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A Step towards Dark Energy and Dark Matter

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Abstract

The paper proposes an "*ab initio*" theoretical contribution to the understanding of dark matter and dark energy. The model has quantum basis and correlates the Einstein cosmological constant to the vacuum energy density; the cosmological parameters are expressed as a self-consistent set of data as a function of age and size of the universe, in agreement with the values of the fundamental constants of nature. The MOND equation is inferred from first principles, the a_0 interpolation constant is calculated from cosmological data.

Keywords

Cosmology, Dark Matter, Dark Energy

1. Introduction

Indeterminism and non-locality, key concepts of the quantum world, imply an agnostic way to redefine the reality. The basic idea is that any experiment is in fact a form of interaction between observer and observed object, which implies that the reality we detect is the perturbed state of a previous unperturbed and unknown state prior the measurement process. A reasonable way of modeling the physical reality as we know it after the observation process should someway simulate this idea, *i.e.* to introduce first the indeterminacy of the physical parameters of interest and next extract information about the perturbed state through those parameters. This consideration has stimulated the idea of a possible way to formulate the physical models based on an extended concept of quantum uncertainty implementing the relationship $\delta x \delta p_x = n\hbar = \delta\varepsilon\delta t$: the number n of quantum states allowed for a system of particles is formulated via arbitrary ranges of fundamental dynamical variables, as a reasonable alternative to the deterministic trust in their local values. The agnostic assumption inherent the uncertainty represents the unperturbed reality to be observed, the way to extract all possible information form the uncertainty ranges is the conceptual equivalent of the observation process.

On the one hand, this approach to understand the reality regards the local values of the physical observables as precursors of the actual values we can detect with the experiment; on the other hand, this conceptual perspective bypasses in a natural way the determinism of input local values.

Once introducing as a sole source of information energy and time ranges linked to conjugate space and momentum ranges, then the space time reality becomes mere combination of dynamical parameters that translate their symbolic meaning into actual information to be confirmed by comparison with the measurement process.

For this reason, this way to formulate any physical problem pays crucial attention to the physical definition and physical dimensions of the dynamical variables, rather than to their representative specificity in a prefixed theoretical frame. For clarity, these ideas, highlighted in the next sections, are exemplified here. Write $\delta\epsilon/\delta p_x = v_x = \delta x/\delta t$; it defines v_x without specifying what is moving and why. But if one specifies in particular $\delta\varepsilon = \delta(\hbar\omega)$ and $\delta p_x = \delta(h/\lambda)$, one trivial step only is enough to define $v_x \equiv v_{gx} = \delta\omega/\delta\kappa$ with $\kappa = 2\pi/\lambda$; in this particular case v_x is the x-component of the group velocity of a wave packet. Also, $\delta p = -h\delta\lambda/\lambda^2 \equiv -(h/c^2)(c^2/\lambda^2) = -(h\nu/c^2)(\nu\delta\lambda) = -(\epsilon\delta v/c^2)$; it yields the relativistic momentum $p = \epsilon v/c^2$, with δp expressed as a function of $\delta v = \nu\delta\lambda$ and $\delta\lambda$. The minus sign is obvious because p increases with v but decreases with λ. This result merges quantum and relativistic definitions of momentum. With this kind of approach, attention is due to the concepts, not to the formulas: here we see that velocity is not mere ratio of space and time intervals, it inherently implies also energy and momentum changes. The next two sections exemplify how the dynamical variables take themselves an autonomous physical meaning regardless of the specific physical problem; it opens the way to a variety of interesting outcomes. The previous example shows how to proceed: the task was not to calculate purposely the group velocity, but to extract information on v from its own physical meaning via the uncertainty.

These general considerations are the reading key of the theoretical model proposed in this paper.

2. Preliminary Considerations: The Π Probability Function

In this introductory section consider again the concept of velocity v in a more systematic way introducing the function $\Pi(v) = v^2/c^2$ and regarding v as a dynamical variable characterized by its own physical properties in an arbitrary reference system, where are also defined its change δv and acceleration $\delta v/\delta t$. The quantum uncertainty is not implemented, to show that the ideas exposed in the section 1 have also classical validity. Give Π the meaning of probability: thus let be by definition $v \le c$ in order that $0 \le \Pi \le 1$, *i.e.*

$$\frac{\delta\Pi}{\Pi} = \frac{2v\delta v}{v^2} = 2\delta\log(v) = 2\log\frac{v}{c} \Rightarrow \frac{\delta\Pi}{\Pi} = \log\left(\frac{v^2}{c^2}\right) = \log\Pi. \tag{2.1}$$

Let us show that $\log\Pi < 0$ is consistent with the statistical definition of entropy. Think a set of N particles each one with its own set of allowed v_j; in this example v takes specific physical meaning if regarded as j-th dynamical state v_{ji} of the i-th particle among the many in a complex system of particles. As no hypothesis is necessary about v_{ji}, the step from (2.1) to (2.2) holds regardless of whether the j-th state results in a set of interacting or non-interacting particles: it is enough to refer each v_{ji} to the probability of an allowed state of a system to infer statistical information about the whole system. Summing first over j one finds

$$\delta\Pi_{ji} = \Pi_{ji}\log\Pi_{ji} \Rightarrow \sum_{j=1}^{j_i}\delta\Pi_{ji} = \delta\Pi_i = \sum_{j=1}^{j_i}\left(\Pi_{ji}\log\Pi_{ji}\right) = -S_i, \qquad (2.2)$$

being $-S_i > 0$ because $v < c$ implies $\Pi_{ji} < 1$; moreover define also

$$\overline{\delta\Pi_i} = X_i^{-1}\int_0^{X_i}\Pi_{ji}\log\Pi_{ji}\mathrm{d}\Pi_{ji}, \qquad (2.3)$$

where X_i is the parameter to calculate average value of $\delta\Pi_{ji}$ of the i-th particle. Since $\int(x\log x)\mathrm{d}x = x^2\log(x)/2 - x^2/4 + const$, where the right hand side does not diverge even for $x \to 0$, summing as before over all j-th allowed states, the result of (2.3) yields

$$\overline{\delta\Pi_i} = \frac{1}{2}X_i\log X_i - \frac{1}{4}X_i + \frac{const}{X_i} \Rightarrow \overline{\delta\Pi_i} \ge \frac{1}{2}X_i\log X_i - \frac{1}{4}X_i + const \quad X_i \le 1;$$

owing to (2.2), S and $\overline{\delta\Pi}$ obtained summing $-S_i$ and $\overline{\delta\Pi_i}$ over the number of particles, reads

$$S \ge \pm Y \pm Z \quad S = \sum_{i=1}^{N} - S_i \quad Y = \frac{1}{2}\sum_{i=1}^{N}X_i\log X_i + C_y \quad Z = C_z - \frac{1}{4}\sum_{i=1}^{N}X_i, \qquad (2.4)$$

where C_y and C_z are arbitrary constants resulting from $C_y + C_z = N \times const$; in principle C_y and C_z justify the double signs of Y and Z. However, being positive the left hand side of (2.4), at least one addend at the right hand side must be positive. As the double signs of Y and Z are independent, this condition is easily identified: Z is positive simply defining $C_z > \sum X_i/2$, in which case $S = \pm Y + Z$. As concerns the sign of Y, in principle nothing excludes that $Y \gtreqless Z$ with $-Y$; the realistic and unforeseeable chance of negative sign of Y due to C_y requires considering in particular $S \ge -Y + Z$. Thus a general notation to express (2.4) including both signs $\pm Y$ allowed for a given value of $|Y|$ is

$$S \ge Y + Z \Rightarrow E_0 S \ge E_0 Y + E_0 Z,$$

with both sides multiplied by a positive constant energy E_0 to convert the relationship between dimensionless quantities into a relationship between different forms of energy. This result appears more familiar rewriting $E_0 = k_B T$ and calculating $\delta(\,)$ of the functions to eliminate the constants at the right hand side; *i.e.* one finds

$$k_B T\delta S \ge \delta U + \delta W \quad U = k_B T\sum_{i=1}^{N}X_i \quad W = k_B T\sum_{i=1}^{N}X_i\log X_i \quad \delta S = S - const_s \qquad (2.5)$$

which does not need further comments; the constants remind that in fact any energy is defined an arbitrary constant apart, which can be put equal to zero. The importance of the first law, here obtained as a corollary of Π, has full statistical meaning: it does not depend upon the single X_i but upon sums over a total number of initial $\overline{\delta\Pi_{ji}}$-th terms.

Return now to (2.2) where v has in general mere meaning of dynamical variable, regardless of any specific reference to a particular physical system. Consider the following steps

$$\delta\Pi = \frac{2v\delta v}{c^2} = \frac{2v\delta v\delta t}{c^2\delta t} = \frac{2a\delta r}{c^2} \quad a = \frac{\delta v}{\delta t} \quad \delta r = v\delta t, \tag{2.6}$$

An approximate solution of (2.6) is found by replacing the finite differentials with

$$\frac{\partial\Pi}{\partial r} \approx \frac{2a}{c^2} \tag{2.7}$$

and then integrating with respect to r. Let hold for a the same considerations introduced for v, *i.e.* regard a as a mere dynamical parameter defined by the change rate dv during the arbitrary time lapse dt as a function of a space parameter r. To prove that (2.7) is sensible, define

$$a = a(r) = \frac{v_0^2}{r} \quad v_0 = const. \tag{2.8}$$

Replacing in (2.7) one finds $d\left(v^2/2v_0^2\right) = dr/r$, whose solution is $v^2/2v_0^2 + C = \log\left(r/r_0\right)$ and then

$$\frac{r}{r_0} = \exp\left(\frac{v^2/2 + Cv_0^2}{v_0^2}\right) \Rightarrow r' = \frac{r_0'}{\exp\left(\frac{v^2/2 + Cv_0^2}{v_0^2}\right) \mp 1} \quad \frac{r}{r_0} = \frac{r_0' \pm r'}{r'} \quad r_-' \le r_{0-}'; \tag{2.9}$$

the last inequality is required for the minus sign in the expression of r/r_0, which corresponds to the positive sign in r'. Divide both sides of (2.9) by an arbitrary length ℓ_0 and acknowledge in the exponential the ratio of different energies; this result with negative value of the integration constant C reads

$$N' = \frac{N_0'}{\exp\left(\frac{\epsilon - \epsilon_F}{\epsilon_0}\right) \mp 1} \quad N_0' = \frac{r_0'}{\ell_0} \quad N' = \frac{r'}{\ell_0} \quad \epsilon = \frac{mv^2}{2} \quad \epsilon_F = mv_0^2 C \quad \epsilon_0 = mv_0^2, \tag{2.10}$$

whereas (2.9) becomes

$$\frac{r'}{r_0'} \le 1 \quad N' \le N_0'. \tag{2.11}$$

Let N_0' be integer number, which is possible with an appropriate ℓ_0: taking in particular the minimum value $N_0' = 1$, the plus sign in (2.9) implies $N' = 0,1$ for any state defined by its own quantum numbers; this restrictive condition is not required for the minus sign of N' in (2.9). The physical meaning of (2.10) is well known, for brevity further considerations are omitted.

Rather it is significant to generalize this result obtained form the definition of Π in one particular case only, where a was specifically identified by (2.8). To this aim, calculate (2.7) writing in general

$$a(r)=\sum_j \frac{\xi_j}{r^j} \quad r>0; \tag{2.12}$$

the inequality excludes the divergence for $r=0$. This chance could be a series expansion of a or, more rationally, a linear combination of possible definitions of $a_j(r)$ via appropriate coefficients ξ_j sharing the physical dimensions of (2.6). In quantum physics a linear combination of allowed states of a system via complex coefficients defines a possible state of a macroscopic system. This idea is here extended thinking that various parts of a macroscopic system are describable by several classical states $a_1=\xi_1/r$ or $a_2=\xi_2/r^2$ and so on; the real coefficients mean that, for example, various parts of a whole system are described by different dynamical states. Although (2.12) is in fact a particular way to concern a as a function of various r^{-j}, short considerations exemplify the outcomes of these concepts. Write thus (2.16) reads

$$\Pi=\frac{2}{c^2}\int a\mathrm{d}r=\frac{2}{c^2}\sum_j \xi_j\int a_j\mathrm{d}r \tag{2.13}$$

the sole requirement being that anyway $a_j=length/time^2$; e.g. $a_1=v_0^2/r$ or $a_2=mG/r^2$ or $a_3=r_0 mG/r^3$ and so on. Since Π is dimensionless, define thus dimensionless coefficients ξ_j as follows

$$\Pi=\sum_j \Pi_j=C+\frac{2}{c^2}\xi_1 v_0^2\int\frac{\mathrm{d}r}{r}+\frac{2}{c^2}\xi_2 mG\int\frac{\mathrm{d}r}{r^2}+\frac{2}{c^2}\xi_3 mGr_0'\int\frac{\mathrm{d}r}{r^3}+\frac{2}{c^2}\xi_4\int\frac{\mathrm{d}r}{r^4}+\cdots, \tag{2.14}$$

being C the integration constant. Seemingly this reasoning is a conceptual loop: starting from Π one defines its differential $\delta\Pi$ and next integrates a linear combination of $\delta\Pi_j$ to concern the respective a_j via the approximation (2.7) of (2.6), which in turn bring back again to Π_j. But precisely the principle of combination of several classical states yields physical information about the various a_j of the possible j_{th} states, some of which are exemplified at the right hand side of (2.14) and read

$$\Pi=C+\xi_1\frac{v_0^2}{c^2}\log\left(\frac{r^2}{r_0^2}\right)-\xi_2\frac{r_{bh}}{r}-\xi_3\frac{4r_0'}{m'c^2}F_N+\frac{2}{3c^2}\frac{\xi_4}{r^3}+\cdots \quad r_{bh}=\frac{2mG}{c^2} \quad F_N=G\frac{m'm}{r^2}. \tag{2.15}$$

The probabilistic meaning of Π corresponds to the various Π_j-th states definable with values between 0 and 1 by normalizing appropriately the coefficients ξ_j.

The various addends Π_j of (2.15) concurring to $\Pi=v^2/c^2$ are now considered separately in order to examine their content of physical information.

-The first term of (2.15) reads

$$\Pi_1=\frac{v_1^2}{c^2}=C+\xi_1 const\log\left(\frac{r^2}{r_0^2}\right) \quad const=\frac{v_0^2}{c^2} \Rightarrow \frac{r^2}{r_0^2}=\exp\left(-\frac{C}{\xi_1 const}\right)\exp\left(\frac{v_1^2/c^2}{\xi_1 const}\right).$$

In this example $v_1^2/c^2=a_1 r_1/c^2=r_1/r'$, having put $a_1=v_1^2/r_1$ and $r'=c^2/a_1$ by dimensional reasons; then, depending on the actual sign of ξ_1 in the first (2.15), write

$$\frac{r^2}{r_0^2}=U_0\exp\left(\pm\frac{r_1/\xi_0}{r'}\right)\quad U_0=\exp\left(-\frac{C}{\xi_0}\right)\quad \xi_0=|\xi_1|const;$$

taking the minus sign of ξ_1 to avoid divergent exponential, this result reads

$$\frac{r}{r_0}=\pm U_0'\exp\left(-\frac{r_1/\xi_0}{2r'}\right)\quad U_0'=\exp\left(\frac{C}{2\xi_0}\right). \tag{2.16}$$

In this equation there are three independent variables r, r', r_1, because the initial v_1 was arbitrary; it implies that the chance of two links between them is still compatible with one freedom degree, e.g. for r at the left hand side, to fulfill the exponential of r_1 and r' at the right hand side. Let us concern the way to express these links in the case where the exponential takes form $\exp(-r_1/constant)$ simply taking r' constant, which means $2r'\xi_0=r_0'$ and thus $a_1=c^2/r'$ constant too; this particular condition aims to obtain quickly recognizable results. Indeed dividing both sides of (2.16) by r_1 one finds

$$\frac{r}{r_0}=\pm U_0'\exp\left(-\frac{r_1}{r_0'}\right)\Rightarrow\frac{r/r_0}{r_1}=\pm\frac{U_0'}{r_1}\exp\left(-\frac{r_1}{r_0'}\right), \tag{2.17}$$

which reads

$$U_Y=\pm\frac{U_0'}{r_1}\exp\left(-\frac{r_1}{r_0'}\right) \tag{2.18}$$

Moreover, multiplying both sides of (2.17) by r_1^2/r_0', one finds

$$\frac{r_1}{r_0'}\frac{r}{r_0}=\frac{r''^2}{r_0'r_0}=\pm U_0'\frac{r_1}{r_0'}\exp\left(-\frac{r_1}{r_0'}\right)\quad r''^2=r_1r;$$

owing to the fact that r_1 and r are independent variables, r''^2 is in principle a new variable. The right hand side has a maximum at $r_{1_{max}}=r_0'$, thus

$$\left.\frac{r''^2}{r_0r_0'}\right|_{max}=U_0'\exp(-1)\Rightarrow\frac{1}{r''^2}=\frac{Constant}{r_0r_0'}; \tag{2.19}$$

i.e. the right hand side explains the meaning of the left hand side.

-With the minus sign, (2.18) has the form of the Yukawa potential; the independent r and r_1 define in fact via the proportionality factor r_0 a new variable taking the values calculable at the right hand side as a function of r_1.

