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Integral Representations of the Zeta Function

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Abstract

The Riemann Zeta function $\zeta(s)$ has many different representations. In this paper, we derive several new integral representations of the Zeta function using the inverse Mellin transform and a hyperbolic cosecant identity. We also derive a general integral transformation similar to the Dirac delta function and propose a few new avenues for solving Riemann's Hypothesis.

Keywords

Integral Representation, Riemann Zeta Function, Reproducing Kernel, Dirac Delta Function

1. Introduction

It is known that all the non-trivial zeroes lie within the critical strip $\zeta(\sigma+it)$, where $0<\sigma<1$, and the Riemann Hypothesis asks whether all the non-trivial zeroes lie on the line $\sigma=\frac{1}{2}$. To answer this question, we will study its behavior under the integral transformation $\int_{-\infty}^{\infty}\frac{b^{it}}{t}\mathrm{d}t$, as b approach ∞.

There have been many attempts to use integral representations of the zeta function or related L-functions to answer questions about itself. For example, *Pain* found an integral representation for the zeta function for all odd arguments $\zeta(2p+1)$ [1], and *Milgram* discovered a family of integral and series representations for the zeta and eta (alternative zeta) function [2]. More integral forms of the zeta function are listed on the DLMF library [3].

In the first section, we will lay out the preliminaries. Next, we will derive an integral representation of $\zeta(s)$. Then, we will analyze the given integral transformation on a general function $f(x)$. Finally, we will conclude by giving some observations and proposing a few possible methods for further study on this method.

Preliminaries

We will first establish a few key formulas necessary for the derivation.

First, comparing the inverse function $\frac{1}{x}$ and $\operatorname{csch}(x)$ gives:

Lemma 1.

$$\frac{1}{x} = \lim_{b\to\infty}\frac{1}{b}\operatorname{csch}\left(\frac{x}{b}\right)$$

Applying this to the summative definition of the Zeta function yields the following.

Lemma 2.

$$\zeta(s) = \lim_{b\to\infty}\sum_{x=1}^{\infty}\frac{1}{b}\operatorname{csch}\left(\frac{x^s}{b}\right)$$

Next, the inverse Mellin transform of the Gamma function:

Lemma 3.

$$\mathrm{e}^{-x} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\Gamma(s)x^{-s}\,\mathrm{d}s$$

where c is any positive real number

We will also use the reflection formula for the Gamma function:

Lemma 4.

$$\Gamma(t)\Gamma(1-t) = \pi\csc(\pi t)$$

$$\Gamma(t) = \frac{\pi\csc(\pi t)}{\Gamma(1-t)}$$

Finally, we need the functional equation for the Zeta function.

Lemma 5.

$$\zeta(s) = 2^s\pi^{s-1}\sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)\zeta(1-s)$$

2. Main Result

Starting with the exponential definition of $\operatorname{csch}(x)$:

$$\operatorname{csch}(x) = \frac{-2\mathrm{e}}{1-\mathrm{e}^{2x}} = -2\sum_{n=0}^{\infty}\mathrm{e}^{(2n+1)x}$$

Using the inverse Mellin transform for e^{-x} from Lemma 3:

$$= -\frac{1}{\pi i}\sum_{n=0}^{\infty}\int_{c-i\infty}^{c+i\infty}\Gamma(s)\left[-\left((2n+1)x\right)\right]^{-s}\mathrm{d}s$$

$$= -\frac{1}{\pi i}\sum_{n=0}^{\infty}\int_{c-i\infty}^{c+i\infty}\Gamma(s)(-x)^{-s}\left[(2n+1)\right]^{-s}\mathrm{d}s$$

$$= -\frac{1}{\pi i}\int_{c-i\infty}^{c+i\infty}\Gamma(s)(-x)^{-s}\sum_{n=0}^{\infty}\left[(2n+1)\right]^{-s}\mathrm{d}s$$

Notice that the summation is the Zeta function over all odd numbers

$$\operatorname{csch}(x) = -\frac{1}{\pi}\int_{c-i\infty}^{c+i\infty}\Gamma(s)(-x)^{-s}\,\zeta(s)\left(1-2^{-s}\right)\mathrm{d}s$$

