Scientific Research Publishing

Journal of Applied Mathematics and Physics, 2026, 14(2), 587-608
https://www.scirp.org/journal/jamp
ISSN Online: 2327-4379
ISSN Print: 2327-4352

Existence of Solutions for Shallow Water Equatorial Waves

Ranis Nail Ibragimov[1], Svetlin Georgiev[2], Deliang Bao[3], Joy Williams[1]

[1]Department of Mathematics, Hampton University, Hampton, VA, USA
[2]Department of Mathematics, Sorbonne University, Paris, France
[3]Department of Physics, Hampton University, Hampton, VA, USA

Email: Ranis.Ibragimov@hamptonu.edu, svetlingeorgiev1@gmail.com, deliang.bao@hamptonu.edu, joy.williams2@my.hamptonu.edu

How to cite this paper: Ibragimov, R.N., Georgiev, S., Bao, D.L. and Williams, J. (2026) Existence of Solutions for Shallow Water Equatorial Waves. *Journal of Applied Mathematics and Physics*, **14**, 587-608.
https://doi.org/10.4236/jamp.2026.142032

Received: November 19, 2025
Accepted: February 7, 2026
Published: February 10, 2026

Abstract

The study focuses on the modeling of nonlinear shallow water equatorial free boundary waves. It has been demonstrated that there exists at least one solution, at least two nonnegative solutions, and at least three nonnegative solutions within the time scales considered. To achieve this, the original nonlinear model is transformed into a linear system using the hodograph method. Subsequently, the mapped system is represented through a new integral formulation that involves two operators. These operators are designed to ensure that any fixed point of their sum corresponds to a solution of the problem at hand.

Keywords

Shallow Water, Equatorial Waves, Existence of Solutions, Integral Presentation

1. Introduction

Planetary waves are large-scale atmospheric disturbances that circulate around entire longitude circles. Equatorial waves, including Kelvin waves and Rossby-gravity modes, play a vital role in linking the ocean-atmosphere system and shaping tropical climate dynamics. These waves are affected by the Coriolis parameter, which changes sign at the equator. They significantly contribute to long-term average upwelling at the tropical tropopause. Recently, interpretations of temperature variations in the tropical lower stratosphere have emphasized the role of upwelling related to the breakdown of planetary waves in the extratropical stratosphere [1] [2].

The oscillation of equatorial planetary waves offers a different perspective on how tropical circulation responds to climate phenomena, particularly during various phases of the El Niño-Southern Oscillation and Madden-Julian Oscillation. These waves also influence the variability of the tropical belt's width, though their exact role in tropical upwelling remains uncertain. Equatorial waves are intriguing components of meteorology, acting as geophysical fluid waves confined near the Equator. They propagate both horizontally and vertically, affecting pressure, temperature, and winds, which can alter large-scale weather patterns. Additionally, these waves can be triggered by dynamic weather events, such as latent heating from tropical convection or cold air influx from the extratropics. Their energy transmission can impact localized regions in the tropical atmosphere or ocean, sometimes affecting a significant portion of the Earth's equator.

One notable example of circulating equatorial waves is the atmospheric stream that follows a hexagonal path at Saturn's north pole [3] [4], as shown in **Figure 1**. The path was first identified by Voyager in the 1980s, and then this hexagon was also revealed by Cassini during Saturn's August 2009 equinox after being obscured for years. While observed in infrared by Cassini's VIMS since 2006, the waves along the hexagon remain unexplained (see also [3]). Scientists believe it is a meandering jet stream at 77 degrees north latitude, but the controlling mechanisms are still unknown.

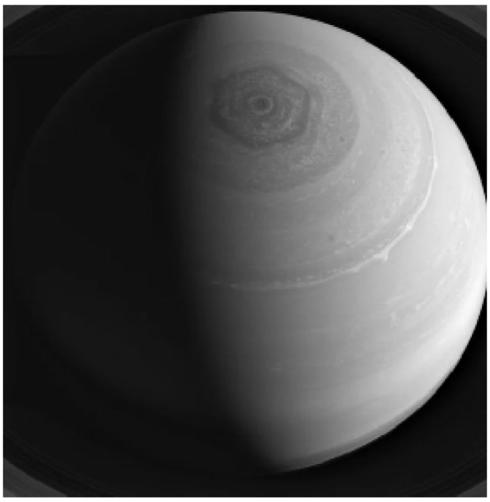

Figure 1. Image from Cassini shows the Saturn's north pole revealing a jet stream that follows a hexagon-shaped path and has long puzzled scientists. Image credit: NASA/JPL.

Recent lab studies involving [5] indicate that the North Polar Hexagon of Saturn may be formed by the stabilization of a standing wave due to angular velocity differences. Nevertheless, due to Saturn's intricate atmospheric composition, the experiments conducted do not yield definitive answers, and the nature of these waves along with the hexagonal shape of the jet stream continues to be an enigma.

Equatorial waves can also be associated with eastward-moving warm water waves known as Kelvin waves, which travel along the equator, particularly in the central and eastern equatorial Pacific, as shown in **Figure 2**.

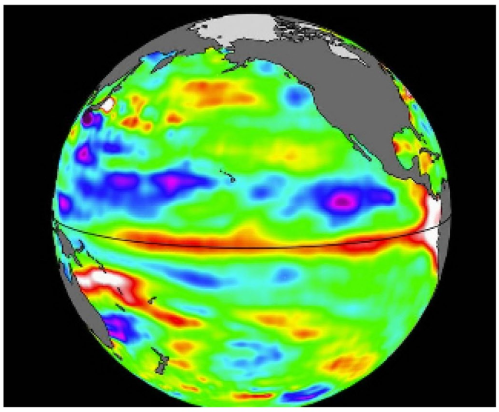

Figure 2. Sea-level height data from November 2009 shows the dynamics of warm water known as Kelvin equatorial waves that can be visualized as traveling eastward along the equator. Image credit: NASA/JPL.

Simulation of Kelvin waves using linearized shallow-water equations for the equatorial plane was done in [6]. This unique solution features zero meridional velocity and a pressure gradient balanced by the Coriolis force. This modeling is essential for understanding significant tropical phenomena, such as the El Niño-Southern Oscillation (ENSO) and the Madden-Julian Oscillation (MJO) (see e.g. [7] [8]).

In essence, equatorial waves are weather-producing waves characterized by decreasing amplitudes at higher latitudes and include both Kelvin waves and Rossby-gravity waves, capable of transmitting energy and momentum in multiple directions.

The present paper is devoted to modeling equatorial waves as a shallow water free boundary problem describing nonstationary motion of a perfect incompressible fluid circulating around a solid circular boundary. In this paper, we will investigate the existence of at least one solution, at least two nonnegative and at least three nonnegative solutions for mathematical model describing equatorial shallow water waves. For this aim, firstly it is given a new integral representation of the solutions of the considered problem and then two operators are constructed so that any fixed point of their sum is a solution to the considered problem. To the best of our knowledge, there are not any results in this direction in the existing references.

The paper is organized as follows. In Section 3.1, we provide some preliminary results on time scales analysis. In Section 3.2, we give some auxiliary results. In Section 4, we prove existence of at least one classical solution for the shallow water model. In Section 5, we prove existence of at least two nonnegative classical solutions. In Section 6, we prove existence of at least three nonnegative classical solutions. In Section 7, we give an example to illustrate our main result.

2. Free Boundary Model

We consider a two dimensional motion of an incompressible perfect fluid which has a free boundary μ and has a solid bottom represented a circle of radius R. So the fluid is circulating around a solid circle and bounded by a free boundary. For the sake of simplicity, the motion of the fluid is supposed to be irrotational and the pressure on a free boundary is constant. It is postulated that that the fluid depth is small compared to the radius of the circle, as shown schematically in Figure 3.

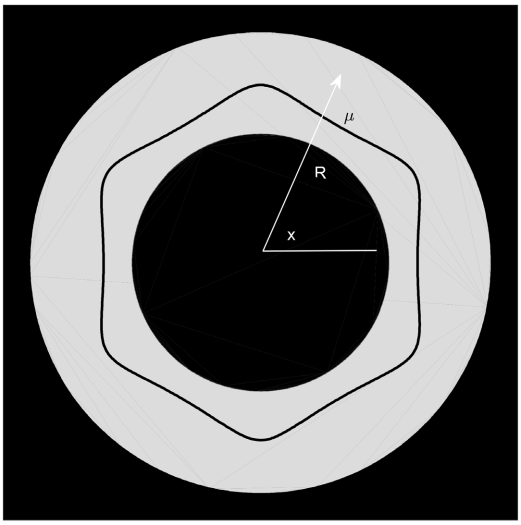

Figure 3. Schematic showing an equatorial motion with unknown free boundary $\mu(x)$.