-As concerns (2.19), the notation emphasizes the physical meaning of r''. Let r''^{-2} be the Gauss curvature $\mathcal{R}_G$, as it is in principle reasonable, and let $(r_0r_0')^{-1}$ consist of two constant values of this curvature. These latter can be regarded as characteristic values of any other $\mathcal{R}_G$ calculable via the arbitrary $(r_0')^{-1}$ and $(r_0)^{-1}$; this idea suggests that the specific $r_0'^{-1}$ and r_0^{-1} are the characteristic minimum and maximum curvatures related to the contextual r''^{-2}. Clearly all of this agrees not only with the interpretation of the constant $(r_0'r_0)|_{max}$ but also with the Gauss "theorema egregium", which appears itself as a natural corollary. Therefore the Yukawa potential is relevant and effective not only because of its short range character but also because it is linked to the space time curvature.

-The second addend Π_2 of (2.15) introduces the relevant result r_{bh} further concerned later; it is remarkable that precisely r_{bh} is obtained through the condition (2.12) of non-divergence. Take in this context the negative sign of the coefficient ξ_2 and consider the cases of integration constant equal to or different from 0. On the one hand this term has anyway the meaning of $(square\ velocity)/c^2$, which implies, taking $\xi_2 < 0$ and the integration constant equal to 0, the kink

$$\frac{v^2}{c^2} \Leftrightarrow \xi_2 \frac{r_{bh}}{r} \quad r_{bh} = \frac{2mG}{c^2}: \tag{2.20}$$

the definition of Π_2 makes v^2/c^2 expressible via the ratio r_{bh}/r.

On the other hand it is possible to write this term again with $\xi_2 < 0$ also as

$$\Pi_2 = \xi_2 r_{bh}\left(\frac{1}{r} + \frac{1}{r_0}\right) \tag{2.21}$$

being now $1/r_0$ integration constant of (2.15). This result involves the classical Young-Laplace curvature radius $\mathcal{R}$, which suggests that Π_2' should have itself a curvature meaning. Write then

$$\Pi_2' = \xi_2 r_{bh}\left(\frac{1}{r} + \frac{1}{r_0}\right) = \xi_2 r_{bh}\mathcal{R} \tag{2.22}$$

and note that

$$\lim_{r\to r_0} \Pi_2' = \frac{2r_{bh}}{r_0} = \frac{4mG/c^2}{r_0}. \tag{2.23}$$

To be short: it is well known that $4mG/c^2 r_0$ is the deflection angle $\delta\phi \equiv \Pi_2'$ of a light beam at a given distance r_0 from the source m of a gravity field.

-The third term Π_3 of (2.15) is regarded along with the higher order terms to obtain

$$F_N^{*gen} = G\frac{m'm}{r^2} + \frac{2}{3c^2}\frac{\xi_4}{r^3} + \cdots \quad \xi_3 = \frac{m'c^2}{4r_0'}: \tag{2.24}$$

the notation emphasizes that this is a generalized form of the classical Newton law F_N^*, which in principle can fit experimental values with any numerical accuracy. It is significant that the coefficient ξ_3 introduces the energy $m'c^2$. Also, putting $\xi_4 = \xi_4' r_0'$ in (2.24), one finds

$$\frac{1}{r_0'} = \frac{\xi_4'}{m'c^2} \quad F = G\frac{m'm}{r^2} + \frac{2}{3c^2}\frac{\xi_4'}{r^3} + \cdots; \tag{2.25}$$

It is interesting to acknowledge that, without introducing additional hypotheses, F is related via the constant length $r_{0'}$ to the rest energy $m'c^2$ of m' that generates the gravity field. Eventually it is worth noticing in particular that $\delta r = v\delta t$ of (2.6) suggests by analogy $r = v't + const$ and thus yields

$$F_N = G\frac{m'm}{(v't + const)^2}. \tag{2.26}$$

This form of Newton's law is more sensible than F_N^* of (2.15): the latter corresponds to an instantaneous action at a distance r, which worried Newton himself, the former implies the propagation rate v' of the gravitational propagation corresponding to the distance between interacting masses. Despite the Newton formula is anyway approximate once neglecting the higher order terms introduced as corollaries from the initial (2.12), (2.26) is physically rational. Nevertheless, as F_N results from a_3 of (2.14) only, trivial considerations on a further term $\xi_4 a_4(r)$ of (2.14) would give

$$\Pi = \cdots + F_N + \frac{\xi_4}{\left(v't + const'\right)^3} + \cdots \tag{2.27}$$

and so on for further terms of the linear combination.

The fact that the early Newton-like (2.24) includes actually further higher order terms, means that the third and fourth allowed states of (2.12) concur to the gravitational interaction state, which in turn contributes to the total probability function Π. This obvious consideration implies defining the corresponding potential energy directly related to (2.24)

$$U = U_0 - \frac{\xi_3}{r} + \frac{\xi_4}{r^2} + \cdots \tag{2.28}$$

In conclusion, even in lack of input information about what is moving and its specific dynamics, these outcomes consist of well known and recognizable results; in particular it is singular that the probabilistic approach yields r_{bh} in the frame of a non-diverging requirement. These short notes have shown that significant results are achieved simply introducing the function Π and implementing m and v as dynamical parameters regardless of their specific reference to a predefined physical problem. The extrapolation of these preliminary results to the cosmological frame is evident, e.g. to the MOND problem concerned later. Note in this respect that an immediate extension of (2.12) is obtained rewriting the coefficients ξ_j without loss of generality as $\xi_2 = \xi_1^2 \xi_2'$ and so on. It follows

$$a = \frac{\xi_1}{r} + \frac{\xi_1^2 \xi_2'}{r^2} + \frac{\xi_1^3 \xi_3'}{r^3} + \cdots \Rightarrow a = a_1 + \xi_2' a_1^2 + \xi_3' a_1^3 + \cdots : \tag{2.29}$$

the first replica of (2.12) merely renames the coefficients, the second one truncated at the second order emphasizes the standard form of the MOND model as a corollary of this section.

To this purpose consider the last (2.29); neglecting for simplicity the higher order terms, rewrite (2.12) more expressively as

$$a = a_1 + \frac{\xi_2''}{a_0} a_1^2 + \cdots \quad \xi_2' \equiv \frac{\xi_2''}{a_0}, \tag{2.30}$$

being a_0 a constant acceleration.

On the one hand, regarding reasonably a_1 as the standard Newtonian acceleration of a rotating gravitational system, $a_N = v_1^2/r$ once more, this result reads

$$a = \frac{v_1^2}{r} + \xi_2'' \frac{\left(v_1^2/r\right)^2}{a_0} + \cdots : \tag{2.31}$$

clearly at large r the first term overcomes the second one, whereas the opposite holds at small r. Note that owing to the first (2.29) even $v_1 = const$ is acceptable. On the other hand, regarding consequently a as due to a circular motion of any mass in the field of a source mass M, (2.31) implies

$$a = \frac{MG}{r^2} = \frac{v_1^2}{r} + \xi_2'' \frac{\left(v_1^2/r\right)^2}{a_0} + \cdots \tag{2.32}$$

First of all rewrite identically (2.32) splitting the second addend as follows

$$\frac{MG}{r^2} = \frac{v_1^2}{r} + \zeta_1 \frac{\left(v_1^2/r\right)^2}{a_0} + \zeta_2 \frac{\left(v_1^2/r\right)^2}{a_0} + \cdots \quad \zeta_1 + \zeta_2 = \xi_2'', \tag{2.33}$$

which in turn implies two equations

$$\frac{MG}{r^2} - \zeta_1 \frac{\left(v_1^2/r\right)^2}{a_0} = \frac{v_1^2}{r} + \zeta_2 \frac{\left(v_1^2/r\right)^2}{a_0} + \cdots \quad \zeta_1 + \zeta_2 = \xi_2''. \tag{2.34}$$

This result reads $\left(MG - \zeta_1 v_1^4/a_0\right)/r^2 = v_1^2/r + \zeta_2 v_1^4/\left(a_0 r^2\right) + \cdots$. The left hand side yields v_1, whereas the right hand side is calculated implementing v_1 just found. The necessity of splitting the coefficient ξ''_2 into ζ_1 and ζ_2 of (2.33) is evident: writing directly (2.32) in the form $MG - \xi_2'' v_1^4/a_0 = 0$ would have implied $v_1^2/r = 0$, which holds at $r = \infty$ only. So, neglecting the higher order terms, trivial calculations yield

$$v_1 = \left(\frac{MGa_0}{\zeta_1}\right)^{1/4} \quad a = \frac{\sqrt{a_0 MG/\zeta_1}}{r} + \frac{2MG}{r^2} + \frac{\zeta_2 MG}{\zeta_1 r^2}, \tag{2.35}$$

i.e. a has the form $const_1/r + const_2/r^2 + \cdots$ expected in agreement with the first (2.29) written with these coefficients: very large r imply constant v_1 and acceleration decreasing like r^{-1}, the last two terms have instead mere Newtonian character prevalent at short r.

Equations (2.30) to (2.35) anticipate the outcomes on the MOND theory of the next section 5.2. It is worth noticing that these results are not mere corollaries able to explain the specific problem of rotational behavior of galaxies; the concept of galaxy is totally out of the conceptual frame hitherto introduced, although it is evident that the space scale of a galaxy is appropriate to make knowledgeable the astrophysical relevance of (MD7). All terms of (2.12) have their own physical meaning: e.g. (2.28) extends the physical meaning of potential energy beyond the Newtonian r^{-1} form, (2.29) is one among a variety of results already exposed above and further extended later.

All of this is interesting for two reasons:

- U of (2.28) includes the expected first order Newtonian potential $U_N = r^{-1}$ plus a second order term r^{-2} and higher order terms. At the first order, (2.28) and (2.31) yield $m'v^2 = -m'mG/r + const$, which fulfills $2\langle \mathrm{T} \rangle = \langle U \rangle$; also, whatever m might be, $v = r/t$ in (2.28) yields $r^3/t^2 \approx const$. Eventually $v^2/r = mG/r^2$, centripetal acceleration of mass m in the force field F_N of m', yields

$\nabla \cdot \boldsymbol{field} = -mG\nabla \cdot \boldsymbol{r}/r^3 = -3Gm/r^3$ *i.e.* $\boldsymbol{\phi} = \boldsymbol{F}/m'$ and $\nabla \cdot \boldsymbol{\phi} = -4\pi G\rho$ with $\rho = m/V$ and $V = (4/3)\pi r^3$. These cornerstones of Newtonian physics, although approximated, deserve however being sketched in view of the MOND approach concerned later.

-The most interesting term of (2.28) is surely r^{-2}. It is known [1] that the Newtonian potential cannot calculate properly the perihelion precession of planets, the correct calculation requires a second order potential term. However the Newton potential cannot account itself for this additional r^{-2} dependence, which instead is justified in the frame of the general relativity. Nevertheless this second order term also appears in (2.28) in a natural way owing to the initial (2.12) and (2.24): this approach does not postulate "a priori" the Newton law, which instead is inferred itself as a first order approximation as shown in (2.15). In fact, the higher order potential terms that in turn imply the correct perihelion precession, which therefore can be regarded as a further corollary of (2.13). For sake of brevity further details on this point, well explained in the quoted textbook, are omitted.

It is instead worth emphasizing that all previous considerations do not necessarily hold for $a = const/r^j$ only or combination of various a_j; rather, it is significant to generalize the basic idea of regarding the dynamical variables via their own physical meaning proposing a further definition of acceleration. After having generalized (2.8) via (2.12), examine now the additional chance to this purpose implementing $a = r^2\omega^3/c$ to explain the Einstein formula on the gravitational orbital energy loss contextual to the emission of gravitational waves. For brevity, start from the well known formula

$$-\frac{\delta E}{\delta t} = \frac{32G}{5c^5}\left(\frac{m_1 m_2}{m_1 + m_2}\right)^2 r^4\omega^6; \tag{2.36}$$

a simple elaboration is useful to better highlight the physical meaning of the equation: for reasons clear shortly, rewrite the integration factor as $32/5 \approx 2\pi$, the deviation being a few % only. Also, rewrite the term in parenthesis as $\mu^{-1} = 1/m_1 + 1/m_2$ and note that

$$a = \frac{r^2\omega^3}{c} \tag{2.37}$$

has physical dimension of acceleration. Recalling eventually that $\delta t = \hbar/\delta\epsilon$, trivial manipulations of this formula yield $-\delta E\delta\epsilon = (\mu a \ell_{Pl})^2$, being $\ell_{Pl}^2 = hG/c^3$, consistent with both signs of a. Therefore this result must be intended as

$$-\delta E = \mu|a|\ell_{Pl} \quad \delta\epsilon = \mu|a|\ell_{Pl} \quad \text{or} \quad \delta E = \mu|a|\ell_{Pl} \quad -\delta\epsilon = \mu|a|\ell_{Pl} \tag{2.38}$$

in agreement with the sign of (2.36). Thus two energy changes, δE and $\delta\epsilon$ with alternate meaning of source and dissipation terms, are related to the effects energy amounts $\mu|a|\ell_{Pl}$ that propagate as waves; this pint is better clarified in the next section 4 with the help of the quantum uncertainty. Instead appears contextually evident the meaning of a: it is clear that implementing the particular a of (2.37) into the definition of energy resulting from the classical definition *reducedmass* × *acceleration* × *Plancklength*, the reverse path from (2.38) to (2.36) brings back via (2.37) to the Einstein formula.

Eventually note how physical information is gained merging the general definitions of velocity $\boldsymbol{v} = \delta \boldsymbol{s}/\delta t$ and acceleration $\dot{\boldsymbol{v}} = \delta \boldsymbol{v}/\delta t$ even without hypothesizing any analytical form of acceleration; the steps consider two components v_x and v_y of $\boldsymbol{v}$ as follows

$$\frac{\delta v_x}{\delta t} = \frac{\delta}{\delta t}\frac{\delta x}{\delta t} = \frac{\delta^2 x}{\delta t^2} \quad v_y = \frac{\delta y}{\delta t} \quad \delta t^2 = \frac{\delta y^2}{v_y^2} \quad \frac{\delta v_x}{\delta t} = \frac{\delta^2 x}{\delta y^2/v_y^2} \Rightarrow 0 = \frac{\delta^2 x}{\delta t^2} - v_y^2 \frac{\delta^2 x}{\delta y^2}: \tag{2.39}$$

the first step defines δv_x, the second and third δv_y, next replacing δt^2 and subtracting side by side one finds the d'Alembert wave equation.

This section has outlined some preliminary considerations introducing the basic essence of the physical model proposed in the following of this paper. This exposition however cannot be considered closed without two further remarks about how to regard the mass in this conceptual frame rooted on Π and how to show that even $1-\Pi$ completes itself the probabilistic physical meaning of the theoretical approach hitherto configured.

Consider the constant e^2/G noting that its physical dimensions are $mass^2$. Define then this mass as $\mathcal{M}_0^2 = e^2/G$ and divide both sides by $\mathcal{M}_0 c^2$; the result is

$$\frac{\mathcal{M}_0 G}{c^2} = \ell_0 = \frac{e^2}{\mathcal{M}_0 c^2} \tag{2.40}$$

The left hand side has physical dimensions of *length*, the result at the right hand side has a well known form; both $\mathcal{M}_0$ and ℓ_0 are fixed values. Multiply now both sides by a proportionality coefficient ξ and rewrite (2.40) as $r_e = e^2/m_e c^2$ with $m_e = \mathcal{M}_0/\xi$: identifying in particular $\mathcal{M}_0/\xi = m_e$ with the electron mass, the right hand side corresponds to the classical electron radius. On the one hand is reasonable the attempt to extend this result to the case of the proton. On the other hand, however, there is no reason to assume that (2.40) still holds for the proton too: the electron is an elementary particle, the proton does not. Hence rises the question: is this approach extensible to the proton and possibly even to neutral particles too, e.g. the neutron?

