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q-Nonuniform Laplace Transform

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Abstract

We first introduce the concepts of *q*-nonuniform exponential and trigonometric functions and then discuss the related *q*-nonuniform difference version of the integral Laplace transform and some of its applications.

Keywords

q-Nonuniform Difference Equations, *q*-Nonuniform Exponential and Trigonometric Functions, *q*-Nonuniform Laplace Transforms

1. Introduction

Considering the most general divided difference derivative [1] [2],

$$\mathcal{D}f\left(t\left(s\right)\right)=\frac{f\left(t\left(s+\frac{1}{2}\right)\right)-f\left(t\left(s-\frac{1}{2}\right)\right)}{t\left(s+\frac{1}{2}\right)-t\left(s-\frac{1}{2}\right)},\tag{1}$$

admitting the property that if $f\left(t\right)=P_n\left(t\left(s\right)\right)$ is a polynomial of degree n in $t\left(s\right)$, then $\mathcal{D}f\left(t\left(s\right)\right)=\tilde{P}_{n-1}\left(t\left(s\right)\right)$ is a polynomial in $t\left(s\right)$ of degree $n-1$, one is led to the following most important canonical forms for $t\left(s\right)$ in order of increasing complexity:

$$t\left(s\right)=t_0;\tag{2}$$

$$t\left(s\right)=s;\tag{3}$$

$$t\left(s\right)=q^{s};\tag{4}$$

$$t\left(s\right)=q^{s}+q^{-s}2,\ \ q\in\mathbb{C},\ s\in\mathbb{Z}.\tag{5}$$

When the function $t\left(s\right)$ is given by (2)-(4), the divided difference derivative (1) leads to the ordinary differential derivative $Df\left(x\right)=\frac{\mathrm{d}}{\mathrm{d}x}f\left(x\right)$, finite difference derivative $\Delta f(t)=f(t+1)-f(t)=\left(\mathrm{e}^{\frac{\mathrm{d}}{\mathrm{d}t}}-1\right)f(t)$ and q-difference derivative (or Jackson derivative [3]) $D_q f(t)=\frac{f(qt)-f(t)}{qt-t}=\frac{q^{\frac{\mathrm{d}}{\mathrm{d}t}}-1}{qt-t}f(t)$, respectively.

When $t(s)$ is given by (5), the corresponding derivative gives

$$\mathcal{D}f(t(s))=\frac{f\left(t\left(s+\frac{1}{2}\right)\right)-f\left(t\left(s-\frac{1}{2}\right)\right)}{t\left(s+\frac{1}{2}\right)-t\left(s-\frac{1}{2}\right)}$$

$$=\frac{f\left(\frac{q^{\left(s+\frac{1}{2}\right)}+q^{\left(-s-\frac{1}{2}\right)}}{2}\right)-f\left(\frac{q^{\left(s-\frac{1}{2}\right)}+q^{\left(-s+\frac{1}{2}\right)}}{2}\right)}{\frac{q^{\left(s+\frac{1}{2}\right)}+q^{\left(-s-\frac{1}{2}\right)}}{2}-\frac{q^{\left(s-\frac{1}{2}\right)}+q^{\left(-s+\frac{1}{2}\right)}}{2}}.$$

It is usually referred to as the *Askey-Wilson* first order divided difference operator [4] that one can write:

$$\mathcal{D}f(x(z))=\frac{f\left(x\left(q^{\frac{1}{2}}z\right)\right)-f\left(x\left(q^{-\frac{1}{2}}z\right)\right)}{x\left(q^{\frac{1}{2}}z\right)-x\left(q^{-\frac{1}{2}}z\right)}, \tag{6}$$

where $x(z)=\frac{z+z^{-1}}{2}$, having in mind that $z=q^{s}$.

The Askey-Wilson polynomials [4], $P_n(x(z)), x(z)=\frac{z+z^{-1}}{2}$ are defined by:

$$P_n(x(z))=\frac{(ab,ac,ad,q)_n}{a^n}\,4^{\phi_3}\left(\begin{matrix}q^{-n},abcdq^{n-1},az,az^{-1}\\ ab,ac,ad\end{matrix}\,|\,q;q\right) \tag{7}$$

where the basic hypergeometric (or *q*-hypergeometric)series r^{ϕ_s} read:

$$r^{\phi_s}\left(\begin{matrix}a_1,\cdots,a_r\\ b_1,\cdots,b_s\end{matrix}\,|\,q;z\right):=\sum_{k=0}^{\infty}\frac{(a_1,\cdots,a_r;q)_k}{(b_1,\cdots,b_s;q)_k}(-1)^{(1+s-r)k}q^{(1+s-r)\binom{k}{2}}\frac{z^k}{(q;q)_k}$$

with

$$(a_1,\cdots,a_r;q)_k:=(a_1;q)_k\cdots(a_r;q)_k,$$

while

$$(\sigma;q)_0:=1,(\sigma;q)_k:=\prod_{i=0}^{k-1}\left(1-\sigma q^{i}\right),k=1,2,3,\cdots$$

They satisfy the following second order *q*-difference equation:

$$£P_n(x(z))=\lambda(n)P_n(x(z)) \tag{8}$$

where

$$£=\left[\frac{(q-1)^2}{4q}\frac{z^2-1}{z\omega(z)}D_q\frac{z^2-1}{z}v\left(q^{-\frac{1}{2}}z\right)\omega\left(q^{-\frac{1}{2}}z\right)\right]D_q,\ \text{with}\ \frac{\omega(qz)}{\omega(z)}=\frac{v(z)}{v\left((qz)^{-1}\right)}$$

is the Askey-Wilson second order q-difference operator and

$$\lambda(n)=-\left(1-q^{-n}\right)\left(1-abcdq^{n-1}\right).$$

The analysis related to (2), (3), (4) and (5) are referred to as the differential, difference, q-difference and q-nonuniform difference analysis respectively. Laplace transform of an exponential type function $f(x)$ is given by

$$F(p)=\mathcal{L}\{f(x)\}\overset{\text{def}}{=}\int_0^{+\infty}\mathrm{e}^{-px}f(x)\,\mathrm{d}x,\ \ p=a+ib\in\mathbb{C},\tag{9}$$

and plays a major role in pure and applied analysis, especially in solving differential equations. If we consider $f(x)$ as a function of a discrete variable i.e. $t\in\mathbb{Z}$, then the transformation (9) reads

$$F(z)=Z\{f(x)\}\overset{\text{def}}{=}\sum_{j=0}^{+\infty}f(j)z^{-j},\ \ z=\mathrm{e}^{-p}.\tag{10}$$

It is referred to as *Z transform* and plays similar role in difference analysis as Laplace transform in continuous analysis, especially in solving difference equations [5].

For a given function $f(x)$ on the lattice (4), we define its *q-Laplace transform* as the function [6]

$$F(p)=\mathcal{L}_q\{f(x)\}\overset{\text{def}}{=}\int_0^{+\infty}e_{q^{-1}}^{-px}f(x)\,\mathrm{d}_qx,\ \ p=s+i\sigma\in\mathbb{C},\tag{11}$$

where the intégral is the Jackson intégral [3]

$$\int_0^x f(x)\,\mathrm{d}_qx=(1-q)x\sum_{i=0}^{\infty}q^if\left(xq^i\right)\tag{12}$$

and

$$\int_0^{\infty} f(x)\,\mathrm{d}_qx=(1-q)\sum_{i=-\infty}^{\infty}q^if\left(q^i\right).\tag{13}$$

It plays similar role in q-difference analysis as Laplace transform in continuous analysis or Z transform in difference analysis, especially in solving q-difference equations [6].

