cases} -(a+bB_1)\Delta u_1 + V_1 u_1 = \lambda_1 u_1 + \nu_1 |u_1|^{p_1-2} u_1 + \alpha q_1 |u_1|^{q_1-2} u_1 |u_2|^{q_2}, \\ -(a+bB_2)\Delta u_2 + V_2 u_2 = \lambda_2 u_2 + \nu_2 |u_2|^{p_2-2} u_2 + \alpha q_2 |u_1|^{q_1} |u_2|^{q_2-2} u_2. \end{cases} \tag{3.1}$$

Its corresponding Pohožaev identity as following:

$$\begin{aligned} J_B(u_1,u_2) := {} & a\sum_{i=1}^{2} |\nabla u_i|_2^2 + b\sum_{i=1}^{2} B_i |\nabla u_i|_2^2 - \frac{1}{2}\sum_{i=1}^{2}\int_{\mathbb{R}^N} W_i u_i^2 \mathrm{d}x \\ & - \sum_{i=1}^{2} \nu_i \gamma_{p_i} |u_i|_{p_i}^{p_i} - \alpha\gamma_{q_1+q_2}(q_1+q_2)\int_{\mathbb{R}^N} |u_1|^{q_1} |u_2|^{q_2} \mathrm{d}x. \end{aligned} \tag{3.2}$$

Lemma 3.2. *Suppose the condition* (M₁) *is satisfied and* $0 < \alpha \leq \alpha_0$, *then we can find a positive radial solution* (u_1,u_2) *to the system* (1.1) *for some* (λ_1,λ_2) *and* $I(u_1,u_2) = \gamma(m_1,m_2)$.

Proof. By Lemma 3.1, it is possible to find a Palais-Smale sequence $\{(u_1^n,u_2^n)\}$ for $I|_{S_r(m_1)\times S_r(m_2)}$ at the level $\gamma(m_1,m_2)$. We first prove that $\{(u_1^n,u_2^n)\}$ is bounded in $H^1_{\text{rad}}(\mathbb{R}^N) \times H^1_{\text{rad}}(\mathbb{R}^N)$. Since $J(u_1^n,u_2^n) \to 0$, we have

$$\begin{aligned} & \frac{1}{\gamma_q q}\left(a\sum_{i=1}^{2} |\nabla u_i^n|_2^2 + b\sum_{i=1}^{2} |\nabla u_i^n|_2^4 - \sum_{i=1}^{2}\int_{\mathbb{R}^N} W_i (u_i^n)^2 \mathrm{d}x - \sum_{i=1}^{2} \nu_i \gamma_{p_i} |u_i^n|_{p_i}^{p_i} \right) \\ & = \alpha\int_{\mathbb{R}^N} |u_1^n|^{q_1} |u_2^n|^{q_2} \mathrm{d}x + o(1). \end{aligned}$$

$$
\begin{aligned}
&\gamma\left(m_1,m_2\right)+o(1)\\
&=\frac{a}{2}\sum_{i=1}^{2}\left|\nabla u_i^n\right|_2^2+\frac{b}{4}\sum_{i=1}^{2}\left|\nabla u_i^n\right|_2^4+\frac{1}{2}\sum_{i=1}^{2}\int_{\mathbb{R}^N}V_i\left(u_i^n\right)^2\mathrm{d}x-\sum_{i=1}^{2}\frac{\nu_i}{p_i}\left|u_i^n\right|_{p_i}^{p_i}-\alpha\int_{\mathbb{R}^N}\left|u_1^n\right|^{q_1}\left|u_2^n\right|^{q_2}\mathrm{d}x\\
&=a\left(\frac{1}{2}-\frac{1}{\gamma_q q}\right)\sum_{i=1}^{2}\left|\nabla u_i^n\right|_2^2+b\left(\frac{1}{4}-\frac{1}{\gamma_q q}\right)\sum_{i=1}^{2}\left|\nabla u_i^n\right|_2^4+\frac{1}{2}\sum_{i=1}^{2}\int_{\mathbb{R}^N}V_i\left(u_i^n\right)^2\mathrm{d}x\\
&\quad-\frac{1}{\gamma_q q}\sum_{i=1}^{2}\int_{\mathbb{R}^N}W_i\left(u_i^n\right)^2\mathrm{d}x-\sum_{i=1}^{2}\nu_i\gamma_{p_i}\left(\frac{1}{\gamma_{p_i}p_i}-\frac{1}{\gamma_q q}\right)\left|u_i^n\right|_{p_i}^{p_i}\\
&\ge\tilde{C}\sum_{i=1}^{2}\left|\nabla u_i^n\right|_2^2+b\left(\frac{1}{4}-\frac{1}{\gamma_q q}\right)\sum_{i=1}^{2}\left|\nabla u_i^n\right|_2^4-\sum_{i=1}^{2}K_i\nu_i\gamma_{p_i}\left(\frac{1}{\gamma_{p_i}p_i}-\frac{1}{\gamma_q q}\right)\left|\nabla u_i^n\right|_2^{\gamma_{p_i}p_i}\\
&\ge\tilde{C}\left(\left|\nabla u_1^n\right|_2^2+\left|\nabla u_2^n\right|_2^2\right)+\frac{b}{2}\left(\frac{1}{4}-\frac{1}{\gamma_q q}\right)\left(\left|\nabla u_1^n\right|_2^2+\left|\nabla u_2^n\right|_2^2\right)^2\\
&\quad-\sum_{i=1}^{2}K_i\nu_i\gamma_{p_i}\left(\frac{1}{\gamma_{p_i}p_i}-\frac{1}{\gamma_q q}\right)\left(\left|\nabla u_1^n\right|_2^2+\left|\nabla u_2^n\right|_2^2\right)^{\frac{\gamma_{p_i}p_i}{2}},
\end{aligned}
$$

where $\tilde{C}:=\left(\frac{a}{2}-\frac{a}{\gamma_q q}-\frac{\max\{\omega_1,\omega_2\}}{2}-\frac{\max\{\rho_1,\rho_2\}}{\gamma_q q}\right), K_i=C_{N,p_i}c_i^{\frac{(1-\gamma_{p_i})p_i}{2}}$, $4<\gamma_q q<2^*$ and $0<\gamma_{p_i}p_i<2$. Therefore, $\left\{\left(u_1^n,u_2^n\right)\right\}$ is bounded in $H_{\mathrm{rad}}^1\left(\mathbb{R}^N\right)\times H_{\mathrm{rad}}^1\left(\mathbb{R}^N\right)$ Consequently, it can be inferred that

$$
\begin{aligned}
&\left(u_1^n,u_2^n\right)\rightharpoonup\left(u_1,u_2\right)\text{ in }H_{\mathrm{rad}}^1\left(\mathbb{R}^N\right)\times H_{\mathrm{rad}}^1\left(\mathbb{R}^N\right),\\
&\left(u_1^n,u_2^n\right)\to\left(u_1,u_2\right)\text{ in }L^s\left(\mathbb{R}^N\right)\times L^s\left(\mathbb{R}^N\right)\text{ for }s\in\left(2,2^*\right).
\end{aligned}
$$

According to Lemma 2.7, we can take a sequence $\left\{\left(\lambda_1^n,\lambda_2^n\right)\right\}\subset\mathbb{R}^2$ such that $\left(\lambda_1^n,\lambda_2^n\right)\to\left(\lambda_1,\lambda_2\right)$ in $\mathbb{R}^2$, $\left(u_1,u_2\right)$ is the solution of the system (3.1) and $J_B\left(u_1,u_2\right)=0$. Since $\left(u_1^n\right)^-\to0$, $\left(u_2^n\right)^-\to0$, then $u_1,u_2\ge0$.

Now, we prove $I\left(u_1,u_2\right)=\gamma\left(m_1,m_2\right)$. The condition $J_B\left(u_1^n,u_2^n\right)\to0$ implies

$$
\begin{aligned}
&a\sum_{i=1}^{2}\left|\nabla u_i^n\right|_2^2+b\sum_{i=1}^{2}\left|\nabla u_i^n\right|_2^4\\
&=\sum_{i=1}^{2}\nu_i\gamma_{p_i}\left|u_i^n\right|_{p_i}^{p_i}+\alpha\gamma_q q\int_{\mathbb{R}^N}\left|u_1^n\right|^{q_1}\left|u_2^n\right|^{q_2}\mathrm{d}x+\sum_{i=1}^{2}\int_{\mathbb{R}^N}W_i\left(u_i^n\right)^2\mathrm{d}x+o(1).
\end{aligned}
\tag{3.3}
$$

By Equation (2.4), we have

$$
\begin{aligned}
&a\sum_{i=1}^{2}\left|\nabla u_i^n\right|_2^2+b\sum_{i=1}^{2}B_i\left|\nabla u_i^n\right|_2^2\\
&=\sum_{i=1}^{2}\nu_i\gamma_{p_i}\left|u_i^n\right|_{p_i}^{p_i}+\alpha\gamma_q q\int_{\mathbb{R}^N}\left|u_1^n\right|^{q_1}\left|u_2^n\right|^{q_2}\mathrm{d}x+\sum_{i=1}^{2}\int_{\mathbb{R}^N}W_i\left(u_i^n\right)^2\mathrm{d}x+o(1)
\end{aligned}
$$

As the sequence $\left(u_1^n,u_2^n\right)$ converges to $\left(u_1,u_2\right)$ in $L^p\left(\mathbb{R}^N\right)\times L^p\left(\mathbb{R}^N\right)$ for $p\in\left(2,2^*\right)$, then it can be inferred that $\sum_{i=1}^{2}\nu_i\gamma_{p_i}\left|u_i^n\right|_{p_i}^{p_i}+\alpha\gamma_q q\int_{\mathbb{R}^N}\left|u_1^n\right|^{q_1}\left|u_2^n\right|^{q_2}\mathrm{d}x+\sum_{i=1}^{2}\int_{\mathbb{R}^N}W_i\left(u_i^n\right)^2\mathrm{d}x+o(1)$ of Equation (3.3) converges to

$$\sum_{i=1}^{2}\nu_i\gamma_{p_i}\left|u_i\right|_{p_i}^{p_i}+\alpha\gamma_q q\int_{\mathbb{R}^N}\left|u_1\right|^{q_1}\left|u_2\right|^{q_2}\,\mathrm{d}x+\sum_{i=1}^{2}\int_{\mathbb{R}^N}W_i\left(u_i\right)^2\,\mathrm{d}x.$$