To generalize (2.40) define $\mathcal{M}_{kin} = \mathcal{M}_0/\beta$, being β an appropriate function that appears here as a proportionality factor replacing the aforesaid constant ξ. Repeating the steps with $\mathcal{M}_{kin}$, the result is now

$$\frac{\mathcal{M}_{kin} G}{c^2} = r = \frac{e^2}{\mathcal{M}_{kin} c^2} \quad \mathcal{M}_{kin} = \frac{\mathcal{M}_0}{\beta} \quad \beta^2 = 1 - \Pi. \tag{2.41}$$

Thus, whatever the initial $\mathcal{M}_0$ and ξ might be, write for electron and proton

$$r_{el} = \frac{e^2}{\mathcal{M}_{el} c^2} \quad \mathcal{M}_{el} = \frac{\mathcal{M}_{0el}}{\beta_{el}} \quad \beta_{el} = \beta(v_{el}) \quad r_{pr} = \frac{e^2}{\mathcal{M}_{pr} c^2} \quad \mathcal{M}_{pr} = \frac{\mathcal{M}_{0pr}}{\beta_{pr}} \quad \beta_{pr} = \beta(v_{pr}), \tag{2.42}$$

reminding once more that according to the previous considerations v_{el} and v_{pr} must be regarded as kinetic parameters having contextualized meaning, and not as properties of something really displacing for any reason to be specified. It follows thus

$$\frac{r_{el}}{r_{pr}} = \frac{\mathcal{M}_{kin,el}}{\mathcal{M}_{kin,pr}} = \frac{\mathcal{M}_{0el}}{\mathcal{M}_{0pr}} \frac{\beta_{pr}}{\beta_{el}}. \tag{2.43}$$

This result suggests that the ratio r_{el}/r_{pr} does not depend upon the dimensional mass ratio only, it is affected by the factor β_{pr}/β_{el}. Moreover, regarding now e^2/c^2 as mere numerical factor, appears clear the chance of calculating (2.43) for the neutron too via M_{0ne} and β_{ne} exactly as done here

$$\frac{r_{ne}}{r_{pr}} = \frac{\mathcal{M}_{kin,ne}}{\mathcal{M}_{kin,pr}} = \frac{\mathcal{M}_{0ne}}{\mathcal{M}_{0pr}} \frac{\beta_{pr}}{\beta_{ne}}. \tag{2.44}$$

It is known that $r_{el}/r_{pr} \approx 3.2$ whereas $r_{ne}/r_{pr} \approx 1$, in agreement with the same orders of magnitude reasonably expected for the right hand sides of (2.43) and (2.44) and with the number of elementary constituents of proton and neutron; this guess, and not the real masses themselves, might be a possible key to explain these ratios. This topic is outside the purposes of this paper; although further considerations about m_{el}/m_{pr} and β_{pr}/β_{el} are omitted for brevity, it appears clear in principle the conceptual worth of this approach to describe the particle classical radii. Instead note that

$$\mathcal{M}_{kin} v = \frac{\mathcal{M}_0}{\beta} v = \frac{\mathcal{M}_0 c^2}{\beta} \frac{v}{c^2} \quad \mathcal{M}_{kin} = \frac{\mathcal{M}_0}{\beta} \Rightarrow \mathcal{P} = \frac{\mathcal{E} v}{c^2} \quad \mathcal{E} = \frac{\mathcal{M}_0 c^2}{\beta} \quad \beta = \beta(v), \tag{2.45}$$

which eventually implies $\partial(\mathcal{P}/\mathcal{E})\partial t = c^{-2}\partial v/\partial t = a/c^2$ and thus

$$v\frac{\partial(\mathcal{P}/\mathcal{E})}{\partial t} = \frac{v\partial v/\partial t}{c^2} = \frac{1}{c^2}\frac{\partial\left(v^2/2\right)}{\partial t} = -\frac{\partial\varphi}{c^2\partial t} \quad \varphi = -\frac{v^2}{2} \tag{2.46}$$

so that

$$av = \frac{\partial\left(v^2/2\right)}{\partial t} = -\frac{\partial\varphi}{\partial t} \Rightarrow a_r = -\frac{\partial\varphi}{\partial r} \quad \mathrm{d}r = v\mathrm{d}t. \tag{2.47}$$

It is worth stressing that $\mathcal{M}$ is mere dimensional mass, not a real measurable mass; P and E are momentum and energy corresponding to the virtual $\mathcal{M}$, analogous to the respective functions of m. Any classical mass can be measured assuming its position somewhere, this is however nonsensical for $\mathcal{M}$. Yet these elementary considerations have shown not only that the virtual mass $\mathcal{M}$ can be identified with and behaves like the ordinary mass, but also that $\mathcal{M}_{kin}$ allows an immediate explanation about why in general the classical radius of composite particles implies a more sophisticated description than that of the elementary particles.

Actually Π and β are the conceptual bases of a self contained physical model, rather than mere intuitions proposed because of their encouraging results. These ideas, in particular (2.47), are further concerned in the next sections aimed to discuss dark matter, dark energy and MOND dynamics. Here however it is worth mentioning an implication of (2.45) considering in particular (2.47) that yields

$$-\frac{\mathrm{d}\varphi}{c^2}=\frac{\mathrm{d}r}{\mathrm{d}t}\frac{\mathrm{d}v_r}{c^2}=\frac{v_r}{c^2}\mathrm{d}v_r=\frac{v_r^2}{c^2}\frac{\mathrm{d}v_r}{v_r}=\frac{v_r^2}{c^2}\frac{\mathrm{d}v_r}{r_0}\frac{r_0}{v_r}\Rightarrow -\frac{\mathrm{d}\varphi}{c^2}=\frac{v_r^2}{c^2}\frac{\mathrm{d}\nu}{\nu}\quad \nu=\frac{v_r}{r_0}; \qquad (2.48)$$

it is interesting to calculate the limit of the result for $v_r \to c$, which reads

$$\lim_{v_r\to c}\frac{\mathrm{d}\varphi}{c^2}=-\frac{\mathrm{d}\nu}{\nu} \qquad (2.49)$$

This result will be obtained again in the next section, without any concern to $\mathcal{M}$; this strategy aims to emphasize that the similarity of (2.45) with the properties of real observable matter is not accidental. For this reason (2.47) is found later. It is anticipated here that φ is the gravitational potential and in fact (2.49) is well known, it is the red/blue shift $\leftrightarrow$ energy loss/gain equation of a photon moving radially in the field of a gravitational real mass in vacuum.

This paper concerns a quantum model and implements quantum approach. The strategy of the paper is to introduce a conceptual basis general enough to infer the necessary information on which is rooted the cosmological nature of the universe. The notation δx_i^2 means $(\delta x_i)^2=(x_i-x_0)^2$, whereas $\delta(x_i^2)=x_i^2-x_0^2$. The text organized in order to be as self-contained as possible.

3. Preliminary Considerations: The β Function

After having examined Π, this section concerns $1-\Pi$ *i.e.*, owing to (2.41), β. It is known [2] that

$$\nabla\varphi=-\dot{\boldsymbol{v}}\quad \dot{\boldsymbol{v}}=\frac{\delta\boldsymbol{v}}{\delta t}, \qquad (3.1)$$

being $\dot{\boldsymbol{v}}$ acceleration and $\varphi=\varphi(r,t)$ a function of space coordinates and time; e.g., if φ is defined by the classical Lagrangian as $\mathcal{L}=mv^2/2-m\varphi$, then (3.1) introduces the gravitational potential per unit mass m subjected to attractive field of another mass M. So it is also possible to define by consequence

$$\boldsymbol{v}\cdot\nabla\varphi=-\boldsymbol{v}\cdot\dot{\boldsymbol{v}} \qquad (3.2)$$

and then

$$\boldsymbol{v}\cdot\dot{\boldsymbol{v}}=\frac{1}{2}\frac{\delta(\boldsymbol{v}\cdot\boldsymbol{v})}{\delta t}=\frac{\delta(v^2/2)}{\delta t}=-\dot{\varphi}\quad \varphi=-\frac{v^2}{2}+const. \qquad (3.3)$$

Note that in fact all quantities of (3.1) to (3.3) are functions of x,y,z,t, as suggested by ∇; however the second (3.1) emphasizes the t-dependence only, which shortens and simplifies the notation $\varphi=\varphi(x,t)$. Also note that (3.1) has been already found in the more general (2.47), whose sign has not been specified without information about $\partial v/\partial t$. The rest of this section deals with the possible implications of (3.1) to (3.3). First of all (3.3) implies

$$\frac{\delta\varphi}{c^2}=-\frac{1}{2}\delta\frac{v^2}{c^2}=\frac{1}{2}\delta\left(1-\frac{v^2}{c^2}\right), \qquad (3.4)$$

which evidences the connection of this section with the section 1, whence

$$\frac{\delta\varphi}{c^2}=\frac{1}{2}\delta\left(\beta^2\right)=\beta\delta\beta \;\Rightarrow\; \frac{\varphi}{c^2}=\frac{\beta^2}{2}+const' \quad \beta=\pm\sqrt{1-\frac{v^2}{c^2}}; \tag{3.5}$$

it also implies

$$\nabla\frac{\varphi}{c^2}=\beta\nabla\beta=-\frac{1}{2}\nabla\frac{v^2}{c^2}, \tag{3.6}$$

while the connection with the results of the previous section is given by

$$\beta^2=1+\frac{2\varphi}{c^2}+const=1-\Pi \;\Rightarrow\; 2\beta\delta\beta=2\frac{\delta\varphi}{c^2}=-\delta\Pi. \tag{3.7}$$

Since $(c\beta)^2=c^2-v^2$ owing to (3.5), multiplying both sides by δt^2 one finds

$$\left(c\beta\delta t\right)^2=\delta\ell_{inv}^2=c^2\delta t^2-\delta x^2 \quad \delta x=v\delta t. \tag{3.8}$$

It is shown in [2] that this invariant is enough to infer the special relativity. The fact of having inferred (3.8) assures in principle that any result of special relativity could be regarded and implemented via the conceptual frame hitherto introduced; it holds of course also for the basic assumptions of the special relativity, e.g. the value of c constant and independent on the motion speed of a light source. Nevertheless the next subsection shows how (3.1) directly brings itself to some more general implications. An example is given here writing

$$\beta^2=\frac{c^2-v^2}{c^2}=\frac{\ell_+\ell_-}{c^2\delta t^2} \quad \ell_+=c\delta t+\delta x \quad \ell_-=c\delta t-\delta x \quad \delta\ell=c\delta t \quad \delta x=v\delta t$$

whence

$$\left(c\beta\delta t\right)^2=\ell_+\ell_-; \tag{3.9}$$

as $(c\beta\delta t)$ of (3.8) is invariant, these results imply

$$\ell_+\ell_-=\ell_{0+}\ell_{0-}=\ell_{0+}\beta\frac{\ell_{0-}}{\beta} \;\Rightarrow\; \ell_+=\ell_{0+}\beta \quad \ell_-=\frac{\ell_{0-}}{\beta}. \tag{3.10}$$

In particular note that $\ell_+/c=time$, so that it is possible to write

$$\frac{\ell_-}{c}=t=\frac{\ell_{0-}/c}{\beta}=\frac{t_0}{\beta}. \tag{3.11}$$

Equations (3.10) and (3.11) introduce space contraction and time dilation with respect to the respective proper values; if in particular $v=const$ in (3.5), then these results are known as the Lorentz transformations in different inertial reference systems moving a relative velocity $v=const$.

Also note that β implies

$$-\frac{\beta\delta\beta}{\delta v}=\frac{v}{c^2}=\frac{(m/\beta)v}{(m/\beta)c^2}=\frac{p}{\epsilon} \quad p=\frac{mv}{\beta} \quad \epsilon=\frac{mc^2}{\beta}, \tag{3.12}$$

being

$$\epsilon\beta=\epsilon_0=mc^2 \quad \beta\delta\epsilon=-\epsilon\delta\beta. \tag{3.13}$$

Let ϵ_0 be the energy in a reference system where one particle is at rest, whereas ϵ_0/β is the kinetic energy in a different reference system; then, owing to (3.8) and (3.13), one finds contextually

$$\left(mc^2\right)^2 = \epsilon^2\beta^2 = \epsilon^2 - \epsilon^2\frac{v^2}{c^2} \Rightarrow \epsilon^2 = \epsilon^2\frac{v^2}{c^2} + \left(mc^2\right)^2 = \left(pc\right)^2 + \left(mc^2\right)^2 \quad p = \frac{\epsilon v}{c^2} \tag{3.14}$$

Consider now any velocity $v_b > v$; defining thus $v_b/\beta = v$ write

$$v_b = \frac{v}{\beta} = \frac{v}{\sqrt{1-v^2/c^2}} \Rightarrow v = \frac{v_b}{\sqrt{1+\frac{v_b^2}{c^2}}} \quad v_b > v, \tag{3.15}$$

having merely solved the first equation with respect to v. The result (3.15) reads, in the particular case of constant acceleration,

$$v = \frac{a_b t}{\sqrt{1+\frac{\left(a_b t\right)^2}{c^2}}} \quad v_v = a_b t. \tag{3.16}$$

To infer further information on φ/c^2, define the dimensional relationship $\epsilon\boldsymbol{v} = \hbar\dot{\boldsymbol{v}} + \boldsymbol{const}$ and express $\dot{\boldsymbol{v}}$ via (3.1), which yields $\epsilon\boldsymbol{v} - \boldsymbol{const} = \hbar\dot{\boldsymbol{v}} = -\hbar\nabla\varphi$; the arbitrary vector $\boldsymbol{const}$ has been added to $\dot{\boldsymbol{v}}$ in order that ϵ does not vanish for $\boldsymbol{v} = const$. Then one finds

$$\varepsilon\boldsymbol{v} = \hbar\left(\boldsymbol{a}_0 + \dot{\boldsymbol{v}}\right) = \hbar\left(\boldsymbol{a}_0 - \nabla\varphi\right). \tag{3.17}$$

Three simple considerations show that (3.17) is sensible.

(i) The right hand side of (3.17) defines the scalars

$$\varepsilon\frac{v^2}{c^2} = \hbar\boldsymbol{v}\cdot\frac{\boldsymbol{a}_0 - \nabla\varphi}{c^2} = \hbar\omega \quad \omega = \boldsymbol{v}\cdot\frac{\boldsymbol{a}_0 - \nabla\varphi}{c^2}. \tag{3.18}$$

On the one hand this result reads

$$\boldsymbol{p} = \varepsilon\frac{\boldsymbol{v}}{c^2} = \hbar\boldsymbol{u}\cdot\frac{\boldsymbol{a}_0 - \nabla\varphi}{c^2} = \frac{\hbar\omega}{v} \quad \boldsymbol{u} = \frac{\boldsymbol{v}}{v} \tag{3.19}$$

and thus also

$$p = \frac{\varepsilon v}{c^2} \Rightarrow p = \frac{h\nu}{v} = \frac{h}{\lambda} \quad v = \lambda\nu; \tag{3.20}$$

on the other hand it implies by dimensional reasons

$$\omega_0 = \boldsymbol{v}\cdot\frac{\boldsymbol{a}_0}{c^2} \tag{3.21}$$

so that (3.17) yields

$$\omega - \omega_0 = -\boldsymbol{v}\cdot\frac{\nabla\varphi}{c^2} = -\omega'\frac{\delta\varphi}{c^2}. \tag{3.22}$$

Indeed, regarding one component of the scalar, e.g. $(\delta x/\delta t)(\delta\varphi/\delta x)$, the last (3.22) reads $\delta\omega \approx -\omega'\delta\varphi/c^2$ with $\omega' = 1/\delta t$, *i.e.*

$$\frac{\delta\omega}{\omega'} = -\frac{\delta\varphi}{c^2}. \tag{3.23}$$

in agreement with (2.49) and related considerations about $\mathcal{M}$.

(ii) It is also immediate to extend this result to the case of any massive particle, or photon moving radially at $v<c$ in a medium with refractive index n. It is convenient to write first

$$t=\frac{t_p}{\beta} \quad \frac{h}{t_p}=\epsilon_0=\frac{h}{t\beta}=\epsilon\beta \quad \epsilon\beta^2=\frac{h}{t} \quad \epsilon=\frac{\epsilon_0}{\beta}, \tag{3.24}$$

whence

$$\frac{h\beta}{t_p}=\frac{h}{t}\epsilon=\epsilon^2\beta^2=\epsilon^2-\epsilon^2\frac{v^2}{c^2}=\epsilon^2-(pc)^2=\left(mc^2\right)^2. \tag{3.25}$$

Thus

$$\delta t=\frac{\delta t_p}{\beta}-\frac{t_p}{\beta^2}\delta\beta=\frac{\hbar}{\delta\varepsilon} \Rightarrow \delta\beta=\frac{\delta t_p}{t_p}\beta-\frac{\hbar/t_p}{\delta\varepsilon}\beta^2=\frac{\delta t_p}{t_p}\beta-\frac{\hbar/t}{\delta\varepsilon}\beta$$

yields

$$\beta^2=\frac{h/\epsilon}{t} \quad \frac{\delta\beta}{\beta}=\frac{\delta t_p}{t_p}-\frac{\varepsilon}{\delta\varepsilon} \Rightarrow \frac{\delta\varphi}{c^2}=\beta^2\frac{\delta\beta}{\beta}=-\beta^2\left(\frac{\varepsilon}{\delta\varepsilon}-\frac{\delta t_p}{t_p}\right)=-\frac{h}{\epsilon}\left(\frac{\varepsilon}{t\delta\varepsilon}-\frac{\delta t_p}{t_p t}\right),$$

which eventually identifies by dimensional reasons

$$\frac{\delta\varphi}{c^2}=-\frac{\nu-\nu'}{\nu''} \quad \nu=\frac{\varepsilon/\delta\varepsilon}{t} \quad \nu'=\frac{\delta t_p/t_p}{t} \quad \nu''=\frac{\epsilon}{h} \tag{3.26}$$

analogous to that already found. Here the frequency $\nu=v/\lambda$ refers to the wave-like behavior of massive particles having momentum h/λ. Note that implementing the uncertainty equation $\delta\epsilon=n\hbar/\delta t$ the actual result would be $\nu=n\varepsilon/(t\delta\varepsilon)$; however, even without introducing the quantization, an acceptable result has been obtained merely considering the dimensional relationship $\varepsilon=h/t$.