In this work, we introduce *q-nonuniform Laplace transform* defined for functions $f(x(z))$ given on the lattice (5), as

$$\begin{aligned}F(p)&=\mathcal{L}_{q^{\frac{1}{2}}}\{f(x(z))\}\\&\overset{\text{def}}{=}\int_{x_0}^{\infty}E_{q^{-\frac{1}{2}}}\left(-p;x_0;x\left(q^{-\frac{1}{2}}\right)\right)f(x(z))\,\mathrm{d}x(z),\end{aligned}\tag{14}$$

$$p=s+i\sigma\in\mathbb{C},\ 0\le z_0,z\le\infty,\ 0<q<1,\ x_0=x(z_0),\ x(\infty)=\infty,$$

where the intégral is defined by [7] [8]

$$\int_{x\left(q^N\right)}^{x\left(q^s\right)} g\left(x(z)\right)\mathrm{d}_q x(z) \overset{\text{def}}{=} \frac{1}{2}(1-q)\sum_{z=q^s}^{q^{N-1}} z\left(1-\frac{1}{qz^2}\right) g\left(x\left(q^{\frac{1}{2}}z\right)\right) \tag{15}$$

and

$$\int_{x(z)}^{\infty} g\left(x(z)\right)\mathrm{d}_q x(z) = \frac{1}{2}(1-q)\, z\sum_{i=0}^{\infty} q^i\left(\frac{1}{q^{2i+1}z^2}-1\right) g\left(x\left(q^{\frac{1}{2}+i}z\right)\right). \tag{16}$$

It is expected to play similar role in q-nonuniform difference analysis as Laplace transform, Z transform and q-Laplace transform in differential, difference and q-difference analysis respectively, especially in solving q-nonuniform difference equations [9].

In the next section, we begin by introducing the concepts of q-nonuniform exponential and trigonometric functions as solutions of first or second order q-nonuniform difference equations.

2. q-Nonuniform Exponential and Trigonometric Functions

Before introducing the concept of q-non nuniform Laplace Transform, we clearly first study these of q-non exponential and trigonometric functions.

Consider first the following equation

$$\mathcal{D}y_0\left(x(z)\right) = a\left(x(z)\right) y_0\left(x\left(zq^{-\frac{1}{2}}\right)\right). \tag{17}$$

Developping it

$$\frac{y_0(x\left(zq^{\frac{1}{2}}\right) - y_0(x\left(zq^{-\frac{1}{2}}\right)}{x\left(zq^{\frac{1}{2}}\right) - x\left(zq^{-\frac{1}{2}}\right)} = a\left(x(z)\right) y_0\left(x\left(zq^{-\frac{1}{2}}\right)\right),$$

we get

$$y_0\left(x\left(zq^{\frac{1}{2}}\right)\right) = p\left(x(z)\right) y_0\left(x\left(zq^{-\frac{1}{2}}\right)\right) \tag{18}$$

where

$$\begin{aligned} p\left(x(z)\right) &= 1 + \left(x\left(zq^{\frac{1}{2}}\right) - x\left(zq^{-\frac{1}{2}}\right)\right) a\left(x(z)\right) \\ &= 1 + \frac{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}{2}\left(z - z^{-1}\right) a\left(x(z)\right). \end{aligned} \tag{19}$$

By the recursion

$$y_0\left(x(z)\right) = \left[p\left(x\left(zq^{\frac{1}{2}}\right)\right)\right]^{-1} y_0\left(x(zq)\right), \tag{20}$$

we get

$$y_0\left(x(z)\right)=y_0\left(x(z_0)\right)\prod_{t=z}^{q^{-1}z_0}\left[p\left(x\left(tq^{\frac{1}{2}}\right)\right)\right]^{-1}, \tag{21}$$

or

$$y_0\left(x(z)\right)=\left(\prod_{i=0}^{N-1}\left[p\left(x\left(zq^{\frac{1}{2}+i}\right)\right)\right]^{-1}\right)y_0\left(x\left(zq^N\right)\right). \tag{22}$$

Define the q-nonuniform exponential function

$$E_{q^{-\frac{1}{2}}}(a;z_0;z)\overset{\text{def}}{=}\begin{cases}\prod_{t=z}^{q^{-1}z_0}\left[p\left(x\left(tq^{\frac{1}{2}}\right)\right)\right]^{-1}, & z>z_0,\\ 1, & z=z_0>0.\end{cases} \tag{23}$$

Clearly, we have the relation

$$E_{q^{-\frac{1}{2}}}(a;z_0;z)=\left[p\left(x\left(zq^{\frac{1}{2}}\right)\right)\right]^{-1}E_{q^{-\frac{1}{2}}}(a;z_0;qz). \tag{24}$$

This means that $E_{q^{-\frac{1}{2}}}(a;z_0;z)$ is a solution of (20) and consequently, that of (17). Moreover,

$$E_{q^{-\frac{1}{2}}}(a;z_0;z)=\left[p\left(x\left(z_0q^{-\frac{1}{2}}\right)\right)\right]^{-1}E_{q^{-\frac{1}{2}}}\left(a;q^{-1}z_0;z\right). \tag{25}$$

From the definition (23), we can evaluate

$$E_{q^{-\frac{1}{2}}}(a;0;z)=\prod_{i=0}^{\infty}\left[p\left(x\left(q^izq^{\frac{1}{2}}\right)\right)\right]^{-1}, \tag{26}$$

and

$$E_{q^{-\frac{1}{2}}}(a;z_0;\infty)=\prod_{i=0}^{\infty}\left[p\left(x\left(q^{-i}q^{-1}z_0q^{\frac{1}{2}}\right)\right)\right]^{-1}=\prod_{i=-1}^{-\infty}\left[p\left(x\left(q^iz_0q^{\frac{1}{2}}\right)\right)\right]^{-1}. \tag{27}$$

Also, one easily verifies that $E_{q^{-\frac{1}{2}}}(a;z_0;\infty)$ is a solution of (25) for $z=\infty$.

If we set $z=z_0=1$ in (26) and (27), we get

$$E_{q^{-\frac{1}{2}}}(a;0;\infty)=\prod_{i=-\infty}^{\infty}\left[p\left(x\left(q^iq^{\frac{1}{2}}\right)\right)\right]^{-1}. \tag{28}$$

On the other side, since $0<q<1$, then, for $a\left(x(z)\right)=a=const$, $z_0,z\neq 0,\infty$, one can verify that

$$E_{q^{-\frac{1}{2}}}(a;0;z)=0=E_{q^{-\frac{1}{2}}}(a;z_0;\infty). \tag{29}$$

Indeed,

$$\begin{aligned} p\left(x\left(zq^{\frac{1}{2}}\right)\right) &= 1+\frac{q^{\frac{1}{2}}-q^{-\frac{1}{2}}}{2}\left(q^{\frac{1}{2}}z-q^{-\frac{1}{2}}z^{-1}\right)a \\ &= \frac{a(q-1)}{2}\frac{(z-z_1)(z-z_2)}{z}; \end{aligned} \tag{30}$$

with

$$z_1=\frac{1-2q+2\sqrt{q^2-2a^2q^2+a^2q^3+a^2q}}{2\left(aq^2-aq\right)},$$

$$z_2=\frac{1-2q-2\sqrt{q^2-2a^2q^2+a^2q^3+a^2q}}{2\left(aq^2-aq\right)}.$$

Hence

$$\begin{aligned} E_{q^{-\frac{1}{2}}}(a;0;z) &= \prod_{i=0}^{\infty}\left[p\left(x\left(q^{i}zq^{\frac{1}{2}}\right)\right)\right]^{-1} \\ &= \frac{2z}{a(q-1)z_1z_2}\frac{1}{\left(z_1^{-1}z;q\right)_\infty\left(z_2^{-1}z;q\right)_\infty}\cdot\prod_{i=0}^{\infty}q^{i}=0, \end{aligned} \tag{31}$$

where $(u;q)_\infty \overset{\text{def}}{=} \prod_{i=0}^{\infty}\left(1-uq^{i}\right)$. Similarly, one proves that $E_{q^{-\frac{1}{2}}}(a;z_0;\infty)=0$.