We introduce polar coordinates, in which x is a polar angle, r is the distance from the origin, and we denote $z = z_0 + \mu(x)$, where z_0 is undisturbed level of the atmospheric level above the radius R of the planet, and $\mu(x)$ is the unknown perturbation level of disturbance of the atmospheric equatorial layer. This means so that the atmospheric layer is contained within annular domain $r \in [R, R + x_0 + \mu(x)]$, which is also shown in Figure 3. We assume that the motion is irrotational and pressure on a free boundary $r = R + z(x,t)$ is constant, where we denote $z = x_0 + \mu(x)$. We also assume that the unperturbed level of atmospheric layer x_0 is much smaller than R and the radial component of the gravity g is directed toward the center of the planet. The dimension of the problem is decreased if we introduce the stream function $\psi(t,r,x)$ by formulae

$$v_r = -\frac{1}{r}\frac{\partial \varphi}{\partial x} \text{ and } v_\theta = \frac{\partial \varphi}{\partial r}, \tag{2.1}$$

in which v_r and v_θ are the radial and the angular components of the velocity vector.

Finally, we introduce the average velocity $y(x,t)$ by the integral relation

$$y(x,t) = \frac{1}{z}\int_R^{R+z} v_\theta(r,x,t)\,\mathrm{d}r. \tag{2.2}$$

In fact, as follows from the definition of the average velocity (2.2), it follows that $y(x,t) = z^{-1}\varphi(R+z,x,t)$.

Then the mathematical model describing planetary equatorial waves is the equatorial plane is described by the Laplace equation in the domain $\left(R < r < R + z(x)\right)$

$$\frac{\partial^2 \varphi}{\partial x^2} + r^2 \frac{\partial^2 \varphi}{\partial r^2} + r \frac{\partial \varphi}{\partial r} = 0 \tag{2.3}$$

subject to the boundary condition at the solid bottom $r = R$:

$$\varphi = 0, \tag{2.4}$$

also the boundary condition at the unknown free boundary: $r = z(x,t)$

$$\varphi = y(x,t) z, \tag{2.5}$$

and the dynamic condition at the unknown boundary $r = z(x,t)$

$$\frac{\partial^2 \varphi}{\partial t \partial r} - \frac{1}{r^2} \frac{\partial z}{\partial x} \frac{\partial^2 \varphi}{\partial t \partial x} + \frac{1}{2r} \frac{\partial}{\partial x} \left[\frac{1}{r^2} \left(\frac{\partial \varphi}{\partial x} \right)^2 + \left(\frac{\partial \varphi}{\partial r} \right)^2 \right] + \frac{g}{r} \frac{\partial z}{\partial x} = 0, \tag{2.6}$$

supplemented by the kinematic condition at the unknown boundary $r = z(x,t)$

$$r \frac{\partial z}{\partial t} + \frac{\partial}{\partial x}(yz) = 0. \tag{2.7}$$

The model (2.3) - (2.7) can be simplified further if we write it in the nondimensional variables:

$$\begin{gathered} x = x', \ r = R + x_0 r', \ z = z_0 z', \ t = \frac{R t'}{\sqrt{g z_0}}, \\ \varphi = z_0 \sqrt{g z_0} \varphi', \ y = \sqrt{g z_0} y'. \end{gathered} \tag{2.8}$$

Since we assumed that the unperturbed level of atmospheric layer x_0 is much smaller than R, introducing a small parameter ε by the relation

$$\varepsilon = \frac{z_0}{R}, \tag{2.9}$$

allows to represent the stream function by the series expansion:

$$\varphi = \sum_n \varepsilon^n \varphi^{(n)}.$$

Then we can reduce the model (2.3) - (2.7) to the nondimensional system of nonlinear shallow water equations, which represents a higher-order of the *Su-Gardner equations* [9]:

$$\begin{gathered} \frac{\partial y}{\partial t} + y \frac{\partial y}{\partial x} + \frac{\partial z}{\partial x} + \frac{\varepsilon}{2} \left(3z \frac{\partial y}{\partial t} - y \frac{\partial z}{\partial t} - y^2 \frac{\partial z}{\partial x} + 2z \frac{\partial z}{\partial x} \right) = 0, \\ \frac{\partial z}{\partial t} + y \frac{\partial z}{\partial x} + z \frac{\partial y}{\partial x} + \varepsilon z \frac{\partial z}{\partial t} = 0. \end{gathered} \tag{2.10}$$

The system (2.10) can be simplified further if we eliminate $\frac{\partial y}{\partial t}$ and $\frac{\partial z}{\partial t}$ from the terms of Equation (2.10) with ε by substituting there

$$\frac{\partial y}{\partial t} = -y \frac{\partial y}{\partial x} - \frac{\partial z}{\partial x} - \varepsilon \xi_1, \ \frac{\partial z}{\partial t} = -y \frac{\partial z}{\partial x} - z \frac{\partial y}{\partial x} - \varepsilon \xi_2,$$

where we denote

$$\xi_1 = \frac{1}{2}\left(3z\frac{\partial y}{\partial t} - y\frac{\partial z}{\partial t} - y^2\frac{\partial z}{\partial x} + 2z\frac{\partial z}{\partial x}h\right) \text{ and } \xi_2 = z\frac{\partial z}{\partial t}.$$

Ignoring terms with ε and ε^2, the system (2.10) becomes

$$\begin{aligned} &\frac{\partial y}{\partial t} + y\frac{\partial y}{\partial x} + \frac{\partial z}{\partial x} - \varepsilon z\left(y\frac{\partial y}{\partial x} + \frac{1}{2}\frac{\partial h}{\partial x}\right) = 0, \\ &\frac{\partial z}{\partial t} + y\frac{\partial h}{\partial x} + z\frac{\partial y}{\partial x} - \varepsilon z\left(y\frac{\partial z}{\partial x} + z\frac{\partial y}{\partial x}\right) = 0. \end{aligned} \tag{2.11}$$

In this paper, we are interested in proving the existence of classical solutions of the shallow water model (2.11) in the case $\varepsilon = 0$. So, our main focus is the unperturbed system:

$$\frac{\partial y}{\partial t} + y\frac{\partial y}{\partial x} + \frac{\partial z}{\partial x} = 0, \quad \frac{\partial z}{\partial t} + y\frac{\partial z}{\partial x} + z\frac{\partial y}{\partial x} = 0. \tag{2.12}$$

Reduction to a Linear System

We apply the hodograph transformation $(t,x) \Leftrightarrow (z,y)$ for the system (2.12) by introducing new independent variables $\hat{t},\hat{x}$ and new dependent variables $\hat{z},\hat{y}$ as follows:

$$\hat{t} = z, \quad \hat{x} = y; \tag{2.13}$$

$$\hat{z} = t, \quad \hat{y} = x. \tag{2.14}$$

A hodograph transformation is a mathematical method that is used to linearize nonlinear partial differential equations by exchanging the dependent and independent variables. This technique is frequently applied in fluid dynamics to make problems easier by transforming physical coordinates into a "hodograph plane," where the new coordinates correspond to physical quantities such as velocity components (see e.g. [10]).

We next introduce two differential operators $\frac{D}{Dt}$ and $\frac{D}{Dx}$ by the rule

$$\begin{aligned} \frac{D}{Dt} &= \frac{D\hat{t}}{Dt}\frac{D}{D\hat{t}} + \frac{D\hat{x}}{Dt}\frac{D}{D\hat{x}}, \\ \frac{D}{Dx} &= \frac{D\hat{t}}{Dx}\frac{D}{D\hat{t}} + \frac{D\hat{x}}{Dx}\frac{D}{D\hat{x}} \end{aligned} \tag{2.15}$$

Whence, using the expressions for $\hat{t}$ and $\hat{x}$, we obtain

$$\begin{aligned} \frac{D}{Dt} &= \frac{\partial z}{\partial t}\frac{D}{Dz} + \frac{\partial y}{\partial t}\frac{D}{Dy}, \\ \frac{D}{Dx} &= \frac{\partial z}{\partial x}\frac{D}{Dz} + \frac{\partial y}{\partial x}\frac{D}{Dy}. \end{aligned} \tag{2.16}$$