(iii) Equation (3.17) yields contextually

$$-\hbar\frac{\nabla\varphi}{c^2}=\boldsymbol{p}-\boldsymbol{const} \quad \boldsymbol{p}=\frac{\epsilon\boldsymbol{v}}{c^2} \quad \boldsymbol{const}=\frac{\hbar\boldsymbol{a}_0}{c^2}, \tag{3.27}$$

while being also

$$\boldsymbol{v}\cdot\nabla\frac{\varphi}{c^2}=\frac{\xi}{\delta t} \Rightarrow \epsilon=\hbar\boldsymbol{v}\cdot\nabla\frac{\varphi}{c^2} \quad \boldsymbol{p}=\hbar\left(\boldsymbol{v}\cdot\nabla\frac{\varphi}{c^2}\right)\frac{\boldsymbol{v}}{c^2}. \tag{3.28}$$

At this point is also interesting the fact that (3.1) implies itself

$$\dot{\boldsymbol{v}}\cdot\nabla\varphi=-\dot{v}^2 \Rightarrow \hbar^2\dot{\boldsymbol{v}}\cdot\nabla\frac{\varphi}{c^2}=-\hbar^2\frac{\dot{v}^2}{c^2}=-(\hbar\omega)^2 \quad \omega=\frac{\dot{v}}{c}; \tag{3.29}$$

thus, taking the square root of this result, one finds

$$\hbar\frac{\delta(v/c)}{\delta t}=\pm\epsilon \quad \epsilon=\hbar\omega. \tag{3.30}$$

On the one hand owing to (3.27) the second (3.29) implies

$$p=\frac{\epsilon}{c^2/v}=\frac{\hbar\omega}{c^2/v}=\frac{h\nu}{c^2/v}=\frac{h}{\lambda} \quad \lambda=\frac{c^2}{v\nu}=\mathrm{n}\frac{c}{\nu} \quad \mathrm{n}=\frac{c}{v}, \tag{3.31}$$

where the definition of λ is justified in turn putting $\mathrm{n}=c/v$. As a check, implementing this wave formulation of momentum (3.31), one finds again (3.23)

$$-\frac{\delta\varphi}{c^2}=\frac{\delta p}{p}=\frac{h\delta(c/c\lambda)}{p}=\frac{h\delta(v/c)}{p}=\frac{\delta(hv/c)}{h/\lambda}=\frac{\delta v}{c/\lambda}=\frac{\delta v}{v}. \quad (3.32)$$

Implement these results to calculate $\delta(pc)$ and $\delta\epsilon$, noting that $\delta\epsilon/\delta p$ has physical dimensions of velocity, say v_g. Merging (3.30) and (3.31), one finds

$$\frac{c}{v_g}=\frac{\delta(pc)}{\delta\epsilon}=\frac{h\delta(c/\lambda)}{h\delta\nu}=\frac{h\delta(c/v)(v/\lambda)}{h\delta\nu}=\frac{\delta(\mathrm{n}\nu)}{\delta\nu} \quad \nu=\frac{v}{\lambda}$$

whence

$$v_g=\frac{c}{\delta(\mathrm{n}\nu)/\delta\nu}, \quad (3.33)$$

which agrees with the result introduced in the section 1.

On the other hand, look now for a more general expression of $\boldsymbol{v}(t)$ solving the first (3.17) with respect to $\dot{\boldsymbol{v}}$ and $\boldsymbol{v}$ of (3.27) to calculate the components $v=\boldsymbol{v}\cdot\boldsymbol{u}$ and $\dot{v}=\dot{\boldsymbol{v}}\cdot\boldsymbol{u}$ of $\boldsymbol{v}$ and $\dot{\boldsymbol{v}}$ along an arbitrary direction defined by the constant unit vector $\boldsymbol{u}$: multiplying both sides of the first equality (3.17) by $\boldsymbol{u}$, elementary calculations yield

$$\begin{gathered}\hbar\frac{\delta\boldsymbol{v}}{\delta t}=\epsilon\boldsymbol{v}-\varepsilon_0\boldsymbol{v}_0 \quad \epsilon=\epsilon(x,t)\\ \Rightarrow \boldsymbol{v}(x,t)=\left(-\frac{\varepsilon_0\boldsymbol{v}_0}{\hbar}\int \mathrm{e}^{-\zeta(t)}\mathrm{d}t+\boldsymbol{v}_0'\right)\mathrm{e}^{\zeta(t)} \quad \zeta(t)=\int\frac{\epsilon(t)}{\hbar}\mathrm{d}t,\end{gathered} \quad (3.34)$$

being $\boldsymbol{v}_0'$ integration constant. Moreover (3.34) rewritten as

$$\frac{\delta\boldsymbol{v}}{\delta t}=\frac{\epsilon(t)}{\hbar}\left(-\frac{\varepsilon_0\boldsymbol{v}_0}{\hbar}\int \mathrm{e}^{-\zeta(t)}\mathrm{d}t+\boldsymbol{v}_0'\right)\mathrm{e}^{\zeta(t)}-\frac{\epsilon_0}{\hbar}\boldsymbol{v}_0 \quad (3.35)$$

is quantum equation of acceleration. Check for the validity of these last results noting that neither $\boldsymbol{v}_0$ nor ζ are "a priori" determined; rather in principle v and its time derivative are determined by $\epsilon(x,y,z,t)$, which is a free parameter as a function of which are calculable further results of physical interest as corollaries. In particular, nothing hinders to think that (3.34) can fit the standard expression of relativistic acceleration, e.g. [2], or even v_g itself of (3.33), via an appropriate best fit function $\epsilon(x,y,z,t)$. The notation emphasizes the time dependence of $\zeta(x,y,z,t)$ necessary to calculate the time integral coming from (3.3); of course the time integration constant $\boldsymbol{v}_0=\boldsymbol{v}_0(x,y,z)$ can actually be function of the coordinates likewise $\boldsymbol{v}_0'=\boldsymbol{v}_0'(x,y,z)$.

In this respect it is worth calculating the general result (3.34) in the following particular cases.

(i) The first case concerns $\epsilon(t)=const$, thus (3.34) becomes

$$\begin{gathered}\epsilon(t)\equiv-\epsilon_0 \Rightarrow \zeta=\frac{-\epsilon_0 t}{\hbar}\\ \Rightarrow v(t)=v_0\exp(-\epsilon_0 t/\hbar)+v_0\frac{\hbar/t_0}{\epsilon_0} \quad \dot{v}(t)=-\frac{v_0\epsilon_0}{\hbar}\exp(-\epsilon_0 t/\hbar);\end{gathered} \quad (3.36)$$

i.e. $v(t)$ does not diverge, it changes in the interval $v(t=0)=v_0'+\hbar const/\epsilon_0$ to $v(t=\infty)=\hbar const/\epsilon_0$. This result shows that defining appropriately the constants in (3.36), at least this one case does in fact exist that implies reasonably

$$0\le v(t)\le c; \tag{3.37}$$

moreover (3.36) also shows the necessity of taking the minus sign of (3.30) to avoid divergent $v(t)$. Actually this conclusion could be expected in general via (3.4) and (3.5) itself. Consider indeed two arbitrary velocities v and v' such that $\delta v'\le\delta v$ and write the identity

$$\delta v\equiv\frac{\beta^2\delta v}{\beta^2}=\frac{\delta v'}{\beta^2}\quad \delta v=v_2-v_1\quad \delta v'=v_2'-v_1'=v_2\beta^2-v_1\beta^2;$$

putting $v^2=v_1'v_2'$, these definitions of v_2' and v_1' yield

$$\delta v=\frac{v_1'-v_2'}{1-v_1'v_2'/c^2}: \tag{3.38}$$

reasonable conditions on v_1' and v_2' are enough to make β compliant with a well known formula of addition of velocities. If either $v_1'=c$ or $v_2'=c$, then $\delta v=\pm c$; of course this holds even though $\delta v\equiv v$ when in particular $v_1=0,v_2=c$ or $v_1=c,v_2=0$.

(ii) imaginary ζ.

Calculate $\zeta(x,t)$ of (3.34) regarding $\epsilon(x,t)$ as an imaginary energy with either possible chance $\pm i\epsilon(x,t)$. It is immediate to show by direct calculation that even this replacement can be interpreted as a sensible solution the first (3.34); the result of this replacement is indeed the following equation to be fulfilled

$$\pm\hbar i\frac{\delta \boldsymbol{v}(x,t)}{\delta t}=\boldsymbol{v}\epsilon(x,t)\pm i\boldsymbol{v}_0\epsilon_0. \tag{3.39}$$

It is instructive to make now an approximation concerning this result: put $v_0=0$, which means $a_0=0$ in (3.17). Replace next $\delta\to\partial$, to obtain the following differential equations corresponding to either sign of (3.39)

$$\hbar i\frac{\partial v(x,t)}{\partial t}=\boldsymbol{v}\epsilon(x,t)\quad -\hbar i\frac{\partial v(x,t)}{\partial t}=\boldsymbol{v}\epsilon(x,t).$$

Now multiply both sides of the first equation by an appropriate $\boldsymbol{f}=\boldsymbol{f}(x)$ to define the scalar $\boldsymbol{v}\cdot\boldsymbol{f}=\psi$; the result is

$$\hbar i\frac{\partial\psi}{\partial t}=\epsilon(x,t)\psi. \tag{3.40}$$

Proceed analogously with the second equation; (3.34) and (3.39) yield

$$-\hbar i\frac{\delta\left(\boldsymbol{v}(x,t)/c^2\right)}{\delta t}=\frac{\boldsymbol{v}(x,t)\epsilon(x,t)}{c^2}\ \Rightarrow\ -\hbar i\frac{\delta\left(\boldsymbol{v}(x,t)/c\right)}{c\delta t}=\frac{\boldsymbol{v}(x,t)}{c}\frac{\epsilon(x,t)}{c};$$

now multiplying again $\boldsymbol{v}/c$ by $\boldsymbol{f}$ at both sides, one finds

$$\frac{\hbar}{i}\frac{\partial\psi}{\partial\ell}=p(\ell,t)\psi\quad \delta\ell=c\delta t\quad p(x,t)\approx\frac{\epsilon(x,t)}{c}. \tag{3.41}$$

Clearly (3.40) and (3.41) are inherently approximate, being obtained replacing $\delta \to \partial$; moreover the last (3.41) is justified by (3.14), as $\epsilon^2 \approx (pc)^2$ once neglecting $(mc^2)^2$ and expressing the arbitrary x space coordinate via ct in order that $\psi(x,t) \to \psi(\ell,t)$.

The results (3.40) and (3.41) are well known and confirm the general validity of (3.34).

(iii) Quantum meaning of β.

As concerns the quantum implications of the β function, consider the following steps to infer further corollaries. Write

$$\beta^2 = 1-\frac{v^2}{c^2} = 1-\frac{v^2\delta t^2}{c^2\delta t^2} = 1-\frac{\delta r^2}{\delta\ell^2} = \frac{\delta\ell^2-\delta r^2}{\delta\ell^2} = \frac{\delta\ell-\delta r}{\delta\ell}\frac{\delta\ell+\delta r}{\delta\ell}$$
$$\Rightarrow \beta^2 = \frac{\delta\ell-\delta r}{\delta\ell}\left(1+\frac{\delta r}{\delta\ell}\right),$$

which in turn yields

$$\beta^2 = \frac{\delta\ell-\delta r}{\delta\ell}+\frac{\delta\ell-\delta r}{\delta\ell}\frac{\delta r}{\delta\ell} \quad \delta\ell = c\delta t \quad \delta r = v\delta t. \tag{3.42}$$

Let $\delta\ell = \ell_2-\ell_1$ be the total size of the uncertainty range in an arbitrary reference system where is delocalized one quantum particle, and let $\delta\ell$ consist of two ideal sub-ranges $\delta\ell-\delta r$ and $\delta r = r_2 - r_1$: the first addend of (3.42) is the probability for the particle to be found in $\delta\ell-\delta r$ only, the second addend is the probability that the particle is simultaneously in $\delta\ell-\delta r$ and in δr. Is interesting the fact that β^2 introduces separate probabilities for a particle to be found in either sub-range or in both sub-ranges simultaneously. The latter is intuitively evident: if one particle is delocalized in the total $\delta\ell$, then it must be somewhere in both sub-ranges by definition; yet is interesting that also appears the probability for the particle to be in either of them as a specific event separately definable. The physical meaning of this reasoning is clear: it suggests the chance that the particle is a corpuscle, located somewhere in $\delta\ell$, but also a wave that spreads through and pervades all available $\delta\ell$: a wave is everywhere, the corpuscle is here or there.

On the one hand these probabilities imply their zero point energies $\epsilon_{(\ell-r)} = \delta p_{(\ell-r)}^2/2m$ and $\epsilon_r = \delta p_r^2/2m$ being m the mass of the corpuscle: indeed the uncertainty principle links δp_r and $\delta p_{(\ell-r)}$ to the respective δr and $\delta\ell-\delta r$.

On the other hand it happens regardless of any additional information, *i.e.* this would hold even though there is a potential wall of value ϵ_{thr} somewhere inside $\delta\ell$ and regardless of the actual energy of the particle, e.g. located at the boundary between $\delta\ell-\delta r$ and δr.

Clearly this reasoning has to do with the tunnel effect: e.g. if the arbitrary size of δr tends to zero, the local delocalization energy of the corpuscle diverges. In other words (3.42) is explainable if $\epsilon_r > \epsilon_{thr}$, which in fact is allowed by an appropriate δr even in the presence of a threshold energy ϵ_{thr} between $\delta\ell-\delta r$ and δr. In turn this conclusion means that the corpuscle, owing to its own zero point energy, has a non-null probability of being found in both sub-regions of $\delta\ell$ regardless of its initial energy before penetrating from $\delta\ell-\delta r$ into δr. As it holds even regardless of any information about ϵ, this result is the well known quantum tunneling of particles overcoming finite potential barrier, here inferred from as a straightforward corollary of the typical relativistic function $0\le\beta\le 1$ regarded via its probabilistic meaning without any "ad hoc" hypothesis.

Eventually it is also worth mentioning what kind of additional information can be deducted from φ/c^2 : as it is physically significant to examine not only $\boldsymbol{v}\cdot\nabla$ but also $\boldsymbol{v}\times\nabla$, write then the identity $\boldsymbol{v}\times\nabla\varphi\equiv e\boldsymbol{v}\times\nabla\varphi/e$. The physical dimension of $\nabla\varphi$ is acceleration by definition, *i.e.* $m\nabla\varphi$ defines force; in turn, as *force/charge* and *force/mass* define *field*, one finds

$$\boldsymbol{v}\times\nabla\varphi\equiv\frac{e\boldsymbol{v}}{m}\times\nabla\frac{m\varphi}{e}=\frac{e\boldsymbol{v}}{m}\times\boldsymbol{field}\ \Rightarrow\ m\boldsymbol{v}\times\nabla\varphi=e\boldsymbol{v}\times\boldsymbol{field}.$$

Since dividing both sides by c, the left hand side has physical dimensions of force, this result is therefore acknowledged as the Lorentz force

$$\boldsymbol{F}=e\frac{\boldsymbol{v}}{c}\times\boldsymbol{H}. \tag{3.43}$$

Define now by dimensional reasons $\boldsymbol{v}/c\times\boldsymbol{H}=\boldsymbol{F}/e=e\boldsymbol{r}/r^3$, in principle possible with appropriate values at the right side; the physical motivation of this step is the chance of defining

$$\frac{\boldsymbol{F}}{e}=\frac{e\boldsymbol{r}}{r^3}=\begin{cases}\nearrow\ \boldsymbol{F}\cdot\boldsymbol{r}=\mathcal{E}=\dfrac{e^2}{r}\\[2ex] \searrow\ \boldsymbol{E}=e\dfrac{\boldsymbol{r}}{r^3}\end{cases}, \tag{3.44}$$

i.e. a new field $\boldsymbol{E}=\boldsymbol{F}/e$ with $\boldsymbol{r}$ oriented like $(\boldsymbol{v}/c)\times\boldsymbol{H}$ and thus $\boldsymbol{H}\cdot\boldsymbol{E}=0$, e.g. like the orthogonal fields of an e.m. wave.

Consider (3.43) to calculate $\nabla\cdot\boldsymbol{E}$ of (3.43); as $\boldsymbol{E}=e\boldsymbol{r}/r^3$, then $\nabla\cdot\boldsymbol{r}/r^3=3/r^3$ yields the Gauss result

$$\nabla\cdot\boldsymbol{E}=4\pi(e/V)=4\pi\rho\quad V=\frac{4\pi r^3}{3}. \tag{3.45}$$

Implement now the shortened notation $\nabla\cdot\boldsymbol{E}\propto\rho$ to skip factors like $4\pi,\epsilon_0,\mu_0$ of the SI system equations and consider now (3.43) to calculate $\nabla\times\boldsymbol{E}\propto\nabla\times(\boldsymbol{v}\times\boldsymbol{H})=(\nabla\cdot\boldsymbol{H})\boldsymbol{v}-(\nabla\cdot\boldsymbol{v})\boldsymbol{H}$, having skipped again the proportionality constant e^2/c. Assuming $\nabla\cdot\boldsymbol{H}=0$ and noting the addends of the scalar $\nabla\cdot\boldsymbol{v}$ have the form $(\partial/\partial x_i)(\partial x_i/\partial t)=\partial/\partial t$, one infers $\nabla\times\boldsymbol{E}\propto-\boldsymbol{H}$. As concerns (3.43), owing to $\boldsymbol{H}\cdot\boldsymbol{E}=0$, note that multiplying (3.43) by an appropriate vector operator $\boldsymbol{X}$ one finds $\boldsymbol{X}\cdot\boldsymbol{E}\propto\boldsymbol{X}\cdot(\boldsymbol{v}\times\boldsymbol{H})=\boldsymbol{v}\cdot(\boldsymbol{X}\times\boldsymbol{H})$. Let be $\boldsymbol{X}=\nabla$: then this result reads $\nabla\cdot\boldsymbol{E}=\boldsymbol{v}\cdot(\nabla\times\boldsymbol{H})$. Next, multiplying again by $\boldsymbol{v}'$ one finds $\boldsymbol{J}=\rho'\boldsymbol{v}'=(\boldsymbol{v}\cdot(\nabla\times\boldsymbol{H}))\boldsymbol{v}'$. The fact that appear in this result two velocities, in fact no reason requires that $\boldsymbol{v}'$ is equal $\boldsymbol{v}$, suggests that $\boldsymbol{v}$ defines ρ via the scalar $\boldsymbol{v}\cdot(\nabla\times\boldsymbol{H})$ as already found, whereas $\boldsymbol{v}'$ defines $\boldsymbol{J}$ via $\rho\boldsymbol{v}'$: it is enough to introduce $\rho_{tot} = \rho + \rho'$ and split the two addends of the total charge into the respective displacement rates related to two displacement mechanisms. As a result one finds starting from (3.43)

$$\nabla \cdot \boldsymbol{E} \propto \rho \quad \nabla \cdot \boldsymbol{H} = 0 \quad \nabla \times \boldsymbol{E} \propto -\dot{\boldsymbol{H}} \quad \nabla \times \boldsymbol{H} \propto \boldsymbol{J} + \dot{\boldsymbol{E}} \tag{3.46}$$

i.e. the Maxwell equations. In this respect it is worth emphasizing that a recent paper has introduced an interaction mechanism involving the quantum vacuum, as a consequence of which even magnetic monopoles can exist at an appropriate threshold energy [3].