Consider now the following equation

$$\mathcal{D}\tilde{y}(x(z))=\tilde{a}(x(z))\,\tilde{y}\left(x\left(zq^{\frac{1}{2}}\right)\right). \tag{32}$$

Developping, we get the relation

$$\tilde{y}\left(x\left(zq^{\frac{1}{2}}\right)\right)\tilde{p}(x(z))=\tilde{y}\left(x\left(zq^{-\frac{1}{2}}\right)\right). \tag{33}$$

where

$$\begin{aligned} \tilde{p}(x(z)) &= 1-\left(x\left(zq^{\frac{1}{2}}\right)-x\left(zq^{-\frac{1}{2}}\right)\right)a(x(z)) \\ &= 1-\frac{q^{\frac{1}{2}}-q^{-\frac{1}{2}}}{2}\left(z-z^{-1}\right)\tilde{a}(x(z)). \end{aligned} \tag{34}$$

Using the recursion

$$\tilde{y}_0(x(z))=\tilde{p}\left(x\left(zq^{\frac{1}{2}}\right)\right)\tilde{y}_0(x(zq)), \tag{35}$$

we get

$$\tilde{y}_0(x(z))=\tilde{y}_0(x(z_0))\prod_{t=z}^{q^{-1}z_0}\tilde{p}\left(x\left(tq^{\frac{1}{2}}\right)\right), \tag{36}$$

or

$$\tilde{y}_0\left(x(z)\right)=\left(\prod_{i=0}^{N-1}\tilde{p}\left(x\left(zq^{\frac{1}{2}+i}\right)\right)\right)\tilde{y}_0\left(x\left(zq^N\right)\right). \tag{37}$$

Define the other $q-$nonuniform exponential function

$$E_{q^{\frac{1}{2}}}\left(\tilde{a};z_0;z\right)\overset{\text{def}}{=}\begin{cases}\prod_{t=z}^{q^{-1}z_0}\tilde{p}\left(x\left(tq^{\frac{1}{2}}\right)\right), & z>z_0,\\ 1, & z=z_0>0.\end{cases} \tag{38}$$

Here also, the function $E_{q^{\frac{1}{2}}}\left(\tilde{a};z_0;z\right)$ admet representations and properties similar to that of $E_{q^{-\frac{1}{2}}}\left(a;z_0;z\right)$. In particular, $E_{q^{\frac{1}{2}}}\left(\tilde{a};z_0;z\right)$ verifies the relation

$$E_{q^{\frac{1}{2}}}\left(\tilde{a};z_0;z\right)=\tilde{p}\left(x\left(zq^{\frac{1}{2}}\right)\right)E_{q^{\frac{1}{2}}}\left(\tilde{a};z_0;zq\right). \tag{39}$$

This means that $E_{q^{\frac{1}{2}}}\left(\tilde{a};z_0;z\right)$ is a solution of (35) and consequently, that of (32). Moreover,

$$E_{q^{\frac{1}{2}}}\left(\tilde{a};z_0;z\right)=\tilde{p}\left(x\left(z_0q^{-\frac{1}{2}}\right)\right)E_{q^{\frac{1}{2}}}\left(\tilde{a};q^{-1}z_0;z\right). \tag{40}$$

Similarly, we evaluate

$$E_{q^{\frac{1}{2}}}\left(\tilde{a};0;z\right)=\prod_{i=0}^{\infty}p\left(x\left(q^izq^{\frac{1}{2}}\right)\right), \tag{41}$$

and

$$E_{q^{\frac{1}{2}}}\left(\tilde{a};z_0;\infty\right)=\prod_{i=-1}^{-\infty}\tilde{p}\left(x\left(q^iz_0q^{\frac{1}{2}}\right)\right). \tag{42}$$

Also, one easily verifies that $E_{q^{\frac{1}{2}}}\left(\tilde{a};z_0;\infty\right)$ is a solution of (40) for $z=\infty$. Also

$$E_{q^{\frac{1}{2}}}\left(\tilde{a};0;\infty\right)=\prod_{i=-\infty}^{\infty}\tilde{p}\left(x\left(q^iz_0q^{\frac{1}{2}}\right)\right), \tag{43}$$

and

$$E_{q^{\frac{1}{2}}}\left(\tilde{a};0;z\right)=\infty=E_{q^{\frac{1}{2}}}\left(\tilde{a};z_0;\infty\right). \tag{44}$$

Consider now the product $h(z)=E_{q^{-\frac{1}{2}}}\left(a;z_0;z\right)\cdot E_{q^{\frac{1}{2}}}\left(\tilde{a};z_0;z\right)$. Define $K_{a,\tilde{a}}$ by

$$K_{a,\tilde{a}}=h(\infty)=E_{q^{-\frac{1}{2}}}\left(a;z_0;\infty\right)\cdot E_{q^{\frac{1}{2}}}\left(\tilde{a};z_0;\infty\right)=\prod_{i=1}^{\infty}\frac{\tilde{p}\left(x\left(q^{-i}z_0q^{\frac{1}{2}}\right)\right)}{p\left(x\left(q^{-i}z_0q^{\frac{1}{2}}\right)\right)}. \tag{45}$$

Simple computations give

$$K_{a,\tilde{a}}=\frac{-\tilde{a}}{a}\frac{\left(1-z_0^{-1}z_1\right)\left(1-z_0^{-1}z_2\right)}{\left(1-z_0^{-1}\tilde{z}_1\right)\left(1-z_0^{-1}\tilde{z}_2\right)}\frac{\left(z_0^{-1}\tilde{z}_1;q\right)_\infty\left(z_0^{-1}\tilde{z}_2;q\right)_\infty}{\left(z_0^{-1}z_1;q\right)_\infty\left(z_0^{-1}z_2;q\right)_\infty}, \tag{46}$$

where z_1, z_2 and $\tilde{z}_1, \tilde{z}_2$ are finite roots of $p\left(q^{\frac{1}{2}}z\right)=0$ and $\tilde{p}\left(q^{\frac{1}{2}}z\right)=0$, respectively, knowing that $\tilde{p}(x(z))$ is obtained from $p(x(z))$ by replacing a by $-\tilde{a}$.

From (19), (23), (34) and (38), we get the following

Theorem 2.1

$$E_{q^{-\frac{1}{2}}}(a;z_0;z)\cdot E_{q^{\frac{1}{2}}}(-a;z_0;z)=1=E_{q^{-\frac{1}{2}}}(-a;z_0;z)\cdot E_{q^{\frac{1}{2}}}(a;z_0;z). \tag{47}$$

More generally, we have the following

Theorem 2.2 *If* $y(x(z))$ *and* $\tilde{y}(x(z))$ *are respective solutions of the equations*

$$\mathcal{D}y(x(z))=a(x(z))\,y\left(x\left(zq^{-\frac{1}{2}}\right)\right),$$

$$\mathcal{D}\tilde{y}(x(z))=-a(x(z))\,\tilde{y}\left(x\left(zq^{-\frac{1}{2}}\right)\right), \tag{48}$$

satifying the conditions

$$y(x(z_0))\,\tilde{y}(x(z_0))=1, \tag{49}$$

then

$$y(x(z))\,\tilde{y}(x(z))=1. \tag{50}$$

Proof.

$$\begin{aligned}\mathcal{D}(y\tilde{y})&=y\left(x\left(q^{-\frac{1}{2}}z\right)\right)\mathcal{D}\tilde{y}+\tilde{y}\left(x\left(zq^{\frac{1}{2}}\right)\right)\mathcal{D}y\\&=y\left(x\left(q^{-\frac{1}{2}}z\right)\right)(-a)\,\tilde{y}\left(x\left(zq^{\frac{1}{2}}\right)\right)+\tilde{y}\left(x\left(zq^{\frac{1}{2}}\right)\right)(a)\,y\left(x\left(q^{-\frac{1}{2}}z\right)\right)\\&=0.\end{aligned}$$

This means that $y\tilde{y}=const$, which, by (49) gives (50).