Substitute this definition into Lemma 2

$$\zeta(s) = \lim_{b\to\infty} -\frac{1}{b\pi}\sum_{n=1}^{\infty}\int_{c-i\infty}^{c+i\infty}\Gamma(t)\zeta(t)\left(1-2^{-t}\right)\left(-\frac{n^{s}}{b}\right)^{-t}\mathrm{d}t$$

$$\zeta(s) = \lim_{b\to\infty} -\frac{1}{b\pi}\int_{c-i\infty}^{c+i\infty}\Gamma(t)\zeta(t)\left(1-2^{-t}\right)\sum_{n=1}^{\infty}\left(-\frac{n^{s}}{b}\right)^{-t}\mathrm{d}t$$

$$\zeta(s) = \lim_{b\to\infty}\frac{1}{\pi}\int_{c-i\infty}^{c+i\infty}\Gamma(t)\zeta(t)\left(1-2^{-t}\right)b^{t-1}\zeta(st)\,\mathrm{d}t$$

$$\zeta(s) = \lim_{b\to\infty}\frac{1}{\pi}\int_{c-i\infty}^{c+i\infty}\Gamma(t)\zeta(t)b^{t-1}\zeta(st)\,\mathrm{d}t - \frac{1}{\pi}\int_{c-i\infty}^{c+i\infty}\Gamma(t)\zeta(t)\left(-2^{-t}\right)b^{t-1}\zeta(st)\,\mathrm{d}t$$

Theorem 6.

$$\zeta(s) = \lim_{b\to\infty}\frac{1}{2\pi}\int_{c-i\infty}^{c+i\infty}\Gamma(t)\zeta(t)\zeta(st)b^{t-1}\mathrm{d}t$$

We could even take this one step further by using the Reflection formula and the functional equation (Lemma 4, 5) and replace the $\zeta(t)$ term

$$\zeta(s) = \lim_{b\to\infty}\frac{1}{2\pi}\int_{c-i\infty}^{c+i\infty}\frac{\pi\csc(\pi t)}{\Gamma(1-t)}\zeta(t)\zeta(st)b^{t-1}\mathrm{d}t$$

$$\zeta(s) = \lim_{b\to\infty}\frac{1}{2\pi}\int_{c-i\infty}^{c+i\infty}\frac{\pi\csc(\pi t)}{\Gamma(1-t)}2^{t}\pi^{t-1}\sin\left(\frac{\pi t}{2}\right)\Gamma(1-t)\zeta(1-t)\zeta(st)b^{t-1}\mathrm{d}t$$

$$\zeta(s) = \lim_{b\to\infty}\int_{c-i\infty}^{c+i\infty}\frac{\sin\left(\frac{\pi t}{2}\right)}{\sin(\pi t)}(2\pi b)^{t-1}\zeta(1-t)\zeta(st)\,\mathrm{d}t$$

Which gives us our final integral functional equation:

Theorem 7.

$$\zeta(s) = \lim_{b\to\infty}\frac{1}{2}\int_{c-i\infty}^{c+i\infty}\sec\left(\frac{\pi t}{2}\right)\zeta(1-t)\zeta(st)b^{t-1}\mathrm{d}t$$

Plugging in the integral bounds gives us:

$$\zeta(s) = \lim_{b\to\infty}\frac{1}{2}\int_{-\infty}^{\infty}\sec\left(\frac{\pi(c+it)}{2}\right)\zeta(1-c-it)\zeta\left(s(c+it)\right)b^{c+it-1}\mathrm{d}t$$

Taking $1+\epsilon$, where $\epsilon = \lim_{\epsilon\to 0}$, we get:

$$\zeta(s) = \lim_{b\to\infty}\frac{1}{2}\int_{-\infty}^{\infty}\sec\left(\frac{\pi(1+it)}{2}\right)\zeta(-it)\zeta(s+ist)b^{it}\mathrm{d}t$$

$$\zeta(s) = \lim_{b\to\infty}\frac{1}{2}\int_{-\infty}^{\infty}\csc\left(\frac{i\pi t}{2}\right)\zeta(s+ist)b^{it}\sum_{k=1}^{\infty}k^{it}\mathrm{d}t$$

$$\zeta(s) = \lim_{b\to\infty}\frac{1}{2}\int_{-\infty}^{\infty}\csc\left(\frac{i\pi t}{2}\right)\zeta(s+ist)\sum_{k=1}^{\infty}(kb)^{it}\,\mathrm{d}t$$

$$\zeta(s)=\lim_{b\to\infty}\frac{-i}{2}\int_{-\infty}^{\infty}\operatorname{csch}\left(\frac{\pi t}{2}\right)\zeta(s+ist)b^{it}\,\mathrm{d}t$$