We next act on Equation (2.12) by the operators (2.15) to we obtain

$$\frac{D\hat{t}}{Dt}\frac{\partial \hat{z}}{\partial \hat{t}} + \frac{D\hat{x}}{Dt}\frac{\partial \hat{z}}{\partial \hat{x}} = 1 \text{ and } \frac{D\hat{t}}{Dx}\frac{\partial \hat{z}}{\partial \hat{t}} + \frac{D\hat{x}}{Dx}\frac{\partial \hat{z}}{\partial \hat{x}} = 0. \tag{2.17}$$

Also,

$$\frac{D\hat{t}}{Dt}\frac{\partial \hat{y}}{\partial \hat{t}}+\frac{D\hat{x}}{Dt}\frac{\partial \hat{y}}{\partial \hat{x}}=0 \text{ and } \frac{D\hat{t}}{Dx}\frac{\partial \hat{y}}{\partial \hat{t}}+\frac{D\hat{x}}{Dx}\frac{\partial \hat{y}}{\partial \hat{x}}=1. \tag{2.18}$$

Alternatively, using the the notation (2.13) - (2.14), we can rewrite Equations (2.17) - (2.18) as

$$\frac{\partial z}{\partial t}\frac{\partial t}{\partial z}+\frac{\partial y}{\partial t}\frac{\partial t}{\partial y}=1,\ \frac{\partial z}{\partial x}\frac{\partial t}{\partial y}+\frac{\partial y}{\partial x}\frac{\partial t}{\partial y}=0, \tag{2.19}$$

$$\frac{\partial z}{\partial t}\frac{\partial x}{\partial z}+\frac{\partial y}{\partial t}\frac{\partial x}{\partial y}=0,\ \frac{\partial z}{\partial x}\frac{\partial x}{\partial z}+\frac{\partial y}{\partial x}\frac{\partial x}{\partial y}=1. \tag{2.20}$$

The Equations (2.19) - (2.20) can be solved now for $\frac{\partial t}{\partial z},\frac{\partial t}{\partial y},\frac{\partial x}{\partial z}$, and $\frac{\partial x}{\partial y}$ to get the change under the hodograph transformation as follows:

$$\frac{\partial t}{\partial z}=-\frac{1}{\frac{\partial y}{\partial t}\frac{\partial z}{\partial x}-\frac{\partial z}{\partial t}\frac{\partial y}{\partial x}}\frac{\partial y}{\partial x},\ \frac{\partial t}{\partial y}=\frac{1}{\frac{\partial y}{\partial t}\frac{\partial z}{\partial x}-\frac{\partial z}{\partial t}\frac{\partial y}{\partial x}}\frac{\partial z}{\partial x} \tag{2.21}$$

In a way similar to the above, Equations (2.19) - (2.20) can be solved now for $\frac{\partial z}{\partial z},\frac{\partial z}{\partial x},\frac{\partial y}{\partial zt}$, and $\frac{\partial y}{\partial x}$ to get the change under the hodograph transformation as follows:

$$\frac{\partial z}{\partial t}=-\frac{1}{\frac{\partial t}{\partial y}\frac{\partial x}{\partial z}-\frac{\partial t}{\partial z}\frac{\partial x}{\partial y}}\frac{\partial x}{\partial y},\ \frac{\partial z}{\partial x}=\frac{1}{\frac{\partial t}{\partial y}\frac{\partial x}{\partial z}-\frac{\partial t}{\partial z}\frac{\partial x}{\partial y}}\frac{\partial t}{\partial y} \tag{2.22}$$

Thus the change of the derivatives (2.22) transforms the original nonlinear model (2.12) to the linear system of equations

$$\frac{\partial x}{\partial z}-y\frac{\partial t}{\partial z}+\frac{\partial t}{\partial y}=0,\ \frac{\partial x}{\partial y}+z\frac{\partial t}{\partial z}-y\frac{\partial t}{\partial y}=0. \tag{2.23}$$

3. Existence of Solutions

Here we will investigate the existence of classical solutions for the shallow water model (2.23) in the time scales introduced via:

(A1) $\mathbb{T}$ and $\mathbb{T}_1$ are time scales with forward jump operators and delta differentiation operators σ, σ_1 and Δ, Δ_1, respectively, $0\in\mathbb{T},\mathbb{T}_1$, $u_0,\theta_0\in\mathcal{C}^1_{rd}\left(\mathbb{T}_1\right)$, $0\le u_0\le B$, $0<\theta_0\le B$ on $\mathbb{T}_1$, p is a nonnegative constant.

We rewrite the model (2.23) in the time scales as follows:

$$\begin{aligned}&y_t^{\Delta}+yy_x^{\Delta_1}+z_x^{\Delta_1}=0\\&z_t^{\Delta}+yz_x^{\Delta_1}+zy_x^{\Delta_1}=0,\ (t,x)\in[0,\sup\mathbb{T})\times\mathbb{T}_1,\\&y(0,x)=y_0(x),\ z(0,x)=z_0(x),\ x\in\mathbb{T}_1,\end{aligned} \tag{3.1}$$

In this paper, under the conditions (A1) we will investigate Equations (3.1) for existence of at least one solution, at least two nonnegative and at least three nonnegative solutions. For this aim, firstly it is given a new integral representation of the solutions of the considered problem and then they are constructed two operators so that any fixed point of their sum is a solution to the considered problem. To the best of our knowledge, there are not any results in this direction in the existing references.

3.1. Preliminary Results

Throughout this paper, we assume that the reader is familiar with the basics of the time scale calculus. A detailed introduction to the time scale calculus is given in [11]. Here, we collect the definitions and theorems that are most useful in this paper.

Definition 3.1. *A time scale, denoted by* $\mathbb{T}$, *is a nonempty, closed subset of* $\mathbb{R}$. *For* $a,b\in\mathbb{T}$, *we let* $[a,b]$ *denote the set* $[a,b]\cap\mathbb{T}$.

Definition 3.2. *Let* $\mathbb{T}$ *be a time scale. For* $t\in\mathbb{T}$, *we define the forward jump operator* $\sigma:\mathbb{T}\to\mathbb{R}$ *by* $\sigma(t)=\inf\{s\in\mathbb{T}:s>t\}$, *and the backward jump operator* $\rho:\mathbb{T}\to\mathbb{T}$ *is given by* $\rho(t)=\sup\{s\in\mathbb{T}:s<t\}$.

By convention, we take $\inf\varnothing=\sup\mathbb{T}$, $\sup\varnothing=\inf\mathbb{T}$. For a function $f:\mathbb{T}\to\mathbb{R}$, we use the notation $f^{\sigma}(t)$ for the composition $f(\sigma(t))$.

Definition 3.3. *The graininess function* $\mu:\mathbb{T}\to[0,\infty)$ *is defined by* $\mu(t)=\sigma(t)-t$, $t\in\mathbb{T}$.

Definition 3.4. *Let* $t\in\mathbb{T}$. *If* $\sigma(t)=t$ *and* $t<\sup\mathbb{T}$, *then* t *is right-dense. If* $\sigma(t)>t$, *then* t *is right-scattered. Similarly, if* $\rho(t)=t$ *and* $t>\inf\mathbb{T}$, *then* t *is left-dense. If* $\rho(t)<t$, *then* t *is left-scattered.*

Definition 3.5. *If* $\sup\mathbb{T}=m$ *such that* m *is left-scattered, then define* $\mathbb{T}^{\kappa}=\mathbb{T}\setminus\{m\}$, *otherwise, define* $\mathbb{T}^{\kappa}=\mathbb{T}$.

Definition 3.6. *A function* $f:\mathbb{T}\to\mathbb{R}$ *is rd-continuous provided it is continuous at right-dense points in* $\mathbb{T}$ *and its left-sided limits exist and are finite at all left-dense points in* $\mathbb{T}$. *A function* $p:\mathbb{T}\to\mathbb{R}$ *is regressive provided* $1+\mu(t)p(t)\neq 0$, $t\in\mathbb{T}^{\kappa}$. *The set of all regressive and rd-continuous functions on a time scale* $\mathbb{T}$ *is denoted by* $\mathcal{R}=\mathcal{R}(\mathbb{T})$. *We use the notation* $\mathcal{R}^{+}$ *to denote the subgroup of those* $p\in\mathcal{R}$ *for which* $1+\mu(t)p(t)>0$ *for all* $t\in\mathbb{T}^{\kappa}$.

Definition 3.7. *The delta derivative of* $f:\mathbb{T}\to\mathbb{R}$ *at* $t\in\mathbb{T}^{\kappa}$, *is defined to be*

$$f^{\Delta}(t)=\lim_{s\to t}\frac{f(\sigma(t))-f(s)}{\sigma(t)-s}$$

provided this limit exists.

