Scientific Research Publishing

Journal of Applied Mathematics and Physics, 2025, 13(12), 4415-4428
https://www.scirp.org/journal/jamp
ISSN Online: 2327-4379
ISSN Print: 2327-4352

The Long-Time Behavior of 2D Cahn-Hilliard-Stokes Model

Yao Tan, Fengqun Tang

School of Mathematics and Statistics, Changsha University of Science and Technology, Changsha, China
Email: tanyao@stu.csust.edu.cn

How to cite this paper: Tan, Y. and Tang, F.Q. (2025) The Long-Time Behavior of 2D Cahn-Hilliard-Stokes Model. *Journal of Applied Mathematics and Physics*, **13**, 4415-4428.
https://doi.org/10.4236/jamp.2025.1312242

Received: November 23, 2025
Accepted: December 19, 2025
Published: December 22, 2025

Abstract

This paper considers the long-time behavior of solutions for the Cahn-Hilliard-Stokes model on a bounded domain with a smooth boundary. The results show that when the initial data $(\phi_0, \boldsymbol{u}_0)$ belong to $H^2(\Omega)$, the solution of the model exists globally in time, and $\boldsymbol{u}$ converges to 0 as time tends to infinity.

Keywords

Cahn-Hilliard-Stokes Model, Initial-Boundary Value Problem, Long-Time Behavior, Partial Differential Equation

1. Introduction

In this paper, we consider the following 2D Cahn-Hilliard-Stokes model

$$\begin{cases} \partial_t \phi + \nabla \cdot (\phi \boldsymbol{u}) = \Delta \mu, \\ \mu = -\varepsilon \Delta \phi + F'(\phi), \\ \partial_t \boldsymbol{u} - \Delta \boldsymbol{u} + \boldsymbol{u} + \nabla P = -\gamma \phi \nabla \mu, \\ \nabla \cdot \boldsymbol{u} = 0. \end{cases} \tag{1}$$

where model (1) is subject to the following initial and boundary conditions

$$\begin{cases} \boldsymbol{u} = 0, \ \nabla \phi \cdot \boldsymbol{n} = \nabla \Delta \phi \cdot \boldsymbol{n} = 0, & \boldsymbol{x} \in \partial\Omega, t > 0, \\ (\phi, \boldsymbol{u})(\boldsymbol{x}, 0) = (\phi_0, \boldsymbol{u}_0)(\boldsymbol{x}), & \boldsymbol{x} \in \bar{\Omega}. \end{cases} \tag{2}$$

Here $\boldsymbol{x} \in \Omega \subset \mathbb{R}^2$ is a bounded domain with smooth boundary $\partial\Omega$. The unknown function ϕ represents the concentration field, μ represents the chemical potential, and the positive parameter ε is the interface thickness parameter. The phase equilibria are represented by the pure phases $\phi = \pm 1$. The unknown function $\boldsymbol{u}$ denotes the advective velocity, P is the pressure, and the positive parameter γ model the surface tension. Besides, $\boldsymbol{n}$ is the unit normal vector.

The phase separation and interface evolution problems of multiphase fluids are of great significance in materials science, fluid mechanics, and biophysics. The Cahn-Hilliard equation was initially proposed by Cahn J. W. and Hilliard J. E. [1] [2] in 1958 to describe the dynamic behavior of phase separation in binary mixtures. Its core idea is to characterize the spatiotemporal evolution of the concentration field through the gradient flow structure of the free energy functional.

With the continuous deepening of research, the single Cahn-Hilliard equation has become insufficient to fully describe the physical mechanisms in complex systems. Therefore, scholars have coupled it with other fluid mechanics equations or physical models, forming a series of representative extended models. For instance, the Cahn-Hilliard-Navier-Stokes model is used to describe the phase separation dynamics of compressible or incompressible two-phase fluids [3]-[11], the Cahn-Hilliard-Brinkman model is often employed to depict the phase flow evolution in porous media with viscous damping effects [12]-[14], and the Cahn-Hilliard-Hele-Shaw model is suitable for studying thin-layer fluids or interface-driven seepage problems [15]-[17]. These coupled models have made significant progress in both theoretical analysis and numerical simulation, providing new ideas and research frameworks for further studying the dynamic behavior and mathematical properties of coupled Cahn-Hilliard systems.

In [18], Ľubomír Baňas *et al.* considered a strongly coupled transient Stokes-Cahn-Hilliard system

$$\begin{cases} \partial_t \phi + \nabla \cdot (\phi \boldsymbol{u}) = \Delta \mu, \\ \mu = -\lambda^2 \Delta \phi + f(\phi), \\ \partial_t \boldsymbol{u} - \omega \Delta \boldsymbol{u} + \nabla p = -\phi \nabla \mu, \\ \nabla \cdot \boldsymbol{u} = 0. \end{cases} \tag{3}$$

Here, ω denotes the fluid viscosity, and λ is a parameter characterizing the interface width. The variables $\boldsymbol{u}$ and μ represent the velocity field and the chemical potential, respectively. The order parameter ϕ, which serves as the microscopic concentration (or volume fraction), approaches values near −1 and 1 within the pure fluid phases. In the thin interfacial layer separating the two phases, it satisfies $|\phi| < 1$. The nonlinear term in the model, defined as $f(\phi) = F'(\phi)$, is derived from the homogeneous free energy functional F. This term acts to penalize deviations from the physical constraint $|\phi| \le 1$. A standard and frequently used form for F is the quadratic double-well free energy

$$F(\phi) = \frac{1}{2}\left(\phi^2 - 1\right)^2.$$

Ľubomír Baňas *et al.* conducted a rigorous homogenization analysis of this equation using the two-scale convergence theory and obtained an effective macroscopic model for two-phase flow in porous media.

In [19], Kelong Cheng *et al.* used the Galerkin method to study the global well-posedness of model (1), where ($\phi, \boldsymbol{u}$) satisfy conditions $\boldsymbol{u} \cdot \boldsymbol{n}|_{\partial\Omega} = 0$ and

$\nabla\phi\cdot\boldsymbol{n}|_{\partial\Omega}=\nabla\Delta\phi\cdot\boldsymbol{n}|_{\partial\Omega}=0$. Model (1) can be regarded as introducing a damping term $\boldsymbol{u}|\boldsymbol{u}|^{\beta-1}$ into the Stokes equation of model (3), with $\beta=1$. This damping term can be physically understood as the frictional resistance that a fluid experiences when moving in a porous medium or a highly viscous environment. It is also often used in mathematics to enhance the dissipative structure of the system, thereby improving the decay properties of the velocity field and the overall dynamic behavior. However, although there is a certain foundation for the research on the well-posedness of model (1), there is still a lack of systematic analysis of the long-time behavior of its solutions.

Therefore, based on the global well-posedness results obtained in [19], this paper considers the long-time behavior of the model under the conditions $\boldsymbol{u}|_{\partial\Omega}=0$ and $\nabla\phi\cdot\boldsymbol{n}|_{\partial\Omega}=\nabla\Delta\phi\cdot\boldsymbol{n}|_{\partial\Omega}=0$. In the proof, we will require that the diffuse interface parameter ε satisfies $\varepsilon-F_3c_0>0$, and the constant $F_3>0$ is chosen such that $F''(\cdot)\geq -F_3$, and c_0 denotes the constant appearing in the Poincare inequality on the domain Ω, which can be explicitly computed for given F and Ω. This condition plays a crucial role in our estimate of $\|\phi-\bar{\phi}\|_{H^2}^2$.