Combining $J_B\left(u_1,u_2\right)=0$, we have

$$\lim_{n\to+\infty}a\sum_{i=1}^{2}\left|\nabla u_i^n\right|_2^2+b\sum_{i=1}^{2}B_i\left|\nabla u_i^n\right|_2^2=a\sum_{i=1}^{2}\left|\nabla u_i\right|_2^2+b\sum_{i=1}^{2}B_i^2.$$

Hence, $I\left(u_1^n,u_2^n\right)\to I\left(u_1,u_2\right)$, and then, $I\left(u_1,u_2\right)=\gamma\left(m_1,m_2\right)$. □

***Proof of Theorem* 1.1.** By Lemma 3.2, we only need to show $\left(u_1,u_2\right)\in S_r\left(m_1\right)\times S_r\left(m_2\right)$. Since $\left(u_1,u_2\right)$ satisfies Equation (1.1), it follows that

$$\begin{aligned}\lambda_1\left|u_1\right|_2^2+\lambda_2\left|u_2\right|_2^2&=a\sum_{i=1}^{2}\left|\nabla u_i\right|_2^2+b\sum_{i=1}^{2}\left|\nabla u_i\right|_2^4+\sum_{i=1}^{2}\int_{\mathbb{R}^N}V_iu_i^2\mathrm{d}x\\&\quad-\sum_{i=1}^{2}\nu_i\left|u_i\right|_{p_i}^{p_i}-\alpha r\int_{\mathbb{R}^N}\left|u_1\right|^{q_1}\left|u_2\right|^{q_2}\,\mathrm{d}x.\end{aligned}$$

Let $F\left(u_1,u_2\right)=\max\left\{\sum_{i=1}^{2}\left|u_i\right|_{p_i}^{p_i},\int_{\mathbb{R}^N}\left|u_1\right|^{q_1}\left|u_2\right|^{q_2}\,\mathrm{d}x\right\}$, since

$$\begin{aligned}\sum_{i=1}^{2}\left(a-\rho_i\right)\left|\nabla u_i\right|_2^2&\le a\sum_{i=1}^{2}\left|\nabla u_i\right|_2^2-\sum_{i=1}^{2}\int_{\mathbb{R}^N}W_iu_i^2\mathrm{d}x\\&\le a\sum_{i=1}^{2}\left|\nabla u_i\right|_2^2+b\sum_{i=1}^{2}\left|\nabla u_i\right|_2^4-\sum_{i=1}^{2}\int_{\mathbb{R}^N}W_iu_i^2\mathrm{d}x,\end{aligned}$$

combining $J\left(u_1,u_2\right)=0$, we have

$$\begin{aligned}\sum_{i=1}^{2}\left|\nabla u_i\right|_2^2&\le\frac{1}{\left(a-\max\left\{\rho_1,\rho_2\right\}\right)}\left(\sum_{i=1}^{2}\nu_i\gamma_{p_i}\left|u_i\right|_{p_i}^{p_i}+\alpha\gamma_q q\int_{\mathbb{R}^N}\left|u_1\right|^{q_1}\left|u_2\right|^{q_2}\,\mathrm{d}x\right)\\&\le\frac{2\left(\sum_{i=1}^{2}\nu_i\gamma_{p_i}+\alpha\gamma_q q\right)F\left(u_1,u_2\right)}{\left(a-\max\left\{\rho_1,\rho_2\right\}\right)}\end{aligned}$$

and then,

$$\begin{aligned}\lambda_1\left|u_1\right|_2^2+\lambda_2\left|u_2\right|_2^2&=\sum_{i=1}^{2}\int_{\mathbb{R}^N}V_iu_i^2\mathrm{d}x+\sum_{i=1}^{2}\int_{\mathbb{R}^N}W_iu_i^2\mathrm{d}x+\sum_{i=1}^{2}\nu_i\left(\gamma_{p_i}-1\right)\left|u_i\right|_{p_i}^{p_i}\\&\quad+\alpha r\left(\gamma_q-1\right)\int_{\mathbb{R}^N}\left|u_1\right|^{q_1}\left|u_2\right|^{q_2}\,\mathrm{d}x\\&\le\sum_{i=1}^{2}\rho_i\left|\nabla u_i\right|_2^2+\sum_{i=1}^{2}\nu_i\left(\gamma_{p_i}-1\right)\left|u_i\right|_{p_i}^{p_i}+\alpha r\left(\gamma_q-1\right)\int_{\mathbb{R}^N}\left|u_1\right|^{q_1}\left|u_2\right|^{q_2}\,\mathrm{d}x\\&\le C_2F\left(u_1,u_2\right)+\alpha r\left(\gamma_q-1\right)\sum_{i=1}^{2}\int_{\mathbb{R}^N}\left|u_1\right|^{q_1}\left|u_2\right|^{q_2}\,\mathrm{d}x<0,\end{aligned}$$

where $2<p_i,r<2^*$, $C_2=\dfrac{\max\left\{\rho_1,\rho_2\right\}\left(\sum_{i=1}^{2}\nu_i\gamma_{p_i}+\alpha\gamma_q q\right)}{\left(a-\max\left\{\rho_1,\rho_2\right\}\right)}+\nu_i\left(\gamma_{p_i}-1\right)<0$ when $a>a^*$.

Consequently, it follows that at least one of the λ_1 or λ_2 is negative. For the sake of argument, let us assume $\lambda_1<0$. According to Lemma 2.8, we have $u_1\in S_r\left(m_1\right)$ and $u_1^n\to u_1$ in $H_{\mathrm{rad}}^1\left(\mathbb{R}^N\right)$. If $\lambda_2\ge 0$, we have

$$-\left(a+b\int_{\mathbb{R}^N}\left|\nabla u_2\right|^2\,\mathrm{d}x\right)\Delta u_2=\lambda_2u_2+\nu_2\left|u_2\right|^{p_2-2}u_2+\alpha q_2\left|u_1\right|^{q_1}\left|u_2\right|^{q_2-2}u_2-V_2u_2\ge 0.$$

From [[24], lemma A.2], we can know that $u_2 = 0$. Therefore, $I(u_1,u_2) = I(u_1,0)$ and $u_1 \in S_r(m_1)$ is the solution of the following equation

$$-\left(a + b\int_{\mathbb{R}^N} |\nabla u_1|^2 \,\mathrm{d}x\right)\Delta u_1 = \lambda_1 u_1 + \nu_1 |u_1|^{p_1-1} u_1 - V_1 u_1. \tag{3.4}$$

By lemma 2.2, u_1 is unique and $I(u_1,0) = l(m_1,\nu_1) < 0$, which contradicts $I(u_1,0) = \gamma(m_1,m_2) > 0$. Consequently, it follows that λ_2 is negative, which implies that $u_2 \in S_r(m_2)$. In the end, using the maximum principle, it can be infer that $u_1,u_2 > 0$ in $\mathbb{R}^N$. ☐

4. Proof of Theorem 1.1(ii)

Let $\tau = |\nabla u_1|_2^2 + |\nabla u_2|_2^2$, then for each $(u_1,u_2) \in S(m_1)\times S(m_2)$

$$\begin{aligned}
I(u_1,u_2) &= \frac{a}{2}\sum_{i=1}^{2}|\nabla u_i|_2^2 + \frac{b}{4}\sum_{i=1}^{2}|\nabla u_i|_2^4 + \frac{1}{2}\sum_{i=1}^{2}\int_{\mathbb{R}^N} V_i u_i^2 \,\mathrm{d}x - \sum_{i=1}^{2}\frac{\nu_i}{p_i}|u_i|_{p_i}^{p_i} \\
&\quad - \alpha\int_{\mathbb{R}^N} |u_1|^{q_1} |u_2|^{q_2} \,\mathrm{d}x \\
&\ge \frac{a}{2}\tau + \frac{b}{4}\tau^2 - \max\{\omega_1,\omega_2\}\tau - \sum_{i=1}^{2}\frac{\nu_i}{p_i} C_{N,p_i} c_i^{\frac{(1-\gamma_{p_i})p_i}{2}} |\nabla u_i|_2^{\gamma_{p_i} p_i} - \alpha K_3 \tau^{\frac{\gamma_q q}{2}} \\
&\ge K_0\tau^2 - K_1\tau^{\frac{\gamma_{p_1} p_1}{2}} - K_2\tau^{\frac{\gamma_{p_2} p_2}{2}} - \alpha K_3 \tau^{\frac{\gamma_q q}{2}} \\
&=: g(\tau),
\end{aligned} \tag{4.1}$$

where $a > a^*$, $K_0 = \frac{b}{4}$, $K_i = \frac{\nu_i}{p_i} C_{N,p_i} c_i^{\frac{(1-\gamma_{p_i})p_i}{2}}$ $(i = 1,2)$, $K_3 = C_{N,r} m_1^{\frac{(1-\gamma_q)q_1}{2}} m_2^{\frac{(1-\gamma_q)q_2}{2}}$, $0 < \gamma_{p_i} p_i < 2$ and $4 < \gamma_q q < 2^*$. One can easily verify that $\lim_{\tau\to+\infty} g(\tau) \to -\infty$ and $\lim_{\tau\to 0^+} g(\tau) \to 0^-$.