Definition 3.8. *For* $p\in\mathcal{R}$, *the generalized exponential function* $e_p:\mathbb{T}\times\mathbb{T}\to\mathbb{R}$ *is defined by*

$$e_p(t,s)=\exp\left(\int_s^t\xi_{\mu(\tau)}\left(p(\tau)\right)\Delta\tau\right),$$

for $s,t\in\mathbb{T}$, *where the cylinder transformation,* $\xi_h(z)$, *is defined by*

$$\xi_h(z)=\begin{cases}\dfrac{1}{h}\operatorname{Log}(1+zh), & h>0,\\ z, & h=0.\end{cases}$$

Definition 3.9. *For* $p,q\in\mathcal{R}$, *we define the operation* $\oplus$ *and* $\ominus$ *as follows*

$$(p\oplus q)(t)=p(t)+q(t)+\mu(t)p(t)q(t),\ (\ominus p)(t)=-\frac{p(t)}{1+\mu(t)p(t)}.$$

The proof of the next theorem is given in [11].

Theorem 3.1. *If* $p,q\in\mathcal{R}$ *and* $t,s,r\in\mathbb{T}$, *then*

1) $e_0(t,s)=1$, $e_p(t,t)=1$.

2) $e_p^{\sigma}(t,s)=\left(1+\mu(t)p(t)\right)e_p(t,s)$.

3) $e_p(s,t)=\dfrac{1}{e_p(t,s)}=e_{\ominus p}(t,s)$.

4) $e_p(t,s)e_p(s,r)=e_p(t,r)$.

5) $e_p(t,s)e_q(t,s)=e_{p\oplus q}(t,s)$.

6) $e_p(t,0)>0$ *for any* $0,t\in\mathbb{T}$ *if* $p\in\mathcal{R}$ *and* $1+\mu(t)p(t)>0$ *for any* $t\in\mathbb{T}^{\kappa}$.

Definition 3.10. *For* $h>0$, *the Hilger complex plane is defined by* $\mathbb{C}_h=\mathbb{C}\setminus\left\{-\frac{1}{h}\right\}$ *and we take* $\mathbb{C}_0=\mathbb{C}$ *and* $\mathbb{C}_\infty=\mathbb{C}\setminus\{0\}$.

Definition 3.11. *For given* $h\in[0,\infty)$, *the Hilger real part of a number* $z\in\mathbb{C}$ *is given by the formula*

$$\operatorname{Re}_h(z)=\begin{cases}\operatorname{Re}(z), & h=0,\\ \dfrac{|1+hz|-1}{h}, & 0<h<\infty,\\ |z|, & h=\infty.\end{cases}$$

It is known, see [12], that for a fixed z and $0<h<\infty$, $\operatorname{Re}_h(z)$ is a nondecreasing function of h. This relationship extends to $h=\infty$ because for any $0<h<\infty$,

$$\operatorname{Re}_h(z)=\frac{|1+hz|-1}{h}\le\frac{1+h|z|-1}{h}=|z|=\operatorname{Re}_\infty(z).$$

3.2. Auxiliary Results

Below, assume that X is a real Banach space. Now, we recall the definition for a completely continuous operator in a Banach space.

Definition 3.12. *Let* $K:M\subset X\to X$ *be a map. We say that* K *is compact if* $K(M)$ *is contained in a compact subset of* X. K *is called a completely continuous map if it is continuous and it maps any bounded set into a relatively compact set.*

The concept for k-set contraction is related to that of the Kuratowski measure of noncompactness which we recall for completeness.

Definition 3.13. *Let* Ω_X *be the class of all bounded sets of* X. *The Kuratowski measure of noncompactness* $\alpha:\Omega_X\to[0,\infty)$ *is defined by*

$$\alpha(Y)=\inf\left\{\delta>0:Y=\bigcup_{j=1}^{m}Y_j \text{ and } diam(Y_j)\le\delta, j\in\{1,\cdots,m\}\right\},$$

where $diam(Y_j)=\sup\{\|x-y\|_X:x,y\in Y_j\}$ *is the diameter of* Y_j, $j\in\{1,\cdots,m\}$.

For the main properties of measure of noncompactness we refer the reader to [13].

Definition 3.14. *A mapping* $K : X \to X$ *is said to be* k *-set contraction if there exists a constant* $k \geq 0$ *such that*

$$\alpha\left(K\left(Y\right)\right) \leq k\alpha\left(Y\right)$$

for any bounded set $Y \subset X$ *.*

Obviously, if $K : X \to X$ is a completely continuous mapping, then K is 0-set contraction (see [14]).

To prove our first existence result we will use the following fixed point theorem. For its proof, we refer the reader to [15] or [16].

Theorem 3.2. *Let* E *be a Banach space,* Y *a closed, convex subset of* E *,* U *be any open subset of* Y *with* $0 \in U$ *. Consider two operators* T *and* S *, where*

$$Tx = \varepsilon x, \quad x \in \bar{U},$$

for $\varepsilon > 1$ *and* $S : \bar{U} \to E$ *be such that*

1) $I - S : \bar{U} \to Y$ *continuous, compact and*

2) $\left\{x \in \partial U : x = \lambda\left(I - S\right)x\right\} = \varnothing$ *, for any* $\lambda \in \left(0, \dfrac{1}{\varepsilon}\right)$.

Then there exists $x^* \in \bar{U}$ *such that*

$$Tx^* + Sx^* = x^*.$$

Definition 3.15. *Let* X *and* Y *be real Banach spaces. A map* $K : X \to Y$ *is called expansive if there exists a constant* $h > 1$ *for which one has the following inequality*

$$\left\|Kx - Ky\right\|_Y \geq h\left\|x - y\right\|_X$$

for any $x, y \in X$ *.*

Now, we will recall the definition for a cone in a Banach space.

Definition 3.16. *A closed, convex set* $\mathcal{P}$ *in* X *is said to be cone if*

1) $\alpha x \in \mathcal{P}$ *for any* $\alpha \geq 0$ *and for any* $x \in \mathcal{P}$ *,*

2) $x, -x \in \mathcal{P}$ *implies* $x = 0$ *.*

Denote $\mathcal{P}^* = \mathcal{P} \setminus \{0\}$. The next result is a fixed point theorem which we will use to prove existence of at least two nonnegative global classical solutions of the IVP (3.1). For its proof, we refer the reader to [17] and [18].

Theorem 3.3. *Let* $\mathcal{P}$ *be a cone of a Banach space* E *;* Ω *a subset of* $\mathcal{P}$ *and* U_1, U_2 *and* U_3 *three open bounded subsets of* $\mathcal{P}$ *such that* $\bar{U}_1 \subset \bar{U}_2 \subset U_3$ *and* $0 \in U_1$ *. Assume that* $T : \Omega \to \mathcal{P}$ *is an expansive mapping,* $S : \bar{U}_3 \to E$ *is a completely continuous map and* $S\left(\bar{U}_3\right) \subset \left(I - T\right)\left(\Omega\right)$ *. Suppose that* $\left(U_2 \setminus \bar{U}_1\right) \cap \Omega \neq \varnothing$ *,* $\left(U_3 \setminus \bar{U}_2\right) \cap \Omega \neq \varnothing$ *, and there exists* $u_0 \in \mathcal{P}^*$ *such that the following conditions hold:*

1) $Sx \neq \left(I - T\right)\left(x - \lambda u_0\right)$ *, for all* $\lambda > 0$ *and* $x \in \partial U_1 \cap \left(\Omega + \lambda u_0\right)$ *;*

2) *There exists* $\varepsilon \geq 0$ *such that* $Sx \neq \left(I - T\right)\left(\lambda x\right)$ *, for all* $\lambda \geq 1 + \varepsilon$ *,* $x \in \partial U_2$ *and* $\lambda x \in \Omega$ *;*

3) $Sx\neq(I-T)(x-\lambda u_0)$, *for all* $\lambda>0$ *and* $x\in\partial U_3\cap(\Omega+\lambda u_0)$.

Then $T+S$ *has at least two non-zero fixed points* $x_1,x_2\in\mathcal{P}$ *such that*

$$x_1\in\partial U_2\cap\Omega\quad\text{and}\quad x_2\in\left(\bar U_3\setminus\bar U_2\right)\cap\Omega$$

or

$$x_1\in\left(U_2\setminus U_1\right)\cap\Omega\quad\text{and}\quad x_2\in\left(\bar U_3\setminus\bar U_2\right)\cap\Omega.$$

The following result will be used to prove the existence of three nonnegative solutions of our problem. For the proof, we use the same arguments used in [17].