Throughout the paper, we denote by $\|\cdot\|_{L^p}$, $\|\cdot\|_{L^\infty}$, and $\|\cdot\|_{W^{s,p}}$ the norms of the standard Lebesgue spaces L^p (for $1\leq p<\infty$), L^∞, and Sobolev spaces $W^{s,p}$, respectively. When $p=2$, we simplify the notation by writing $\|\cdot\|$ for $\|\cdot\|_{L^2}$ and $\|\cdot\|_{H^s}$ for $\|\cdot\|_{W^{s,2}}$.

The principal function spaces considered in this work are

$$L^\infty\left((0,t);H^r(\Omega)\right) \text{ and } L^2\left((0,t);H^s(\Omega)\right),$$

which are equipped with the norms

$$\operatorname*{ess\,sup}_{t>0}\|\Psi(\cdot,t)\|_{H^r} \text{ and } \left(\int_0^T\|\Psi(\cdot,\tau)\|_{H^s}^2\,\mathrm{d}\tau\right)^{1/2},$$

respectively, where r and s are positive integers.

Unless otherwise stated, the letter C denotes a generic positive constant which may depend on Ω, the initial data, but is independent of the unknown functions ϕ and $\boldsymbol{u}$.

Our main result can be stated in the following theorem.

Theorem 1. Let Ω be a bounded domain with smooth boundary. Consider the initial-boundary value problem (1)-(2). Assume that the initial data $(\phi_0,\boldsymbol{u}_0)\in H^2(\Omega)$ are compatible with the boundary condition. And the constant $\alpha=\varepsilon-F_3c_0>0$, where c_0 is the constant in the Poincaré inequality on the domain.

Furthermore, assume that $F(\cdot)$ satisfy the following conditions

1) $F(\cdot)$ is of C^3 class and $F(\cdot)\geq 0$.

2) There exist constants $F_1,F_2>0$ such that $\left|F^{(n)}(\phi)\right|\leq F_1|\phi|^{p-n}+F_2$, $n=1,2,3$, $\forall 3\leq p<\infty$ and $\phi\in\mathbb{R}$.

3) There exist a constant $F_3\geq 0$ such that $F''\geq -F_3$.

Then there exists a unique and global-in-time solution $(\phi,\boldsymbol{u})$ to (1)-(2), such that $\phi\in L^\infty\left((0,t);H^2(\Omega)\right)\cap L^2\left((0,t);H^4(\Omega)\right)$,

$\boldsymbol{u} \in L^\infty\left((0,t);H^2(\Omega)\right) \cap L^2\left((0,t);H^3(\Omega)\right)$ for any $t>0$. Moreover, the function $\boldsymbol{u}$ obeys the long-time behavior as $t \to \infty$

$$\left\|\boldsymbol{u}(\cdot,t)\right\|_{H^2} \to 0,\ \left\|\partial_t \boldsymbol{u}(\cdot,t)\right\| \to 0. \tag{4}$$

This paper is organized as follows. In Section 2, we provide the key lemmas necessary for the subsequent analysis. Then, we complete the proof of Theorem 1 by using energy estimates to study the regularity and long-time behavior of the solutions in Section 3.

2. Preliminaries

In this section, in order to study the the theorem 1, we present the following lemmas. The first lemma concerns some inequalities of Sobolev and Ladyzhenskaya type, while the second one summarizes classical results for elliptic equations.

Lemma 2. [20]-[22] Let $\Omega \subset \mathbb{R}^2$ be any bounded domain with smooth boundary. Then

(i) $\|f\|_{L^\infty}^2 \le C\|f\|_{H^2}^2$;

(ii) $\|f\|_{L^\infty}^2 \le C\|f\|\|f\|_{H^2}$;

(iii) $\|f\|_{L^p}^2 \le C\|f\|_{H^1}^2,\ \forall 1 \le p < \infty$;

(iv) $\|f\|_{L^4}^2 \le C\left(\|f\|\|\nabla f\| + \|f\|^2\right)$;

(v) $\|f-\bar{f}\|^2 \le c_0\|\nabla f\|^2$, where $\bar{f} = \frac{1}{|\Omega|}\int_\Omega f \mathrm{d}\boldsymbol{x}$.

Lemma 3. [23] Let $\Omega \subset \mathbb{R}^2$ be any bounded domain with smooth boundary $\partial\Omega$. Then, for any function $H^s(\Omega) \ni f : \Omega \to \mathbb{R}$ satisfying $\nabla f \cdot \boldsymbol{n}|_{\partial\Omega} = 0$, it follows that

$$\|f-\bar{f}\|_{H^s}^2 \le C\|\Delta f\|_{H^{s-2}}^2,$$

where $\bar{f} = \frac{1}{|\Omega|}\int_\Omega f \mathrm{d}\boldsymbol{x}$, $C = C(s,\Omega)$ and integer $s \ge 2$.

3. Proof of the Theorem 1

In this section, we will systematically prove Theorem 1. The proof of Theorem 1 mainly consists of the results of the following lemmas.

Lemma 4. Under the assumptions of Theorem 1, it follows that

$$\|\nabla\phi(\cdot,t)\|^2 + \|\boldsymbol{u}(\cdot,t)\|^2 + \int_0^t \left(\|\nabla\mu\|^2 + \|\boldsymbol{u}\|^2 + \|\nabla\boldsymbol{u}\|^2\right)(\tau)\mathrm{d}\tau \le C.$$

Proof. In the section, we prove the theorem 1. For the reader's convenience, we recall the system of equations

$$\begin{cases} \partial_t\phi + \nabla\cdot(\phi\boldsymbol{u}) = \Delta\mu, \\ \mu = -\varepsilon\Delta\phi + F'(\phi), \\ \partial_t\boldsymbol{u} - \Delta\boldsymbol{u} + \boldsymbol{u} + \nabla P = -\gamma\phi\nabla\mu, \\ \nabla\cdot\boldsymbol{u} = 0, \end{cases} \tag{5}$$

Firstly, we take the L^2 inner product of $(5)_1$ with μ, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\varepsilon\|\nabla\phi\|^2+2\int_\Omega F(\phi)\mathrm{d}\boldsymbol{x}\right)+\|\nabla\mu\|^2=-\int_\Omega\mu(\boldsymbol{u}\cdot\nabla\phi)\mathrm{d}\boldsymbol{x}. \tag{6}$$

Taking the L^2 inner product of $(5)_3$ with $\boldsymbol{u}$, we can get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{u}\|^2+\|\boldsymbol{u}\|^2+\|\nabla\boldsymbol{u}\|^2=\gamma\int_\Omega\mu(\boldsymbol{u}\cdot\nabla\phi)\mathrm{d}\boldsymbol{x}. \tag{7}$$

Multiply (6) by γ and together with (7), which yields

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\varepsilon\gamma\|\nabla\phi\|^2+2\gamma\int_\Omega F(\phi)\mathrm{d}\boldsymbol{x}+\|\boldsymbol{u}\|^2\right)+\gamma\|\nabla\mu\|^2+\|\boldsymbol{u}\|^2+\|\nabla\boldsymbol{u}\|^2=0. \tag{8}$$

Integrating (8) over t, we obtain

$$\frac{\varepsilon\gamma}{2}\|\nabla\phi\|^2+\gamma\int_\Omega F(\phi)\mathrm{d}\boldsymbol{x}+\frac{1}{2}\|\boldsymbol{u}\|^2+\int_0^t\left(\gamma\|\nabla\mu\|^2+\|\boldsymbol{u}\|^2+\|\nabla\boldsymbol{u}\|^2\right)(\tau)\mathrm{d}\tau\le C.$$

Therefore, we can get

$$\|\nabla\phi(\cdot,t)\|^2+\|\boldsymbol{u}(\cdot,t)\|^2+\int_0^t\left(\|\nabla\mu\|^2+\|\boldsymbol{u}\|^2+\|\nabla\boldsymbol{u}\|^2\right)(\tau)\mathrm{d}\tau\le C. \tag{9}$$

This completes the proof of the Lemma 4.