Consider next the following definitions

1)

$$\cos_{q^{\frac{1}{2}}}(a;z_0;z)\overset{\text{def}}{=}\frac{E_{q^{\frac{1}{2}}}(ai;z_0;z)+E_{q^{\frac{1}{2}}}(-ai;z_0;z)}{2};$$

$$\sin_{q^{\frac{1}{2}}}(a;z_0;z)\overset{\text{def}}{=}\frac{E_{q^{\frac{1}{2}}}(ai;z_0;z)-E_{q^{\frac{1}{2}}}(-ai;z_0;z)}{2i};$$

$$\cosh_{q^{\frac{1}{2}}}(a;z_0;z)\overset{\text{def}}{=}\frac{E_{q^{\frac{1}{2}}}(a;z_0;z)+E_{q^{\frac{1}{2}}}(-a;z_0;z)}{2};$$

$$\sinh_{q^{\frac{1}{2}}}(a;z_0;z)\overset{\text{def}}{=}\frac{E_{q^{\frac{1}{2}}}(a;z_0;z)-E_{q^{\frac{1}{2}}}(-a;z_0;z)}{2}. \tag{51}$$

2)

$$\cos_{q^{-\frac{1}{2}}}\left(a;z_0;z\right)\overset{\text{def}}{=}\frac{E_{q^{-\frac{1}{2}}}\left(ai;z_0;z\right)+E_{q^{-\frac{1}{2}}}\left(-ai;z_0;z\right)}{2};$$

$$\sin_{q^{-\frac{1}{2}}}\left(a;z_0;z\right)\overset{\text{def}}{=}\frac{E_{q^{-\frac{1}{2}}}\left(ai;z_0;z\right)-E_{q^{-\frac{1}{2}}}\left(-ai;z_0;z\right)}{2i};$$

$$\cosh_{q^{-\frac{1}{2}}}\left(a;z_0;z\right)\overset{\text{def}}{=}\frac{E_{q^{-\frac{1}{2}}}\left(a;z_0;z\right)+E_{q^{-\frac{1}{2}}}\left(-a;z_0;z\right)}{2};$$

$$\sinh_{q^{-\frac{1}{2}}}\left(a;z_0;z\right)\overset{\text{def}}{=}\frac{E_{q^{-\frac{1}{2}}}\left(a;z_0;z\right)-E_{q^{-\frac{1}{2}}}\left(-a;z_0;z\right)}{2}. \tag{52}$$

3)

$$\cos_{q^{\frac{1}{2}}q^{-\frac{1}{2}}}\left(a;z_0;z\right)\overset{\text{def}}{=}\frac{E_{q^{\frac{1}{2}}}\left(ai;z_0;z\right)+E_{q^{-\frac{1}{2}}}\left(-ai;z_0;z\right)}{2};$$

$$\sin_{q^{\frac{1}{2}}q^{-\frac{1}{2}}}\left(a;z_0;z\right)\overset{\text{def}}{=}\frac{E_{q^{\frac{1}{2}}}\left(ai;z_0;z\right)-E_{q^{-\frac{1}{2}}}\left(-ai;z_0;z\right)}{2i};$$

$$\cosh_{q^{\frac{1}{2}}q^{-\frac{1}{2}}}\left(a;z_0;z\right)\overset{\text{def}}{=}\frac{E_{q^{\frac{1}{2}}}\left(a;z_0;z\right)+E_{q^{-\frac{1}{2}}}\left(-a;z_0;z\right)}{2};$$

$$\sinh_{q^{\frac{1}{2}}q^{-\frac{1}{2}}}\left(a;z_0;z\right)\overset{\text{def}}{=}\frac{E_{q^{\frac{1}{2}}}\left(a;z_0;z\right)-E_{q^{-\frac{1}{2}}}\left(-a;z_0;z\right)}{2}. \tag{53}$$

We get

$$\mathcal{D}\cos_{q^{\frac{1}{2}}}\left(a;z_0;z\right)=-a\sin_{q^{\frac{1}{2}}}\left(a;z_0;zq^{\frac{1}{2}}\right);$$

$$\mathcal{D}^{-}\cos_{q^{\frac{1}{2}}}\left(a;z_0;z\right)=-a\sin_{q^{\frac{1}{2}}}\left(a;z_0;z\right),$$

$$\mathcal{D}\sin_{q^{\frac{1}{2}}}\left(a;z_0;z\right)=a\cos_{q^{\frac{1}{2}}}\left(a;z_0;zq^{\frac{1}{2}}\right);$$

$$\mathcal{D}^{-}\sin_{q^{\frac{1}{2}}}\left(a;z_0;z\right)=a\cos_{q^{\frac{1}{2}}}\left(a;z_0;z\right), \tag{54}$$

where $\mathcal{D}^{-}f\left(x\left(z\right)\right)=Df\left(x\left(z\right)\right)\Big|_{z:=zq^{-\frac{1}{2}}}$. Hence

$$\left[\mathcal{D}^{-}\right]^2\cos_{q^{\frac{1}{2}}}\left(a;z_0;z\right)=-a^2\cos_{q^{\frac{1}{2}}}\left(a;z_0;z\right),$$

$$\left[\mathcal{D}^{-}\right]^2\sin_{q^{\frac{1}{2}}}\left(a;z_0;z\right)=-a^2\sin_{q^{\frac{1}{2}}}\left(a;z_0;z\right). \tag{55}$$

Similarly

$$\mathcal{D}\cosh_{q^{\frac{1}{2}}}(a;z_0;z) = a\sinh_{q^{\frac{1}{2}}}\left(a;z_0;zq^{\frac{1}{2}}\right);$$

$$\mathcal{D}^{-}\cosh_{q^{\frac{1}{2}}}(a;z_0;z) = a\sinh_{q^{\frac{1}{2}}}(a;z_0;z),$$

$$\mathcal{D}\sinh_{q^{\frac{1}{2}}}(a;z_0;z) = a\cosh_{q^{\frac{1}{2}}}\left(a;z_0;zq^{\frac{1}{2}}\right);$$

$$\mathcal{D}^{-}\sinh_{q^{\frac{1}{2}}}(a;z_0;z) = a\cosh_{q^{\frac{1}{2}}}(a;z_0;z), \tag{56}$$

and consequently

$$\left[\mathcal{D}^{-}\right]^2\cosh_{q^{\frac{1}{2}}}(a;z_0;z) = a^2\cosh_{q^{\frac{1}{2}}}(a;z_0;z),$$

$$\left[\mathcal{D}^{-}\right]^2\sinh_{q^{\frac{1}{2}}}(a;z_0;z) = a^2\sinh_{q^{\frac{1}{2}}}(a;z_0;z). \tag{57}$$

On the other side, we have

$$\mathcal{D}\cos_{q^{-\frac{1}{2}}}(a;z_0;z) = -a\sin_{q^{-\frac{1}{2}}}\left(a;z_0;zq^{-\frac{1}{2}}\right);$$

$$\mathcal{D}^{+}\cos_{q^{-\frac{1}{2}}}(a;z_0;z) = -a\sin_{q^{-\frac{1}{2}}}(a;z_0;z),$$

$$\mathcal{D}\sin_{q^{-\frac{1}{2}}}(a;z_0;z) = a\cos_{q^{-\frac{1}{2}}}\left(a;z_0;zq^{-\frac{1}{2}}\right);$$

$$\mathcal{D}^{+}\sin_{q^{-\frac{1}{2}}}(a;z_0;z) = a\cos_{q^{-\frac{1}{2}}}(a;z_0;z), \tag{58}$$

where $\mathcal{D}^{+}f(x(z)) = Df(x(z))\big|_{z:=zq^{\frac{1}{2}}}$. Hence

$$\left[\mathcal{D}^{+}\right]^2\cos_{q^{-\frac{1}{2}}}(a;z_0;z) = -a^2\cos_{q^{-\frac{1}{2}}}(a;z_0;z),$$

$$\left[\mathcal{D}^{+}\right]^2\sin_{q^{-\frac{1}{2}}}(a;z_0;z) = -a^2\sin_{q^{-\frac{1}{2}}}(a;z_0;z). \tag{59}$$