Lemma 4.1. *Suppose the condition* (M$_1$) *is satisfied, then we can find a constant* $\alpha_1 > 0$, *ensuring that* $g(\tau)$ *possesses a global maximum point at the positive level and a local minimum at the negative level, which are uniqe when* $0 < \alpha < \alpha_1$. *Furthermore, it is observed that* $0 < \tau_0 < \tau_1$ *depending on* α *such that* $g(\tau_0) = g(\tau_1) = 0$ *and* $g(\tau) > 0$ *if and only if* $\tau \in (\tau_0,\tau_1)$.

Proof. Without loss of generality, we may assume that $p_1 \le p_2$. We only show the proof for $p_1 = p_2$, the proof for case $p_1 < p_2$ is similar. Let $p = p_1 = p_2$, for $\tau > 0$, we have

$$\begin{aligned}
g(\tau) &= K_0\tau^2 - (K_1 + K_2)\tau^{\frac{\gamma_p p}{2}} - \alpha K_3 \tau^{\frac{\gamma_q q}{2}} \\
&= \tau^{\frac{\gamma_p p}{2}}\left(K_0\tau^{\frac{4-\gamma_p p}{2}} - \alpha K_3 \tau^{\frac{\gamma_q q - \gamma_p p}{2}} - (K_1 + K_2)\right).
\end{aligned}$$

Let $\chi(\tau) = K_0\tau^{\frac{4-\gamma_p p}{2}} - \alpha K_3 \tau^{\frac{\gamma_q q - \gamma_p p}{2}}$, then $g(\tau) > 0$ if and only if $\chi(\tau) > K_1 + K_2$. Clearly, $\chi(\tau)$ possesses a maximum point which is unique

$$\overline{\tau} = \left(\frac{K_0\left(4-\gamma_p p\right)}{\alpha K_3\left(\gamma_q q-\gamma_p p\right)}\right)^{\frac{2}{\gamma_q q-4}}$$

and

$$\chi\left(\overline{\tau}\right) = C\alpha^{-\frac{4-\gamma_p p}{\gamma_q q-4}},$$

where $0<\gamma_{p_i} p_i<2, 4<\gamma_q q<2^*$ and $C>0$. Therefore, we are able to make α as small as necessary, ensuring that $\chi\left(\overline{\tau}\right)>K_1+K_2$. Then, we can find a constant $\alpha_1>0$, ensuring that $g\left(\tau\right)>0$ on $\tau\in\left(\tau_0,\tau_1\right)$ when $0<\alpha<\alpha_1$. Since $g\left(\tau\right)\to 0^-$ as $\tau\to 0^+$, it is evident that $g\left(\tau\right)$ can find a local minimum on $\left(0,\tau_0\right)$. Therefore, $g\left(\tau\right)$ must possess a minimum of two critical points. Set $h\left(\tau\right):=2K_0\tau^{\frac{4-\gamma_p p}{2}}-\frac{\gamma_q q}{2}\alpha K_3\tau^{\frac{\gamma_q q-\gamma_p p}{2}}$, then

$$\begin{aligned} g'\left(\tau\right) &= 2K_0\tau-\frac{\gamma_p p}{2}\left(K_1+K_2\right)\tau^{\frac{\gamma_p p-2}{2}}-\frac{\gamma_q q}{2}\alpha K_3\tau^{\frac{\gamma_q q-2}{2}} \\ &= \tau^{\frac{\gamma_p p-2}{2}}\left(h\left(\tau\right)-\frac{\gamma_p p}{2}\left(K_1+K_2\right)\right). \end{aligned}$$

So $g'\left(\tau\right)=0$ if and only if $h\left(\tau\right)=\frac{\gamma_p p}{2}\left(K_1+K_2\right)$. As the presence of a sole global maximum for $h\left(\tau\right)$, it leads to $h\left(\tau\right)=\frac{\gamma_p p}{2}\left(K_1+K_2\right)$ possesses no more than two solutions, *i.e.*, $g\left(\tau\right)$ possesses no more than two points. □

Remark 4.1. *If* $p_1<p_2$, *we have*

$$\begin{aligned} g\left(\tau\right) &= K_0\tau^2-K_1\tau^{\frac{\gamma_{p_1}p_1}{2}}-K_2\tau^{\frac{\gamma_{p_2}p_2}{2}}-\alpha K_3\tau^{\frac{\gamma_q q}{2}} \\ &= \tau^{\frac{\gamma_{p_1}p_1}{2}}\left(K_0\tau^{\frac{4-\gamma_{p_1}p_1}{2}}-K_2\tau^{\frac{\gamma_{p_2}p_2-\gamma_{p_1}p_1}{2}}-\alpha K_3\tau^{\frac{\gamma_q q-\gamma_{p_1}p_1}{2}}-K_1\right) \\ &=: \tau^{\frac{\gamma_{p_1}p_1}{2}}\left(\chi\left(\tau\right)-K_1\right). \end{aligned}$$

Through computational analysis, it is easy to know that $\chi\left(\tau\right)$ exists a global maximum point $\overline{\tau}$ with

$$\overline{\tau}>\left(\frac{C_1}{C_2\alpha}\right)^{\frac{2}{\gamma_q q-4}}=:\tilde{\tau},$$

where $C_1=\frac{K_0\left(4-\gamma_{p_1}p_1\right)\left(4-\gamma_{p_2}p_2\right)}{4}, C_2=\frac{K_3\left(\gamma_q q-\gamma_{p_1}p_1\right)\left(\gamma_q q-\gamma_{p_2}p_2\right)}{4}$.

Moreover,

$$\chi\left(\overline{\tau}\right)>\chi\left(\tilde{\tau}\right)>\tilde{\tau}^{\frac{\gamma_{p_2}p_2-\gamma_{p_1}p_1}{2}},$$

when α is sufficiently small. Therefore, we are able to make α as small as necessary, ensuring that $\chi(\overline{\tau}) > K_1$. The case of $p_1 > p_2$ is similar.

Lemma 4.2. *Suppose the condition* (M_1) *is satisfied, then we can find a constant* $\alpha_2 > 0$*, ensuring that* $\mathcal{M}_0 = \varnothing$ *when* $0 < \alpha < \alpha_2$*. The* $\mathcal{M}$ *is a submanifold of* C^1 *with a codimension of three in* $H^1_{\mathrm{rad}}(\mathbb{R}^N) \times H^1_{\mathrm{rad}}(\mathbb{R}^N)$.

Proof. Assume that there exists $(u_1, u_2) \in \mathcal{M}_0$, then

$$\begin{aligned}\left(\Upsilon_{(u_1,u_2)}\right)(0) &= a\sum_{i=1}^{2}|\nabla u_i|_2^2 + b\sum_{i=1}^{2}|\nabla u_i|_2^4 - \sum_{i=1}^{2}\int_{\mathbb{R}^N} W_i u_i^2 \mathrm{d}x \\ &= \sum_{i=1}^{2}\nu_i\gamma_{p_i}|u_i|_{p_i}^{p_i} + \alpha\gamma_q q\int_{\mathbb{R}^N}|u_1|^{q_1}|u_2|^{q_2}\,\mathrm{d}x,\end{aligned} \tag{4.2}$$

and

$$\begin{aligned}\left(\Upsilon_{(u_1,u_2)}\right)(0) &= 2a\sum_{i=1}^{2}|\nabla u_i|_2^2 + 4b\sum_{i=1}^{2}|\nabla u_i|_2^4 + \sum_{i=1}^{2}\int_{\mathbb{R}^N} Z_i u_i^2 \mathrm{d}x \\ &= \sum_{i=1}^{2}\nu_i\gamma_{p_i}^2 p_i|u_i|_{p_i}^{p_i} + \alpha(\gamma_q q)^2\int_{\mathbb{R}^N}|u_1|^{q_1}|u_2|^{q_2}\,\mathrm{d}x.\end{aligned} \tag{4.3}$$

Combine (4.2) and (4.3), we have

$$\begin{aligned}&(\gamma_q q - 2)a\sum_{i=1}^{2}|\nabla u_i|_2^2 + (\gamma_q q - 4)b\sum_{i=1}^{2}|\nabla u_i|_2^4 \\ &= \sum_{i=1}^{2}\nu_i\gamma_{p_i}(\gamma_q q - \gamma_{p_i}p_i)|u_i|_{p_i}^{p_i} + \sum_{i=1}^{2}\int_{\mathbb{R}^N} Y_i u_i^2 \mathrm{d}x\end{aligned}$$

and then

$$\begin{aligned}\frac{b}{2}\left(|\nabla u_1|_2^2 + |\nabla u_2|_2^2\right)^2 &\le b\left(|\nabla u_1|_2^4 + |\nabla u_2|_2^4\right) + \frac{a(\gamma_q q - 2)}{\gamma_q q - 4}\left(|\nabla u_1|_2^2 + |\nabla u_2|_2^2\right) \\ &= \frac{1}{\gamma_q q - 4}\sum_{i=1}^{2}\nu_i\gamma_{p_i}(\gamma_q q - \gamma_{p_i}p_i)|u_i|_{p_i}^{p_i} + \frac{1}{\gamma_q q - 4}\sum_{i=1}^{2}\int_{\mathbb{R}^N} Y_i u_i^2 \mathrm{d}x \\ &\le C_3\left(|\nabla u_1|_2^2 + |\nabla u_2|_2^2\right)^{\frac{\gamma_{p_i}p_i}{2}} + C_4\left(|\nabla u_1|_2^2 + |\nabla u_2|_2^2\right),\end{aligned}$$

where $C_4 = \dfrac{\max\{\sigma_1, \sigma_2\}}{\gamma_q q - 4}$. Hence, there exists $C_5 > 0$, independent of α such that $|\nabla u_1|_2^2 + |\nabla u_2|_2^2 \le C_5$. By (4.2) and (4.3), we can get

$$\begin{aligned}\sum_{i=1}^{2}\nu_i\gamma_{p_i}(2 - \gamma_{p_i}p_i)|u_i|_{p_i}^{p_i} &\le 2b\sum_{i=1}^{2}|\nabla u_i|_2^4 + \sum_{i=1}^{2}\nu_i\gamma_{p_i}(2 - \gamma_{p_i}p_i)|u_i|_{p_i}^{p_i} \\ &= \alpha\gamma_q q(\gamma_q q - 2)\int_{\mathbb{R}^N}|u_1|^{q_1}|u_2|^{q_2}\,\mathrm{d}x - \sum_{i=1}^{2}\int_{\mathbb{R}^N}(2W_i - Z_i)u_i^2 \mathrm{d}x \\ &\le \alpha\gamma_q q(\gamma_q q - 2)\int_{\mathbb{R}^N}|u_1|^{q_1}|u_2|^{q_2}\,\mathrm{d}x + \sum_{i=1}^{2}(2\rho_i + \sigma_i)|\nabla u_i|_2^2.\end{aligned}$$