We now proceed with a more detailed estimate of (ϕ, $\boldsymbol{u}$). For the sake of estimation convenience, we write $\tilde{\phi}=\phi-\bar{\phi}$, then is equivalent to

$$\begin{cases}\partial_t\tilde{\phi}+\nabla\cdot\left(\tilde{\phi}\boldsymbol{u}\right)=\Delta\mu,\\ \mu=-\varepsilon\Delta\tilde{\phi}+F'(\phi),\\ \partial_t\boldsymbol{u}-\Delta\boldsymbol{u}+\boldsymbol{u}+\nabla P=-\gamma\left(\tilde{\phi}+\bar{\phi}\right)\nabla\mu,\\ \nabla\cdot\boldsymbol{u}=0,\end{cases} \tag{10}$$

where $\bar{\phi}$ is defined as

$$\bar{\phi}=\frac{1}{|\Omega|}\int_\Omega\phi(\boldsymbol{x},t)\mathrm{d}\boldsymbol{x}=\frac{1}{|\Omega|}\int_\Omega\phi_0(\boldsymbol{x})\mathrm{d}\boldsymbol{x}.$$

Lemma 5. Under the assumptions of Theorem 1, it follows that

$$\left\|\tilde{\phi}(\cdot,t)\right\|_{H^2}^2+\int_0^t\left\|\tilde{\phi}(\tau)\right\|_{H^4}^2\mathrm{d}\tau\le C.$$

Proof. Taking the L^2 inner product of $(10)_1$ with $\tilde{\phi}$, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\tilde{\phi}\right\|^2+\varepsilon\left\|\Delta\tilde{\phi}\right\|^2=-\int_\Omega F''(\phi)\left|\nabla\tilde{\phi}\right|^2\mathrm{d}\boldsymbol{x}.$$

Using the Cauchy-Schwarz inequality and Lemma 2 (v), which implies that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\tilde{\phi}\right\|^2+\varepsilon\left\|\Delta\tilde{\phi}\right\|^2\le F_3\left\|\nabla\tilde{\phi}\right\|^2\le c_0F_3\left\|\Delta\tilde{\phi}\right\|^2. \tag{11}$$

It follows that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\tilde{\phi}\right\|^2+\alpha\left\|\Delta\tilde{\phi}\right\|^2\le 0,\ \left(\text{where } \alpha=\varepsilon-c_0F_3>0\right). \tag{12}$$

Thus, by integrating (12) over t, we get

$$\left\|\tilde{\phi}(\cdot,t)\right\|^2+\int_0^t\left\|\Delta\tilde{\phi}(\tau)\right\|^2\mathrm{d}\tau\le C. \tag{13}$$

Using (9) and Lemma 3, which implies that

$$\left\|\tilde{\phi}(\cdot,t)\right\|_{H^1}^2+\int_0^t\left\|\tilde{\phi}(\tau)\right\|_{H^2}^2\mathrm{d}\tau\le C. \tag{14}$$

Since $\mu=-\varepsilon\Delta\tilde{\phi}+F'(\phi)$, we have

$$\left\|\tilde{\phi}\right\|_{H^3}^2\le C\left\|\nabla\Delta\tilde{\phi}\right\|^2\le C\left(\left\|\nabla\mu\right\|^2+\left\|F''(\phi)\nabla\tilde{\phi}\right\|^2\right),$$

where

$$\begin{aligned}\left\|F''(\phi)\nabla\tilde{\phi}\right\|^2&\le C\left(\left\|\phi\right\|_{L^{4(p-2)}}^{2(p-2)}\left\|\nabla\tilde{\phi}\right\|_{L^4}^2+\left\|\nabla\tilde{\phi}\right\|^2\right)\\&\le C\left\|\phi\right\|_{H^1}^{2(p-2)}\left(\left\|\nabla\tilde{\phi}\right\|\left\|\nabla^2\tilde{\phi}\right\|+\left\|\nabla\tilde{\phi}\right\|^2\right)+C\left\|\nabla\tilde{\phi}\right\|^2\\&\le C\left\|\Delta\tilde{\phi}\right\|^2.\end{aligned}$$

Using (9) and (13), we deduce that

$$\int_0^t\left\|\tilde{\phi}(\tau)\right\|_{H^3}^2\mathrm{d}\tau\le C\int_0^t\left(\left\|\nabla\mu(\tau)\right\|^2+\left\|\Delta\tilde{\phi}(\tau)\right\|^2\right)\mathrm{d}\tau\le C. \tag{15}$$

Then, multiply $(10)_1$ by $\Delta^2\tilde{\phi}$ and integrate over the domain, and applying the Cauchy-Schwarz and Young's inequalities, we obtain

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\Delta\tilde{\phi}\right\|^2+\varepsilon\left\|\Delta^2\tilde{\phi}\right\|^2&=-\int_\Omega\left(\boldsymbol{u}\cdot\nabla\tilde{\phi}\right)\Delta^2\tilde{\phi}\mathrm{d}\boldsymbol{x}+\int_\Omega\Delta F'(\phi)\Delta^2\tilde{\phi}\mathrm{d}\boldsymbol{x}\\&\le\frac{\varepsilon}{2}\left\|\Delta^2\tilde{\phi}\right\|^2+C\left(\left\|\boldsymbol{u}\cdot\nabla\tilde{\phi}\right\|^2+\left\|\Delta F'(\phi)\right\|^2\right).\end{aligned}$$

By employing Lemma 2 (ii), Lemma 3 and (9), we can derive

$$\begin{aligned}\left\|\boldsymbol{u}\cdot\nabla\tilde{\phi}\right\|^2+\left\|\Delta F'(\phi)\right\|^2&\le C\left(\left\|F'''(\phi)\right\|^2\left\|\nabla\tilde{\phi}\right\|_{L^\infty}^4+\left\|F''(\phi)\right\|_{L^4}^2\left\|\Delta\tilde{\phi}\right\|_{L^4}^2\right)+\left\|\boldsymbol{u}\right\|^2\left\|\nabla\tilde{\phi}\right\|_{L^\infty}^2\\&\le C\left(\left\|\nabla\Delta\phi\right\|^2+\left\|\nabla\tilde{\phi}\right\|_{H^2}^2\left\|\nabla\tilde{\phi}\right\|^2+\left\|\Delta\tilde{\phi}\right\|_{L^4}^2\right)\\&\le C\left\|\nabla\Delta\tilde{\phi}\right\|^2.\end{aligned}$$

which gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\|\Delta\tilde{\phi}\right\|^2+\varepsilon\left\|\Delta^2\tilde{\phi}\right\|^2\le C\left\|\nabla\Delta\tilde{\phi}\right\|^2. \tag{16}$$

Integrating (16) over t, and using (15), we have

$$\left\|\Delta\tilde{\phi}\right\|^2+\varepsilon\int_0^t\left\|\Delta^2\tilde{\phi}(\tau)\right\|^2\mathrm{d}\tau\le C.$$

Namely,

$$\left\|\tilde{\phi}(\cdot,t)\right\|_{H^2}^2+\int_0^t\left\|\tilde{\phi}(\tau)\right\|_{H^4}^2\mathrm{d}\tau\le C. \tag{17}$$

Hence, which implies that

$$\tilde{\phi}\in L^\infty\left((0,t);H^2(\Omega)\right)\cap L^2\left((0,t);H^4(\Omega)\right).$$

This completes the proof of the Lemma 5.