To complete the list of corollaries of (3.1) note the chance that

$$\dot{\boldsymbol{v}} = \frac{\delta \boldsymbol{v}}{\delta t} = \frac{\delta}{\delta t}\frac{\delta \boldsymbol{x}}{\delta t} = \frac{\delta^2 \boldsymbol{x}}{\delta t^2} = -\nabla \varphi,$$

whence

$$\frac{\dot{\boldsymbol{v}}}{c^2} = -\nabla \frac{\varphi}{c^2} = -\frac{\delta^2 \boldsymbol{x}}{\delta \ell^2} \quad \delta \ell = c \delta t \ \Rightarrow \ -c^2 \frac{\delta^2 \boldsymbol{x}}{\delta \ell^2} = -\frac{\delta^2 \boldsymbol{x}}{\delta t^2}. \tag{3.47}$$

In general, given any function $f = f(x)$ with $x = x(t)$, note that owing to (3.3)

$$\frac{\delta f}{\delta t} = \frac{\delta f}{\delta x}\frac{\delta x}{\delta t} = v_x \frac{\delta f}{\delta x}$$

which implies

$$\frac{\delta^2 f}{\delta t^2} = \frac{\delta\left(v_x \frac{\delta f}{\delta x}\right)}{\delta t} = v_x \frac{\delta\left(v_x \frac{\delta f}{\delta x}\right)}{\delta x} = v_x^2 \frac{\delta\left(\frac{\delta f}{\delta x}\right)}{\delta x} + v_x \frac{\delta v_x}{\delta x}\frac{\delta f}{\delta x} = v_x^2 \frac{\delta^2 f}{\delta x^2} - \frac{\delta f}{\delta x}\frac{\delta \varphi}{\delta x}$$

and thus yields

$$\frac{\delta^2 f}{\delta t^2} = v_x^2 \frac{\delta^2 f}{\delta x^2} - \frac{\delta f}{\delta x}\frac{\delta \varphi}{\delta x}; \tag{3.48}$$

if in particular v_x does not depend upon x, e.g. vibrating homogeneous spring or e.m. wave propagating in a homogeneous dispersive medium, then the second addend at the right hand side vanishes. This particular the case implies the D'Alembert equation

$$\frac{\delta^2 f(x,t)}{\delta t^2} = v_x^2 \frac{\delta^2 f(x,t)}{\delta x^2}, \tag{3.49}$$

which generalizes (3.47) to any propagation rate v_x of the wave.

Note that all information hitherto inferred has been extracted uniquely from the mere definition (3.1) of gravitational potential φ only, without additional hypotheses to explain the link between classical physics, quantum physics, electromagnetism and relativity.

The link of this preliminary section with the introductory section 1 is summarized by (3.7). The essential points of this section are:

-All considerations hitherto introduced generalize the concept of Lorentz transformations, according which v/c constant concerns the displacement rate of inertial reference systems only.

-The actual analytical forms of $\boldsymbol{v}(x,y,z,t)$ and $\dot{\boldsymbol{v}}(x,y,z,t)$ are specified by $\epsilon(x,y,z,t)$, *i.e.* the results hold for $\boldsymbol{v}$ and $\dot{\boldsymbol{v}}$ governed by an appropriate $\epsilon(x,y,z,t)$, arbitrary input.

-The first step (3.2) defines φ in (3.3), *i.e.* $\boldsymbol{v}\cdot\dot{\boldsymbol{v}}$, deserves more attention; this point is again concerned later.

-The physical meaning of (3.4), and thus of (3.10) and (3.11), are further considered in the next sections to explain why stretching/shrinking of the space time extends these preliminary results directly to the cosmological conceptual frame.

4. Further Considerations on β: Quantum Uncertainty

The results of the section 2 suggest that the physical meaning of the β function of (3.5) is well beyond that of mere Lorentz factor of special relativity, where in particular $v = constant$ accounts for inertial reference systems in relative motion. This section concerns three corollaries of β aimed to investigate this point with the help of the quantum uncertainty equation inferred in [4]

$$\delta x \delta p_x = n\hbar = \delta\varepsilon\delta t. \tag{4.1}$$

To show (4.1) for completeness, consider the Planck constant n times, e.g. for $n\hbar/t$ to describe the huge energy of a macroscopic system. Since by definition $\hbar$ is defined by the Planck energy and time, $\hbar = \varepsilon_{Pl} t_{Pl}$, it follows $n\hbar = n\varepsilon_{Pl} t_{Pl}$. Yet at the right hand side n multiplies two quantities; so write $n\hbar = n'\varepsilon_{Pl} n'' t_{Pl}$ with the condition $n'n'' = n$. By definition n is integer, whereas n' and n'' do not. Rewrite thus this result as $n\hbar = \delta\varepsilon\delta t$, with $\delta\varepsilon = n'\varepsilon_{Pl}$ and $\delta t = n'' t_{Pl}$; *i.e.*, simply admitting that n' and n'' are defined by arbitrary boundary values as $n_{1'} \le n' \le n_{2'}$ and $n_{1''} \le n'' \le n_{2''}$, the result is $\delta\varepsilon\delta t = n\hbar$. Moreover it is also true that $\hbar = x_{Pl} p_{Pl}$ via Planck length and momentum; so an identical reasoning yields $n\hbar = \delta x \delta p_x$. These results merge into (4.1).

Actually the number of possible space extra dimensions δx_j and conjugate momenta p_{xj} concerned by the scalar is in principle arbitrary; in this paper the number of space-momentum components is the usual 3, however also the space extra-dimensions, if any, fulfill the time product at the right hand side. It is also worth emphasizing that any uncertainty range is not related to a specific reference frame because neither boundary coordinate is known or knowable by fundamental assumption. Eventually (4.1) shows that the time is inherently linked to any calculation involving explicitly space and momenta coordinates only: (4.1) implies by corollary the concept of space time, whatever the number of space coordinates at the left hand side might be. In lack of information about size and position of any range with the respect to the origin of a system of coordinates R, once having waived the local values of dynamical variables it follows that any other R' is indistinguishable from R. In other words, δx and δp_x enclosing any x, p_x in a given reference system R imply different $\delta x'$ and $\delta p'_x$ enclosing x', p'_x in R' while being $\delta x \delta p_x = n\hbar$ and $\delta x' \delta p'_x = n'\hbar$; however R and R' are in fact indistinguishable, because by definition the numbers n and n' of allowed states are by definition sequences of arbitrary integers and not specific values identifiable for any R. Hence the physical meaning of quantum curvature radius δr^{-1} is conceptually different from the deterministic r^{-1} of the classical physics. In the classical physics both x and x' are exactly known and subjected to the Lorentz transformation in different inertial reference systems in the special relativity. Of course here all of this still holds, but in fact δr^{-1} and $\delta r'^{-1}$ are indistinguishable, because nothing in known about them: this a conceptual requirement not a simplifying approximation for numerical purposes. In fact different reference systems are themselves not distinguishable "per se", once waiving the local values of dynamical variables because of indistinguishable n and n'. Once acknowledging the uncertainty

space time background $\Rightarrow$ momentum change and space gap
$\leftrightarrow$ energy change and time lapse

are contextually configured by the number n of allowed states of the physical system. According to the quantum uncertainty: (i) no instantaneous changes, ∂/∂, but ratios of finite ranges δ/δ of dynamical variables are introduced by the uncertainty; (ii) unspecified reference system in lack of information about the range boundaries.

Equation (4.1) has classical, quantum and relativistic implications.

(i) Classical corollary.

Multiply $\delta x \delta p_x = \delta\epsilon\delta t$ by v_x; as $v_x = \delta x/\delta t$ and $v_x \delta p_x = m v_x \delta v_x = \delta\left(m v_x^2\right)/2$ by definition, then

$$v_x \delta x \delta p_x = v_x \delta\epsilon\delta t \;\Rightarrow\; \delta\left(m v_x^2\right)/2 = \delta\epsilon \quad \delta x = v_x \delta t \;\Rightarrow\; \frac{1}{2} m v_x^2 + const = \epsilon.$$

(ii) Quantum corollary.

Multiply $n\hbar = \delta\epsilon\delta t$ by δt^{-1}; thus one finds

$$\frac{n\hbar}{\delta t} = \delta\epsilon \;\Rightarrow\; \delta\frac{n\hbar}{\delta t} - \hbar\frac{\delta n}{\delta t} = \delta\epsilon;$$

if n does not depend upon δt, then $\delta\left(n\hbar\right)/\delta t = \delta\epsilon$ reads $n\hbar\omega + const = \epsilon$. This well known energy quantization is actually approximate, it holds if n is time independent.

(iii) Relativistic corollary.

Identify $\epsilon = m\varphi$ and calculate according to (3.3) $\delta\epsilon = \epsilon_2 - \epsilon_1 = -m v_2^2/2 + m v_1^2/2$; thus, owing to the attractive nature of the gravitational field, one expects that $\delta\epsilon = -\delta\varphi$. Write then $\delta\epsilon = -\left(mc^2\right)\delta\left(\varphi/c^2\right)$, *i.e.*

$$-\frac{\delta\epsilon}{\epsilon_0} = \delta\frac{\varphi}{c^2} \rightarrow -\frac{\delta\hbar\omega}{\hbar\omega_0} = \frac{\delta\varphi}{c^2} \quad \epsilon_0 = mc^2 \;\Rightarrow\; -\frac{\delta\omega}{\omega_0} = \frac{\delta\varphi}{c^2}:$$

this result, agrees with (3.23) previously found, does not depend on m because $\delta\epsilon/\epsilon_0$ turns into $\delta\omega/\omega_0$ and thus still holds even for $v \rightarrow c$, *i.e.* for a photon displacing radially in the vacuum with respect to a gravitational source. Owing to the general considerations carried out in the section 1, the aforesaid three points are the natural consequence of the theoretical approach followed in this paper.

Note that the scalar $\delta x \delta p_x$ of (4.1) does not exclude and thus implies also the vector forms

$$\delta \boldsymbol{x} \cdot \delta \boldsymbol{p} = n\hbar = \delta\varepsilon\delta t \tag{4.2}$$

and

$$\boldsymbol{u} \cdot \delta \boldsymbol{x} \times \delta \boldsymbol{p} = n\hbar = \delta\varepsilon\delta t, \tag{4.3}$$

being $\boldsymbol{u}$ unit vector. If (4.1) has physical meaning, the same must hold even for both (4.2) and (4.3). In fact (4.2) and (4.3), although conceptually identical to (4.1) with mere vector notation, deserve further attention.

Regard the classical definition $\boldsymbol{M} = \boldsymbol{r} \times \boldsymbol{p}$ according to the uncertainty, *i.e.* replacing the deterministic $\boldsymbol{r}$ and $\boldsymbol{p}$ as $\delta \boldsymbol{r}$ and $\delta \boldsymbol{p}$; thus $\delta \boldsymbol{M} = \delta \boldsymbol{r} \times \delta \boldsymbol{p}$ is the range of $\boldsymbol{M}$ corresponding to all $\boldsymbol{r}$ and all $\boldsymbol{p}$ falling in the respective $\delta \boldsymbol{r}$ and $\delta \boldsymbol{p}$. Starting from the component $\mathrm{M}_u = \boldsymbol{M} \cdot \boldsymbol{u}$ of $\boldsymbol{M}$, write $\mathrm{M}_u = \boldsymbol{u} \cdot (\delta \boldsymbol{r} \times \delta \boldsymbol{p}) \equiv \delta \boldsymbol{p} \cdot (\boldsymbol{u} \times \delta \boldsymbol{r})$; then $\delta \boldsymbol{k} = \boldsymbol{u} \times \delta \boldsymbol{r}$ yields $\mathrm{M}_u = \delta \boldsymbol{p} \cdot \delta \boldsymbol{k}$, *i.e.*

$$\mathrm{M}_u = \delta \boldsymbol{p} \cdot \delta \boldsymbol{k} = \pm \delta p_k \delta k = \pm n\hbar \quad \delta k = |\delta \boldsymbol{k}| \quad n = 0, integer. \tag{4.4}$$

It is also immediate to calculate $\mathrm{M}^2 = \mathrm{M}_x^2 + \mathrm{M}_y^2 + \mathrm{M}_z^2$ assuming $\langle \mathrm{M}_x^2 \rangle = \langle \mathrm{M}_y^2 \rangle = \langle \mathrm{M}_z^2 \rangle$ in an isotropic space. If $-N \le n \le N$, with N arbitrary boundary allowed to n, then the average of each square component is

$$\sum_{-N}^{N} \mathrm{M}_i^2 = N(N+1)(2N+1)/3 \tag{4.5}$$

in $\hbar^2$ units; so the average of the $2N+1$ terms of each sum is $\langle \mathrm{M}_i^2 \rangle = N(N+1)/3$ for each square component. Eventually, summing also over the three components the result is

$$\langle \mathrm{M}^2 \rangle = N(N+1)\hbar^2. \tag{4.6}$$

Eventually note that $N(N+1) \equiv (N+1/2)^2 - 1/4$ reads $\langle \mathrm{M}^2 \rangle = (2N+1)^2 \hbar^2/4 - \hbar^2/4$, *i.e.* the right hand side is square angular momentum; the fractional notation has a meaning analogous to that of (4.5). Then, summing $(N+1/2)\hbar^2$ to both sides of $\langle \mathrm{M}^2 \rangle + \hbar^2/4 = (N+1/2)^2 \hbar^2$, the result is

$$\langle \mathrm{M}^2 \rangle + \left(\frac{1}{2}\right)^2 + \left(L + \frac{1}{2}\right) \equiv \left(L + \frac{1}{2}\right)\left(L + \frac{1}{2} + 1\right) = \mathcal{M}_{LS}(\mathcal{M}_{LS} + 1) \quad \mathcal{M}_{LS} = L \pm \frac{1}{2}. \tag{4.7}$$

It is crucial to note that (2.38) was previously obtained implementing $\delta t = \hbar/\delta\epsilon$; implementing instead $\delta t = n\hbar/\delta\epsilon$, *i.e.* accounting for the uncertainty, the actual result reads now

$$-\delta E = n\mu|a|\ell_{Pl} \quad \delta\epsilon = \mu|a|\ell_{Pl} \quad \text{or} \quad \delta E = \mu|a|\ell_{Pl} \quad -\delta\epsilon = n\mu|a|\ell_{Pl}; \tag{4.8}$$

the amounts of energy $\mu|a|\ell_{Pl}$ become now quanta $n\mu|a|\ell_{Pl}$ of energy released to the space time. The quantization of the original (2.36) avoids that the fate of any orbiting system is, sooner or later, the collapse. Similarly to electromagnetic field, the energy exchanges occur via quantized wave packets: the decay of one gravitational system, *i.e.* the change of its n, can excite another resonant system, which in turn changes its own actual n'. The universe becomes more interconnected than predicted by the original Einstein formula. Follow now some corollaries of the quantum uncertainty.