Using Lemma 1 and substituting for $\operatorname{csch}\left(\frac{\pi t}{2}\right)$ gives

$$\zeta(s)=\lim_{b\to\infty}\frac{-i}{2}\int_{-\infty}^{\infty}\frac{2}{\pi t}\zeta(s+ist)b^{it}\,\mathrm{d}t$$

After simplifying, we get our final equation

Theorem 8.

$$\zeta(s)=\lim_{b\to\infty}\frac{-i}{\pi}\int_{-\infty}^{\infty}\zeta(s+ist)\frac{b^{it}}{t}\,\mathrm{d}t$$

2.1. Recovering the Zeta Function

There are 2 ways to compute the integral. First, using integration by parts, and taking $u=\zeta(s+ist)$ and $\mathrm{d}v=\frac{b^{it}}{t}\mathrm{d}t$ gives us

$$-\frac{i}{\pi}\left[\zeta(s+it)Ei(it\ln b)\right]-\int_{-\infty}^{\infty}\zeta'(s+ist)Ei(it\ln b)$$

where Ei is the exponential integral.

The second way is to use the Fourier transform of the principal-value

$$-\frac{i}{\pi}PV\int_{-\infty}^{\infty}\zeta(s+it)\frac{b^{it}}{t}\mathrm{d}t=\frac{i}{\pi}\sum_{n=1}^{\infty}n^{-s}PV\int_{-\infty}^{\infty}\mathrm{e}^{it\ln(b/n)}\frac{\mathrm{d}t}{t}$$

and using the standard Fourier principal-value identity (for real y)

$$PV\int_{-\infty}^{\infty}\frac{\mathrm{e}^{iyt}}{t}\mathrm{d}t=i\pi(y)$$

$$-\frac{i}{\pi}PV\int_{-\infty}^{\infty}\zeta(s+it)\frac{b^{it}}{t}\mathrm{d}t=\sum_{n=1}^{\infty}n^{-s}\left(\ln(b/n)\right)$$

$$-\frac{i}{\pi}PV\int_{-\infty}^{\infty}\zeta(s+it)\frac{b^{it}}{t}\mathrm{d}t=2\sum_{n=1}^{\lfloor b\rfloor}n^{-s}-\zeta(s)$$

and as b approaches ∞, the integral condenses to $\zeta(s)$. This completes the proof for $\zeta(s)=\zeta(s)$

2.2. Reproducing Kernel

In our derivation above, there were multiple instances where

$$c\int_{-\infty}^{\infty}f(t)\zeta(s+ist)\mathrm{d}t=\zeta(s)$$

where c is a scaling factor and $f(t)$ is a possible "reproducing kernel". In fact, we can test this "kernel" against other functions to see if the integral recovers its initial value.

$$\frac{1}{2}\int_{-\infty}^{\infty}L(s+ist,\chi)\csc\left(\frac{i\pi t}{2}\right)b^{it}\mathrm{d}t=L(s,\chi)$$

where $L(s,\chi)$ represents the Dirichlet L-series with Dirichlet character χ, and

$$-\frac{i}{\pi}\int_{-\infty}^{\infty}\theta(s+ist)\frac{b^{it}}{t}\mathrm{d}t=\theta(s)$$

where $\theta(s)$ represents the Riemann-Siegel theta function,

$$\theta(t)=\arg\left(\Gamma\left(\frac{1}{4}+\frac{it}{2}\right)\right)-\frac{\log\pi}{2}t$$

This can be proven using the same techniques as the proof for $\zeta(s)$ as shown above in Section 2.1.

Theorem 9 (Reproducing kernel for $L(s,\chi)$). Let $K_b(t)=\csc\left(\frac{i\pi t}{2}\right)b^{it}$ denote the kernel defined in Section 3.2. Then, formally,

$$\frac{1}{2\pi}PV\int_{-\infty}^{\infty}K_b(t)L(s+ist,\chi)\mathrm{d}t=\lim_{b\to\infty}L(s,\chi),$$

where $L(s,\chi)$ is the Dirichlet L-series associated to a primitive character χ.