Theorem 3.4. *Let* $\mathcal{P}$ *be a cone of a Banach space* E; Ω *a subset of* $\mathcal{P}$ *and* U_1,U_2 *and* U_3 *three open bounded subsets of* $\mathcal{P}$ *such that* $\bar U_1\subset\bar U_2\subset U_3$ *and* $0\in U_1$. *Assume that* $T:\Omega\to E$ *is an expansive mapping,* $S:\bar U_3\to E$ *is a completely continuous one and* $S\left(\bar U_3\right)\subset(I-T)(\Omega)$. *Suppose that* $\left(U_2\setminus\bar U_1\right)\cap\Omega\neq\varnothing$, $\left(U_3\setminus\bar U_2\right)\cap\Omega\neq\varnothing$, *and there exist* $w_0\in\mathcal{P}^*$ *and* $\varepsilon>0$ *small enough such that the following conditions hold*:

1) $Sx\neq(I-T)(\lambda x)$, *for all* $\lambda\geq1+\varepsilon$, $x\in\partial U_1$ *and* $\lambda x\in\Omega$;

2) $Sx\neq(I-T)(x-\lambda w_0)$, *for all* $\lambda\geq0$ *and* $x\in\partial U_2\cap(\Omega+\lambda w_0)$;

3) $Sx\neq(I-T)(\lambda x)$, *for all* $\lambda\geq1+\varepsilon$, $x\in\partial U_3$ *and* $\lambda x\in\Omega$.

Then $T+S$ *has at least three non trivial fixed points* $x_1,x_2,x_3\in\mathcal{P}$ *such that*

$$x_1\in\bar U_1\cap\Omega\quad\textit{and}\quad x_2\in\left(U_2\setminus\bar U_1\right)\cap\Omega\quad\textit{and}\quad x_3\in\left(\bar U_3\right)\setminus\bar U_2)\cap\Omega.$$

In $X_1=\mathcal{C}^1_{rd}\left([0,\sup\mathbb{T})\times\mathbb{T}_1\right)$ we introduce the norm

$$\|u\|_1=\max\left\{\sup_{(t,x)\times[0,\sup\mathbb{T})\times\mathbb{T}_1}|u(t,x)|,\ \sup_{(t,x)\in[0,\sup\mathbb{T})\times\mathbb{T}_1}\left|u_x^{\Delta_1}(t,x)\right|,\ \sup_{(t,x)\in[0,\sup\mathbb{T})\times\mathbb{T}_1}\left|u_t^{\Delta}(t,x)\right|\right\},$$

provided it exists. Let $X=X_1^2$ be endowed with the norm.

$$\|u\|=\max\left\{\|u_1\|_1,\|u_2\|_1\right\},\ u=(u_1,u_2).$$

For $u\in X$, $u=(u_1,u_2)$, and $C\in\mathbb{R}$, when we write $u\geq(\leq)C$ we have in mind $u_1,u_2\geq(\leq)C$.

4. Existence of at Least One Solution

In this section, we will prove that the problem (3.1) has at least one solution. Let $0\in\mathbb{T}_1$ be arbitrarily chosen and fixed. For $u\in X$, define the operators

$$S_1^1(u)(t,x)=u_1(t,x)-y_0(x)+\int_0^t\left(u_1(\tau,x)u_{1x}^{\Delta_1}(\tau,x)+u_{2x}^{\Delta_1}(\tau,x)\right)\Delta\tau,$$

$$S_1^2(u)(t,x)=u_2(t,x)-z_0(x)+\int_0^t\left(u_1(\tau,x)u_{2x}^{\Delta_1}(\tau,x)+u_2(\tau,x)u_{1x}^{\Delta_1}(\tau,x)\right)\Delta\tau,$$

$$S_1(u)(t,x)=\left(S_1^1(u)(t,x),S_1^2(u)(t,x)\right),\ (t,x)\in[0,\sup\mathbb{T})\times\mathbb{T}_1.$$

Lemma 4.1. *If* $u\in X$ *satisfies the equation*

$$S_1(u)(t,x)=0,\ (t,x)\in[0,\sup\mathbb{T})\times\mathbb{T}_1,$$

then u *is a solution to the problem* (3.1).

Proof. We have

$$u_1(t,x)-y_0(x)+\int_0^t\left(u_1(\tau,x)u_{1x}^{\Delta_1}(\tau,x)+u_{2x}^{\Delta_1}(\tau,x)\right)\Delta\tau=0,$$

$$u_2(t,x)-z_0(x)+\int_0^t\left(u_1(\tau,x)u_{2x}^{\Delta_1}(\tau,x)+u_2(\tau,x)u_{1x}^{\Delta_1}(\tau,x)\right)\Delta\tau=0,$$

$(t,x)\in[0,\sup\mathbb{T})\times\mathbb{T}_1$. We differentiate the last system with respect to t and we find the first two equations of (3.1). We put $t=0$ and we arrive at

$$u_1(0,x)-y_0(x)=0,$$

$$u_2(0,x)-z_0(x)=0,\ x\in\mathbb{T}_1.$$

Therefore (u_1,u_2) is a solution to the problem (3.1). This completes the proof. □

Let

$$B_1=\max\left\{2B,2B^2,B^2+B\right\}.$$

Lemma 4.2. *Suppose* (A1). *If* $u\in X$, $\|u\|\le B$, *then*

$$\left|S_1^j(u)(t,x)\right|\le B_1(1+t)(1+|x|),\ (t,x)\in[0,\sup\mathbb{T})\times\mathbb{T}_1,\ j\in\{1,2\}.$$

Proof. We have

$$\begin{aligned}\left|S_1^1(u)(t,x)\right|&=\left|u_1(t,x)-y_0(x)+\int_0^t\left(u_1(\tau,x)u_{1x}^{\Delta_1}(\tau,x)+u_{2x}^{\Delta_1}(\tau,x)\right)\Delta\tau\right|\\&\le\left|u_1(t,x)\right|+y_0(x)+\int_0^t\left(\left|u_1(\tau,x)\right|\left|u_{1x}^{\Delta_1}(\tau,x)\right|+\left|u_{2x}^{\Delta_1}(\tau,x)\right|\right)\Delta\tau\\&\le 2B+\left(B^2+B\right)t\\&\le B_1(1+t)\\&\le B_1(1+t)(1+|x|),\ (t,x)\in[0,\sup\mathbb{T})\times\mathbb{T}_1,\end{aligned}$$

and

$$\begin{aligned}\left|S_1^2(u)(t,x)\right|&=\left|u_2(t,x)-z_0(x)+\int_0^t\left(u_1(\tau,x)u_{2x}^{\Delta_1}(\tau,x)+u_2(\tau,x)u_{1x}^{\Delta_1},(\tau,x)\right)\Delta\tau\right|\\&\le\left|u_2(t,x)\right|+z_0(x)+\int_0^t\left(\left|u_1(\tau,x)\right|\left|u_{2x}^{\Delta_1}(\tau,x)\right|+\left|u_2(\tau,x)\right|\left|u_{1x}^{\Delta_1}(\tau,x)\right|\right)\Delta\tau\\&\le 2B+2B^2t\\&\le B_1(1+t)\\&\le B_1(1+t)(1+|x|),\ (t,x)\in[0,\sup\mathbb{T})\times\mathbb{T}_1.\end{aligned}$$

This completes the proof.

In addition, we suppose:

(A2) There exist a nonnegative function $g\in\mathcal{C}_{rd}\left([0,\sup\mathbb{T})\times\mathbb{T}_1\right)$ and a nonnegative constant A such that

1) $g>0$ on $[0,\sup\mathbb{T})\times\left(\mathbb{T}_1\setminus\{0\}\right)$.