4.1. Corollary 1

Let $r = r(t)$ be the radius of a spherical zone of the space time and assume that the size of this space region can expand as a function of time, e.g. because the whole universe expands itself. The function describing the expansion of the range δr from its initial size r during δt at rate $\delta r/\delta t$ reads

$$H(r,t) = \frac{\delta v_r}{r} = \frac{\delta r/r}{\delta t}. \tag{4.9}$$

Actually H is described in three ways, pertinent to radius, surface and volume of the growing sphere: since volume and boundary surface of this environment increase along with the enclosed diametric size, the right hand side of (4.9) yields

$$\frac{1}{2\pi r}\frac{\delta(2\pi r)}{\delta t} \to \frac{1}{r}\frac{\delta r}{\delta t} = H. \tag{4.10}$$

Moreover for the spherical surface of this zone

$$\frac{1}{4\pi r^2}\frac{\delta(4\pi r^2)}{\delta t} \to \frac{2r}{r^2}\frac{\delta r}{\delta t} = 2\frac{1}{r}\frac{\delta r}{\delta t} = 2H. \tag{4.11}$$

Eventually an identical reasoning for a spherical volume implies also

$$\frac{1}{(4\pi/3)r^3}\frac{\delta((4\pi/3)r^3)}{\delta t} \to \frac{3r^2}{r^3}\frac{\delta r}{\delta t} = 3\frac{1}{r}\frac{\delta r}{\delta t} = 3H. \tag{4.12}$$

These results define the function

$$\frac{1}{r}\frac{\delta r}{H\delta t} = k \quad k = 1,2,3$$

compatible with the driving force F and energy ϵ of the growth process via $P = F/S$ and $\eta = \epsilon/V$ having the same physical dimensions. Since P/η is dimensionless like k, let be $F/S \propto \epsilon/V$. As the energy PV cannot overcome ϵ, by definition enclosed in V, let be $(PV)_{\max} = \epsilon$ *i.e.* $PV = (k/k_{\max})\epsilon$; thus

$$\frac{F}{S} = \frac{k}{3}\frac{\epsilon}{V} \;\Rightarrow\; P = \frac{k}{3}\eta. \tag{4.13}$$

This well known result and (HH7) support the validity of this reasoning. It is implicitly evident the usefulness of considering not only (4.10) but also (4.11) and (4.12), of course consistent with both chances (4.15).

To proceed further, note that $r^{-1}\delta r/\delta t$ of (4.9) consists of two functions $r_2/(r\delta t)$ and $r_1/(r\delta t)$ once expressing explicitly $\delta r = r_2 - r_1$; if so, then it is convenient to consider δr as

$$\delta r = r_2 - r_1 \equiv r_2 - r' + r' - r_1 = (r_2 - r') + (r' - r_1) \tag{4.14}$$

in order to obtain from δr two terms $\delta r_1 = r_2 - r'$ and $\delta r_1 = r' - r_1$ formally analogous to the ones just considered in (4.10) to (4.12). Consider that in principle even r' could be regarded as a range $\delta r' = r' - 0$, *i.e.* with its lower boundary centered on the origin of its own reference system R. This emphasizes why the deterministic coordinate r' requires specifying its own R, whereas δr does not; δr could be defined as $r - r_0'$ with r_0' related to its own R', or as $r - r_0''$ with r_0'' related to R'', and so on. However, if r_0' and r_0'' are undefined, then the respective links to R' and R'' are undefinable as well. The same holds for the upper boundary r. Clearly the lack of any information about both range boundaries implies the lack of information about their related R. The same holds for the range sizes: in fact $n\hbar$ in R is indistinguishable from any $n'\hbar$ in R', because the integers n and n' symbolize sequences of numbers of states rather than specific allowed states, *i.e.* $\delta x \delta p_x$ and $\delta x' \delta p_x'$ are in fact indistinguishable. The step (4.14) aims to convert $r_2/(r\delta t)$ and $r_1/(r\delta t)$ into $\delta r_2/(r\delta t)$ and $\delta r_1/(r\delta t)$ analogous to that defining H of the initial (4.9), in agreement with the formalism of (4.1). Therefore rewrite identically the first (4.9) according to the last (4.14)

$$H_2 = \frac{1}{r}\frac{\delta r_2}{\delta t} \quad H_1 = \frac{1}{r}\frac{\delta r_1}{\delta t} \quad \delta r_2 = r_2 - r' \quad \delta r_1 = r' - r_1, \tag{4.15}$$

so that (4.15) yields owing to (4.14)

$$\frac{1}{r}\frac{\delta r}{\delta t} = H = H_2 + H_1 : \tag{4.16}$$

in other words, the splitting of $r^{-1}\delta r/\delta t$ into two levels H_1 and H_2 rewrites the initial $r_2/(r\delta t)$ and $r_1/(r\delta t)$ in order to preserve the form (4.15) at the right hand side. Thus two values are allowed to H. Note that if $r_1 < r' < r_2$, *i.e.* r' is inside δr, then both H_1 and H_2 are positive. If instead $r' < r_1$, *i.e.* r' is outside δr, then in principle (4.14) still holds with $H = H_2 - H_1$ and $H_1 > 0$; in fact it means that H splits into two values H_2 and H_1 greater and smaller than the initial H. In summary, simply considering the possible ways (H09) or (4.14) to regard $r^{-1}\delta r$, from the initial function $H(r,t)$ of (4.9) have been inferred from (4.1) without additional hypotheses two functions H_2 and H_1 of (4.15) formally equivalent to (H00) and concurring to H_u of (4.16).

4.2. Corollary 2

Equation (4.11) reeads

$$\frac{\delta A}{A} = 2\frac{\delta r}{r} \Rightarrow \frac{A\delta A}{A_0^2} = 2\frac{A^2}{A_0^2}\frac{\delta r}{r}$$

being A the arbitrary surface concerned by $4\pi r^2$; the second equation is inferred via trivial algebraic steps after having multiplied both sides of the first equation by A^2/A_0^2, with A_0 constant surface. Since $A\delta A = \delta(A^2)/2$ it follows next

$$\delta\frac{A^2}{4A_0^2}=\frac{A^2}{A_0^2}\log\frac{r}{r_0}\;\Rightarrow\;\delta\frac{A^2}{4A_0^2}=\frac{1}{2}\frac{A^2}{A_0^2}\log\frac{r^2}{r_0^2}=\frac{1}{2}\frac{A^2}{A_0^2}\log\frac{A}{A_0}.$$

Define $\delta\left(A^2\right)$ as $A^2-A_0^2$, so that the left hand side becomes $A^2/A_0^2-const$; thus the last result reads

$$\frac{A^2}{4A_0^2}-const=\frac{1}{2}\frac{A^2}{A_0^2}\log\frac{A}{A_0}.$$

If $const\ll A^2/4A_0^2$, then one finds

$$\frac{A^2}{4A_0^2}\approx\frac{A}{A_0}\frac{S}{2}\quad\text{i.e.}\;\frac{A}{2A_0}\approx S\quad S=\frac{A}{A_0}\log\frac{A}{A_0}. \tag{4.17}$$

The last equation identifies S, whatever the initial $\delta r/r$ and A_0 might be. In the particular case where $r\equiv r_{bh}$ and A_0 is of the order of the *Plank length*[2], then (4.17) is compatible with the Hawking entropy

$$\frac{A}{4\ell_{Pl}^2}=S\quad A_0=2\ell_{Pl}^2.$$

Although this topic is outside the purposes of this paper, (4.17) has been shortly sketched owing to its conceptual importance. For sake of brevity is also waived the chance of implementing (4.12) in an analogous way, as strongly suggested by this result; exploring the problem of black body volume entropy is certainly an amazing extension of the Hawking result. It would be interesting to extend this reasoning to (4.12) too.

4.3. Corollary 3

From (4.2) in radial form write $\delta r\delta p_r=\delta\varepsilon\delta t$ via (3.3) and (3.12)

$$\delta r\delta\frac{mv_r}{\beta}=n\hbar=\delta\varepsilon\delta t\;\Leftrightarrow\;v_r\delta\frac{mv_r}{\beta}=n\hbar\omega=\delta\varepsilon\quad v_r=\frac{\delta r}{\delta t} \tag{4.18}$$

Next note that the differential $mv_r\delta\left(v_r/\beta\right)=\delta\left(mc^2/\beta\right)$, so that (3.23) yields

$$\delta\frac{mc^2}{\beta}=\delta\varepsilon\;\Rightarrow\;\frac{mc^2}{\beta}=\varepsilon+const. \tag{4.19}$$

This result, coherent with (3.12), identifies ε as the energy corresponding to the given definition of momentum p_r simply linking the respective δp_r and $\delta\varepsilon$. Regard now this preliminary reasoning about the correlation between p_r and ε as a significant hint to infer a new and more interesting result: the correlation between kinetic energy ε and gravitational potential φ, due to the position of a body in a gravitational field. In principle this task is reasonable as ε and φ are two forms of energy a body can possess. Try thus to relate $m\varphi$ to the analytical form of the gravitational field acting on the body replacing p_r with $m\varphi f$, where f is proportionality factor allowing ε to be expressed as a function of $m\varphi$. In other words, the previous correlation $p_r\leftrightarrow\varepsilon$ via δp_r and $\delta\varepsilon$ reads now $\varepsilon\leftrightarrow m\varphi f$ via $\delta\varepsilon$ and $\delta\left(m\varphi f\right)$. Repeating the same steps of

(4.18) and (4.19), rewrite

$$\delta\frac{mc^2}{\beta}=\frac{n\hbar}{\delta t}=\delta\left(m\varphi f\right);$$

next, since $\varphi=\beta^2c^2/2+const$ owing to (3.5), explain $\delta\left(mc^2/\beta\right)=\delta\left(m\varphi f\right)$ as

$$\frac{mc^2}{\beta}+const'=\frac{mc^2\beta^2 f}{2}\ \Rightarrow\ f=\frac{2}{\beta^3}+\frac{2const'}{\beta^2mc^2}. \qquad (4.20)$$

To check the validity of this reasoning, it is useful to expand f in series of v around $v=0$; owing to (3.3) trivial calculations yield

$$f=k_0+k_1v^2+k_2v^4+\cdots\ \Rightarrow\ m\varphi f=U=\xi_0-\xi_1v^2+\xi_2v^4+\xi_3v^6+\cdots, \quad (4.21)$$

being ξ_j the coefficients of the series. This result is more conveniently expressed in the equivalent form

$$\frac{U_r}{F_{Pl}}=const_0+r_0\left(-\frac{\xi_1}{r}+\frac{\xi_2}{r^2}+\cdots\right)\quad F_{Pl}=\frac{c^4}{G}\quad v^2=\frac{mG}{r}, \qquad (4.22)$$

as in fact it has been already found in (2.28).

4.4. Corollary 4

Consider (3.17)

$$\boldsymbol{p}-\boldsymbol{p}_0=\hbar\nabla\frac{\varphi}{c^2}\ \Rightarrow\ \frac{\delta\boldsymbol{p}}{\delta t}=-\boldsymbol{F}=-\frac{\hbar}{\delta t}\nabla\frac{\varphi}{c^2}=\nabla U:$$

since ∇ operates on the space coordinates only, one would expect classically $\boldsymbol{F}=-\nabla U$ with $U=\hbar v^2/2c^2\delta t$ an arbitrary constant apart. Thus U should be defined by v^2 only, which however is inconsistent with (2.31). In fact δt must be regarded as $\delta t=\delta x\delta p_x/\delta\varepsilon$, so that this last step is wrong; in principle this disagreement excludes the idea of absolute time, independent upon the coordinate system where are defined the space coordinates. In other words it is true that not necessarily holds $\boldsymbol{F}=-\nabla U$ in general; however, even though this condition is actually fulfilled in a conservative force field, $\delta t^{-1}\nabla\varphi\neq\nabla\left(\varphi/\delta t\right)$ does not contradict U of (2.31).

Before commenting these results, note eventually the further chance of defining β via (3.3) and (2.15) as

$$\beta^2=const+\frac{2\varphi}{c^2}=const-\frac{2mG}{c^2r}=const-\frac{r_{bh}}{r}\quad r_{bh}=\frac{2mG}{c^2} \qquad (4.23)$$

which implies according to (3.11) and (2.15)

$$\beta=\frac{t_0}{t'}=\sqrt{1-\frac{r_{bh}}{r}}; \qquad (4.24)$$

As t' is equal to t_0 at $r\to\infty$ and at $m\to0$, whereas t' is the time expressed at finite distance from m, this result accounts for the gravitational time dilation when replacing v as a function of (4.22). It also appears that the local time at $r=r_{bh}$ is $t_0=0$.

Hitherto v and c defining β have been regarded separately, e.g. to obtain (4.24). Regard now the ratio c/v with the physical meaning of refraction index n of an e.m. wave propagating at speed $v < c$ through a transparent medium. In this case $v/c \le 1$ is concerned by $1/\mathrm{n} \ge 1$, while β reads

$$\beta = \sqrt{1 - \frac{v^2}{c^2}} = \sqrt{1 - \sin^2\theta} = \cos\theta \quad \sin\theta = \frac{v}{c}:$$

i.e. $0 \le v \le c$ for $0 \le \theta \le \pi/2$. Therefore $\mathrm{n} = c/v$ implies for two different media 1 and 2 through which the beam propagates

$$\mathrm{n}\sin\theta = 1 \;\Rightarrow\; \mathrm{n}_1 \sin\theta_1 = \mathrm{n}_2 \sin\theta_2.$$

Appears in this result the Snell refraction law, which can be extended to the reflection law if $\mathrm{n}_1 = \mathrm{n}_2$, as it is known. Try thus to express β as a function of this physical parameter, the refractive index $\mathrm{n} = c/v \ge 1$; if so, express then $\varphi/c^2 = -1/2\mathrm{n}^2 = -\beta/2\sqrt{\mathrm{n}^2 - 1}$ and $\beta = \sqrt{\mathrm{n}^2 - 1}/\mathrm{n}$. Even the basic laws of geometrical optics fit the conceptual frame of the previous sections.

It is worth emphasizing at this point that the Equations (3.22), (4.24), and (4.22) go well beyond the mere domain of the special relativity; the freedom of introducing and implementing $v(x, y, z, t)$ not necessarily constant to define β implies results typical of the general relativity, in agreement with the ability of β to account for the space time deformation evidenced in (CCC) and (4.29) and (4.32). In this respects appear significant the steps from (3.13) to (3.14). In this respect are significant the next three subsections.

4.5. Corollary 5

As concerns the uncertainty ranges of (4.1) note that hold the notations

$$\delta\sqrt{\epsilon} = \frac{\delta\epsilon}{2\sqrt{\epsilon}} \quad \delta\sqrt{x} = \frac{\delta x}{2\sqrt{x}}$$

and so on for the other ranges. Then trivial algebraic manipulations yield identically the following notations

$$\delta x \delta p_x = n\hbar = \delta\epsilon\delta t \;\Leftrightarrow\; \delta\sqrt{x} = \frac{\delta x}{2\sqrt{x}},\; \delta\sqrt{\epsilon} = \frac{\delta\epsilon}{2\sqrt{\epsilon}} \quad 2\sqrt{x}\delta\left(\sqrt{x}\right)\delta p_x = n\hbar = 2\sqrt{\epsilon}\delta\left(\sqrt{\epsilon}\right)\delta t$$

which in turn imply

$$\delta p_x = \frac{n\hbar}{\delta x} = \frac{\delta\left(\sqrt{\epsilon}\right)}{\delta\left(\sqrt{x}\right)}\sqrt{\frac{\epsilon}{x}}\delta t.$$

Let ϵ be defined as the energy of two charges at rest; then

$$\epsilon = \frac{e^2}{x} \quad \sqrt{\frac{\epsilon}{x}} = \sqrt{\frac{e^2}{x^2}} = \pm\frac{e}{x} \;\Rightarrow\; \delta p_x = \frac{n\hbar}{\delta x} = \pm\frac{\delta\left(\sqrt{e^2/x}\right)}{\delta\left(\sqrt{x}\right)}\frac{e}{x}\delta t.$$

So the replacement $\delta p_x \to -\delta p_x$ implies $\delta x \to -\delta x$. This does not require $x \to -x$, but $x_2 - x_1 \to x_1 - x_2$ only; *i.e.* the mirror image of δx implies the way of regarding the choice of the upper and lower range boundaries, not the signs of the respective coordinates. Regard thus the last equation as

$$\frac{n\hbar}{\delta x} = \pm e\left(\frac{\delta\left(\sqrt{e^2/x}\right)}{\delta\left(\sqrt{x}\right)}\frac{1}{x}\right)\delta t; \tag{4.25}$$

Consider this relationship with either sign, e.g. positive; then $\delta x \to -\delta x$ requires $e \to -e$ and concurrently $\delta t \to -\delta t$ to leave unchanged the whole equation. This is the quantum CPT theorem.

4.6. Corollary 6

Introduce a surface element $\mathrm{d}A = \mathrm{d}x'\mathrm{d}x$ defined by two infinitesimal lengths $\mathrm{d}x$ and $\mathrm{d}x'$ in an arbitrary reference system R; this value of $\mathrm{d}A$ is uniquely determined at any time t with respect to an initial time t_0. Implement now a non-deterministic quantum approach, where is definable a finite interval δA of surface values where $\mathrm{d}x$ and $\mathrm{d}x'$ are replaced by finite ranges δx and $\delta x'$ non-uniquely fixed: it is enough to consider an arbitrary number of elementary surface elements $\mathrm{d}A_j = \mathrm{d}x_j\mathrm{d}x'_j$ defining $\delta x = \sum_i \mathrm{d}x_j$ and $\delta x' = \sum_j \mathrm{d}x'_j$ such that $\delta A = \sum_j \mathrm{d}x_j\mathrm{d}x'_j = \delta x\delta x'$ at the current time range $\delta t = t - t_0$. Write anyway

$$\frac{\delta A}{\delta x'} = \delta x. \tag{4.26}$$

Consider now the further chance that, for any reason, δx and $\delta x'$ are themselves time dependent: e.g. because the universe expands, so that any length linking two space coordinates is subjected to change itself as a function of time. This implies the necessity of introducing $\delta(\delta A)$ to account for the time dependence of variable $\delta A(t)$. For simplicity, but without loss of generality, it is enough to express the time dependence of δA via either $\delta x'$ or δx only; also, in this general framework, it is possible to assume the initial $\delta x = \delta x'$ at $t = t_0$.