Lemma 6. Under the assumptions of Theorem 1, it follows that

$$\|\boldsymbol{u}(\cdot,t)\|_{H^1}^2+\int_0^t\left(\|\partial_t\boldsymbol{u}\|^2+\|\boldsymbol{u}\|_{H^2}^2\right)(\tau)\,\mathrm{d}\tau\le C.$$

Proof. Taking the L^2 inner product of $(10)_2$ with $\partial_t\boldsymbol{u}$, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\boldsymbol{u}\|^2+\|\nabla\boldsymbol{u}\|^2\right)+\|\partial_t\boldsymbol{u}\|^2=-\gamma\int_\Omega\left(\tilde{\phi}+\bar{\phi}\right)\nabla\mu\cdot\partial_t\boldsymbol{u}\mathrm{d}\boldsymbol{x}. \tag{18}$$

For the right hand side(RHS) of (18), using Young's inequality and (14), we can arrive that

$$\begin{aligned}\left|-\gamma\int_\Omega\left(\tilde{\phi}+\bar{\phi}\right)\nabla\mu\cdot\partial_t\boldsymbol{u}\mathrm{d}\boldsymbol{x}\right|&\le\gamma\int_\Omega\left|\left(\tilde{\phi}+\bar{\phi}\right)\nabla\mu\cdot\partial_t\boldsymbol{u}\right|\mathrm{d}\boldsymbol{x}\\&\le\frac{1}{2}\|\partial_t\boldsymbol{u}\|^2+C\left\|\tilde{\phi}+\bar{\phi}\right\|_{L^\infty}^2\|\nabla\mu\|^2\\&\le\frac{1}{2}\|\partial_t\boldsymbol{u}\|^2+C\left\|\tilde{\phi}+\bar{\phi}\right\|_{H^2}^2\|\nabla\mu\|^2\\&\le\frac{1}{2}\|\partial_t\boldsymbol{u}\|^2+C\|\nabla\mu\|^2.\end{aligned}$$

So we can update as

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\boldsymbol{u}\|^2+\|\nabla\boldsymbol{u}\|^2\right)+\|\partial_t\boldsymbol{u}\|^2\le C\|\nabla\mu\|^2.$$

It is readily verified that

$$\|\boldsymbol{u}(\cdot,t)\|_{H^1}^2+\int_0^t\|\partial_t\boldsymbol{u}(\tau)\|^2\,\mathrm{d}\tau\le C. \tag{19}$$

From $(10)_3$, an application of the triangle inequality, Lemma 2 (i) and (17) yields

$$\begin{aligned}\|\boldsymbol{u}\|_{H^2}^2&\le C\left(\|\partial_t\boldsymbol{u}\|^2+\|\boldsymbol{u}\|^2+\left\|\left(\tilde{\phi}+\bar{\phi}\right)\nabla\mu\right\|^2\right)\\&\le C\left(\|\partial_t\boldsymbol{u}\|^2+\|\boldsymbol{u}\|^2+\left\|\tilde{\phi}+\bar{\phi}\right\|_{L^\infty}^2\|\nabla\mu\|^2\right)\\&\le C\left(\|\partial_t\boldsymbol{u}\|^2+\|\boldsymbol{u}\|^2+\|\nabla\mu\|^2\right).\end{aligned} \tag{20}$$

Using (9) and (19), we have

$$\int_0^t\|\boldsymbol{u}(\tau)\|_{H^2}^2\,\mathrm{d}\tau\le C\int_0^t\left(\|\partial_t\boldsymbol{u}\|^2+\|\boldsymbol{u}\|^2+\|\nabla\mu\|^2\right)(\tau)\,\mathrm{d}\tau\le C. \tag{21}$$

This completes the proof of the Lemma 6.

Lemma 7. Under the assumptions of Theorem 1, it follows that

$$\int_0^t\left(\|\Delta\mu\|^2+\|\partial_t\tilde{\phi}\|^2\right)(\tau)\,\mathrm{d}\tau\le C.$$

Proof. Next, taking Δ on both sides of (10), and then by applying the triangle inequality, we can calculate

$$\begin{aligned}\|\Delta\mu\|^2&\le\left\|\Delta\left(-\varepsilon\Delta\tilde{\phi}+F'(\phi)\right)\right\|^2\\&\le\varepsilon\left\|\Delta^2\tilde{\phi}\right\|^2+\|F''(\phi)\|_{L^4}^2\left\|\Delta\tilde{\phi}\right\|_{L^4}^2\|F'''(\phi)\|^2\left\|\nabla\tilde{\phi}\right\|_{L^\infty}^4\\&\le C\left(\left\|\Delta^2\tilde{\phi}\right\|^2+\left\|\nabla\Delta\tilde{\phi}\right\|^2\right).\end{aligned}$$

Using (15) and (17), we have

$$\int_0^t \left\|\Delta\mu(\tau)\right\|^2 \mathrm{d}\tau \le C. \tag{22}$$

From (10), together with Lemma 3 and (9) we have

$$\begin{aligned}\left\|\partial_t\tilde{\phi}\right\|^2 &\le \left\|-\boldsymbol{u}\cdot\nabla\tilde{\phi}+\Delta\mu\right\|^2\\ &\le C\left(\|\boldsymbol{u}\|^2\left\|\nabla\tilde{\phi}\right\|_{L^\infty}^2+\|\Delta\mu\|^2\right)\\ &\le C\left(\left\|\nabla\Delta\tilde{\phi}\right\|^2+\|\Delta\mu\|^2\right).\end{aligned}$$

Then we have

$$\int_0^t \left\|\partial_t\tilde{\phi}(\tau)\right\|^2 \mathrm{d}\tau \le C. \tag{23}$$

This completes the proof of the Lemma 7.

Lemma 8. Under the assumptions of Theorem 1, it follows that

$$\left\|\partial_t\tilde{\phi}(\cdot,t)\right\|^2+\left\|\nabla\mu(\cdot,t)\right\|^2+\int_0^t\left(\left\|\Delta\partial_t\tilde{\phi}\right\|^2+\left\|\nabla\partial_t\tilde{\phi}\right\|^2\right)(\tau)\,\mathrm{d}\tau \le C.$$

Proof. Taking L^2 inner product of $(10)_1$ with $\partial_t\mu$ and applying Hölder's and Young's inequalities, we can get

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\mu\|^2+\varepsilon\left\|\nabla\partial_t\tilde{\phi}\right\|^2 &= -\int_\Omega\left(\boldsymbol{u}\cdot\nabla\tilde{\phi}\right)\partial_t\mu\,\mathrm{d}\boldsymbol{x}-\int_\Omega F''(\phi)\left|\partial_t\tilde{\phi}\right|^2\mathrm{d}\boldsymbol{x}\\ &\le \|\boldsymbol{u}\|\left\|\nabla\tilde{\phi}\right\|_{L^\infty}\|\partial_t\mu\|+\|F''(\phi)\|_{L^\infty}\left\|\partial_t\tilde{\phi}\right\|^2\\ &\le \frac{1}{8}\left\|F''(\phi)\partial_t\tilde{\phi}-\varepsilon\Delta\partial_t\tilde{\phi}\right\|^2+C\left(\left\|\nabla\Delta\tilde{\phi}\right\|^2+\left\|\partial_t\tilde{\phi}\right\|^2\right)\\ &\le \frac{\varepsilon}{4}\left\|\Delta\partial_t\tilde{\phi}\right\|^2+C\left(\left\|\nabla\Delta\tilde{\phi}\right\|^2+\left\|\partial_t\tilde{\phi}\right\|^2\right),\end{aligned}$$

which implies that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\mu\|^2+\varepsilon\left\|\nabla\partial_t\tilde{\phi}\right\|^2 \le \frac{\varepsilon}{4}\left\|\Delta\partial_t\tilde{\phi}\right\|^2+C\left(\left\|\nabla\Delta\tilde{\phi}\right\|^2+\left\|\partial_t\tilde{\phi}\right\|^2\right). \tag{24}$$