Suppose $\int_{\mathbb{R}^N}|u_1|^{q_1}|u_2|^{q_2}\,\mathrm{d}x = 0$, we can infer that $u_1 = u_2 = 0$. There is a conflict. So $\int_{\mathbb{R}^N}|u_1|^{q_1}|u_2|^{q_2}\,\mathrm{d}x \ne 0$, then

$$
\begin{aligned}
a\left(|\nabla u_1|_2^2+|\nabla u_2|_2^2\right) &\le b\left(|\nabla u_1|_2^4+|\nabla u_2|_2^4\right)+a\left(|\nabla u_1|_2^2+|\nabla u_2|_2^2\right)\\
&=\sum_{i=1}^{2}\nu_i\gamma_{p_i}|u_i|_{p_i}^{p_i}+\alpha\gamma_q q\int_{\mathbb{R}^N}|u_1|^{q_1}|u_2|^{q_2}\,\mathrm{d}x+\sum_{i=1}^{2}\int_{\mathbb{R}^N}W_iu_i^2\mathrm{d}x\\
&\le\left(\frac{\gamma_q q-2}{2-\max\left\{\gamma_{p_1}p_1,\gamma_{p_2}p_2\right\}}+1\right)\alpha\gamma_q q\int_{\mathbb{R}^N}|u_1|^{q_1}|u_2|^{q_2}\,\mathrm{d}x+\sum_{i=1}^{2}\rho_i|\nabla u_i|_2^2\\
&\quad+\left(\frac{1}{2-\max\left\{\gamma_{p_1}p_1,\gamma_{p_2}p_2\right\}}\right)\sum_{i=1}^{2}\left(2\rho_i+\sigma_i\right)|\nabla u_i|_2^2\\
&\le\alpha C_6\left(|\nabla u_1|_2^2+|\nabla u_2|_2^2\right)^{\frac{\gamma_q q}{2}}+C_7\left(|\nabla u_1|_2^2+|\nabla u_2|_2^2\right),
\end{aligned}
$$

where $C_6>0$, $C_7=\left(\frac{2\max\{\rho_1,\rho_2\}+\max\{\sigma_1,\sigma_2\}}{2-\max\left\{\gamma_{p_1}p_1,\gamma_{p_2}p_2\right\}}\right)+\max\{\rho_1,\rho_2\}$ and then,

$|\nabla u_1|_2^2+|\nabla u_2|_2^2\ge\left(\frac{a-C_7}{\alpha C_6}\right)^{\frac{2}{\gamma_q q-2}}$ where $a>a^*$. Hence

$$
\left(\frac{a-C_7}{\alpha C_6}\right)^{\frac{2}{\gamma_q q-2}}\le|\nabla u_1|_2^2+|\nabla u_2|_2^2\le C_5.
$$

There is unachievable when α is sufficiently small.

Next, we verify that $\mathcal{M}$ is a submanifold of C^1 with a codimension of three in $H_{\text{rad}}^1\left(\mathbb{R}^N\right)\times H_{\text{rad}}^1\left(\mathbb{R}^N\right)$. Note that

$$
\mathcal{M}:=\left\{\left(u_1,u_2\right)\in H_{\text{rad}}^1\left(\mathbb{R}^N\right)\times H_{\text{rad}}^1\left(\mathbb{R}^N\right):J\left(u_1,u_2\right)=0,D_1\left(u_1\right)=0,D_2\left(u_2\right)=0\right\},
$$

where $D_1\left(u_1\right)=|u_1|_2^2-m_1,D_2\left(u_2\right)=|u_2|_2^2-m_2$. We just have to attest this map

$$
d\left(J,D_1,D_2\right):H_{\text{rad}}^1\left(\mathbb{R}^N\right)\times H_{\text{rad}}^1\left(\mathbb{R}^N\right)\to\mathbb{R}^3
$$

is a surjective. Otherwise, by the independence of $dD_1\left(u_1\right)$ and $dD_2\left(u_2\right)$, $dJ\left(u_1,u_2\right)$ must be a linear combination of $dD_1\left(u_1\right)$ and $dD_2\left(u_2\right)$, *i.e.*, there existn $\nu_1,\nu_2\in\mathbb{R}$ such that

$$
dJ\left(u_1,u_2\right)=\nu_1dD_1\left(u_1\right)+\nu_2dD_2\left(u_2\right),
$$

that is, $\left(u_1,u_2\right)$ satisfies the following system

$$
\begin{cases}
-\left(a+2b\int_{\mathbb{R}^N}|\nabla u_1|^2\,\mathrm{d}x\right)\Delta u_1+W_1u_1=\nu_1u_1+\dfrac{\nu_1\gamma_{p_1}p_1}{2}|u_1|^{p_1-2}u_1+\dfrac{\alpha q_1\gamma_q q}{2}|u_1|^{q_1-2}u_1|u_2|^{q_2},\\
-\left(a+2b\int_{\mathbb{R}^N}|\nabla u_2|^2\,\mathrm{d}x\right)\Delta u_2+W_2u_2=\nu_2u_2+\dfrac{\nu_2\gamma_{p_2}p_2}{2}|u_2|^{p_2-2}u_2+\dfrac{\alpha q_2\gamma_q q}{2}|u_1|^{q_1}|u_2|^{q_2-2}u_2.
\end{cases}
$$

According to the Pohožaev identity, we can get

$$
2a\sum_{i=1}^{2}|\nabla u_i|_2^2+4b\sum_{i=1}^{2}|\nabla u_i|_2^4+\sum_{i=1}^{2}\int_{\mathbb{R}^N}Z_iu_i^2\mathrm{d}x=\sum_{i=1}^{2}\nu_i\gamma_{p_i}^2p_i|u_i|_{p_i}^{p_i}+\alpha\left(\gamma_q q\right)^2\int_{\mathbb{R}^N}|u_1|^{q_1}|u_2|^{q_2}\,\mathrm{d}x
$$

and then $\left(u_1,u_2\right)\in\mathcal{M}_0$, there is a conflict. □

Remark 4.2. *We can observe that* $\mathcal{M}=\mathcal{M}_+\cup\mathcal{M}_-$, *where* $\mathcal{M}_+\cap\mathcal{M}_-=\varnothing$

and $\alpha \in (0,\alpha_2)$.

We define $\alpha_0 = \min\{\alpha_1,\alpha_2\}$,

$\mathcal{V}_+ := \left\{(u_1,u_2) \in H^1_{\text{rad}}(\mathbb{R}^N)\times H^1_{\text{rad}}(\mathbb{R}^N) : \int_{\mathbb{R}^N} |u_1|^{q_1}\cdot|u_2|^{q_2}\,\mathrm{d}x > 0\right\}$ and $\mathcal{V}_0 := \left\{(u_1,u_2) \in H^1_{\text{rad}}(\mathbb{R}^N)\times H^1_{\text{rad}}(\mathbb{R}^N) : \int_{\mathbb{R}^N} |u_1|^{q_1}|u_2|^{q_2} = 0\right\}$. Next, we begin to investigate the properties of $\Upsilon_{(u_1,u_2)}(s)$.

Lemma 4.3. *Suppose the condition* (M$_1$) *is satisfied. For all* $(u_1,u_2) \in (S_r(m_1)\times S_r(m_2))\cap\mathcal{V}_+$, *we can infer that the function* $\Upsilon_{(u_1,u_2)}(\theta)$ *possesses two zeros* $s_{(u_1,u_2)} < t_{(u_1,u_2)}$ *and two critical points* $c_{(u_1,u_2)} < d_{(u_1,u_2)}$ *satisfying the inequality* $c_{(u_1,u_2)} < s_{(u_1,u_2)} < d_{(u_1,u_2)} < t_{(u_1,u_2)}$, *where* $0 < \alpha < \alpha_0$. *Additionally*

(i) $\theta * (u_1,u_2) \in \mathcal{M}_+ \Leftrightarrow \theta = c_{(u_1,u_2)}$ and $\theta * (u_1,u_2) \in \mathcal{M}_- \Leftrightarrow \theta = d_{(u_1,u_2)}$.

(ii) $\left|\nabla(\theta*(u_1,u_2))\right|_2^2 < \tau_0$ for every $\theta < s_{(u_1,u_2)}$ and

$$I\left(c_{(u_1,u_2)} * (u_1,u_2)\right) = \min\left\{I(\theta*(u_1,u_2)) : \theta\in\mathbb{R}, \left|\nabla(\theta*(u_1,u_2))\right|_2^2 < \tau_0\right\} < 0.$$

(iii) $I\left(d_{(u_1,u_2)} * (u_1,u_2)\right) = \max_{s\in\mathbb{R}} I(\theta*(u_1,u_2))$.

(iv) The maps $(u_1,u_2) \mapsto c_{(u_1,u_2)}$ and $(u_1,u_2)\mapsto d_{(u_1,u_2)}$ belongs to C^1.