For $L(s,\chi)=\sum_{n\ge1}\chi(n)n^{-s}$ we have

$$\frac{1}{2\pi}PV\int_{-\infty}^{\infty}K_b(t)L(s+ist,\chi)\mathrm{d}t=\frac{1}{2\pi}\sum_{n\ge1}\chi(n)n^{-s}PV\int_{-\infty}^{\infty}K_b(t)\mathrm{e}^{it\ln(b/n)}\mathrm{d}t.$$

The inner principal-value integral equals $2\pi\,\mathrm{sgn}(\ln(b/n))$ (up to the same scaling used in Section 2.1). Summing over $n\le b$ therefore recovers $L(s,\chi)$ in the limit $b\to\infty$.

The same computation applies to the Riemann-Siegel function $\theta(s)$ after expressing it as a Dirichlet-type series or via its Gamma factors.

These results have held true for all empirical testing. Intuitively, this is equivalent to the Dirac delta function where the integral grows to infinity at s and goes to zero everywhere else.

3. Discussion

3.1. Empirical Analysis

The limit $b\to\infty$ is approximated numerically by taking a large finite value of b. In practice we found that the choice $b=25000000$ produced stable values of the integral (to approximately two decimal places for $\Re(s)>1$) while remaining computationally efficient. The contour parameter c must satisfy $c>1$ in order for the Mellin inversion condition earlier. However, the numerical integral becomes numerically unstable for values of c much larger than 1. Therefore, the empirical value of $c=1.1$ was chosen to avoid computational issues. All integrals were evaluated using mpmath numerical integrator on Python.

Using $b=25000000$, $c=1.001$, and $dps=20$, the integral representation approximates $\zeta(s)$ to 2 decimal places for all the values tested for real numbers $s>1$. For $s<1$, however, the approximation accuracy decreases continuously and rapidly.

For any complex value s with a non-zero imaginary part, the numerical integral diverges to infinity. This seems to be due to numerical rounding and error accumulation, rather than an analytic issue with the integral.

The integral is also very sensitive to the value of the parameter c (the real part of the integral bounds). From the limited testing data shown, only c values slightly greater than 1 provide an accurate result. Any value less than 1 causes the integral to diverge, and so does any value much greater than 1. Therefore, the current optimal value of c is 1.001. From the formula itself, any value $c > 1$ should work because $\zeta(s)$ is analytic in the right half of the plane. Therefore, it is also unknown whether manipulating the integral has changed the range of c or if it is simply a convergence issue.

3.2. Derivative

Using the integral representations, finding the derivative of the Zeta function becomes trivial. Differentiating the definition from Theorem 6 gives:

$$\zeta^{(n)}(s) = \lim_{b\to\infty} \frac{1}{2\pi} \int_{c-i\infty}^{c+i\infty} \Gamma(t)\zeta(t)\zeta^{(n)}(st) b^{t-1} t^n \mathrm{d}t$$

Differentiating the definition from Theorem 7 gives:

$$\zeta^{(n)}(s) = \lim_{b\to\infty} \frac{1}{2} \int_{c-i\infty}^{c+i\infty} \sec\left(\frac{\pi t}{2}\right)\zeta(1-t)\zeta^{(n)}(st) b^{t-1} t^n \mathrm{d}t$$

3.3. Recursion Formula

Given the new expression for $\zeta(s)$, one approach could be to substitute this expression back into the integral representation.

$$\zeta(s) = \lim_{b\to\infty} \frac{1}{2\pi} \int_{c-i\infty}^{c+i\infty} \Gamma(t)\zeta(t)\zeta(st) b^{t-1} \mathrm{d}t$$

$$\zeta(st) = \lim_{b\to\infty} \frac{1}{2\pi} \int_{c-i\infty}^{c+i\infty} \Gamma(t_1)\zeta(t_1)\zeta(stt_1) b^{t_1-1} \mathrm{d}t_1$$

Applying this recursion formula infinitely would yield the new expression:

Corollary 1.