To further improve the estimate of $\partial_t\tilde{\phi}$, we differentiate both sides of $(10)_1$ with respect to t, that

$$\partial_{tt}\tilde{\phi}+\partial_t\boldsymbol{u}\cdot\nabla\tilde{\phi}+\boldsymbol{u}\cdot\nabla\partial_t\tilde{\phi}=\Delta\partial_t\mu. \tag{25}$$

Taking L^2 inner product of (25) with $\partial_t\tilde{\phi}$ and applying the divergence theorem, we have

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\partial_t\tilde{\phi}\right\|^2+\varepsilon\left\|\Delta\partial_t\tilde{\phi}\right\|^2\\ &= -\int_\Omega\partial_t\tilde{\phi}\left(\partial_t\boldsymbol{u}\cdot\nabla\tilde{\phi}\right)\mathrm{d}\boldsymbol{x}+\int_\Omega F''(\phi)\partial_t\tilde{\phi}\Delta\partial_t\tilde{\phi}\,\mathrm{d}\boldsymbol{x}\\ &= \int_\Omega\left(\nabla\partial_t\tilde{\phi}\cdot\partial_t\boldsymbol{u}\right)\tilde{\phi}\,\mathrm{d}\boldsymbol{x}+\int_\Omega F''(\phi)\partial_t\tilde{\phi}\Delta\partial_t\tilde{\phi}\,\mathrm{d}\boldsymbol{x}.\end{aligned} \tag{26}$$

For the first term on the RHS of (26), by using (17) we obtain

$$\begin{aligned}\int_\Omega\left(\nabla\partial_t\tilde{\phi}\cdot\partial_t\boldsymbol{u}\right)\tilde{\phi}\,\mathrm{d}\boldsymbol{x} &\le \frac{\varepsilon}{2}\left\|\nabla\partial_t\tilde{\phi}\right\|^2+C\left\|\tilde{\phi}\right\|_{L^\infty}^2\|\partial_t\boldsymbol{u}\|^2\\ &\le \frac{\varepsilon}{2}\left\|\nabla\partial_t\tilde{\phi}\right\|^2+C\|\partial_t\boldsymbol{u}\|^2.\end{aligned} \tag{27}$$

Similarly, for the second term on the RHS of (26), we can show that

$$\begin{aligned}\int_\Omega F''(\phi)\partial_t\tilde{\phi}\Delta\partial_t\tilde{\phi}\mathrm{d}\boldsymbol{x} &\le \frac{\varepsilon}{4}\left\|\Delta\partial_t\tilde{\phi}\right\|^2+\left\|F''(\phi)\right\|_{L^\infty}^2\left\|\partial_t\tilde{\phi}\right\|^2\\ &\le \frac{\varepsilon}{4}\left\|\Delta\partial_t\tilde{\phi}\right\|^2+C\left\|\partial_t\tilde{\phi}\right\|^2.\end{aligned} \tag{28}$$

Substituting (27) and (28) into (26) yields

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\partial_t\tilde{\phi}\right\|^2+\frac{3\varepsilon}{4}\left\|\Delta\partial_t\tilde{\phi}\right\|^2\le\frac{\varepsilon}{2}\left\|\nabla\partial_t\tilde{\phi}\right\|^2+C\left(\left\|\partial_t\tilde{\phi}\right\|^2+\left\|\partial_t\boldsymbol{u}\right\|^2\right). \tag{29}$$

Combining (24) and (29), we obtain

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left(\left\|\partial_t\tilde{\phi}\right\|^2+\left\|\nabla\mu\right\|^2\right)+\varepsilon\left(\left\|\Delta\partial\tilde{\phi}\right\|^2+\left\|\nabla\partial_t\tilde{\phi}\right\|^2\right)\\ &\le C\left(\left\|\partial_t\tilde{\phi}\right\|^2+\left\|\partial_t\boldsymbol{u}\right\|^2+\left\|\nabla\Delta\tilde{\phi}\right\|^2\right).\end{aligned}$$

Integrating both sides of the above equation with respect to time t we find that

$$\left\|\partial_t\tilde{\phi}(\cdot,t)\right\|^2+\left\|\nabla\mu(\cdot,t)\right\|^2+\int_0^t\left(\left\|\Delta\partial_t\tilde{\phi}\right\|^2+\left\|\nabla\partial_t\tilde{\phi}\right\|^2\right)(\tau)\mathrm{d}\tau\le C, \tag{30}$$

where we have applied (15), (19) and (23).

This completes the proof of the Lemma 8.

Lemma 9. Under the assumptions of Theorem 1, it follows that

$$\left\|\nabla\partial_t\tilde{\phi}(\cdot,t)\right\|^2+\int_0^t\left\|\nabla\partial_t\mu(\tau)\right\|^2\mathrm{d}\tau\le C.$$

Proof. Taking L^2 inner product of (25) with $\partial_t\mu$, we have

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\varepsilon\left\|\nabla\partial_t\tilde{\phi}\right\|^2+\int_\Omega F''(\phi)\left|\partial_t\tilde{\phi}\right|^2\mathrm{d}\boldsymbol{x}\right)+\left\|\nabla\partial_t\mu\right\|^2\\ &=\int_\Omega\left(\tilde{\phi}\partial_t\boldsymbol{u}+\partial_t\tilde{\phi}\boldsymbol{u}\right)\cdot\nabla\partial_t\mu\mathrm{d}\boldsymbol{x}+\frac{1}{2}\int_\Omega F'''(\phi)\left|\partial_t\tilde{\phi}\right|^3\mathrm{d}\boldsymbol{x}.\end{aligned} \tag{31}$$

For the first term on the RHS of (31), using Lemma 2 (i) and (30) we calculate that

$$\begin{aligned}&\int_\Omega\left(\tilde{\phi}\partial_t\boldsymbol{u}+\partial_t\tilde{\phi}\boldsymbol{u}\right)\cdot\nabla\partial_t\mu\mathrm{d}\boldsymbol{x}\\ &\le\frac{1}{2}\left\|\nabla\partial_t\mu\right\|^2+C\left(\left\|\partial_t\boldsymbol{u}\right\|^2\left\|\tilde{\phi}\right\|_{L^\infty}^2+\left\|\boldsymbol{u}\right\|_{L^\infty}^2\left\|\partial_t\tilde{\phi}\right\|^2\right)\\ &\le\frac{1}{2}\left\|\nabla\partial_t\mu\right\|^2+C\left(\left\|\partial_t\boldsymbol{u}\right\|^2\left\|\tilde{\phi}\right\|_{H^2}^2+\left\|\boldsymbol{u}\right\|_{H^2}^2\right)\\ &\le\frac{1}{2}\left\|\nabla\partial_t\mu\right\|^2+C\left(\left\|\partial_t\boldsymbol{u}\right\|^2+\left\|\boldsymbol{u}\right\|_{H^2}^2\right).\end{aligned} \tag{32}$$

We can estimate the second term on the RHS of (31) as

$$\begin{aligned}\frac{1}{2}\int_\Omega F'''(\phi)\left|\partial_t\tilde{\phi}\right|^3\mathrm{d}x&\le\left\|F'''(\phi)\right\|_{L^\infty}\left\|\partial_t\tilde{\phi}\right\|_{L^3}^3\\ &\le C\left\|\partial_t\tilde{\phi}\right\|_{L^\infty}^2\left\|\partial_t\tilde{\phi}\right\|\\ &\le C\left\|\partial_t\tilde{\phi}\right\|_{H^2}^2\\ &\le C\left\|\Delta\partial_t\tilde{\phi}\right\|^2.\end{aligned} \tag{33}$$