Proof. Clearly, we can get

$$\begin{aligned}\left(\Upsilon_{(u_1,u_2)}\right)(\theta) &= a\mathrm{e}^{2\theta}\sum_{i=1}^{2}|\nabla u_i|_2^2 + b\mathrm{e}^{4\theta}\sum_{i=1}^{2}|\nabla u_i|_2^4 - \sum_{i=1}^{2}\int_{\mathbb{R}^N} W_i\left(\mathrm{e}^{-\theta}x\right)u_i^2\mathrm{d}x \\ &\quad - \sum_{i=1}^{2}\nu_i\gamma_{p_i}\mathrm{e}^{\gamma_{p_i}p_i\theta}|u_i|_{p_i}^{p_i} - \alpha\gamma_q q\mathrm{e}^{\gamma_q q\theta}\int_{\mathbb{R}^N}|u_1|^{q_1}|u_2|^{q_2}\,\mathrm{d}x \\ &= a\sum_{i=1}^{2}\left|\nabla(\theta*u_i)\right|_2^2 + b\sum_{i=1}^{2}\left|\nabla(\theta*u_i)\right|_2^4 - \sum_{i=1}^{2}\int_{\mathbb{R}^N} W_i(\theta*u_i)^2\,\mathrm{d}x \\ &\quad - \sum_{i=1}^{2}\nu_i\gamma_{p_i}|\theta*u_i|_{p_i}^{p_i} - \alpha\gamma_q q\int_{\mathbb{R}^N}|\theta*u_1|^{q_1}|\theta*u_2|^{q_2}\,\mathrm{d}x \\ &= J(\theta*u_1,\theta*u_2),\ \forall(u_1,u_2)\in(S_r(m_1)\times S_r(m_2))\cap\mathcal{V}_+.\end{aligned}$$

Thus, $\theta*(u_1,u_2)\in\mathcal{M}\Leftrightarrow\left(\Upsilon_{(u_1,u_2)}\right)'(\theta)=0$. According to Equation (4.1)

$$\Upsilon_{(u_1,u_2)}(\theta) = I(\theta*u_1,\theta*u_2) \geq g\left(\mathrm{e}^{2\theta}\tau\right),$$

where $\tau = \sum_{i=1}^{2}|\nabla u_i|_2^2$. Hence, if $\mathrm{e}^{2\theta}\tau\in(\tau_0,\tau_1)$, *i.e.*, $\theta\in\left(\frac{1}{2}\ln\frac{\tau_0}{\tau},\frac{1}{2}\ln\frac{\tau_1}{\tau}\right)$, we get $\Upsilon_{(u_1,u_2)}(\theta) > 0$. By $\Upsilon_{(u_1,u_2)}(-\infty) = 0^-$ and $\Upsilon_{(u_1,u_2)}(+\infty) = -\infty$, we can infer that $\Upsilon_{(u_1,u_2)}(\theta)$ possesses two critical points. Here $c_{(u_1,u_2)}$ represents the local minimum, $d_{(u_1,u_2)}$ denotes the global maximum and $c_{(u_1,u_2)} < d_{(u_1,u_2)}$. In addition, they satisfy the following

$$c_{(u_1,u_2)} < \frac{1}{2}\ln\frac{\tau_0}{\tau} < d_{(u_1,u_2)} < \frac{1}{2}\ln\frac{\tau_1}{\tau}.$$

As with Lemma 4.1, we can prove that $\Upsilon_{(u_1,u_2)}(\theta)$ possesses no more than two critical points. Hence, $\Upsilon_{(u_1,u_2)}(\theta)$ possesses precisely two points of criticality.

Observe the fact that $\theta*(u_1,u_2)\in\mathcal{M}\Leftrightarrow\left(\Upsilon_{(u_1,u_2)}\right)(\theta)=0$, then

$\theta * (u_1, u_2) \in \mathcal{M} \Leftrightarrow \theta \in \left\{ c_{(u_1,u_2)}, d_{(u_1,u_2)} \right\}$. As we know that $c_{(u_1,u_2)}$ is a local minimum point, so

$$\left(\Upsilon_{c_{(u_1,u_2)} * (u_1,u_2)} \right)'' (0) = \left(\Upsilon_{(u_1,u_2)} \right)'' \left(c_{(u_1,u_2)} \right) \geq 0.$$

Since $\mathcal{M}_0 = \varnothing$, so $\left(\Upsilon_{c_{(u_1,u_2)} * (u_1,u_2)} \right)'' (0) > 0$, and then $c_{(u_1,u_2)} * (u_1, u_2) \in \mathcal{M}_+$. Similarly, $d_{(u_1,u_2)} * (u_1, u_2) \in \mathcal{M}'_-$. Furthermore, due to the monotonic nature and considering the asymptotic behavior, $\Upsilon_{(u_1,u_2)}(\theta)$ possesses precisely two zeros $c_{(u_1,u_2)} < d_{(u_1,u_2)}$ and they satisfy $c_{(u_1,u_2)} < s_{(u_1,u_2)} < d_{(u_1,u_2)} < t_{(u_1,u_2)}$.

It remains to show that the maps $(u_1, u_2) \mapsto c_{(u_1,u_2)}$ and $(u_1, u_2) \mapsto d_{(u_1,u_2)}$ are of class C^1. Utilizing the theorem of implicit function on $T(\theta, u_1, u_2) := \left(\Upsilon_{(u_1,u_2)} \right)' (\theta)$, and using facts that

$$T\left(c_{(u_1,u_2)}, u_1, u_2 \right) = T\left(d_{(u_1,u_2)}, u_1, u_2 \right) = 0,$$

$$\partial_\theta T\left(c_{(u_1,u_2)}, u_1, u_2 \right) = \left(\Upsilon_{(u_1,u_2)} \right) \left(c_{(u_1,u_2)} \right) > 0,$$

$$\partial_\theta T\left(d_{(u_1,u_2)}, u_1, u_2 \right) = \left(\Upsilon_{(u_1,u_2)} \right)'' \left(d_{(u_1,u_2)} \right) < 0$$

and the reality that a continuous transition from $\mathcal{M}_+$ to $\mathcal{M}_-$ is impossible. Thus, the analysis shows that $(u_1, u_2) \mapsto c_{(u_1,u_2)}$ and $(u_1, u_2) \mapsto d_{(u_1,u_2)}$ belong to the C^1 class. □

Lemma 4.4. *Suppose the condition* (M_1) *is satisfied. For all* $(u_1, u_2) \in (S_r(m_1) \times S_r(m_2)) \cap \mathcal{V}_0$, *we have* $\Upsilon_{(u_1,u_2)}\left(s_{(u_1,u_2)} \right) = 0$ *and* $\left(\Upsilon_{(u_1,u_2)} \right)' \left(c_{(u_1,u_2)} \right) = 0$ *such that* $s_{(u_1,u_2)} < c_{(u_1,u_2)}$. *Additionally*

(i) $\mathcal{M} = \mathcal{M}_+$ and $\theta * (u_1, u_2) \in \mathcal{M}_+ \Leftrightarrow \theta = s_{(u_1,u_2)}$.

(ii) $I\left(c_{(u_1,u_2)} * (u_1, u_2) \right) = \min_{\theta \in \mathbb{R}} I\left(\theta * (u_1, u_2) \right)$.

(iii) $\left| \nabla(\theta * u_1) \right|_2^2 + \left| \nabla(\theta * u_2) \right|_2^2 < \tau_0$ where $\theta < s_{(u_1,u_2)}$.

Proof. We can suppose that $p_1 \leq p_2$ which is without generality loss. Clearly, $\Upsilon_{(u_1,u_2)}(\theta) \to +\infty$ as $\theta \to +\infty$ and $\Upsilon_{(u_1,u_2)}(\theta) \to 0^-$ as $\theta \to -\infty$ where $(u_1, u_2) \in (S_r(m_1) \times S_r(m_2)) \cap \mathcal{V}_0$. Hence, the function $\Upsilon_{(u_1,u_2)}(\theta)$ attains its global minimum at the point $c_{(u_1,u_2)}$, which is below the zero level. In order to prove the critical point of $\Upsilon_{(u_1,u_2)}(\theta)$ is unique, we can see that $\left(\Upsilon_{(u_1,u_2)} \right)' (\theta) = 0$ is equivalent to

$$\begin{aligned} & a\mathrm{e}^{\left(2-\gamma_{p_1} p_1\right)\theta} \sum_{i=1}^{2} \left| \nabla u_i \right|_2^2 + b\mathrm{e}^{\left(4-\gamma_{p_1} p_1\right)\theta} \sum_{i=1}^{2} \left| \nabla u_i \right|_2^4 \\ & - \mathrm{e}^{\left(-\gamma_{p_1} p_1\right)\theta} \sum_{i=1}^{2} \int_{\mathbb{R}^N} W_i\left(\mathrm{e}^{-\theta} x \right) u_i^2 \mathrm{d}x - \nu_2 \gamma_{p_2} \mathrm{e}^{\left(\gamma_{p_2} p_2 - \gamma_{p_1} p_1\right)\theta} \left| u_2 \right|_{p_2}^{p_2} \\ & = \nu_1 \gamma_{p_1} \left| u_1 \right|_{p_1}^{p_1}. \end{aligned}$$

Through some calculation analysis, it becomes evident that the equation possesses a single solution. Therefore, $\theta * (u_1, u_2) \in \mathcal{M} \Leftrightarrow \theta = s_{(u_1,u_2)}$. By minimality $\left(\Upsilon_{(u_1,u_2)} \right)'' \left(s_{(u_1,u_2)} \right) \geq 0$, and since $\mathcal{M}_0 = \varnothing$, we deduce that

$\left(\Upsilon_{c_{(u_1,u_2)}*(u_1,u_2)}\right)''(0)>0$, and then $c_{(u_1,u_2)}*(u_1,u_2)\in\mathcal{M}_+$. Furthermore, due to the monotonic nature and considering the asymptotic behavior, the function $\Upsilon_{(u_1,u_2)}(\theta)$ possesses a sole zero point $s_{(u_1,u_2)}$ and satisfying $c_{(u_1,u_2)}<s_{(u_1,u_2)}$. As $\Upsilon_{(u_1,u_2)}(\theta)\geq g\left(\mathrm{e}^{2\theta}\left(|\nabla u_1|_2^2+|\nabla u_2|_2^2\right)\right)$, then $\Upsilon_{(u_1,u_2)}(\theta)\geq g(\tau_0)=0$ at $s=\frac{1}{2}\ln\frac{\tau_0}{|\nabla u_1|_2^2+|\nabla u_2|_2^2}$. Hence, $c_{(u_1,u_2)}\leq\frac{1}{2}\ln\frac{\tau_0}{|\nabla u_1|_2^2+|\nabla u_2|_2^2}$. It follows that $|\nabla(\theta*u_1)|_2^2+|\nabla(\theta*u_2)|_2^2<\tau_0$ where $\theta<c(u_1,u_2)$. □

Remark 4.3. *By Lemma* 4.3 *and* 4.4, *we can get* $\theta*(u_1,u_2)\in\mathcal{M}_+\Leftrightarrow\theta=s_{(u_1,u_2)}$ *and* $s_{(u_1,u_2)}*(u_1,u_2)\in\mathcal{B}(\tau_0)$ *where* $(u_1,u_2)\in S_r(m_1)\times S_r(m_2)$.