Similarly

$$\mathcal{D}\cosh_{q^{-\frac{1}{2}}}(a;z_0;z) = a\sinh_{q^{-\frac{1}{2}}}\left(a;z_0;zq^{-\frac{1}{2}}\right);$$

$$\mathcal{D}^{-}\cosh_{q^{-\frac{1}{2}}}(a;z_0;z) = a\sinh_{q^{-\frac{1}{2}}}(a;z_0;z),$$

$$\mathcal{D}\sinh_{q^{-\frac{1}{2}}}(a;z_0;z) = a\cosh_{q^{-\frac{1}{2}}}\left(a;z_0;zq^{-\frac{1}{2}}\right);$$

$$\mathcal{D}^{-}\sinh_{q^{-\frac{1}{2}}}(a;z_0;z) = a\cosh_{q^{-\frac{1}{2}}}(a;z_0;z),$$

and consequently

$$\left[\mathcal{D}^{+}\right]^{2}\cosh_{q^{-\frac{1}{2}}}\left(a;z_0;z\right)=a^{2}\cosh_{q^{-\frac{1}{2}}}\left(a;z_0;z\right),$$

$$\left[\mathcal{D}^{+}\right]^{2}\sinh_{q^{-\frac{1}{2}}}\left(a;z_0;z\right)=a^{2}\sinh_{q^{-\frac{1}{2}}}\left(a;z_0;z\right). \tag{60}$$

From the preceding, one easily verifies that

a)

$$\cos^{2}_{q^{\frac{1}{2}}q^{-\frac{1}{2}}}\left(a;z_0;z\right)+\sin^{2}_{q^{\frac{1}{2}}q^{-\frac{1}{2}}}\left(a;z_0;z\right)=1, \tag{61}$$

b)

$$\cosh^{2}_{q^{\frac{1}{2}}q^{-\frac{1}{2}}}\left(a;z_0;z\right)-\sinh^{2}_{q^{\frac{1}{2}}q^{-\frac{1}{2}}}\left(a;z_0;z\right)=1, \tag{62}$$

c)

$$\cos_{q^{\frac{1}{2}}}\left(a;z_0;z\right)\cos_{q^{-\frac{1}{2}}}\left(a;z_0;z\right)+\sin_{q^{\frac{1}{2}}}\left(a;z_0;z\right)\sin_{q^{-\frac{1}{2}}}\left(a;z_0;z\right)=1, \tag{63}$$

d)

$$\cosh_{q^{\frac{1}{2}}}\left(a;z_0;z\right)\cosh_{q^{-\frac{1}{2}}}\left(a;z_0;z\right)-\sinh_{q^{\frac{1}{2}}}\left(a;z_0;z\right)\sinh_{q^{-\frac{1}{2}}}\left(a;z_0;z\right)=1. \tag{64}$$

3. *q*-Nonuniform Laplace Transform

For a given function $f\left(x\left(z\right)\right)$, we define its *q-nonuniform Laplace transform* or *Laplace transform on the q-nonuniform lattice* $x\left(z\right)=\left(z+z^{-1}\right)/2$, $z=q^{s}$, $0<q<1$, $-\infty\le s\le+\infty$, as the function

$$\begin{aligned}F\left(p\right)&=\mathcal{L}_{q^{\frac{1}{2}}}\left\{f\left(x\left(z\right)\right)\right\}\\&\overset{\text{def}}{=}\int_{x\left(z_0\right)=x_0}^{x\left(\infty\right)=\infty}E_{q^{-\frac{1}{2}}}\left(-p;z_0;zq^{-\frac{1}{2}}\right)f\left(x\left(z\right)\right)\mathrm{d}x\left(z\right);\ \ p=s+i\sigma\in\mathbb{C}.\end{aligned}$$

We denote $f\left(x\left(z\right)\right)\rightleftharpoons_{q^{\frac{1}{2}}}F\left(p\right)$, and we say that $f\left(x\left(z\right)\right)$ is the *original* of $F\left(p\right)$ while $F\left(p\right)$ is the *image* of $f\left(x\left(z\right)\right)$ by the Laplace transform on *q*-nonuniform lattices.

3.1. Properties of the *q*-Nonuniform Laplace Transform

1) **Linearity.** By the linearity of the integral, it becomes clear that

$$\mathcal{L}_{q^{\frac{1}{2}}}\left\{\alpha f\left(x\left(z\right)\right)+\beta g\left(x\left(z\right)\right)\right\}=\alpha\mathcal{L}_{q^{\frac{1}{2}}}\left\{f\left(x\left(z\right)\right)\right\}+\beta\mathcal{L}_{q^{\frac{1}{2}}}\left\{g\left(x\left(z\right)\right)\right\}.$$

2) **Transform of derivatives.** We have

$$\begin{aligned}\mathcal{L}_{q^{\frac{1}{2}}}\left\{Df\left(x(z)\right)\right\} &= \int_{x_0}^{\infty} E_{q^{-\frac{1}{2}}}\left(-p;z_0;zq^{-\frac{1}{2}}\right)\mathcal{D}f\left(x(z)\right)\mathrm{d}x(z)\\ &= \left[E_{q^{-\frac{1}{2}}}\left(-p;z_0;z\right)f\left(x(z)\right)\right]_{z_0}^{\infty}\\ &\quad -\int_{x_0}^{\infty} f\left(x\left(zq^{\frac{1}{2}}\right)\right)\mathcal{D}E_{q^{-\frac{1}{2}}}\left(-p;z_0;z\right)\mathrm{d}x(z)\\ &= -f\left(x(z_0)\right)+p\int_{x_0}^{\infty} f\left(x\left(zq^{\frac{1}{2}}\right)\right)E_{q^{-\frac{1}{2}}}\left(-p;z_0;zq^{-\frac{1}{2}}\right)\mathrm{d}x(z)\\ &= p\mathcal{L}_{q^{\frac{1}{2}}}\left\{f\left(x\left(zq^{\frac{1}{2}}\right)\right)\right\}-f\left(x(z_0)\right),\end{aligned} \tag{65}$$

where we used the q-nonuniform integration by parts [8]:

$$\begin{aligned}&\int_{x_0}^{x(z)} g\left(x\left(zq^{-\frac{1}{2}}\right)\right)\mathcal{D}f\left(x(z)\right)\mathrm{d}x(z)\\ &= \left[g\left(x(z)\right)f\left(x(z)\right)\right]_{x_0}^{x(z)} - \int_{x_0}^{x(z)} f\left(x\left(zq^{\frac{1}{2}}\right)\right)\mathcal{D}g\left(x(z)\right)\mathrm{d}x(z),\end{aligned} \tag{66}$$

and (29). Thus

i)

$$\mathcal{L}_{q^{\frac{1}{2}}}\left\{\mathcal{D}f\left(x(z)\right)\right\} = p\mathcal{L}_{q^{\frac{1}{2}}}\left\{f\left(x\left(zq^{\frac{1}{2}}\right)\right)\right\}-f\left(x(z_0)\right), \tag{67}$$

and

$$\mathcal{L}_{q^{\frac{1}{2}}}\left\{\mathcal{D}f\left(x\left(zq^{\alpha}\right)\right)\right\} = p\mathcal{L}_{q^{\frac{1}{2}}}\left\{f\left(x\left(zq^{\alpha+\frac{1}{2}}\right)\right)\right\}-f\left(x\left(q^{\alpha}z_0\right)\right). \tag{68}$$

Hence

$$\mathcal{D}f\left(x\left(zq^{-\frac{1}{2}}\right)\right) \rightleftharpoons_{q^{\frac{1}{2}}} pF(p)-f\left(x\left(z_0q^{-\frac{1}{2}}\right)\right), \tag{69}$$

and

$$\mathcal{D}f\left(x\left(zq^{\frac{1}{2}}\right)\right) \rightleftharpoons_{q^{\frac{1}{2}}} p\mathcal{L}_{q^{\frac{1}{2}}}\left\{f\left(x(zq)\right)\right\}-f\left(x\left(z_0q^{\frac{1}{2}}\right)\right). \tag{70}$$