2) if $0 \in \mathbb{T}_1$ is right-dense, then

$$g(t,0)=0, \; t \in [0,\sup\mathbb{T}).$$

3) if $0 \in \mathbb{T}_1$ is right-scattered, then

$$g(t,0)>0, \; t \in [0,\sup\mathbb{T}).$$

4)

$$\left|\int_0^t\int_0^x (1+\tau)(1+|s|)\,g(\tau,s)\,\Delta_1 s\Delta\tau\right| \le A,$$

$$\left|\int_0^t (1+\tau)(1+|x|)\,g(\tau,x)\,\Delta\tau\right| \le A,$$

$$\left|\int_0^x (1+t)(1+|s|)\,g(t,s)\,\Delta_1 s\right| \le A, \; (t,x) \in [0,\sup\mathbb{T})\times\mathbb{T}_1.$$

In the last section, we will give an example for a function g and a constant A that satisfy (A2). For $u \in X$, define the operator

$$S_2^j(u)(t,x)=\int_0^t\int_0^x g(s,y)\,S_1^j(u)(s,y)\,\Delta_1 y\Delta s, \; j=1,2,$$

$$S_2(u)(t,x)=\left(S_2^1(u)(t,x),S_2^2(u)(t,x)\right), \; (t,x) \in [0,\sup\mathbb{T})\times\mathbb{T}_1.$$

Lemma 4.3. *Suppose* (A1) *and* (A2). *If* $u \in X$ *and* $\|u\| \le B$, *then*

$$\|S_2 u\| \le AB_1.$$

Proof. We have

$$\begin{aligned}\left|S_2^j(u)(t,x)\right| &= \left|\int_0^t\int_0^x g(s,y)\,S_1^j(u)(s,y)\,\Delta_1 y\Delta s\right| \\ &\le \left|\int_0^t\int_0^x g(s,y)\left|S_1^j(u)(s,y)\right|\Delta_1 y\Delta s\right| \\ &\le B_1\left|\int_0^t\int_0^x (1+s)(1+|y|)\,g(s,y)\,\Delta_1 y\Delta s\right| \\ &\le AB_1, \; j=1,2, \; (t,x) \in [0,\sup\mathbb{T})\times\mathbb{T}_1,\end{aligned}$$

and

$$\begin{aligned}\left|S_{2x}^{j\Delta_1}(u)(t,x)\right| &= \left|\int_0^t g(s,x)\,S_1^j(u)(s,x)\,\Delta s\right| \\ &\le \left|\int_0^t\int_0^x g(s,x)\left|S_1^j(u)(s,x)\right|\Delta s\right| \\ &\le B_1\left|\int_0^t (1+s)(1+|x|)\,g(s,x)\,\Delta s\right| \\ &\le AB_1, \; j=1,2, \; (t,x) \in [0,\sup\mathbb{T})\times\mathbb{T}_1,\end{aligned}$$

and

$$\begin{aligned}\left|S_{2t}^{j\Delta}(u)(t,x)\right| &= \left|\int_0^x g(t,y)S_1^j(u)(t,y)\Delta_1 y\right| \\ &\le \left|\int_0^x g(t,y)\left|S_1^j(u)(t,y)\right|\Delta_1 y\right| \\ &\le B_1\left|\int_0^x (1+t)(1+|y|)g(t,y)\Delta_1 y\right| \\ &\le AB_1,\ j=1,2,\ (t,x)\in[0,\sup\mathbb{T})\times\mathbb{T}_1,\end{aligned}$$

whereupon we get the desired result. This completes the proof. ☐

Lemma 4.4. *Suppose* (A1) *and* (A2). *If* $u\in X$ *satisfies the equation*

$$S_2(u)(t,x)=C,\ (t,x)\in[0,\sup\mathbb{T})\times\mathbb{T}_1, \tag{4.1}$$

for some constant C, *then* u *is a solution to the problem* (3.1).

Proof. We differentiate with respect to t and x Equation (4.1) and we find

$$g(t,x)S_1(u)(t,x)=0,\ (t,x)\in[0,\sup\mathbb{T})\times\mathbb{T}_1,$$

whereupon

$$S_1(u)(t,x)=0,\ (t,x)\in[0,\sup\mathbb{T})\times(\mathbb{T}_1\setminus\{0\}).$$

If $0\in\mathbb{T}_1$ is right-scattered, then

$$S_1(u)(t,x)=0,\ (t,x)\in[0,\sup\mathbb{T})\times\mathbb{T}_1.$$

If $0\in\mathbb{T}_1$ is right-dense, using that $S_1(u)$ is a continuous function on $[0,\sup\mathbb{T})\times\mathbb{T}_1$, we get

$$S_1(u)(t,0)=\lim_{x\to 0}S_1(u)(t,x)=0,\ t\in[0,\sup\mathbb{T}).$$

Therefore

$$S_1(u)(t,x)=0\ \ (t,x)\in[0,\sup\mathbb{T})\times\mathbb{T}_1.$$

Hence, we conclude that u is a solution to the problem (3.1). This completes the proof. ☐

Below, suppose:

(A3) $\varepsilon>1$.

In the last section, we will give an example for the constants ε, A, B and B_1. Our main result in this section is as follows.

Theorem 4.1. *Suppose* (A1) - (A3). *Then Equation* (3.1) *has at least one solution in* X.

Proof. Let $\tilde{Y}$ denote the set of all equi-continuous families in X with respect to the norm $\|\cdot\|$. Let also,

$$\tilde{\tilde{Y}}=\left\{u\in\tilde{Y}:u(t,x)\ge\frac{\|u\|}{2},(t,x)\in[0,\sup\mathbb{T})\times\mathbb{T}_1\right\}$$

and $Y=\overline{\tilde{\tilde{Y}}}$ be the closure of $\tilde{Y}$,

$$U=\{u\in Y:\|u\|<B\}.$$

For $u \in \bar{U}$ and $\varepsilon > 0$, define the operators

$$Tu(t,x) = \varepsilon u(t,x),$$

$$Su(t,x) = u(t,x) - \varepsilon u(t,x) - \varepsilon S_2(u)(t,x),\ (t,x) \in [0, \sup \mathbb{T}) \times \mathbb{T}_1.$$

For $u \in \bar{U}$, we have

$$\begin{aligned}\|(I-S)u\| &= \|\varepsilon u + \varepsilon S_2(u)\| \\ &\le \varepsilon\|u\| + \varepsilon\|S_2(u)\| \\ &\le \varepsilon B + \varepsilon A B_1.\end{aligned}$$

Thus, $S : \bar{U} \to X$ is continuous and $(I-S)(\bar{U})$ resides in a compact subset of Y. Now, suppose that there is a $u \in \bar{U}$ so that $\|u\| = B$ and

$$u = \lambda(I-S)u$$

or

$$u = \lambda\varepsilon(u + S_2(u)), \tag{4.2}$$

for some $\lambda \in \left(0, \frac{1}{\varepsilon}\right)$. Then, using that $\|u\| \ge \frac{B}{2}$, we get $u(0,x) > \frac{B}{2}$, $x \in \mathbb{T}_1$, and

$$u(0,x) = \lambda\varepsilon u(0,x),\ x \in \mathbb{T}_1,$$

whereupon $\lambda\varepsilon = 1$, which is a contradiction. Consequently

$$\left\{u \in \bar{U} : u = \lambda_1(I-S)u, \|u\| = B\right\} = \varnothing$$

for any $\lambda_1 \in \left(0, \frac{1}{\varepsilon}\right)$. Then, from Theorem 3.2, it follows that the operator $T+S$ has a fixed point $u^* \in Y$. Therefore

$$\begin{aligned}u^*(t,x) &= Tu^*(t,x) + Su^*(t,x) \\ &= \varepsilon u^*(t,x) + u^*(t,x) - \varepsilon u^*(t,x) - \varepsilon S_2(u^*)(t,x),\ (t,x) \in [0, \sup \mathbb{T}) \times \mathbb{T}_1,\end{aligned}$$

whereupon

$$S_2(u^*)(t,x) = 0,\ (t,x) \in [0, \sup \mathbb{T}) \times \mathbb{T}_1.$$

From here, u^* is a solution to the problem (3.1) and from Lemma 4.4, it follows that u is a solution to Equation (3.1). This completes the proof. ☐

5. Existence of at Least Two Solutions

Let X be the space used in the previous section. Suppose:

(A4) Let m, r, L, R_1 be positive constants that satisfy the following conditions

$$r < L < R_1 \le B.$$

Our main result in this section is as follows.

Theorem 5.1. *Suppose that* (A1), (A2) *and* (A4) *hold. Then Equation* (3.1) *has at least two nonnegative solutions in* X.

Proof. Let

$$\tilde{P} = \left\{ u \in X : u \geq 0 \text{ on } \left[0, \sup \mathbb{T}\right) \times \mathbb{T}_1 \right\}.$$

With $\mathcal{P}$ we will denote the set of all equi-continuous families in $\tilde{P}$. For $v \in X$, define the operators

$$T_1 v(t,x) = (1 + m\varepsilon) v(t,x),$$

$$S_3 v(t,x) = -\varepsilon \left| S_2(v)(t,x) \right| - m\varepsilon v(t,x),$$

$(t,x) \in [0, \sup \mathbb{T}) \times \mathbb{T}_1$. Note that any fixed point $v \in X$ of the operator $T_1 + S_3$ is a solution to Equation (3.1). Define

$$\Omega = P,$$

$$U_1 = \mathcal{P}_r = \left\{ v \in \mathcal{P} : \|v\| < r \right\},$$

$$U_2 = \mathcal{P}_L = \left\{ v \in \mathcal{P} : \|v\| < L \right\},$$

$$U_3 = \mathcal{P}_{R_1} = \left\{ v \in \mathcal{P} : \|v\| < R_1 \right\}.$$

1) For $v_1, v_2 \in \Omega$, we have

$$\|T_1 v_1 - T_1 v_2\| = (1 + m\varepsilon) \|v_1 - v_2\|,$$

whereupon $T_1 : \Omega \to X$ is an expansive operator with a constant $h = 1 + m\varepsilon > 1$.