According to Equation (1.11), we have $c(m_1,m_2)=\inf_{\mathcal{B}(\tau_0)}I(u_1,u_2)<0$, we will study the properties of $c(m_1,m_2)$ as follows.

Lemma 4.5. *Suppose the condition* (M$_1$) *is satisfied. When* $0<\alpha<\alpha_0$, *we have*

$$c(m_1,m_2)=\inf_{\mathcal{M}}I(u_1,u_2)=\inf_{\mathcal{M}_+}I(u_1,u_2),$$

and there is a positive ε_0 sufficiently small, ensuring that

$$c(m_1,m_2)<\inf_{\overline{\mathcal{B}(\tau_0)\backslash\mathcal{B}(\tau_0-\varepsilon_0)}}I(u_1,u_2).$$

Proof. We first prove $c(m_1,m_2)=\inf_{\mathcal{M}_+}I(u_1,u_2)$. For every $(u_1,u_2)\in\mathcal{M}_+$, $c_{(u_1,u_2)}=0$. It follows from Lemma 4.3 and Lemma 4.4 that $0<\frac{1}{2}\ln\frac{\tau_0}{|\nabla u_1|_2^2+|\nabla u_2|_2^2}$, namely, $|\nabla u_1|_2^2+|\nabla u_1|_2^2<\tau_0$. Thus, $\mathcal{M}_+\subset\mathcal{B}(\tau_0)$, and then $c(m_1,m_2)\leq\inf_{\mathcal{M}_+}I(u_1,u_2)$. For every $(u_1,u_2)\in\mathcal{B}(\tau_0)$, there exists a unique $c_{(u_1,u_2)}\in\mathbb{R}$, such that

$$c_{(u_1,u_2)}*(u_1,u_2)\in\mathcal{M}_+\subset\mathcal{B}(\tau_0).$$

According to Lemma 4.3 (ii) and Lemma 4.4, it can infer that

$$\begin{aligned}I\left(c_{(u_1,u_2)}*(u_1,u_2)\right)&=\min\left\{I(\theta*(u_1,u_2)):s\in\mathbb{R},|\nabla(\theta*(u_1,u_2))|_2^2<\tau_0\right\}\\&\leq I(u_1,u_2).\end{aligned}$$

Thus, $\inf_{\mathcal{M}_+}I(u_1,u_2)\leq c(m_1,m_2)$. To sum up, we have $c(m_1,m_2)=\inf_{\mathcal{M}_+}I(u_1,u_2)$. By 4.3 (iii) and Lemma 4.4, we get that

$$\inf_{\mathcal{M}}I(u_1,u_2)=\inf_{\mathcal{M}_+}I(u_1,u_2).$$

Next, we have to proof $c(m_1,m_2)<\inf_{\overline{\mathcal{B}(\tau_0)\backslash\mathcal{B}(\tau_0-\varepsilon_0)}}I(u_1,u_2)$. As $g(\tau_0)=0$, considering the function g is continuous, there is a positive ε_0 sufficiently small, ensuring that $g(\tau)\geq\frac{c(m_1,m_2)}{2}$ where $\tau\in[\tau_0-\varepsilon_0,\tau_0]$. Therefore,

$$I(u_1,u_2)\geq g(\tau)\geq\frac{c(m_1,m_2)}{2}>c(m_1,m_2)$$

for every $(u_1,u_2)\in\overline{\mathcal{B}(\tau_0)}\setminus\mathcal{B}(\tau_0-\varepsilon_0)$. ☐

Lemma 4.6. *Suppose the condition* (M_1) *is satisfied. When* $0<\alpha<\alpha_0$, *we have* $c(m_1,m_2)<\min\{l(m_1,\nu_1),l(m_2,\nu_2)\}$.

Proof. We only prove $c(m_1,m_2)<l(m_1,\nu_1)$. For every $(u_1,u_2)\in\mathcal{B}(\tau_0)$

$$\begin{aligned}I(u_1,u_2)&=E_{\nu_1}(u_1)+E_{\nu_2}(u_2)-\alpha\int_{\mathbb{R}^N}|u_1|^{q_1}|u_2|^{q_2}\,\mathrm{d}x\\&\le E_{\nu_1}(u_1)+E_{\nu_2}(u_2).\end{aligned}$$

Thus, $c(m_1,m_2)\le\inf_{\mathcal{B}(\tau_0)}\left(E_{\nu_1}(u_1)+E_{\nu_2}(u_2)\right)$. For all $u_1\in S_r(m_1)$ satisfying $|\nabla u_1|_2^2=\tau_0$, as defined in Lemma 4.1, we have

$$\begin{aligned}E_{\nu_1}(u_1)&=\frac{a}{2}|\nabla u_1|_2^2+\frac{b}{4}|\nabla u_1|_2^4+\frac{1}{2}\int_{\mathbb{R}^N}V_1u_1^2\mathrm{d}x-\frac{\nu_1}{p_1}|u_1|_{p_1}^{p_1}\\&\ge K_0\tau_0^2-K_1\tau_0^{\frac{\gamma_{p_1}p_1}{2}}\ge g(\tau_0)=0,\end{aligned}$$

where K_0,K_1 are given by equation (4.1). Thus

$$\inf_{S_r(m_1)}E_{\nu_1}=\inf_{B(m_1,\tau_0)}E_{\nu_1}<0,$$

where $B(c,\tau):=\left\{u\in S_r(c):|\nabla u|_2^2<\tau\right\}$. In addition, because $\tau\mapsto K_0\tau^2-K_1\tau^{\frac{\gamma_p p}{2}}$ is continuous, by using the proof from Lemma 4.5 to show that we can find a sufficiently small constant $\varepsilon_0>0$, ensuring that

$$l(m_1,\nu_1)<\inf_{\overline{B(m_1,\tau_0)}\setminus B(m_1,\tau_0-\varepsilon_0)}E_{\nu_1}(u_1),$$

and one can easily confirm that $\inf_{B(m_2,\varepsilon_0)}E_{\nu_2}(u_2)<0$. Let

$$\mathbb{U}=\left\{(u_1,u_2):u_1\in B(m_1,\tau_0-\varepsilon_0),u_2\in B(m_2,\varepsilon_0)\right\},$$

then $\mathbb{U}\subset\mathcal{B}(\tau_0)$. Hence

$$\begin{aligned}\inf_{\mathcal{B}(\tau_0)}\left(E_{\nu_1}(u_1)+E_{\nu_2}(u_2)\right)&\le\inf_{\mathbb{U}}\left(E_{\nu_1}(u_1)+E_{\nu_2}(u_2)\right)\\&=\inf_{B(m_1,\tau_0-\varepsilon_0)}E_{\nu_1}(u_1)+\inf_{B(m_2,\varepsilon_0)}E_{\nu_2}(u_2)\\&<\inf_{B(m_1,\tau_0-\varepsilon_0)}E_{\nu_1}(u_1)\\&=l(m_1,\nu_1).\end{aligned}$$

Therefore, $c(m_1,m_2)<l(m_1,\nu_1)$. Similarly, $c(m_1,m_2)<l(m_2,\nu_2)$. ☐

Lemma 4.7. *Take the sequence* $\left\{(u_1^n,u_2^n)\right\}\subset S_r(m_1)\times S_r(m_2)$ *as a* $(PS)_{c(m_1,m_2)}$ *sequence for* $I|_{S_r(m_1)\times S_r(m_2)}$ *and* $J(u_1^n,u_2^n)\to0$ *as* $n\to+\infty$. *Then,* $(u_1^n,u_2^n)\to(u_1,u_2)$ *in* $H_{rad}^1(\mathbb{R}^N)\times H_{rad}^1(\mathbb{R}^N)$.