Using (67), one easily obtains

ii)

$$\begin{aligned}&\mathcal{L}_{q^{\frac{1}{2}}}\left\{D^2f\left(x(z)\right)\right\} = \mathcal{L}_{q^{\frac{1}{2}}}\left\{\mathcal{D}\left(\mathcal{D}f\left(x(z)\right)\right)\right\}\\ &= p^2\mathcal{L}_{q^{\frac{1}{2}}}\left\{f\left(x(zq)\right)\right\}-pf\left(x\left(q^{\frac{1}{2}}z_0\right)\right)-\mathcal{D}f\left(x(z_0)\right).\end{aligned} \tag{71}$$

Hence

$$\mathcal{L}_{q^{\frac{1}{2}}}\left\{D^2 f\left(x\left(zq^{\frac{1}{2}}\right)\right)\right\} = p^2\mathcal{L}_{q^{\frac{1}{2}}}\left\{f\left(x\left(zq^{\frac{3}{2}}\right)\right)\right\} - pf\left(x\left(qz_0\right)\right) - \mathcal{D}f\left(x\left(q^{\frac{1}{2}}z_0\right)\right). \tag{72}$$

iii)

$$\mathcal{L}_{q^{\frac{1}{2}}}\left\{\mathcal{D}^3 f\left(x(z)\right)\right\} = \mathcal{L}_{q^{\frac{1}{2}}}\left\{\mathcal{D}\left(\mathcal{D}^2 f\left(x(z)\right)\right)\right\} = p^3\mathcal{L}_{q^{\frac{1}{2}}}\left\{f\left(x\left(zq^{\frac{3}{2}}\right)\right)\right\} - \left[p^2 f\left(x\left(qz_0\right)\right) + p\mathcal{D}f\left(x\left(q^{\frac{1}{2}}z_0\right)\right) + \mathcal{D}^2 f\left(x\left(z_0\right)\right)\right]. \tag{73}$$

…

n)

$$\mathcal{L}_{q^{\frac{1}{2}}}\left\{D^n f\left(x(z)\right)\right\} = p^n\mathcal{L}_{q^{\frac{1}{2}}}\left\{f\left(x\left(zq^{\frac{n}{2}}\right)\right)\right\} - \left[p^{n-1} f\left(x\left(q^{\frac{n-1}{2}}z_0\right)\right) + p^{n-2}\mathcal{D}f\left(x\left(q^{\frac{n-2}{2}}z_0\right)\right) + \cdots + \mathcal{D}^{n-1} f\left(x\left(z_0\right)\right)\right], \tag{74}$$

or equivalently

$$\mathcal{L}_{q^{\frac{1}{2}}}\left\{D^n f\left(x\left(q^{-\frac{n}{2}}z\right)\right)\right\} = p^n\mathcal{L}_{q^{\frac{1}{2}}}\left\{f\left(x(z)\right)\right\} - \left[p^{n-1} f\left(x\left(q^{-\frac{1}{2}}z_0\right)\right) + p^{n-2}\mathcal{D}f\left(x\left(q^{-1}z_0\right)\right) + \cdots + \mathcal{D}^{n-1} f\left(x\left(q^{-\frac{n}{2}}z_0\right)\right)\right]. \tag{75}$$

For example, for $n = 2$, we have

$$\mathcal{L}_{q^{\frac{1}{2}}}\left\{D^2 f\left(x\left(zq^{-1}\right)\right)\right\} = p^2\mathcal{L}_{q^{\frac{1}{2}}}\left\{f\left(x(z)\right)\right\} - \left[pf\left(x\left(q^{-\frac{1}{2}}z_0\right)\right) + \mathcal{D}f\left(x\left(q^{-1}z_0\right)\right)\right]. \tag{76}$$

3) Transform of integrals. In the equation

$$\mathcal{L}_{q^{\frac{1}{2}}}\left\{\mathcal{D}f\left(x(z)\right)\right\} = p\mathcal{L}_{q^{\frac{1}{2}}}\left\{f\left(x\left(q^{\frac{1}{2}}z\right)\right)\right\} - f\left(x\left(z_0\right)\right),$$

set

$$f\left(x(z)\right) = \int_{x_0}^{x(z)} g\left(x(z)\right)\mathrm{d}x(z). \tag{77}$$

We have

$$\mathcal{L}_{q^{\frac{1}{2}}}\left\{g\left(x(z)\right)\right\} = p\mathcal{L}_{q^{\frac{1}{2}}}\left\{\int_{x_0}^{x\left(zq^{\frac{1}{2}}\right)} g\left(x(z)\right)\mathrm{d}x(z)\right\}. \tag{78}$$

Hence

$$\mathcal{L}_{q^{\frac{1}{2}}}\left\{\int_{x_0}^{x\left(zq^{\frac{1}{2}}\right)} g\left(x\left(z\right)\right)\mathrm{d}x\left(z\right)\right\}=\frac{1}{p}\mathcal{L}_{q^{\frac{1}{2}}}\left\{g\left(x\left(z\right)\right)\right\}. \tag{79}$$

3.2. *q*-Nonuniform Laplace Transform for Some Elementary Functions

1) $f\left(x\left(z\right)\right)=1$. We have

$$\begin{aligned}\mathcal{L}_{q^{\frac{1}{2}}}\left\{1\right\} &= \int_{x_0}^{\infty} E_{q^{-\frac{1}{2}}}\left(-p;z_0;q^{-\frac{1}{2}}z\right)\mathrm{d}x\left(z\right)\\ &= -\frac{1}{p}\int_{x_0}^{\infty}\mathcal{D}E_{q^{-\frac{1}{2}}}\left(-p;z_0;z\right)\mathrm{d}x\left(z\right)\\ &= -\frac{1}{p}E_{q^{-\frac{1}{2}}}\left(-p;z_0;z\right)\Big|_{z_0}^{\infty}\\ &= \frac{1}{p}.\end{aligned} \tag{80}$$

2) $f\left(x\left(z\right)\right)=x\left(zq^{\frac{1}{2}}\right)$. We have

$$\begin{aligned}\mathcal{L}_{q^{\frac{1}{2}}}\left\{x\left(zq^{\frac{1}{2}}\right)\right\} &= \int_{x_0}^{\infty} x\left(zq^{\frac{1}{2}}\right)E_{q^{-\frac{1}{2}}}\left(-p;z_0;q^{-\frac{1}{2}}z\right)\mathrm{d}x\left(z\right)\\ &= -\frac{1}{p}\int_{x_0}^{\infty} x\left(zq^{\frac{1}{2}}\right)\mathcal{D}E_{q^{-\frac{1}{2}}}\left(-p;z_0;z\right)\mathrm{d}x\left(z\right)\\ &= -\frac{1}{p}\left[x\left(z\right)E_{q^{-\frac{1}{2}}}\left(-p;z_0;z\right)\right]_{z_0}^{\infty}+\frac{1}{p}\int_{x_0}^{\infty}E_{q^{-\frac{1}{2}}}\left(-p;z_0;q^{-\frac{1}{2}}z\right)\mathrm{d}x\left(z\right)\\ &= \frac{x\left(z_0\right)}{p}-\frac{1}{p^2}\int_{x_0}^{\infty}\mathcal{D}E_{q^{-\frac{1}{2}}}\left(-p;z_0;z\right)\mathrm{d}x\left(z\right)\\ &= \frac{x\left(z_0\right)}{p}-\frac{1}{p^2}\left[E_{q^{-\frac{1}{2}}}\left(-p;z_0;z\right)\right]_{z_0}^{\infty}\\ &= \frac{x\left(z_0\right)}{p}+\frac{1}{p^2}.\end{aligned} \tag{81}$$