Plugging (32) and (33) into (31), we can show that

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left(\varepsilon\left\|\nabla\partial_t\tilde{\phi}\right\|^2+\int_\Omega F''(\phi)\left|\partial_t\tilde{\phi}\right|^2\mathrm{d}\boldsymbol{x}\right)+\left\|\nabla\partial_t\mu\right\|^2\\&\le C\left(\left\|\Delta\partial_t\tilde{\phi}\right\|^2+\left\|\partial_t\boldsymbol{u}\right\|^2+\left\|\boldsymbol{u}\right\|_{H^2}^2\right).\end{aligned}\tag{34}$$

Then we can get

$$\left\|\nabla\partial_t\tilde{\phi}(\cdot,t)\right\|^2+\int_0^t\left\|\nabla\partial_t\mu(\tau)\right\|^2\mathrm{d}\tau\le C.\tag{35}$$

This completes the proof of the Lemma 9.

Lemma 10. Under the assumptions of Theorem 1, it follows that

$$\left\|\partial_t\boldsymbol{u}(\cdot,t)\right\|^2+\left\|\boldsymbol{u}(\cdot,t)\right\|_{H^2}^2+\int_0^t\left\|\partial_t\boldsymbol{u}(\tau)\right\|_{H^1}^2\mathrm{d}\tau\le C\ \text{ and }\ \lim_{t\to\infty}\left\|\boldsymbol{u}(\cdot,t)\right\|_{H^1}^2=0.$$

Proof. We now continue to improve the estimate for $\boldsymbol{u}$. Firstly, differentiating $(10)_3$ with t to get

$$\partial_{tt}\boldsymbol{u}-\Delta\partial_t\boldsymbol{u}+\partial_t\boldsymbol{u}+\nabla\partial_t P=-\gamma\partial_t\tilde{\phi}\nabla\mu-\gamma\left(\tilde{\phi}+\bar{\phi}\right)\nabla\partial_t\mu.\tag{36}$$

Firstly, taking the L^2 inner product of (36) with $\partial_t\boldsymbol{u}$, we have

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\partial_t\boldsymbol{u}\right\|^2+\left\|\partial_t\boldsymbol{u}\right\|^2+\left\|\nabla\partial_t\boldsymbol{u}\right\|^2\\&=-\gamma\int_\Omega\partial_t\tilde{\phi}\nabla\mu\cdot\partial_t\boldsymbol{u}\mathrm{d}\boldsymbol{x}-\gamma\int_\Omega\left(\tilde{\phi}+\bar{\phi}\right)\nabla\partial_t\mu\cdot\partial_t\boldsymbol{u}\mathrm{d}\boldsymbol{x}.\end{aligned}\tag{37}$$

For the first term on the RHS of (37), using Lemma 2 (iii), (iv) and (30) we have

$$\begin{aligned}\left|-\gamma\int_\Omega\partial_t\tilde{\phi}\nabla\mu\cdot\partial_t\boldsymbol{u}\mathrm{d}\boldsymbol{x}\right|&\le\gamma\int_\Omega\left|\partial_t\tilde{\phi}\nabla\mu\cdot\partial_t\boldsymbol{u}\right|\mathrm{d}\boldsymbol{x}\\&\le\gamma\left\|\partial_t\tilde{\phi}\right\|_{L^4}\left\|\nabla\mu\right\|\left\|\partial_t\boldsymbol{u}\right\|_{L^4}\\&\le C\left\|\nabla\partial_t\tilde{\phi}\right\|\left(\left\|\partial_t\boldsymbol{u}\right\|^{\frac{1}{2}}\left\|\nabla\partial_t\boldsymbol{u}\right\|^{\frac{1}{2}}+\left\|\partial_t\boldsymbol{u}\right\|\right)\\&\le C\left\|\nabla\partial_t\tilde{\phi}\right\|\left(\left\|\partial_t\boldsymbol{u}\right\|+\left\|\nabla\partial_t\boldsymbol{u}\right\|\right)\\&\le\frac{1}{4}\left\|\partial_t\boldsymbol{u}\right\|^2+\frac{1}{2}\left\|\nabla\partial_t\boldsymbol{u}\right\|^2+C\left\|\nabla\partial_t\tilde{\phi}\right\|^2.\end{aligned}\tag{38}$$

By applying Lemma 2 (i) and (17), the second term on the RHS of (37) can be estimated that

$$\begin{aligned}\left|-\gamma\int_\Omega\left(\tilde{\phi}+\bar{\phi}\right)\nabla\partial_t\mu\cdot\partial_t\boldsymbol{u}\mathrm{d}\boldsymbol{x}\right|&\le\gamma\int_\Omega\left|\left(\tilde{\phi}+\bar{\phi}\right)\nabla\partial_t\mu\cdot\partial_t\boldsymbol{u}\right|\mathrm{d}\boldsymbol{x}\\&\le\gamma\left\|\tilde{\phi}+\bar{\phi}\right\|_{L^\infty}\left\|\nabla\partial_t\mu\right\|\left\|\partial_t\boldsymbol{u}\right\|\\&\le C\left\|\tilde{\phi}+\bar{\phi}\right\|_{H^2}\left\|\nabla\partial_t\mu\right\|\left\|\partial_t\boldsymbol{u}\right\|\\&\le C\left\|\nabla\partial_t\mu\right\|\left\|\partial_t\boldsymbol{u}\right\|\\&\le\frac{1}{4}\left\|\partial_t\boldsymbol{u}\right\|^2+C\left\|\nabla\partial_t\mu\right\|^2.\end{aligned}\tag{39}$$

Together (38) and (39), the Equation (37) can be updated that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_t \boldsymbol{u}\|^2 + \|\partial_t \boldsymbol{u}\|^2 + \|\nabla \partial_t \boldsymbol{u}\|^2 \le C\left(\|\nabla \partial_t \tilde{\phi}\|^2 + \|\nabla \partial_t \mu\|^2\right). \tag{40}$$

Using (30) and (35), we get

$$\|\partial_t \boldsymbol{u}\|^2 + \int_0^t \left(\|\partial_t \boldsymbol{u}\|^2 + \|\nabla \partial_t \boldsymbol{u}\|^2\right)(\tau)\,\mathrm{d}\tau \le C. \tag{41}$$

Since

$$\begin{aligned}\int_0^t \left|\frac{\mathrm{d}}{\mathrm{d}\tau}\|\boldsymbol{u}(\cdot,\tau)\|_{H^1}^2\right|\mathrm{d}\tau &\le 2\int_0^t \|\boldsymbol{u}(\cdot,\tau)\|_{H^1}^2 \,\|\partial_t \boldsymbol{u}(\cdot,\tau)\|_{H^1}^2\,\mathrm{d}\tau \\ &\le \int_0^t \left(\|\boldsymbol{u}(\cdot,\tau)\|_{H^1}^2 + \|\partial_t \boldsymbol{u}(\cdot,\tau)\|_{H^1}^2\right)\mathrm{d}\tau \\ &\le C.\end{aligned}$$

So we derive that

$$\|\boldsymbol{u}(\cdot,t)\|_{H^1}^2 \in W^{1,1}(0,\infty). \tag{42}$$

Thus, we can show that

$$\lim_{t\to\infty}\|\boldsymbol{u}(\cdot,t)\|_{H^1}^2 = 0. \tag{43}$$

Besides, from (20), we get

$$\|\boldsymbol{u}\|_{H^2}^2 \le C\left(\|\partial_t \boldsymbol{u}\|^2 + \|\boldsymbol{u}\|^2 + \|\nabla \mu\|^2\right) \le C. \tag{44}$$

Thus, we can get

$$\|\partial_t \boldsymbol{u}(\cdot,t)\|^2 + \|\boldsymbol{u}(\cdot,t)\|_{H^2}^2 + \int_0^t \|\partial_t \boldsymbol{u}(\tau)\|_{H^1}^2\,\mathrm{d}\tau \le C. \tag{45}$$

This completes the proof of the Lemma 10.