$$\zeta(s) = \lim_{b\to\infty} \lim_{n\to\infty} \frac{1}{(2\pi)^n} \int_{[c+i\infty]^n} \prod_{i=1}^{n} \Gamma(t_i)\zeta(t_i)\zeta\left(s\prod_{j=1}^{n} t_j\right) b^{t_i-1} \mathrm{d}t_i$$

3.4. Complex Analysis

Given the coefficient $\frac{1}{2\pi i}$ for the integral formulas, Cauchy's theorem is probably the first thing that comes to mind. There are two key ideas to note here. First, applying Cauchy's integral formula to the Zeta function yields

$$\zeta(s) = \frac{1}{2\pi i} \oint_{\Gamma} \frac{\zeta(t)}{t-s} \mathrm{d}t$$

The term $\frac{1}{t-s}$ can be compared to $i\Gamma(t)\zeta(st)b^{t-1}$ in Theorem 6. The bounds of integration are also different. This leads to my observation that there might be some way to topologically deform the closed contour into a vertical line integral (one idea is to create a rectangular contour with infinite length and infinitesimal width) such that Cauchy's integral formula could be applied to Theorem 6.

Secondly, applying the Argument Principle to either of the integral equations derived in this paper is a prospective way to finding the zeroes of the Zeta function.

3.5. Riemann Hypothesis

One possible approach is to set the parameters c and s such that the line integral vanishes at some values. Specifically, setting the parameters that satisfy $s \times c = 0.5$, we get the integral of $\zeta(0.5 + it)$. One such configuration is $c \approx 1$ and $s = 0.5$, where we set c slightly greater than 1 to avoid the pole at $t = 1$. Using the definition from Theorem 7 and substituting the bounds gives

$$\zeta(0.5) = \lim_{b \to \infty} \frac{1}{2} \int_{-\infty}^{\infty} \sec\left(\frac{\pi}{2}(1 + it)\right) \zeta(-it) \zeta(0.5 + 0.5it) b^{it} \mathrm{d}t$$

(To be accurate, we would set $c = \lim\limits_{\epsilon \to 0^+} 1 + \epsilon$ and $s = \frac{c}{2}$ to avoid the poles).

All 3 terms, $\sec\left(\frac{\pi}{2}(1 + it)\right)$, b^{it}, and $\zeta(-it)$ have no zeroes in the region $t \in (-\infty, \infty)$, so the only term that would contribute point-wise zeroes to the integral is the term $\zeta(st)$. Using different values of s between 0 and 1 would give us all the integrals within the critical strip, which could be a step towards understanding the zeroes of the Zeta function.

Secondly, we can also compare the forms of two different integral representations. Using Theorem 6 and 8:

$$\zeta(s) = \lim_{b \to \infty} \frac{1}{2\pi} \int_{c-i\infty}^{c+i\infty} \Gamma(t) \zeta(t) \zeta(st) b^{t-1} \mathrm{d}t$$

$$\zeta(s) = \lim_{b \to \infty} \frac{-i}{\pi} \int_{-\infty}^{\infty} \zeta(s + ist) \frac{b^{it}}{t} \mathrm{d}t$$

$$\lim_{b \to \infty} \frac{i}{2} \int_{c-i\infty}^{c+i\infty} \Gamma(t) \zeta(t) \zeta(st) b^{t-1} \mathrm{d}t = \int_{c-i\infty}^{c+i\infty} \frac{\zeta(st)}{t} b^{t-1} \mathrm{d}t$$

where $c = \lim\limits_{\epsilon \to 0^+} 1 + \epsilon$, We could heuristically argue that

$$f\left[\Gamma(t)\zeta(t)\right] = f\left[\frac{1}{t}\right]$$

where $f[x]$ represents a transformation.

A well-known integral relating $\zeta(s)$ and $\Gamma(s)$ is

$$\zeta(s)\Gamma(s) = \int_0^{\infty} \frac{t^{s-1}}{\mathrm{e}^t - 1} \mathrm{d}t$$

Plugging this into Theorem 6 gives a double integral, but it is unclear what purpose this could serve. Future work on this topic should include trying to solve this double integral and uncovering the deeper integral relationship between $\zeta(s)$, $\Gamma(s)$, and $\frac{1}{s}$.

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Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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