3) $f\left(x\left(z\right)\right)=E_{q^{-\frac{1}{2}}}\left(a;z_0;q^{\frac{1}{2}}z\right)$. We calculate

$$
\begin{aligned}
I &= \mathcal{L}_{q^{\frac{1}{2}}}\left\{E_{q^{-\frac{1}{2}}}\left(a;z_0;q^{\frac{1}{2}}z\right)\right\} \\
&= \int_{x_0}^{\infty} E_{q^{-\frac{1}{2}}}\left(-p;z_0;q^{-\frac{1}{2}}z\right) E_{q^{-\frac{1}{2}}}\left(a;z_0;q^{\frac{1}{2}}z\right)\mathrm{d}x(z) \\
&= -\frac{1}{p}\int_{x_0}^{\infty} E_{q^{-\frac{1}{2}}}\left(a;z_0;q^{\frac{1}{2}}z\right)\mathcal{D}E_{q^{-\frac{1}{2}}}\left(-p;z_0;z\right)\mathrm{d}x(z) \\
&= -\frac{1}{p}\left[E_{q^{-\frac{1}{2}}}\left(a;z_0;z\right)E_{q^{-\frac{1}{2}}}\left(-p;z_0;z\right)\right]_{z_0}^{\infty} \\
&\quad +\frac{1}{p}\int_{x_0}^{\infty} E_{q^{-\frac{1}{2}}}\left(-p;z_0;q^{-\frac{1}{2}}z\right)\mathcal{D}E_{q^{\frac{1}{2}}}\left(a;x_0;z\right)\mathrm{d}x(z) \\
&= \frac{1}{p}+\frac{a}{p}\int_{x_0}^{\infty} E_{q^{-\frac{1}{2}}}\left(-p;z_0;q^{-\frac{1}{2}}z\right)E_{q^{\frac{1}{2}}}\left(a;z_0;q^{\frac{1}{2}}z\right)\mathrm{d}x(z). \\
&= \frac{1}{p}+\frac{a}{p}I.
\end{aligned}
$$

Hence

$$I = \frac{1}{p-a}. \tag{82}$$

4) $f\left(x(z)\right) = \cos_{q^{-\frac{1}{2}}}\left(ai;z_0;q^{\frac{1}{2}}z\right) = \dfrac{E_{q^{-\frac{1}{2}}}\left(ai;z_0;q^{\frac{1}{2}}z\right)+E_{q^{-\frac{1}{2}}}\left(-ai;z_0;q^{\frac{1}{2}}z\right)}{2}.$

Using the preceding case, we obtain

$$\mathcal{L}_{q^{\frac{1}{2}}}\left\{\cos_{q^{-\frac{1}{2}}}\left(a;z_0;q^{\frac{1}{2}}z\right)\right\} = \frac{1}{2}\left(\frac{1}{p-ai}+\frac{1}{p+ai}\right) = \frac{p}{p^2+a^2}. \tag{83}$$

5) $f\left(x(z)\right) = \sin_{q^{-\frac{1}{2}}}\left(ai;z_0;q^{\frac{1}{2}}z\right) = \dfrac{E_{q^{-\frac{1}{2}}}\left(ai;z_0;q^{\frac{1}{2}}z\right)-E_{q^{-\frac{1}{2}}}\left(-ai;z_0;q^{\frac{1}{2}}z\right)}{2i}.$

Similarly, we get

$$\mathcal{L}_{q^{\frac{1}{2}}}\left\{\sin_{q^{-\frac{1}{2}}}\left(a;z_0;q^{\frac{1}{2}}z\right)\right\} = \frac{1}{2i}\left(\frac{1}{p-ai}-\frac{1}{p+ai}\right) = \frac{a}{p^2+a^2}. \tag{84}$$

6) $f\left(x(z)\right) = \cosh_{q^{-\frac{1}{2}}}\left(a;z_0;q^{\frac{1}{2}}z\right) = \dfrac{E_{q^{-\frac{1}{2}}}\left(a;z_0;q^{\frac{1}{2}}z\right)+E_{q^{-\frac{1}{2}}}\left(-a;z_0;q^{\frac{1}{2}}z\right)}{2}.$ Here also, we obtain

$$\mathcal{L}_{q^{\frac{1}{2}}}\left\{\cosh_{q^{-\frac{1}{2}}}\left(a;z_0;q^{\frac{1}{2}}z\right)\right\} = \frac{1}{2}\left(\frac{1}{p-a}+\frac{1}{p+a}\right) = \frac{p}{p^2-a^2}. \tag{85}$$

7) $f(x(z))=\sinh_{q^{-\frac{1}{2}}}\left(a;z_0;q^{\frac{1}{2}}z\right)=\dfrac{E_{q^{-\frac{1}{2}}}\left(a;z_0;q^{\frac{1}{2}}z\right)-E_{q^{-\frac{1}{2}}}\left(-a;z_0;q^{\frac{1}{2}}z\right)}{2}$.

Similarly,

$$\mathcal{L}_{q^{\frac{1}{2}}}\left\{\sinh_{q^{-\frac{1}{2}}}\left(a;z_0;q^{\frac{1}{2}}z\right)\right\}=\frac{1}{2}\left(\frac{1}{p-a}-\frac{1}{p+a}\right)=\frac{a}{p^2-a^2}. \tag{86}$$

8) $f(x(z))=E_{q^{\frac{1}{2}}}\left(a;z_0;q^{\frac{1}{2}}z\right)$. We calculate

$$\begin{aligned}
I&=\mathcal{L}_{q^{\frac{1}{2}}}\left\{E_{q^{\frac{1}{2}}}\left(a;z_0;q^{\frac{1}{2}}z\right)\right\}\\
&=\int_{x_0}^{\infty}E_{q^{-\frac{1}{2}}}\left(-p;z_0;q^{-\frac{1}{2}}z\right)E_{q^{\frac{1}{2}}}\left(a;z_0;q^{\frac{1}{2}}z\right)\mathrm{d}x(z)\\
&=-\frac{1}{p}\int_{x_0}^{\infty}E_{q^{\frac{1}{2}}}\left(a;z_0;q^{\frac{1}{2}}z\right)\mathcal{D}E_{q^{-\frac{1}{2}}}\left(-p;z_0;z\right)\mathrm{d}x(z)\\
&=-\frac{1}{p}\left[E_{q^{\frac{1}{2}}}\left(a;z_0;z\right)E_{q^{-\frac{1}{2}}}\left(-p;z_0;z\right)\right]_{z_0}^{\infty}\\
&\quad+\frac{1}{p}\int_{x_0}^{\infty}E_{q^{-\frac{1}{2}}}\left(-p;z_0;q^{-\frac{1}{2}}z\right)\mathcal{D}E_{q^{\frac{1}{2}}}\left(a;x_0;z\right)\mathrm{d}x(z)\\
&=-\frac{1}{p}\left[E_{q^{\frac{1}{2}}}\left(a;z_0;z\right)E_{q^{-\frac{1}{2}}}\left(-p;z_0;z\right)\right]_{z_0}^{\infty}\\
&\quad+\frac{a}{p}\int_{x_0}^{\infty}E_{q^{-\frac{1}{2}}}\left(-p;z_0;q^{-\frac{1}{2}}z\right)E_{q^{\frac{1}{2}}}\left(a;z_0;q^{\frac{1}{2}}z\right)\mathrm{d}x(z).\\
&=-\frac{1}{p}\left[E_{q^{\frac{1}{2}}}\left(a;z_0;z\right)E_{q^{-\frac{1}{2}}}\left(-p;z_0;z\right)\right]_{z_0}^{\infty}+\frac{a}{p}I.
\end{aligned} \tag{87}$$

Hence

$$\begin{aligned}
\left(1-\frac{a}{p}\right)I&=-\frac{1}{p}\left[E_{q^{-\frac{1}{2}}}\left(-p;z_0;z\right)E_{q^{\frac{1}{2}}}\left(a;x_0;z\right)\right]_{z_0}^{\infty}\\
&=\frac{1}{p}-\frac{1}{p}E_{q^{-\frac{1}{2}}}\left(-p;z_0;\infty\right)E_{q^{\frac{1}{2}}}\left(a;z_0;\infty\right).
\end{aligned}$$

Consequently

$$I=\frac{1}{p-a}-\frac{1}{p-a}E_{q^{-\frac{1}{2}}}\left(-p;z_0;\infty\right)E_{q^{\frac{1}{2}}}\left(a;z_0;\infty\right)=\frac{1-K_{-p,a}}{p-a}, \tag{88}$$

where $K_{-p,a}=E_{q^{-\frac{1}{2}}}\left(-p;z_0;\infty\right)E_{q^{\frac{1}{2}}}\left(a;z_0;\infty\right)$ (see (45)).