Lemma 11. Under the assumptions of Theorem 1, it follows that

$$\|\boldsymbol{u}(\cdot,t)\|_{H^2}^2 + \int_0^t \|\boldsymbol{u}(\tau)\|_{H^3}^2\,\mathrm{d}\tau \le C \ \text{ and } \ \lim_{t\to\infty}\|\partial_t \boldsymbol{u}\|^2 = 0.$$

Proof. Taking L^2 inner product of (35) with $\Delta\partial_t \boldsymbol{u}$, we can have

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla \partial_t \boldsymbol{u}\|^2 + \|\Delta \partial_t \boldsymbol{u}\|^2 + \|\nabla \partial_t \boldsymbol{u}\|^2 \\ &= \gamma\int_\Omega \partial_t \tilde{\phi}\nabla \mu \cdot \Delta \partial_t \boldsymbol{u}\,\mathrm{d}\boldsymbol{x} + \gamma\int_\Omega \left(\tilde{\phi} + \overline{\phi}\right)\nabla \partial_t \mu \cdot \Delta \partial_t \boldsymbol{u}\,\mathrm{d}\boldsymbol{x} \\ &\le \frac{1}{2}\|\Delta \partial_t \boldsymbol{u}\|^2 + C\left(\|\partial_t \tilde{\phi}\|_{L^\infty}^2 \|\nabla \mu\|^2 + \|\tilde{\phi} + \overline{\phi}\|_{L^\infty}^2 \|\nabla \partial_t \mu\|^2\right) \\ &\le \frac{1}{2}\|\Delta \partial_t \boldsymbol{u}\|^2 + C\left(\|\Delta \partial_t \tilde{\phi}\|^2 + \|\nabla \partial_t \mu\|^2\right).\end{aligned} \tag{46}$$

Then, we can update as

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla \partial_t \boldsymbol{u}\|^2 + \|\Delta \partial_t \boldsymbol{u}\|^2 + 2\|\nabla \partial_t \boldsymbol{u}\|^2 \le C\left(\|\Delta \partial_t \tilde{\phi}\|^2 + \|\nabla \partial_t \mu\|^2\right). \tag{47}$$

We now integrate both sides of (47) with respect to t. This gives

$$\|\nabla \partial_t \boldsymbol{u}\|^2 + \int_0^t \left(\|\Delta \partial_t \boldsymbol{u}\|^2 + \|\nabla \partial_t \boldsymbol{u}\|^2\right)(\tau)\,\mathrm{d}\tau \le C. \tag{48}$$

From the triangle inequality, Lemma 2 (i), Lemma 3 and (30), it follows that

$$
\begin{aligned}
\left\|\partial_{tt}\boldsymbol{u}\right\|^{2} &\le C\left(\left\|\Delta\partial_{t}\boldsymbol{u}\right\|^{2}+\left\|\partial_{t}\boldsymbol{u}\right\|^{2}+\left\|\partial_{t}\tilde{\phi}\right\|_{L^{\infty}}^{2}\left\|\nabla\mu\right\|^{2}+\left\|\tilde{\phi}+\bar{\phi}\right\|_{L^{\infty}}^{2}\left\|\nabla\partial_{t}\mu\right\|^{2}\right)\\
&\le C\left(\left\|\Delta\partial_{t}\boldsymbol{u}\right\|^{2}+\left\|\partial_{t}\boldsymbol{u}\right\|^{2}+\left\|\partial_{t}\tilde{\phi}\right\|_{H^{2}}^{2}+\left\|\tilde{\phi}+\bar{\phi}\right\|_{H^{2}}^{2}\left\|\nabla\partial_{t}\mu\right\|^{2}\right)\\
&\le C\left(\left\|\Delta\partial_{t}\boldsymbol{u}\right\|^{2}+\left\|\partial_{t}\boldsymbol{u}\right\|^{2}+\left\|\Delta\partial_{t}\tilde{\phi}\right\|^{2}+\left\|\nabla\partial_{t}\mu\right\|^{2}\right).
\end{aligned}
$$

Using (30), (35), (41) and (48), we have

$$\int_{0}^{t}\left\|\partial_{tt}\boldsymbol{u}(\tau)\right\|^{2}\mathrm{d}\tau\le C. \tag{49}$$

Since

$$
\begin{aligned}
\int_{0}^{t}\left|\frac{\mathrm{d}}{\mathrm{d}t}\left\|\partial_{t}\boldsymbol{u}(\cdot,t)\right\|^{2}\right|\mathrm{d}\tau &\le 2\int_{0}^{t}\left\|\partial_{t}\boldsymbol{u}(\cdot,\tau)\right\|^{2}\left\|\partial_{tt}\boldsymbol{u}(\cdot,\tau)\right\|^{2}\mathrm{d}\tau\\
&\le \int_{0}^{t}\left(\left\|\partial_{t}\boldsymbol{u}(\cdot,\tau)\right\|^{2}+\left\|\partial_{tt}\boldsymbol{u}(\cdot,\tau)\right\|^{2}\right)\mathrm{d}\tau\\
&\le C.
\end{aligned}
$$

We can get

$$\left\|\partial_{t}\boldsymbol{u}(\cdot,t)\right\|^{2}\in W^{1,1}(0,\infty). \tag{50}$$

Therefore, we can show that

$$\lim_{t\to\infty}\left\|\partial_{t}\boldsymbol{u}\right\|^{2}=0. \tag{51}$$

Next, we can arrive that

$$
\begin{aligned}
\left\|\nabla\Delta\boldsymbol{u}\right\|^{2} &\le C\left(\left\|\nabla\boldsymbol{u}\right\|^{2}+\left\|\nabla\tilde{\phi}\cdot\nabla\mu\right\|^{2}+\left\|\left(\tilde{\phi}+\bar{\phi}\right)\nabla^{2}\mu\right\|^{2}+\left\|\nabla\partial_{t}\boldsymbol{u}\right\|^{2}\right)\\
&\le C\left(\left\|\nabla\boldsymbol{u}\right\|^{2}+\left\|\nabla\tilde{\phi}\right\|_{L^{\infty}}^{2}\left\|\nabla\mu\right\|^{2}+\left\|\tilde{\phi}+\bar{\phi}\right\|_{L^{\infty}}^{2}\left\|\nabla^{2}\mu\right\|^{2}+\left\|\nabla\partial_{t}\boldsymbol{u}\right\|^{2}\right)\\
&\le C\left(\left\|\nabla\boldsymbol{u}\right\|^{2}+\left\|\nabla\Delta\tilde{\phi}\right\|^{2}+\left\|\mu\right\|_{H^{2}}^{2}+\left\|\nabla\partial_{t}\boldsymbol{u}\right\|^{2}\right),
\end{aligned}
$$

which implies that

$$\int_{0}^{t}\left\|\boldsymbol{u}(\tau)\right\|_{H^{3}}^{2}\mathrm{d}\tau\le C\int_{0}^{t}\left\|\nabla\Delta\boldsymbol{u}(\tau)\right\|^{2}\mathrm{d}\tau\le C.$$

Thus, we can get

$$\left\|\boldsymbol{u}(\cdot,t)\right\|_{H^{2}}^{2}+\int_{0}^{t}\left\|\boldsymbol{u}(\tau)\right\|_{H^{3}}^{2}\mathrm{d}\tau\le C. \tag{52}$$

Hence, which yeilds

$$\boldsymbol{u}\in L^{\infty}\left((0,t);H^{2}(\Omega)\right)\cap L^{2}\left((0,t);H^{3}(\Omega)\right).$$

This completes the proof of the Lemma 11.