These considerations plug δA into a general context compliant with the formalism of the uncertainty, which implies the necessity of introducing $\delta(\delta A)$ and $\delta(\delta x)$ to account for the time dependence of δA and δx upon δt once having introduced variable range sizes; via $\delta x = n\hbar/\delta p_x$ and $\delta t = n\hbar/\delta\varepsilon$ calculate thus

$$\delta\dot{x}'_\pm = \pm\left|\frac{\delta x'|_{t=t_2} - \delta x'|_{t=t_1}}{\delta t}\right| \quad \delta t = t_2 - t_1 \;\Rightarrow\; \delta(\delta x') = \delta x'|_{t=t_2} - \delta x'|_{t=t_1} = \delta\dot{x}'_\pm\delta t \tag{4.27}$$

where $\delta\dot{x}'$ is the rate at which changes $\delta x'$ during δt. In an analogous way define the deformation rate $\delta(\delta A)$ of δA during the time lapse δt as $\delta(\delta A) = (\delta t\delta\dot{x}')\delta x$ having put $\delta x' = \delta t\delta\dot{x}'$. The notation emphasizes that in fact the sign of $\delta(\delta x')$ depends on whether $\delta x'|_{t=t_2} \gtreqless \delta x'|_{t=t_1}$, in principle possible because $\delta x'|_{t=t_2}$ and $\delta x'|_{t=t_1}$ are both arbitrary. So

$$\delta A = \delta x\delta x' = \frac{(n\hbar)^2\delta\dot{x}'}{\delta p_x\delta\varepsilon} \tag{4.28}$$

yields

$$\delta(\delta A) = \delta x \delta \dot{x}'_{\pm} \delta t \quad \delta(\delta A) \gtrless 0. \tag{4.29}$$

So $\delta(\delta A)$ is related to and justified by the change of energy and momentum ranges corresponding to $\delta(\delta x')$ during δt. This means that the change $\delta(\delta A)$ of δA describes in general local stretching or shrinking of the initial surface range δA, both in principle intuitively acknowledged and physically allowed by the uncertainty.

In general (4.26) and (4.29) yield

$$\delta x = \frac{\delta(\delta A)}{\delta t \delta \dot{x}'_{\pm}} = \frac{\delta}{\delta t}\frac{\delta A}{\delta \dot{x}'_{\pm}} \quad \delta x = \frac{\delta A}{\delta x'} \tag{4.30}$$

As the left hand side of this result takes both signs according to either stretching or shrinking rate $\delta \dot{x}'$ of $\delta x'$, the two chances are

$$\frac{\delta}{\delta t}\frac{\delta A_+}{\delta \dot{x}'} = \frac{\delta A_+}{\delta x'} \Leftrightarrow -\frac{\delta}{\delta t}\frac{\delta A_-}{\delta \dot{x}'} = \frac{\delta A_-}{\delta x'}. \tag{4.31}$$

Compare eventually these results multiplying both sizes by a constant m_0 / t_0^2: the respective equations read in general as linear combinations of energies after having converted A_+ into $\mathcal{L} = m_0 A_+ / t_0^2$ and A_- into $\mathcal{H} = m_0 A_- / t_0^2$, *i.e.*

$$\frac{\delta}{\delta t}\frac{\delta \mathcal{L}}{\delta \dot{x}'} = \frac{\delta \mathcal{L}}{\delta x'} \Leftrightarrow \frac{\delta}{\delta t}\frac{\delta \mathcal{H}}{\delta \dot{x}'} + \frac{\delta \mathcal{H}}{\delta x'} = 0. \tag{4.32}$$

So $\mathcal{L}$ and $\mathcal{H}$ differ by the sign of the potential energy related to the kinetic energy; the fact that changing the sign of the potential term the equations coincide, $\delta\mathcal{H}/\delta x = U = -\delta\mathcal{L}/\delta x$, suggests that the linear combination of energies is consistent with $\mathcal{L} = T - U$ and $\mathcal{H} = T + U$. In other words, once having defined T and U, (4.32) introduces the appropriate formulas to correlate $\delta(\delta x)$ and $\delta(\delta A)$ to $\delta\epsilon$ of the uncertainty equation.

The crucial point is that these results have been contextually obtained by deforming the space time, as evidenced by $\delta(\delta x')$ and $\delta(\delta A)$ of (4.27) and (4.29).

4.7. Corollary 7

For the purposes of this section, remind first that (3.14) reads

$$\beta^2\epsilon^2 = \epsilon^2 - (pc)^2 = (mc^2)^2 \quad p = \frac{\epsilon v}{c^2} \quad \epsilon = \frac{mc^2}{\beta} \quad \beta = \sqrt{1 - \frac{v^2}{c^2}} \quad p_0 c = mc^2 = \beta\epsilon. \tag{4.33}$$

and that

$$(c\beta\delta t)^2 = (c\delta t)^2 - \delta r^2 \quad \delta r = v\delta t. \tag{4.34}$$

These are the standard equation and the invariant interval of the special relativity. Repeat now an identical reasoning but considering a different β. The same steps yield

$$\varepsilon^2 = \beta'^2 (mc^2)^2 \quad \beta_{\pm} = \sqrt{1 - \frac{v^2}{c^2} \pm \frac{v'^2}{c^2}} \Rightarrow \varepsilon^2 = (pc)^2 + (mc^2)^2 \pm \frac{v'^2}{c^2}(mc^2)^2 \tag{4.35}$$

and thus, subtracting side by side (4.36) and (4.33),

$$\varepsilon^2 = \epsilon^2 \pm \frac{v'^2}{c^2}\left(mc^2\right)^2. \tag{4.36}$$

The reason of having quoted (4.33) and (4.36) is that (3.17), (3.1) and (3.14) yield

$$\boldsymbol{p} = -\hbar\nabla\frac{\varphi}{c^2} - \boldsymbol{const} = \hbar\frac{\dot{\boldsymbol{v}}}{c^2} - \boldsymbol{p}_0 \quad \boldsymbol{const} = \frac{\hbar}{c^2}\boldsymbol{a}_0 = \boldsymbol{p}_0 \ \Rightarrow\ \boldsymbol{p} + \boldsymbol{p}_0 = \hbar\frac{\dot{\boldsymbol{v}}}{c^2}, \tag{4.37}$$

whence, squaring both sides of the last equation,

$$\hbar^2\frac{\dot{v}^2}{c^4} = p^2 + p_0^2 + 2p_0\cdot p \ \Rightarrow\ \hbar^2\frac{\dot{v}^2}{c^2} = \left(pc\right)^2 + \left(p_0c\right)^2 + 2p_0\cdot pc^2 \tag{4.38}$$

and then

$$\varepsilon^2 = \left(pc\right)^2 + \left(mc^2\right)^2 + 2p_0pc^2\cos\theta \quad \varepsilon^2 = \hbar^2\frac{\dot{v}^2}{c^2} \quad mc^2 = p_0c: \tag{4.39}$$

i.e. one finds again the standard (4.33) plus a correction term $2p_0pc^2\cos\theta = 2mc^2pc\cos\theta$ likewise as in (4.36).

The first step to guess $\cos\theta$ is the rational definition

$$\cos\theta = \pm\frac{\varepsilon p}{\varepsilon_\theta p_\theta + \varepsilon p} \quad \varepsilon_\theta p_\theta = \frac{\hbar^2}{\tau_\theta \ell_\theta}, \tag{4.40}$$

which reproduces the expected range of values for $\varepsilon p \to 0$ and $\varepsilon p \to \infty$; the second equation defines the space and time scales of $\cos\theta$, e.g. $\cos\theta \to 0$ for $\ell_\theta \to 0$ or $\tau_\theta \to 0$, in which case (4.39) tends to the standard (4.33) of the special relativity. Then, assuming $\tau_\theta \ell_\theta = const$ *i.e.* a constant scale factor of the correction term, one finds

$$2p_0pc^2\cos\theta = \pm 2p_0C_\theta\frac{\varepsilon\left(pc\right)^2}{1+\varepsilon pC_\theta} \quad C_\theta = \frac{\tau_\theta\ell_\theta}{\hbar^2}; \tag{4.41}$$

thus replacing in (4.39) the result is

$$\varepsilon^2 = \left(pc\right)^2 + \left(mc^2\right)^2 \pm 2p_0C_\theta\frac{\varepsilon\left(pc\right)^2}{1+\varepsilon pC_\theta}. \tag{4.42}$$

In conclusion (4.39) reads

$$\varepsilon^2 - \left(p'c\right)^2 = \left(mc^2\right)^2 \quad p' = pf_\pm = p\sqrt{1 \pm 2p_0C_\theta\frac{\varepsilon}{1+\varepsilon pC_\theta}} \tag{4.43}$$

Here the crucial parameter to infer the standard (4.33) from the modified (4.39) is C_θ: if $C_\theta \to 0$, then $f_\pm \to 1$ so that $p' \to p$ implies $\varepsilon \to \epsilon$, which in turn requires $\beta_\pm \to \beta$ in (4.35) *i.e.* $v' \to 0$ in (4.36). In other words the crucial parameter to link (4.33) and (4.39), the scalar $\boldsymbol{p}\cdot\boldsymbol{p}_0$, is actually the scale factor C_θ via the space and time factors ℓ_θ and τ_θ; the fact that reasonably the correction factor of the special relativity should be small suggests the quantum scale of C_θ and thus that $\ell_\theta \sim \ell_{Pl}$ and $\tau_\theta \sim t_{Pl}$. The endpoint of this reasoning is that ϵ of the standard special relativity splits because of $\beta_\pm \neq 0$ into two energy levels

(4.36) due to $v' \neq 0$, *i.e.*

$$\varepsilon^2 = \begin{cases} \nearrow \epsilon^2 + \dfrac{v'^2}{c^2}\left(mc^2\right)^2 \\ \searrow \epsilon^2 - \dfrac{v'^2}{c^2}\left(mc^2\right)^2 \end{cases} . \tag{4.44}$$

On the one hand is evident the formal analogy between (4.33) of the special relativity and (4.43): in fact their difference is due to $\beta \neq \beta_\pm$ only, in agreement with the hint of the corollary 1: here it appears again that the mere fact of modifying β implies the transition from special relativity to a more general context that can be nothing but general relativity. In effect (4.42) is known in the literature [5], it is acknowledged as a relativistic equation of quantum gravity having cosmological relevance.

One must conclude that in fact (4.33) is an equation of Euclidean flat space of the special relativity, whereas (4.42) is an equation of a non-Euclidean curved space of the general relativity. Note that, owing to (3.5), v defining β is arbitrary; however, whatever its initial value might be, is crucial the change $\delta\beta = \beta_\pm - \beta$. In other words one must conclude that changing the initial β to another value $\beta_\pm$ the gap $v \leftrightarrow v'$ reflects the changing of curvature of the space time. The question arises therefore about the physical meaning of (4.35) in generalizing the special relativity: this is concerned in the next subsection.

4.8. Corollary 8

The previous results rise the question: do $\nabla\beta$ and $\delta\beta/\delta r$ have really to do with the space time curvature? The answer is positive. Consider the geometrical definition of radius of curvature $r_c = |\boldsymbol{v}|^3 / |\boldsymbol{v} \times \dot{\boldsymbol{v}}|$, whence, expressing $\dot{\boldsymbol{v}}$ via (3.1) and (3.17),

$$\mathcal{R} = \frac{|\boldsymbol{v} \times \dot{\boldsymbol{v}}|}{|\boldsymbol{v}|^3} \quad \varepsilon\boldsymbol{v} = \hbar\left(\boldsymbol{a}_0 + \dot{\boldsymbol{v}}\right) \Rightarrow \mathcal{R} = \frac{|\boldsymbol{v} \times \dot{\boldsymbol{v}}|}{|\boldsymbol{v}|^3} = \frac{\left|\left(a_0 + \dot{\boldsymbol{v}}\right)\hbar/\epsilon \times \dot{\boldsymbol{v}}\right|}{\left|\left(a_0 + \dot{\boldsymbol{v}}\right)\hbar/\epsilon\right|^3} = \frac{\left|-a_0 \times \nabla\left(c\beta\right)^2\right|}{\left|a_0 - \nabla\left(c\beta\right)^2\right|^3}\frac{\epsilon^2}{\hbar^2}. \tag{4.45}$$

So (4.45) is the answer to the previous question: changing β means changing $\mathcal{R}$. Regard now $\nabla\left(\beta^2\right)$ as $\delta\left(\beta^2\right)/\delta r$ and thus with finite $\delta\left(\beta^2\right)$ given by $\beta^2 - \beta'^2$ across the finite uncertainty range δr. Of course $\boldsymbol{v}$ is not deterministic ratio of local infinitesimal $\partial\boldsymbol{s}/\partial t$, but ratio of finite uncertainty ranges $\delta\boldsymbol{s}/\delta t$ not related to any specific reference system. Without need of calculations, (4.45) shows that changing β means changing the local curvature $\mathcal{R}$ of the space time. Is crucial the fact that $\boldsymbol{a}_0 \neq 0$, otherwise $\mathcal{R} = 0$ would mean flat space time. The reason is clear: dividing by c^2 the second (4.45) reads $\boldsymbol{p} = \boldsymbol{p}_0 + \epsilon\delta\boldsymbol{v}/c^2$, being $\boldsymbol{p}_0 = \hbar\boldsymbol{a}_0/c^2$ and $\epsilon = \hbar/\delta t$. So $\boldsymbol{a}_0 \neq 0$ allows defining $\delta\boldsymbol{p}$ and then writing $\delta\boldsymbol{p} = m\delta\boldsymbol{v}/\beta$, with $m/\beta = \epsilon/c^2$.

Note that $\nabla\left(\beta c\right)^2 = 2\nabla\varphi$ since $\varphi = -v^2/2$ and $c^2\beta^2 = c^2 - v^2$; so the right hand side of the last $\mathcal{R}$ yields

$$\mathcal{R} = \frac{|-2\boldsymbol{a}_0 \times \nabla\varphi|}{|\boldsymbol{a}_0 - 2\nabla\varphi|^3} \frac{\epsilon^2}{\hbar^2} = \frac{|-\boldsymbol{a}_0 \times \nabla/2m\varphi|}{|m\boldsymbol{a}_0 - \nabla(2m\varphi)|^3} \frac{\epsilon^2}{\hbar^2}.$$

In particular $\dot{\varphi}$ is related via H_u to the universe expansion and thus to the change of its time dependent curvature, which in turn governs gravitational effects. The following step provides an intuitive explanation of $\mathcal{R}$:

$$\mathcal{R} = \frac{|\boldsymbol{v}t|^3}{|\boldsymbol{v}t \times \dot{\boldsymbol{v}}t^2|} = \frac{|\boldsymbol{s}|^3}{|\boldsymbol{s} \times \boldsymbol{s}'|} = \frac{V}{A}.$$

The conceptual background hitherto exposed justifies the attempt to address two cosmological problems of crucial importance, the dark matter and dark energy.

5. Cosmological Data

In this paper are implemented cosmological data introduced in [6], which in turn have been inferred from the estimates quoted in [7] and reported for convenience in Table 1.

$$t_u = 8.7 \times 10^{17}\ \text{s} \quad r_u = 4.35 \times 10^{26}\ \text{m} \quad M_u = 5.9 \times 10^{53}\ \text{kg} \quad m_{ob} = 3 \times 10^{52}\ \text{kg} \quad (5.1)$$

being t_u and r_u universe's age and radius and M_u universe total mass; m_{ob} counts only the mass of observable stars, whose light has reached our point of sight, whereas M_u is calculated through r_u as

Table 1. Estimated cosmological data reported in [7] of today's universe.