Proof. Because $I(|u_1^n|,|u_2^n|)=I(u_1^n,u_2^n)$, we might as well assume $u_1^n,u_2^n\ge0$. To begin with, we demonstrate that the sequence $\left\{(u_1^n,u_2^n)\right\}$ is bounded in $H_{\mathrm{rad}}^1(\mathbb{R}^N)\times H_{\mathrm{rad}}^1(\mathbb{R}^N)^1$. As $\left\{(u_1^n,u_2^n)\right\}$ is a $(PS)_{c(m_1,m_2)}$ sequence for I restricted on $S_r(m_1)\times S_r(m_2)$, we conclude that

$$I(u_1^n,u_2^n)\to c(m_1,m_2),\tag{4.4}$$

$$\left\| \left(I\big|_{S_r(m_1)\times S_r(m_2)} \right)' \left(u_1^n, u_2^n \right) \right\|_{\left(H^1\left(\mathbb{R}^N\right)\right)^*} \to 0, \tag{4.5}$$

$$J\left(u_1^n, u_2^n \right) \to 0, n \to +\infty. \tag{4.6}$$

By Equations (4.4) and (4.6), we get

$$\begin{aligned}
& c\left(m_1, m_2\right) + o(1) \\
& = a\left(\frac{1}{2} - \frac{1}{\gamma_q q}\right)\sum_{i=1}^{2}\left|\nabla u_i^n\right|_2^2 + b\left(\frac{1}{4} - \frac{1}{\gamma_q q}\right)\sum_{i=1}^{2}\left|\nabla u_i^n\right|_2^4 + \frac{1}{2}\sum_{i=1}^{2}\int_{\mathbb{R}^N} V_i u_i^2 \mathrm{d}x \\
& \quad + \frac{1}{\gamma_q q}\sum_{i=1}^{2}\int_{\mathbb{R}^N} W_i u_i^2 \mathrm{d}x \sum_{i=1}^{2} \nu_i \gamma_{p_i}\left(\frac{1}{\gamma_{p_i} p_i} - \frac{1}{\gamma_q q}\right)\left|u_i^n\right|_{p_i}^{p_i} \\
& \geq \left(\frac{a - \max\left\{\omega_1, \omega_2\right\}}{2} - \frac{a - \min\left\{\rho_1, \rho_2\right\}}{\gamma_q q}\right)\tau + \frac{b}{2}\left(\frac{1}{4} - \frac{1}{\gamma_q q}\right)\tau^2 \\
& \quad - \sum_{i=1}^{2} C_{N,p_i}\gamma_{p_i}\left(\frac{1}{\gamma_{p_i} p_i} - \frac{1}{\gamma_q q}\right) c_i^{\frac{\left(1-\gamma_{p_i}\right)p_i}{2}} \nu_i \left|\nabla u_i^n\right|_2^{\gamma_{p_i} p_i} \\
& \geq \left(\frac{a - \max\left\{\omega_1, \omega_2\right\}}{2} - \frac{a - \min\left\{\rho_1, \rho_2\right\}}{\gamma_q q}\right)\tau + \frac{b}{2}\left(\frac{1}{4} - \frac{1}{\gamma_q q}\right)\tau^2 - \sum_{i=1}^{2} C_i \tau^{\frac{\gamma_{p_i} p_i}{2}},
\end{aligned}$$

where $\tau = \sum_{i=1}^{2}\left|\nabla u_i\right|_2^2$, $0 < \gamma_{p_i} p_i < 2$, $4 < \gamma_q q < 2^*$, $C_i > 0 \ (i = 1,2)$. Thus, $\left(u_1^n, u_2^n\right)$ is bounded in $H_{\mathrm{rad}}^1\left(\mathbb{R}^N\right) \times H_{\mathrm{rad}}^1\left(\mathbb{R}^N\right)$, and we may assume that

$$\left(u_1^n, u_2^n\right) \rightharpoonup \left(u_1, u_2\right) \text{ in } H_{\mathrm{rad}}^1\left(\mathbb{R}^N\right) \times H_{\mathrm{rad}}^1\left(\mathbb{R}^N\right),$$

$$\left(u_1^n, u_2^n\right) \to \left(u_1, u_2\right) \text{ in } L^s\left(\mathbb{R}^N\right) \times L^s\left(\mathbb{R}^N\right), s \in \left(2, 2^*\right).$$

According to Equation (4.5), we can find two real-valued sequences $\left\{\lambda_1^n\right\}, \left\{\lambda_2^n\right\}$, such that

$$\begin{aligned}
o(1)\left\|\left(\varphi_1, \varphi_2\right)\right\| = {} & a\sum_{i=1}^{2}\int_{\mathbb{R}^N} \nabla u_i^n \nabla \varphi_i \mathrm{d}x + b\sum_{i=1}^{2}\int_{\mathbb{R}^N}\left|\nabla u_i^n\right|^2 \mathrm{d}x \int_{\mathbb{R}^N} \nabla u_i^n \nabla \varphi_i \mathrm{d}x \\
& + \sum_{i=1}^{2}\int_{\mathbb{R}^N} V_i^n u_i^n \varphi_i \mathrm{d}x - \sum_{i=1}^{2} \nu_i \int_{\mathbb{R}^N}\left|u_i^n\right|^{p_i - 2} u_i^n \varphi_i \mathrm{d}x \\
& - \alpha q_1 \int_{\mathbb{R}^N}\left|u_1^n\right|^{q_1 - 2} u_1^n \left|u_2^n\right|^{q_2} \varphi_1 \mathrm{d}x - \alpha q_2 \int_{\mathbb{R}^N}\left|u_1^n\right|^{q_1}\left|u_2^n\right|^{q_2 - 2} u_2^n \varphi_2 \mathrm{d}x \\
& - \sum_{i=1}^{2}\int_{\mathbb{R}^N} \lambda_i^n u_i^n \varphi_i \mathrm{d}x,
\end{aligned}$$

where $o(1) \to 0$ as $n \to +\infty$. By Lemma 2.7, $\lambda_i^n \to \lambda_i$ for $i = 1,2$ and $\left(u_1, u_2\right)$ is a solution to the system (1.1). Since $I\left(u_1, u_2\right) \leq \lim_{n \to \pm\infty} I\left(u_1^n, u_2^n\right) = c\left(m_1, m_2\right) < 0$, then $\left(u_1, u_2\right) \neq (0,0)$. According to the argument of Theorem 1.1 (i), at least one of λ_1, λ_2 is less than zero. If $\lambda_1 < 0$, based on the information provided in Lemma 2.6, we can deduce that $u_1^n \to u_1$ in $H_{\mathrm{rad}}^1\left(\mathbb{R}^N\right)$. Subsequently, we show that $\lambda_2 < 0$. If λ_2 is not less than zero, then

$$\left(a + b\int_{\mathbb{R}^N}\left|\nabla u_2\right|^2 \mathrm{d}x\right)\Delta u_2 + V_2 u_2 = \lambda_2 u_2 + \nu_2\left|u_2\right|^{p_2 - 2} u_2 + \alpha q_2\left|u_1\right|^{q_1}\left|u_2\right|^{q_2 - 2} u_2 \geq 0$$

and u_1 satisfies the following equation

$$\begin{cases} -\left(a+b\int_{\mathbb{R}^N}\left|\nabla u_1\right|^2 \mathrm{d}x\right)\Delta u_1+V_1u_1=\lambda u_1+\nu\left|u_1\right|^{p-2}u_1 & \text{in } \mathbb{R}^N, \\ \int_{\mathbb{R}^N}\left|u_1\right|^2 \mathrm{d}x=m_1 & \text{in } \mathbb{R}^N, \end{cases}$$

where $2<p_1<2+\frac{4}{N}$. By Lemma 2.1, u_1 is unique and then, $E_{\nu_1}\left(u_1\right)=l\left(m_1,\nu_1\right)$. Thus

$$\begin{aligned} c\left(m_1,m_2\right) &= \lim_{n\to\infty} I\left(u_1^n,u_2^n\right) \\ &= \lim_{n\to\infty}\frac{a}{2}\sum_{i=1}^{2}\left|\nabla u_i^n\right|_2^2+\frac{b}{4}\sum_{i=1}^{2}\left|\nabla u_i^n\right|_2^4+\sum_{i=1}^{2}\int_{\mathbb{R}^N}V_i^n\left(u_i^n\right)^2\mathrm{d}x \\ &\quad -\sum_{i=1}^{2}\frac{\nu_i}{p_i}\left|u_i^n\right|_{p_i}^{p_i}-\alpha\int_{\mathbb{R}^N}\left|u_1^n\right|^{q_1}\left|u_2^n\right|^{q_2}\mathrm{d}x \\ &= \lim_{n\to\infty}\frac{a}{2}\sum_{i=1}^{2}\left|\nabla u_i^n\right|_2^2+\frac{b}{4}\sum_{i=1}^{2}\left|\nabla u_i^n\right|_2^4+\sum_{i=1}^{2}\int_{\mathbb{R}^N}V_i^n\left(u_i^n\right)^2\mathrm{d}x-\frac{\nu_1}{p_1}\left|u_1^n\right|_{p_1}^{p_1} \\ &\geq \frac{a}{2}\left|\nabla u_1\right|_2^2+\frac{b}{4}\left|\nabla u_1\right|_2^4+\int_{\mathbb{R}^N}V_1u_1^2\mathrm{d}x-\frac{\nu_1}{p_1}\left|u_1\right|_{p_1}^{p_1} \\ &= l\left(m_1,\nu_1\right). \end{aligned}$$

This results in a conflict with the conclusions of Lemma 4.6. Consequently, we can infer that $\lambda_2<0$ which implies that $u_2^n\to u_2$ in $H_{\mathrm{rad}}^1\left(\mathbb{R}^N\right)$. □

***Proof of Theorem* 1.1.** (*ii*). Let $\left\{\left(u_1^n,u_2^n\right)\right\}\subset\mathcal{B}\left(\tau_0\right)$ be a minimizing sequence for $c\left(m_1,m_2\right)$, that is

$$I\left(u_1^n,u_2^n\right)\to c\left(m_1,m_2\right).$$

By Lemmas 4.3 and 4.4, $s_{\left(u_1^n,u_2^n\right)}*\left(u_1^n,u_2^n\right)\in\mathcal{M}_+$ for every n,

$$\left|\nabla\left(s_{\left(u_1^n,u_2^n\right)}*\left(u_1^n,u_2^n\right)\right)\right|_2^2<\tau_0 \text{ and}$$

$$I\left(s_{\left(u_1^n,u_2^n\right)}*\left(u_1^n,u_2^n\right)\right)\leq I\left(u_1^n,u_2^n\right).$$