2) For $v \in \overline{\mathcal{P}}_{R_1}$, we get

$$\|S_3 v\| \leq \varepsilon \|S_2(v)\| + m\varepsilon \|v\| \leq \varepsilon (AB_1 + mR_1).$$

Therefore $S_3\left(\overline{\mathcal{P}}_{R_1}\right)$ is uniformly bounded. Since $S_3 : \overline{\mathcal{P}}_{R_1} \to X$ is continuous, we have that $S_3\left(\overline{\mathcal{P}}_{R_1}\right)$ is equi-continuous. Consequently $S_3 : \overline{\mathcal{P}}_{R_1} \to X$ is a 0-set contraction.

3) Let $v_1 \in \overline{\mathcal{P}}_{R_1}$. Set

$$v_2 = v_1 + \frac{1}{m} \left| S_2(v_1) \right|.$$

We have $v_2 \geq 0$ on $[0, \sup \mathbb{T}) \times \mathbb{T}_1$. Therefore $v_2 \in \Omega$ and

$$-\varepsilon m v_2 = -\varepsilon m v_1 - \varepsilon \left| S_2(v_1) \right|$$

or

$$(I - T_1) v_2 = -\varepsilon m v_2 = S_3 v_1.$$

Consequently $S_3\left(\overline{\mathcal{P}}_{R_1}\right) \subset (I - T_1)(\Omega)$.

4) Assume that for any $v_0 \in \mathcal{P}^*$ there exist $\lambda \geq 0$ and $v \in \partial \mathcal{P}_r \cap (\Omega + \lambda v_0)$ or $v \in \partial \mathcal{P}_{R_1} \cap (\Omega + \lambda v_0)$ such that

$$S_3 v = (I - T_1)(v - \lambda v_0).$$

Then

$$-\varepsilon \left| S_2(v) \right| - m\varepsilon v = -m\varepsilon (v - \lambda v_0)$$

or

$$-\left| S_2(v) \right| = \lambda m v_0.$$

This is a contradiction.

5) Let $\varepsilon_1 = \frac{AB_1}{mL}$. Suppose that there exist a $v_1 \in \partial\mathcal{P}_L$ and $\lambda_1 \ge 1+\varepsilon_1$ such that

$$S_3 v_1 = (I - T_1)(\lambda_1 v_1). \tag{5.1}$$

Moreover,

$$-\varepsilon\left|S_2(v_1)\right| - m\varepsilon v_1 = -\lambda_1 m\varepsilon v_1,$$

or

$$\left|S_2(v_1)\right| + mv_1 = \lambda_1 m v_1.$$

From here,

$$\lambda_1 mL = \lambda_1 m\|v_1\| \le \|S_2 v_1\| + m\|v_1\| \le AB_1 + mL$$

and

$$\lambda_1 \le 1+\frac{AB_1}{mL},$$

which is a contradiction.

Therefore, all conditions of Theorem 3.3 hold. Hence, the problem (3.1) has at least two solutions u_1 and u_2 so that

$$\|u_1\| = L < \|u_2\| < R_1$$

or

$$r < \|u_1\| < L < \|u_2\| < R_1.$$

□

6. Existence of at Least Three Solutions

Our main results for existence of at least three solutions of the problem (3.1) are as follows.

Theorem 6.1. *Under the hypotheses* (A1), (A2) *and* (A4), *the problem* (3.1) *has at least three nonnegative solutions* $u_1, u_2, u_3 \in X$.

Proof.

1) Assume that there are $\lambda_1 \ge 1+\frac{2AB_1}{mr}$, $u \in \partial U_1$ and $\lambda_1 u \in \Omega$ so that

$$S_3(u) = (I - T_1)(\lambda_1 u).$$

Then

$$-\epsilon\left|S_2(u)\right| - m\epsilon u = -m\epsilon\lambda_1 u$$

or

$$\left|S_2(u)\right| + mu = m\lambda_1 u$$

Hence,

$$\lambda_1 mr = \lambda_1 m\|u\| \le \left\|S_2(u)\right\| + m\|u\| \le AB_1 + \lambda_1 mr,$$

whereupon

$$\lambda_1 \le 1 + \frac{AB_1}{mr},$$

which is a contradiction. Thus, the condition (1) of Theorem 3.4 holds.

2) Now, assume that there are $\lambda_1 \ge 1 + \frac{2AB_1}{mr}$, $u \in \partial U_3$ and $\lambda_1 u \in \Omega$ so that

$$S_3(u) = (I - T_1)(\lambda_1 u).$$

As above,

$$\lambda_1 m R_1 = \lambda_1 m \|u\| \le \|S_2(u)\| + m\|u\| \le AB_1 + \lambda_1 m R_1,$$

whereupon

$$\lambda_1 \le 1 + \frac{AB_1}{mR_1} \le 1 + \frac{AB_1}{mr},$$

which is a contradiction. Hence, the condition (3) of Theorem 3.4 holds.

3) Assume that for any $u_0 \in \mathcal{P}^*$ there exist $\lambda_1 \ge 0$ and $u \in \partial \mathcal{P}_L \cap (\Omega + \lambda_1 u_0)$ such that

$$S_3(u) = (I - T_1)(u - \lambda_1 u_0).$$

Then

$$-\epsilon |S_2(u)| - m\epsilon u = -m\epsilon(u - \lambda_1 u_0)$$

or

$$-|S_2(u)| = \lambda_1 m u_0.$$

This is a contradiction. Form here, the condition (2) of Theorem 3.4 holds.

Now, by Theorem 3.4, it follows that the problem (3.1) has at least three classical solutions u_1, u_2 and u_3 such that

$$u_1 \in \partial U_1 \cap \Omega \text{ and } u_2 \in (U_2 \setminus \bar{U}_1) \cap \Omega \text{ and } u_3 \in (\bar{U}_3) \setminus \bar{U}_2) \cap \Omega,$$

or

$$u_1 \in U_1 \cap \Omega \text{ and } u_2 \in (U_2 \setminus \bar{U}_1) \cap \Omega \text{ and } u_3 \in (\bar{U}_3) \setminus \bar{U}_2) \cap \Omega .$$

☐

7. An Example

Below, we will illustrate our main results. Let

$$g_0(t) = \frac{1}{(1+t)^2 (1 + \sigma(t))},$$

$$g_1(x) = \frac{\sigma_1(x) + x}{(1+x^2)^2 \left(1 + (\sigma_1(x))^2\right)},$$

$$g(t,x) = A g_0(t) g_1(x), \quad (t,x) \in [0, \sup \mathbb{T}) \times \mathbb{T}_1,$$

where the positive constant A will be determined below. Hence,

$$\begin{aligned}
&\left|\int_0^t\int_0^x (1+s)\left(1+y^2\right)g(s,y)\Delta_1 y\Delta s\right| \\
&= A\left|\int_0^t\int_0^x (1+s)\left(1+y^2\right)\frac{\sigma_1(y)+y}{(1+s)^2(1+\sigma(s))\left(1+y^2\right)^2\left(1+(\sigma_1(y))^2\right)}\Delta_1 y\Delta s\right| \\
&\le A\left(\int_0^t \frac{1}{(1+s)(1+\sigma(s))}\Delta s\right)\left|\int_0^x \frac{\sigma_1(y)+y}{\left(1+y^2\right)\left(1+(\sigma_1(y))^2\right)}\Delta_1 y\right| \\
&= A\left(1-\frac{1}{1+t}\right)\left(1-\frac{1}{1+x^2}\right) \\
&\le A,\ (t,x)\in[0,\sup\mathbb{T})\times\mathbb{T}_1,
\end{aligned}$$

and

$$\begin{aligned}
&\left|\int_0^t (1+s)\left(1+x^2\right)g(s,x)\Delta s\right| \\
&\le A\frac{\sigma_1(x)+x}{1+(\sigma_1(x))^2}\left|\int_0^t (1+s)\frac{1}{(1+s)^2(1+\sigma(s))}\Delta s\right| \\
&\le A\left(\int_0^t \frac{1}{(1+s)(1+\sigma(s))}\Delta s\right) \\
&= A\left(1-\frac{1}{1+t}\right) \\
&\le A,\ (t,x)\in[0,\sup\mathbb{T})\times\mathbb{T}_1,
\end{aligned}$$

and

$$\begin{aligned}
&\left|\int_0^x (1+t)\left(1+y^2\right)g(t,y)\Delta_1 y\right| \\
&= A\left|\int_0^x (1+t)\left(1+y^2\right)\frac{\sigma_1(y)+y}{(1+t)^2(1+\sigma(t))\left(1+y^2\right)^2\left(1+(\sigma_1(y))^2\right)}\Delta_1 y\Delta s\right| \\
&\le A\left|\int_0^x \frac{\sigma_1(y)+y}{\left(1+y^2\right)\left(1+(\sigma_1(y))^2\right)}\Delta_1 y\right| \\
&= A\left(1-\frac{1}{1+x^2}\right) \\
&\le A,\ (t,x)\in[0,\sup\mathbb{T})\times\mathbb{T}_1,
\end{aligned}$$

We have that g satisfies (A2).