Hence, the proof of Theorem 1.1 is completed by applying Lemma 4-Lemma 11.

4. Conclusion

This paper investigates the long-time behavior of 2D Cahn-Hiliiard-Stokes model by using energy method. This result enriches the theoretical research on the coupled Cahn-Hilliard model, which physically indicates that as time goes to infinity, the system approaches a stable equilibrium state. In future research, we could extend the study to examine the well-posedness, regularity, and long-time behavior of solutions under different external force conditions and initial-boundary value conditions.

Funding

Supported by Hunan Provincial Key Research Project on Teaching Reform in Regular Undergraduate Education (Grant No. 202502000461); National Natural Science Foundation of China (Grant No. 12001064); Natural Science Foundation of Hunan Province (Grant No. 2023JJ0007); National First-class Offline Undergraduate Course Com-plex Variable Functions and Integral Transformations and Major Scientific and Technological Innovation Platform Project of Hunan Province (Grant No. 2024JC1003); National First-class Offfine Undergraduate Course Complex Variable Functions and Integral Transformations and the Hunan Provincial Key Laboratory of Mathematical Modeling and Analysis in Engineering (Grant No. 2017TP1017).

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

References

[1] Cahn, J.W. and Hilliard, J.E. (1958) Free Energy of a Nonuniform System. I. Interfacial Free Energy. *The Journal of Chemical Physics*, **28**, 258-267. https://doi.org/10.1063/1.1744102

[2] Cahn, J.W. (1961) On Spinodal Decomposition. *Acta Metallurgica*, **9**, 795-801. https://doi.org/10.1016/0001-6160(61)90182-1

[3] Boyer, F. (1999) Mathematical Study of Multi-Phase Flow under Shear through Order Parameter Formulation. *Asymptotic Analysis*, **20**, 175-212. https://doi.org/10.3233/asy-1999-345

[4] Abels, H. (2008) On a Diffuse Interface Model for Two-Phase Flows of Viscous, Incompressible Fluids with Matched Densities. *Archive for Rational Mechanics and Analysis*, **194**, 463-506. https://doi.org/10.1007/s00205-008-0160-2

[5] Gal, C.G. and Grasselli, M. (2010) Asymptotic Behavior of a Cahn-Hilliard-Navier–stokes System in 2D. *Annales de l'Institut Henri Poincaré C, Analyse Non Linéaire*, **27**, 401-436. https://doi.org/10.1016/j.anihpc.2009.11.013

[6] Gal, C.G. and Grasselli, M. (2010) Longtime Behavior for a Model of Homogeneous Incompressible Two-Phase Flows. *Discrete & Continuous Dynamical Systems*, **28**, 1-39. https://doi.org/10.3934/dcds.2010.28.1

[7] Giorgini, A., Miranville, A. and Temam, R. (2019) Uniqueness and Regularity for the Navier-Stokes-Cahn-Hilliard System. *SIAM Journal on Mathematical Analysis*, **51**, 2535-2574. https://doi.org/10.1137/18m1223459

[8] Abels, H., Garcke, H. and Giorgini, A. (2023) Global Regularity and Asymptotic Stabilization for the Incompressible Navier-Stokes-Cahn-Hilliard Model with Unmatched Densities. *Mathematische Annalen*, **389**, 1267-1321. https://doi.org/10.1007/s00208-023-02670-2

[9] Abels, H. (2012) Strong Well-Posedness of a Diffuse Interface Model for a Viscous,

Quasi-Incompressible Two-Phase Flow. *SIAM Journal on Mathematical Analysis*, **44**, 316-340. https://doi.org/10.1137/110829246

[10] Abels, H., Depner, D. and Garcke, H. (2012) Existence of Weak Solutions for a Diffuse Interface Model for Two-Phase Flows of Incompressible Fluids with Different Densities. *Journal of Mathematical Fluid Mechanics*, **15**, 453-480. https://doi.org/10.1007/s00021-012-0118-x

[11] Abels, H., Depner, D. and Garcke, H. (2013) On an Incompressible Navier-Stokes/Cahn-Hilliard System with Degenerate Mobility. *Annales de l'Institut Henri Poincaré C, Analyse Non Linéaire*, **30**, 1175-1190. https://doi.org/10.1016/j.anihpc.2013.01.002

[12] Ngamsaad, W., Yojina, J. and Triampo, W. (2010) Theoretical Studies of Phase-Separation Kinetics in a Brinkman Porous Medium. *Journal of Physics A: Mathematical and Theoretical*, **43**, Article 202001. https://doi.org/10.1088/1751-8113/43/20/202001

[13] Bosia, S., Conti, M. and Grasselli, M. (2015) On the Cahn-Hilliard-Brinkman System. *Communications in Mathematical Sciences*, **13**, 1541-1567. https://doi.org/10.4310/cms.2015.v13.n6.a9

[14] Li, B., Wang, F., Xue, L., *et al.* (2021) On the Cahn-Hilliard-Brinkman Equations in $\mathbb{R}^4$: Global Well-Posedness. *Annals of Applied Mathematics*, **37**, 513-536. https://doi.org/10.4208/aam.oa-2021-0011

[15] Wang, X. and Wu, H. (2012) Long-Time Behavior for the Hele-Shaw-Cahn-Hilliard System. *Asymptotic Analysis*, **78**, 217-245. https://doi.org/10.3233/asy-2012-1092

[16] Wang, X. and Zhang, Z. (2013) Well-Posedness of the Hele-Shaw-Cahn-Hilliard System. *Annales de l'Institut Henri Poincaré C, Analyse Non Linéaire*, **30**, 367-384. https://doi.org/10.1016/j.anihpc.2012.06.003

[17] Lowengrub, J., Titi, E. and Zhao, K. (2013) Analysis of a Mixture Model of Tumor Growth. *European Journal of Applied Mathematics*, **24**, 691-734. https://doi.org/10.1017/s0956792513000144

[18] Baňas, Ľ. and Mahato, H.S. (2017) Homogenization of Evolutionary Stokes-Cahn–Hilliard Equations for Two-Phase Porous Media Flow. *Asymptotic Analysis*, **105**, 77-95. https://doi.org/10.3233/asy-171436

[19] Cheng, K. and Feng, W. (2017) Global Well-Posedness of Strong Solution for the Three Dimensional Dynamic Cahn-Hilliard-Stokes Model. *The Journal of Nonlinear Sciences and Applications*, **10**, 4209-4221. https://doi.org/10.22436/jnsa.010.08.16

[20] Adams, R.A. and Fournier, J.J.F. (2003) Sobolev Spaces. Elsevier.

[21] Constantin, P. and Foias, C. (1988) Navier-Stokes Equations. University of Chicago Press. https://doi.org/10.7208/chicago/9780226764320.001.0001

[22] Temam, R. (2024) Navier-Stokes Equations: Theory and Numerical Analysis. American Mathematical Society.

[23] Zhao, K. (2011) Large Time Behavior of a Cahn-Hilliard-Boussinesq System on a Bounded Domain. *Electronic Journal of Differential Equations*, **2011**, 1-21. http://ejde.math.txstate.edu