Let $\left(w_1^n,w_2^n\right):=s_{\left(u_1^n,u_2^n\right)}*\left(u_1^n,u_2^n\right)$, then $\left\{\left(w_1^n,w_2^n\right)\right\}\subset\mathcal{B}\left(\tau_0\right)$ serves as a minimizing sequence for $c\left(m_1,m_2\right)$ and $\left(w_1^n,w_2^n\right)\in\mathcal{M}_+$. By Lemma 4.5, $\left\{\left(w_1^n,w_2^n\right)\right\}\subset A\left(\tau_0-\varepsilon_0\right)$. Thus, Ekeland's variational principle yields the existence of a new minimizing sequence $\left\{\left(v_1^n,v_2^n\right)\right\}$, which is also a Palais-Smale sequence for $I|_{S_r\left(m_1\right)\times S_r\left(m_2\right)}$ and $\left\|\left(w_1^n,w_2^n\right)-\left(v_1^n,v_2^n\right)\right\|\to0$. Hence, $\left\{\left(v_1^n,v_2^n\right)\right\}\subset\mathcal{B}\left(\tau_0\right)$ and $J\left(v_1^n,v_2^n\right)\to0$. It follows from Lemma 4.7 that there exists $v_1,v_2>0$ such that $\left(v_1^n,v_2^n\right)\to\left(v_1,v_2\right)$ in $H_{\mathrm{rad}}^1\left(\mathbb{R}^N\right)\times H_{\mathrm{rad}}^1\left(\mathbb{R}^N\right)$, and then, $I|_{A\left(\tau_0\right)}$ attains a local minimum at $\left(v_1,v_2\right)$. Accordingly, $\left(v_1,v_2\right)$ is a solution for Equations (1.1)-(1.2) for some $\lambda_1,\lambda_2<0$, which is positive and radially. □

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

References

[1] Li, G., Luo, X. and Yang, T. (2022) Normalized Solutions to a Class of Kirchhoff Equations with Sobolev Critical Exponent. *Annales Fennici Mathematici*, **47**, 895-925. https://doi.org/10.54330/afm.120247

[2] Carrião, P.C., Miyagaki, O.H. and Vicente, A. (2022) Normalized Solutions of Kirchhoff Equations with Critical and Subcritical Nonlinearities: The Defocusing Case. *Partial Differential Equations and Applications*, **3**, Article No. 64. https://doi.org/10.1007/s42985-022-00201-3

[3] Ye, H. (2014) The Existence of Normalized Solutions for L^2-Critical Constrained Problems Related to Kirchhoff Equations. *Zeitschrift für angewandte Mathematik und Physik*, **66**, 1483-1497. https://doi.org/10.1007/s00033-014-0474-x

[4] Kirchhoff, G. (1883) Mechanik. Teubner.

[5] Lions, J.L. (1978) On Some Questions in Boundary Value Problems of Mathematical Physics. *North-Holland Mathematics Studies*, **30**, 284-346. https://doi.org/10.1016/s0304-0208(08)70870-3

[6] Alves, C.O., Corrêa, F.J.S.A. and Ma, T.F. (2005) Positive Solutions for a Quasilinear Elliptic Equation of Kirchhoff Type. *Computers & Mathematics with Applications*, **49**, 85-93. https://doi.org/10.1016/j.camwa.2005.01.008

[7] Arosio, A. and Panizzi, S. (1996) On the Well-Posedness of the Kirchhoff String. *Transactions of the American Mathematical Society*, **348**, 305-330. https://doi.org/10.1090/s0002-9947-96-01532-2

[8] He, X. and Zou, W. (2012) Existence and Concentration Behavior of Positive Solutions for a Kirchhoff Equation in $\mathbb{R}^3$. *Journal of Differential Equations*, **252**, 1813-1834. https://doi.org/10.1016/j.jde.2011.08.035

[9] Mao, A. and Zhang, Z. (2009) Sign-Changing and Multiple Solutions of Kirchhoff Type Problems without the P.S. Condition. *Nonlinear Analysis*: *Theory*, *Methods & Applications*, **70**, 1275-1287. https://doi.org/10.1016/j.na.2008.02.011

[10] Zhang, Z. and Perera, K. (2006) Sign Changing Solutions of Kirchhoff Type Problems via Invariant Sets of Descent Flow. *Journal of Mathematical Analysis and Applications*, **317**, 456-463. https://doi.org/10.1016/j.jmaa.2005.06.102

[11] Akhmediev, N. and Ankiewicz, A. (1999) Partially Coherent Solitons on a Finite Background. *Physical Review Letters*, **82**, 2661-2664. https://doi.org/10.1103/physrevlett.82.2661

[12] Frantzeskakis, D.J. (2010) Dark Solitons in Atomic Bose-Einstein Condensates: From Theory to Experiments. *Journal of Physics A*: *Mathematical and Theoretical*, **43**, Article ID: 213001. https://doi.org/10.1088/1751-8113/43/21/213001

[13] Pohožaev, S. (1975) A Certain Class of Quasilinear Hyperbolic Equations. *Matematicheskiĭ Sbornik*, **96**, 152-166.

[14] Liu, C. and Yang, X. (2022) Existence of Normalized Solutions for Semilinear Elliptic Systems with Potential. *Journal of Mathematical Physics*, **63**, Article ID: 061504. https://doi.org/10.1063/5.0077931

[15] Liu, M. and Fang, X. (2022) Normalized Solutions for the Schrödinger Systems with Mass Supercritical and Double Sobolev Critical Growth. *Zeitschrift für angewandte Mathematik und Physik*, **73**, Article No. 108.

https://doi.org/10.1007/s00033-022-01757-1

[16] Chen, Z. and Zou, W. (2021) Normalized Solutions for Nonlinear Schrödinger Systems with Linear Couples. *Journal of Mathematical Analysis and Applications*, **499**, Article ID: 125013. https://doi.org/10.1016/j.jmaa.2021.125013

[17] Bartsch, T., Jeanjean, L. and Soave, N. (2016) Normalized Solutions for a System of Coupled Cubic Schrödinger Equations on $\mathbb{R}^3$. *Journal de Mathématiques Pures et Appliquées*, **106**, 583-614. https://doi.org/10.1016/j.matpur.2016.03.004

[18] Bartsch, T. and Soave, N. (2017) A Natural Constraint Approach to Normalized Solutions of Nonlinear Schrödinger Equations and Systems. *Journal of Functional Analysis*, **272**, 4998-5037. https://doi.org/10.1016/j.jfa.2017.01.025

[19] Bartsch, T. and Soave, N. (2019) Multiple Normalized Solutions for a Competing System of Schrödinger Equations. *Calculus of Variations and Partial Differential Equations*, **58**, Article No. 22. https://doi.org/10.1007/s00526-018-1476-x

[20] Li, H. and Zou, W. (2021) Normalized Ground States for Semilinear Elliptic Systems with Critical and Subcritical Nonlinearities. *Journal of Fixed Point Theory and Applications*, **23**, Article No. 43. https://doi.org/10.1007/s11784-021-00878-w

[21] Luo, X., Yang, X.L. and Zou, W. (2021) Positive Normalized Solutions to Nonlinear Elliptic Systems in $\mathbb{R}^4$ with Critical Sobolev Exponent. arXiv:2107.08708

[22] Hu, J. and Mao, A. (2023) Normalized Solutions to Nonlocal Schrödinger Systems with L^2-Subcritical and Supercritical Nonlinearities. *Journal of Fixed Point Theory and Applications*, **25**, Article No. 77. https://doi.org/10.1007/s11784-023-01077-5

[23] Bartsch, T. and Jeanjean, L. (2017) Normalized Solutions for Nonlinear Schrödinger Systems. *Proceedings of the Royal Society of Edinburgh*: *Section A Mathematics*, **148**, 225-242. https://doi.org/10.1017/s0308210517000087

[24] Ikoma, N. (2014) Compactness of Minimizing Sequences in Nonlinear Schrödinger Systems under Multiconstraint Conditions. *Advanced Nonlinear Studies*, **14**, 115-136. https://doi.org/10.1515/ans-2014-0104

[25] Weinstein, M.I. (1983) Nonlinear Schrödinger Equations and Sharp Interpolation Estimates. *Communications in Mathematical Physics*, **87**, 567-576. https://doi.org/10.1007/bf01208265

[26] Qi, S. and Zou, W. (2022) Exact Number of Positive Solutions for the Kirchhoff Equation. *SIAM Journal on Mathematical Analysis*, **54**, 5424-5446. https://doi.org/10.1137/21m1445879

[27] Ye, H. (2014) The Sharp Existence of Constrained Minimizers for a Class of Nonlinear Kirchhoff Equations. *Mathematical Methods in the Applied Sciences*, **38**, 2663-2679. https://doi.org/10.1002/mma.3247

[28] Zeng, X. and Zhang, Y. (2017) Existence and Uniqueness of Normalized Solutions for the Kirchhoff Equation. *Applied Mathematics Letters*, **74**, 52-59. https://doi.org/10.1016/j.aml.2017.05.012

[29] Gou, T. and Jeanjean, L. (2018) Multiple Positive Normalized Solutions for Nonlinear Schrödinger Systems. *Nonlinearity*, **31**, 2319-2345. https://doi.org/10.1088/1361-6544/aab0bf

[30] Willem, M. (1996) Minimax Theorems. Birkhauser.

[31] Luo, X. and Wang, Q. (2017) Existence and Asymptotic Behavior of High Energy Normalized Solutions for the Kirchhoff Type Equations in $\mathbb{R}^N$. *Nonlinear Analysis*: *Real World Applications*, **33**, 19-32. https://doi.org/10.1016/j.nonrwa.2016.06.001

[32] Ghoussoub, N. (1993) Duality and Perturbation Methods in Critical Point Theory. Cambridge University Press. https://doi.org/10.1017/cbo9780511551703