Let $\mathbb{T}=\mathbb{T}_1=\mathbb{Z}$. Then $[0,\sup\mathbb{T})=\mathbb{N}_0$.

Let $y_0(x)=\frac{1}{1+x^2}$, $z_0(x)=\frac{1}{1+2x^2+4x^4}$, $x\in\mathbb{T}_1$ and

$$c=1,\ R_1=1,\ L=\frac{1}{4},\ r=\frac{1}{5},\ m=10^{50},\ B=1,\ p=2,\ A=\frac{1}{10B_1},\ \varepsilon=4.$$

Then $B_1 = 2$, $A = \dfrac{1}{20}$ and $\epsilon > 1$, *i.e.*, (A3) holds. Next,

$$0 < r < L < R_1 = B.$$

i.e., (A4) holds. Therefore for the Cauchy problem of the considered shallow water equations all conditions of Theorem 4.1, Theorem 5.1 and Theorem 6.1 are fulfilled.

8. Concluding Remarks

In order to investigate the behavior of the general solution of the unperturbed model, we map the nonlinear system (2.12) to a linear system by the hodograph method. The linear system is reduced to a single second-order linear equation by an appropriate change of the dependent variables. This paper provides the existence of at least one solution, at least two nonnegative solutions, and at least three nonnegative solutions for the corresponding linear system.

We remark that the linear system (2.23) can also be reduced to a second-order linear equation by introducing the new dependent variables τ and χ defined by

$$\tau = zt, \quad \chi = x - yt. \tag{8.1}$$

Then the system (2.23) becomes

$$\tau_y + y\chi_z = 0, \quad \tau_z + \chi_y = 0 \tag{8.2}$$

and can be replaced by the single linear second-order equation

$$z\tau_{zz} - \tau_{yy} = 0. \tag{8.3}$$

In general, Equation (8.3) has a *mixed type*. It is hyperbolic when $z > 0$ and elliptic when $z < 0$. But in our case Equation (8.3) is hyperbolic because z is positive due to its physical meaning. The model (8.3) can be integrated by using Riemann's method [19] and the invariance principle. The Riemann integration method and the analysis of the perturbed system (2.11) will be considered in the forthcoming studies with the goal to study the influence of the perturbation in terms of approximate symmetries and approximately invariant solutions of the system (2.11). One of the particular focuses will be on using the hodograph method to construct the formation of shock waves in the atmospheric motions. The shock waves deserve particular attention because singularities in solutions of a mathematical model are observable in natural phenomena described by the considered mathematical model.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

References

[1] Holton, J.R., Haynes, P.H., McIntyre, M.E., Douglass, A.R., Rood, R.B. and Pfister, L. (1995) Stratosphere-Troposphere Exchange. *Reviews of Geophysics*, **33**, 403-439. https://doi.org/10.1029/95rg02097

[2] Yulaeva, E. and Wallace, J.M. (1994) The Signature of ENSO in Global Temperature and Precipitation Fields Derived from the Microwave Sounding Unit. *Journal of Climate*, **7**, 1719-1736. https://doi.org/10.1175/1520-0442(1994)007<1719:tsoeig>2.0.co;2

[3] Fletcher, L.N., Irwin, P.G.J., Orton, G.S., Teanby, N.A., Achterberg, R.K., Bjoraker, G.L., *et al.* (2008) Temperature and Composition of Saturn's Polar Hot Spots and Hexagon. *Science*, **319**, 79-81. https://doi.org/10.1126/science.1149514

[4] Vasavada, A.R., Hörst, S.M., Kennedy, M.R., Ingersoll, A.P., Porco, C.C., Del Genio, A.D., *et al.* (2006) Cassini Imaging of Saturn: Southern Hemisphere Winds and Vortices. *Journal of Geophysical Research*: *Planets*, **111**, 5004-5017. https://doi.org/10.1029/2005je002563

[5] Barbosa Aguiar, A.C., Read, P.L., Wordsworth, R.D., Salter, T. and Hiro Yamazaki, Y. (2010) A Laboratory Model of Saturn's North Polar Hexagon. *Icarus*, **206**, 755-763. https://doi.org/10.1016/j.icarus.2009.10.022

[6] Foltz, G.R. and McPhaden, M.J. (2010) Abrupt Equatorial Wave-Induced Cooling of the Atlantic Cold Tongue in 2009. *Geophysical Research Letters*, **37**, L24605. https://doi.org/10.1029/2010gl045522

[7] Illig, S. and Dewitte, B. (2006) Local Coupled Equatorial Variability versus Remote ENSO Forcing in an Intermediate Coupled Model of the Tropical Atlantic. *Journal of Climate*, **19**, 5227-5252. https://doi.org/10.1175/jcli3922.1

[8] Lübbecke, J.F., Böning, C.W., Keenlyside, N.S. and Xie, S. (2010) On the Connection between Benguela and Equatorial Atlantic Niños and the Role of the South Atlantic Anticyclone. *Journal of Geophysical Research*: *Oceans*, **115**, C09015. https://doi.org/10.1029/2009jc005964

[9] Su, C.H. and Gardner, C.S. (1969) Derivation of the Korteweg-de Vries Equations and Burgers Equations. *Journal of Mathematical Physics*, **10**, 536-539. https://doi.org/10.1063/1.1664873

[10] Chan, P.C. and Chow, W.L. (1984) The Study of Gravitational Nozzle Flows by Hodograph Transformations. *Journal of Applied Mechanics*, **51**, 457-464. https://doi.org/10.1115/1.3167657

[11] Bohner, M. and Georgiev, S.G. (2017) Multidimensional Time Scale Calculus. Springer. https://doi.org/10.1007/978-3-319-47620-9_1

[12] Bohner, M., Guseinov, G.S. and Karpuz, B. (2011) Properties of the Laplace Transform on Time Scales with Arbitrary Graininess. *Integral Transforms and Special Functions*, **22**, 785-800. https://doi.org/10.1080/10652469.2010.548335

[13] Deimling, K. (1985) Nonlinear Functional Analysis. Springer-Verlag.

[14] Drabek, P. and Milota, J. (2007) Methods in Nonlinear Analysis, Applications to Differential Equations. Birkhäuser.

[15] Georgiev, S., Kheloufi, A. and Mebarki, K. (2024) Existence of Classical Solutions for a Class of Impulsive Hamilton-Jacobi Equations. *Palestine Journal of Mathematics*, **13**, 1084-1087

[16] Khemmar, K., Mebarki, K. and Georgiev, S. (2024) Existence of Solutions for a Class of Boundary Value Problems for Weighted P(t)-Laplacian Impulsive Systems. *Filomat*, **38**, 7563-7577. https://doi.org/10.2298/fil2421563k

[17] Djebali, S. and Mebarki, K. (2019) Fixed Point Index for Expansive Perturbation of K-Set Contraction Mappings. *Topological Methods in Nonlinear Analysis*, **54**, 613-640.

[18] Mouhous a, A., Georgiev b, S.G. and Mebarki c, K. (2022) Existence of Solutions for

a Class of First Order Boundary Value Problems. *Archivum Mathematicum*, **58**, 141-158. https://doi.org/10.5817/am2022-3-141

[19] Riemann, G. (1860) Über die fortpflanzung ebener luftwellen von endlicher schwingungsweite. *Abhandlungen der Königlichen Gesellschaft der Wissenschaften zu Göttingen*, **8**, 43-65.