9) $f(x(z))=\cos_{q^{\frac{1}{2}}}\left(ai;z_0;q^{\frac{1}{2}}z\right)=\dfrac{E_{q^{\frac{1}{2}}}\left(ai;z_0;q^{\frac{1}{2}}z\right)+E_{q^{\frac{1}{2}}}\left(-ai;z_0;q^{\frac{1}{2}}z\right)}{2}$. Using the preceding case, we obtain

$$\mathcal{L}_{q^{\frac{1}{2}}}\left\{\cos_{q^{\frac{1}{2}}}\left(a;z_0;q^{\frac{1}{2}}z\right)\right\}=\frac{1}{2}\left(\frac{1-K_{-p,ai}}{p-ai}+\frac{1-K_{-p,-ai}}{p+ai}\right). \tag{89}$$

10) $f\left(x(z)\right)=\sin_{q^{\frac{1}{2}}}\left(ai;z_0;q^{\frac{1}{2}}z\right)=\dfrac{E_{q^{\frac{1}{2}}}\left(ai;z_0;q^{\frac{1}{2}}z\right)-E_{q^{\frac{1}{2}}}\left(-ai;z_0;q^{\frac{1}{2}}z\right)}{2i}$.

Similarly, we get

$$\mathcal{L}_{q^{\frac{1}{2}}}\left\{\sin_{q^{\frac{1}{2}}}\left(a;z_0;q^{\frac{1}{2}}z\right)\right\}=\frac{1}{2i}\left(\frac{1-K_{-p,ai}}{p-ai}-\frac{1-K_{-p,-ai}}{p+ai}\right). \tag{90}$$

11) $f\left(x(z)\right)=\cosh_{q^{\frac{1}{2}}}\left(a;z_0;q^{\frac{1}{2}}z\right)=\dfrac{E_{q^{\frac{1}{2}}}\left(a;z_0;q^{\frac{1}{2}}z\right)+E_{q^{\frac{1}{2}}}\left(-a;z_0;q^{\frac{1}{2}}z\right)}{2}$. Here also, we obtain

$$\mathcal{L}_{q^{\frac{1}{2}}}\left\{\cosh_{q^{\frac{1}{2}}}\left(a;z_0;q^{\frac{1}{2}}z\right)\right\}=\frac{1}{2}\left(\frac{1-K_{-p,a}}{p-a}+\frac{1-K_{-p,-a}}{p+a}\right). \tag{91}$$

12) $f\left(x(z)\right)=\sinh_{q^{\frac{1}{2}}}\left(a;z_0;q^{\frac{1}{2}}z\right)=\dfrac{E_{q^{\frac{1}{2}}}\left(a;z_0;q^{\frac{1}{2}}z\right)-E_{q^{\frac{1}{2}}}\left(-a;z_0;q^{\frac{1}{2}}z\right)}{2}$.

Similarly,

$$\mathcal{L}_{q^{\frac{1}{2}}}\left\{\sinh_{q^{\frac{1}{2}}}\left(a;z_0;q^{\frac{1}{2}}z\right)\right\}=\frac{1}{2}\left(\frac{1-K_{-p,a}}{p-a}-\frac{1-K_{-p,-a}}{p+a}\right). \tag{92}$$

3.3. Inverse of the *q*-Nonuniform Laplace Transform

For searching the original function from its image, we have to use, as in other cases of Laplace transforms, the *q*-nonuniform Laplace transform properties of the subsection 3.1 and the *q*-nonuniform Laplace transform of elementary functions of the subsection 3.2.

3.4. Applications of *q*-Nonuniform Laplace Transform on *q*-Nonuniform Difference Equations

In the same way that the Laplace transfrom, *Z*-transform and *q*-Laplace transfrom are applied respectively in differential, difference [5] and *q*-difference equations [6], the *q*-nonuniform Laplace transform is expected to be applied in *q*-nonuniform difference equations [9].

For question of simplicity, we suppose that the order of the equation is $n=2$. So, consider the second order *q*-nonuniform difference equation

$$a_0\mathcal{D}^2y\left(x\left(q^{-1}z\right)\right)+a_1\mathcal{D}y\left(x\left(q^{-\frac{1}{2}}z\right)\right)+a_2y\left(x(z)\right)=g\left(x(z)\right), \tag{93}$$

with the initial conditions

$$y\left(x\left(q^{-\frac{1}{2}}z_0\right)\right)=y_0;\ \mathcal{D}y\left(x\left(q^{-1}z_0\right)\right)=y_1. \tag{94}$$

Applying the q-nonuniform Laplace transform as in the subsection 3.1, we obtain

$$\mathcal{L}_{\frac{1}{q^2}}\left\{\mathcal{D}^2y\left(x\left(q^{-1}z\right)\right)\right\}=p^2\mathcal{L}_{\frac{1}{q^2}}\left\{y\left(x\left(z\right)\right)\right\}-py\left(x\left(q^{-\frac{1}{2}}z_0\right)\right)-\mathcal{D}y\left(x\left(q^{-1}z_0\right)\right), \tag{95}$$

$$\mathcal{L}_{\frac{1}{q^2}}\left\{Dy\left(x\left(q^{-\frac{1}{2}}z\right)\right)\right\}=p\mathcal{L}_{\frac{1}{q^2}}\left\{y\left(x\left(z\right)\right)\right\}-y\left(x\left(q^{-\frac{1}{2}}z_0\right)\right). \tag{96}$$

Applying the q-nonuniform Laplace transform on both sides of the equation (93) and using (94), (95) and (96), we get

$$\begin{aligned}&a_0p^2\mathcal{L}_{\frac{1}{q^2}}\left\{y\left(x\left(z\right)\right)\right\}-a_0y_0p-a_0y_1+a_1p\mathcal{L}_{\frac{1}{q^2}}\left\{y\left(x\left(z\right)\right)\right\}\\&-a_1y_0+a_2\mathcal{L}_{\frac{1}{q^2}}\left\{y\left(x\left(z\right)\right)\right\}=G\left(p\right)\\ \Leftrightarrow\ &a_0p^2\mathcal{L}_{\frac{1}{q^2}}\left\{y\left(x\left(z\right)\right)\right\}+a_1p\mathcal{L}_{\frac{1}{q^2}}\left\{y\left(x\left(z\right)\right)\right\}+a_2\mathcal{L}_{\frac{1}{q^2}}\left\{y\left(x\left(z\right)\right)\right\}\\&=G\left(p\right)+a_0y_0p+a_0y_1+a_1y_0.\end{aligned}$$

Setting $\mathcal{L}_{\frac{1}{q^2}}\left\{y\left(x\left(z\right)\right)\right\}=Y\left(p\right)$, we obtain

$$Y\left(p\right)\left(a_0p^2+a_1p+a_2\right)=G\left(p\right)+a_0y_0p+a_0y_1+a_1y_0.$$

Consequently, $Y\left(p\right)=\dfrac{G\left(p\right)+a_0y_0p+a_0y_1+a_1y_0}{a_0p^2+a_1p+a_2}$ and $y\left(x\left(z\right)\right)=\mathcal{L}^{-1}_{\frac{1}{q^2}}\left\{Y\left(p\right)\right\}$.

4. Conclusion

In this work, basic concepts of exponential and trigonometric functions on q-nonuniform Lattices were introduced and q-nonuniform difference version of the integral Laplace tranform and also some of its applications were given. Clearly, more applications of q-nonuniform Laplace Transform for solving q-nonuniform difference equations are expected.

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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