Property of present day observable universe	Approximate number of Planck units	Equivalents
Age	8.08×10^{60} tp	4.35×10^{17} s or 1.38×10^{10} years
Diameter	5.4×10^{61} Ip	8.7×10^{26} m or 9.2×10^{10} light-years
Mass	approx. 1060 mp	3×10^{52} kg or 1.5×10^{22} solar masses (only counting stars) 10 80 protons (sometimes known as the Eddington number)
Density	1.8×10^{-123} mp·lp^{-3}	9.9×10^{-27} kg·m^{-3}
Temperature	1.9×10^{-32} Tp	2.725 K temperature of the cosmic microwave background radiation
Cosmological constant	10^{-122} lp^{-2}	10^{-52} m^{-2}
Hubble constant	10^{-61} tp^{-1}	10^{-18} s^{-1} 10^2 (km/s)/Mpc

$$r_u = \frac{M_u G}{c^2}. \quad (5.2)$$

Note that these values are slightly different from the mere literature estimates reported in Table 1,whereas in particular t_u is about twice the value usually regarded as today's age of universe; this discrepancy, explained in the quoted paper, agrees with the value calculated in [8]. Also,

$$\Lambda = 1.6 \times 10^{-53}\ \mathrm{m}^{-2} \quad \Lambda_t = c^2\Lambda = 1.4 \times 10^{-36}\ \mathrm{s}^{-2}, \tag{5.3}$$

being Λ the Einstein cosmological constant. Eventually, are useful the density ρ_{vac} and energy density η_{vac} of quantum vacuum calculated in [6]

$$\rho_{vac} = 58 \times 10^{-28} \frac{\mathrm{kg}}{\mathrm{m}^3} \quad \eta_{vac} = 5.2 \times 10^{-10} \frac{\mathrm{J}}{\mathrm{m}^3} \tag{5.4}$$

in agreement with literature data [9]

$$\rho_{vac} = (60.3 \pm 1.3) \times 10^{-31} \frac{\mathrm{g}}{\mathrm{cm}^3} \quad \eta_{vac} = (5.4 \pm 0.1) \times 10^{-9} \frac{\mathrm{erg}}{\mathrm{cm}^3}. \tag{5.5}$$

Is attracting the chance that somehow these cosmological parameters can be rationally organized by means of and in agreement with the fundamental constants of nature. In fact it is easy to verify that all these data result in a coherent set of self-consistent cosmological parameters: from a mere numerical point of view, (5.1) and (5.3) fulfill the following numerical relationships:

$$\Lambda = \frac{1}{(ct_u)^2} = 1.5 \times 10^{-53}\ \mathrm{m}^{-2} \tag{5.6}$$

and the correlations

$$\sqrt{\Lambda_t} = \sqrt{\Lambda c^2} = 1.2 \times 10^{-18}\ \mathrm{s}^{-1} \quad \sqrt{\Lambda_t} = \frac{1}{t_u} = 1.1 \times 10^{-18}\ \mathrm{s}^{-1}. \tag{5.7}$$

Moreover, the importance of (3.2) appears considering the following numerical coincidences that correlate the cosmological parameters directly to t_u; (5.1) and (5.3) yield

$$|v \cdot v| = |\dot{\varphi}| \Rightarrow \frac{1}{t_u^3\Lambda} = 0.1, \quad \frac{c^2}{t_u} = 0.1, \quad c^3\sqrt{\Lambda} = 0.11, \quad \frac{c}{t_u^2\sqrt{\Lambda}} = 0.1\frac{\mathrm{m}^2}{\mathrm{s}^3}. \tag{5.8}$$

These values cannot be merely accidental, rather they reveal significant links with Λ, which becomes fundamental feature of the universe. If this is true, then these values must be someway involve G. Indeed, as concerns (5.1) and (5.3) it is worth noticing that owing to (5.6)

$$r_u = \frac{M_u G}{c^2} = M_u G\Lambda t_u^2 \Rightarrow G = \frac{r_u/\Lambda}{M_u t_u^2} = \frac{length^3}{mass \times time^2} = 6.1 \times 10^{-11}, \tag{5.9}$$

as in fact

$$M_u G\Lambda t_u^2 = 4.7 \times 10^{26}\ \mathrm{m} \leftrightarrow r_u \tag{5.10}$$

with the proposed age of the universe; *i.e.* the numerical value of G is nothing else but the combination of cosmological parameters which in fact define the value of the gravity constant according to its own dimensional meaning.

Also, (5.6) suggests how to combine the cosmological data in order to define the following pure number

$$\frac{(M_u G)(\Lambda t_u^2)}{r_u} = 1.1: \tag{5.11}$$

this result is interesting as it includes total mass, radius and age of the universe via

ΛG. It appears in (5.7) too.

Note that the set of data (5.5) and (5.3) should be optimized to allow a better self-consistency of (5.9) and (5.11): nevertheless the data concerned here are deliberately still those already reported in the previous paper [6] to show the continuity and consistency of the new results obtained here with the results therein exposed. This holds particularly for t_u: this age of the universe, although considerably different from what is commonly acknowledged, fits several correlations with other cosmological parameters. The values (5.8) to (5.11) are relevant as they link reasonably t_u, the cosmological Einstein parameter Λ and the change rate $\dot{\varphi}$ of the gravitational potential φ. Moreover the pure number (5.11) are also consistent with the aforesaid value of M_u, calculated simply via the gravitational radius introduced in (5.2), which fulfills the left hand side equation: *i.e.* the value of G is nothing else but the combination of cosmological parameters linked by the gravity constant according to their own dimensional meaning. In other words is reasonably true that in (5.9) the cosmological parameters surrogate the meaning of the respective physical dimensions. Note in this respect that

$$\eta_{vac} G = a_0^2 \leftrightarrow a_0 = 1.9 \times 10^{-10} \frac{\text{m}}{\text{s}^2}: \tag{5.12}$$

also this result, of interest for the MOND discussed in the next section, fits the age of the universe with the characteristic acceleration a_0.

The validity of (5.2) is verifiable in two ways. Since M_u is total mass, calculate the total surface and volume of the universe via r_u as

$$S_u = 4\pi r_u^2 \quad V_u^{ru} = \frac{4\pi}{3} r_u^3$$

and implement (4.13). Owing to (5.2), one finds

$$\frac{M_u G}{r_u^2} = \frac{c^2}{r_u} = a_0 = 2.1 \times 10^{-10} \frac{\text{m}}{\text{s}^2}$$

and

$$\frac{1}{3} \frac{M_u c^2}{(4\pi/3) r_u^3} = \frac{\eta_u}{3} = P_u = \frac{F_u}{4\pi r_u^2} \Rightarrow \frac{M_u c^2}{r_u} = 1.2 \times 10^{44} \frac{\text{N}}{\text{m}^2} \text{ i.e. } F_u = F_{Pl}. \tag{5.13}$$

The fact of having implemented the coefficient 1/3 will be explained later. Anyway, it is crucial the fact that both results are recognizable in the global frame of (5.1) to (5.5). Is interesting the fact that $\eta_u = 1.5 \times 10^{-10}\ \text{J/m}^3$ is of the order of but considerably smaller than η_{vac} of (5.4): this result is explained in the next subsection.

6. Dark Matter and Dark Energy

The visible mass m_{ob}, as it results in **Table 1** counting the stars only, amounts to $m_{ob} = 3 \times 10^{52}$ kg about. As this value concerns by definition the stars whose light reaches our point of sight, calculate the total star mass M_{ob} considering the radius $r_u = 4.35 \times 10^{26}$ m at the time $t_u = 8.7 \times 10^{17}$ s inferred in [6]. This age of universe is reasonable because it fits the other cosmological parameters (5.8). This result implies an interesting conclusion about the total amount of standard matter in the universe. As the current radius r_u is due to the expansion rate of the universe from the big bang to today's time t_u around an arbitrarily small initial size, the assumption of statistically homogeneous universe implies a total observable mass $M_{ob} = m_{ob}\left(r_u/ct_u\right)^3$. This extrapolated value of observable mass, appropriately definable as ordinary matter, means that a light beam starting from the growth nucleus of universe, wherever it might actually be, reaches the distance ct_u smaller than the actual boundary r_u of the universe; indeed $ct_u/r_u = 0.6$. So M_{ob} scales up the estimate m_{ob} we can afford by observing and counting the number of stars whose light has in fact reached us. The value of m_{ob} yields

$$M_{ob} = 1.4\times10^{53}\ \text{kg}; \tag{6.1}$$

thus the total mass of the universe $M_u = 5.9\times10^{53}\ \text{kg}$, inferred from the gravitational radius $r_u c^2/G$ in agreement with (5.11), still implies a residual mass $M_u - M_{ob} = 4.5\times10^{53}\ \text{kg}$. Try now to calculate the ratios of the masses concerned in this reasoning normalized to the total M_u; one finds that

$$\frac{m_{ob}}{M_u} \quad \frac{M_{ob}}{M_u} = \frac{m_{ob}\left(r_u/ct_u\right)^3}{M_u} \quad \frac{M_u - \left(m_{ob} + M_{ob}\right)}{M_u}$$

yield

$$\begin{aligned} &\frac{m_{ob}}{M_u} = \frac{3\times10^{52}}{5.9\times10^{53}} = 0.05 \quad \frac{M_{ob}}{M_u} = \frac{1.4\times10^{53}}{5.9\times10^{53}} = 0.24 \\ &\frac{M_u - M_{ob} - m_{ob}}{M_u} = \frac{4.2\times10^{53}}{5.9\times10^{53}} = 0.71 \end{aligned} \tag{6.2}$$

In fact these ratios are well known: they fit reasonably the current literature estimates that regard however the three ratios as corresponding to different masses: in particular m_{ob} is related to the amount of visible ordinary matter, M_{ob} corresponds reasonably to the value expected for the so called "dark mass" M_d, whereas the last equation should be related by difference to the dark energy $E_{de}/c^2 = M_u - M_{ob} - m_{ob}$. Actually, however, the physical meaning of the ratios appears to be completely different. More simply, M_{ob} is mere extrapolation of m_{ob}: in principle the ratio 0.24 includes the invisible matter of a universe too large to be fully observable, instead of including a new kind of matter whose identification is in fact beyond our current understanding. If this reasoning is true, then one should more appropriately write

$$\frac{M_{ob}}{M_u} = \frac{1.4\times10^{53}}{5.9\times10^{53}} = 0.24 \quad \frac{M_u - M_{ob}}{M_u} = \frac{4.5\times10^{53}}{5.9\times10^{53}} = 0.76, \tag{6.3}$$

while being

$$\mathcal{M}_{de} = M_u - M_{ob} = 4.5\times10^{53}\ \text{kg}, \tag{6.4}$$

with notation emphasizing that is concerned here not a true mass but E_{de}/c^2.

-On the one hand, (6.3) still supports the validity of total mass of the universe M_u simply inferred from the concept of gravitational radius $r_m = M_u G/c^2$ in

[6] according to (5.11); anyway (6.3) acknowledges $M_{ob} < M_u$ and implies

$$M_{ob}\Lambda^{1.5} = 9\times10^{-27} \quad a_0 = F_{Pl}/M_u = 2\times10^{-10}\ \frac{\text{m}}{\text{s}^2}. \tag{6.5}$$

-On the other hand the second equation calculates an excess mass inherent M_u with respect to the ordinary matter mass M_{ob} in the universe; in other words, the concept of dark energy $E_{de}/c^2 = M_u - M_{ob}$ is still actual even in the whole universe assumed homogeneous. True physical meaning has therefore the dark energy here expressed as E_{de}/c^2. Is obvious the question about where E_{de} comes from. This point has been already concerned in [6] and shortly summarized below for completeness. Nevertheless E_{de} is not mere residual mass, as it could seem in (6.3), it has its own physical meaning highlighted in the following equations. In agreement with the physical dimensions of G write, in agreement with (5.11),

$$\frac{\Lambda^{-1.5}}{\mathcal{M}_{de}\Lambda_t^{-1}} = \frac{c^2}{\mathcal{M}_{de}\sqrt{\Lambda}} = 5\times10^{-11} \quad \frac{c^4}{G} = \mathcal{M}_{de}a_0; \tag{6.6}$$

moreover

$$\sqrt{\Lambda^{-1}} = 2.5\times10^{26}\ \text{m} \quad ct_u = 2.6\times10^{26}\ \text{m}. \tag{6.7}$$

Also, compare the quantum vacuum energy density (5.4) with that calculable via (6.6) and the universe volume of (V8V)

$$\eta_{vac} = 5.2\times10^{-10} \quad \frac{\mathcal{M}_{de}c^2}{V_u} = \frac{c^5 t_u}{V_u G} = 4.9\times10^{-10}\ \frac{\text{J}}{\text{m}^3}: \tag{6.8}$$

this result supports the idea that the dark energy density is in fact the quantum vacuum energy density.

This explains the discrepancy with respect to η_u of (5.13): *i.e.* η_{vac} is not referable to $M_u c^2$, mass energy, but to the vacuum energy having quantum meaning. Therefore the volume to calculate the mass density ρ_u as $M_u/\left(4\pi r_u^3/3\right)$ is different fro that to calculate ρ_{de}. In fact it is possible to define the volume via Λ recalling its physical dimensions: *i.e.* as $V_u^\Lambda = (4\pi/3)\Lambda^{-1.5} = 6.5\times10^{79}\ \text{m}^3$. So

$$\frac{\mathcal{M}_{de}c^2}{V_u^\Lambda} = 6.2\times10^{-10}\ \frac{\text{J}}{\text{m}^3}, \tag{6.9}$$

which fits reasonably η_{vac}; moreover, implementing (4.13), calculate

$$\frac{1}{3}\eta_{vac} = \frac{F_{de}}{4\pi/\Lambda} \ \Rightarrow\ F_{de} = \frac{4\pi}{3}\frac{\eta_{vac}}{\Lambda} = 1.4\times10^{44}. \tag{6.10}$$

Thus once more $F_{de} = F_{Pl}$ like in (5.13). It is known in general that in (4.13) the coefficients $k=1$ or $k=2$ depend on whether or not the matter in the volume V where is defined η bounces back or not at the enclosing surface S. In principle such a conservative situation could be fulfilled in a black hole universe. Unfortunately this does not seem the case. Calculating

$$r^* = \frac{2M_{ob}G}{c^2} = 2.1\times10^{26}\ \text{m}; \tag{6.11}$$

one concludes that the actual $r_u > r^*$ makes M_{ob} insufficient to meet the threshold value to fulfill $r^* \equiv r_{bh}$; *i.e.* M_{ob} should be twice its actual value. In effect, (6.10) and (5.13) are consistent assuming both $k = 1$. The coefficient $k = 1$ of (4.13) accounts for the lack of any particle bouncing back at the boundary of the volume V_u^{de}. It appears that the immaterial length $\sqrt{\Lambda^{-1}}$ implies the internal pressure P_{de} acting inside V_u^{ct} and for the vacuum energy density η_{vac}. If so, then the fact of having assumed $\eta_{vac} \equiv \eta_{de}$ implies that $P_{de} \equiv P_{vac}$ keeps swelled the universe; also, the value of the active force inside V_u^{Λ} corresponds to the Planck force. Note that four consequences are implied by these achievements:

-The dark matter does not exist, it is ordinary matter M_{ob} that extrapolates the visible m_{ob}, whereas the dark energy is nothing else but vacuum energy.

-the MOND option becomes now essential to explain the rotation of galaxies: this conceptual frame is credible if able to infer and explain the MOND approach too, which in effect has been already introduced in (2.31).

-the numerical value of $\mathcal{M}_{de}$, initially introduced as the mere difference in (6.2), can be identified itself in a self-consistent way.

-Is crucial for these conclusions the given value t_u of universe age.

These results are further concerned in the next sections. . The Hubble tension It has been shown in [4] that actually the Hubble factor consists of two separated values around an approximate average $\overline{H_u}$; this point has been shortly explained in (4.16) of section 3.1; for completeness here follows a short remark on the two allowed values of the Hubble function. Two equations are necessary to calculate H_1 and H_2; assuming by definition $H_2 > H_1$, consider the uncertainty equation (4.1) at the today time $t = t_u$ of (5.1). Since H_u has physical dimensions of $time^{-1}$, it is reasonable to define $n\hbar = \delta t_u \delta\varepsilon_u$ putting $\delta\varepsilon_u = h\delta H_u$ with $h\delta H_u \delta t_u = n\hbar$ and $\delta t_u = t_u - t_{bb}$, being t_{bb} the big bang time. It is of interest here to put $\delta H_u = H_2 - H_1$; moreover, as t_{bb} is in fact the beginning of space and time, put without loss of generality $t_{bb} = 0$ with respect to which is defined t_u. The uncertainty takes the form

$$H_2 - H_1 = \frac{n}{2\pi t_u}. \tag{6.12}$$

The second equation is guessed thinking that at the cosmological scale $H \approx t^{-1}$, having approximated $1/\delta t \approx 1/t$. Then the second equation has the form of a boundary condition for H_1^{-1} and H_2^{-1}, *i.e.*

$$\frac{1}{H_1} + \frac{1}{H_2} = t_u. \tag{6.13}$$

For $n = 1$ these two equations yield

$$t_u = 8.7 \times 10^{17}\ \text{s} \Rightarrow H_1 = 2.21 \times 10^{-18} \quad H_2 = 2.39 \times 10^{-18}\ \text{s}^{-1},$$

i.e. 68.2 (Km/s)/Mpc and 73.8 (Km/s)/Mpc with conversion factor 1 (Km/s)/Mpc = $3.24 \times 10^{-20}\ \text{s}^{-1}$. The observed values are 74.3 ± 2 and 68.2 ± 0.6 Mpc [10]: in fact there in the literature are several error ranges around different central values, yet the agreement is reasonable. Once more is evident the reliability of the value of t_u proposed in (5.1).

6.2. The MOND Approach

This section has clearly classical character; it updates the section 3, where the Newton law has been introduced along with non-Newtonian potential terms. Differentiate (3.12) expressed in radial coordinates with respect to v_r and calculate the ratio $\delta p_r/\delta t$ via the uncertainty equation; as $v_r = \delta r/\delta t$ and $\delta(v/\beta) = \delta v/\beta^3$ owing to (3.5) and (3.17), one finds

$$F = \frac{\delta p}{\delta t} = \frac{m}{\beta^3}\frac{\delta v}{\delta t} \quad \beta = \beta(v) \Rightarrow \frac{\hbar}{\beta^3}\frac{\delta v}{\delta t} = \epsilon v - \hbar a_0. \tag{6.14}$$

Approximating $\beta \approx 1$ for $v \ll c$, this result is nothing else but the Newton law; thus (6.14) is a reasonable starting point to concern the MOND approach. Implementing these equations one finds

$$\frac{F}{m} = \frac{1}{\beta^3}\frac{\delta v}{\delta t} = \frac{\epsilon v}{\hbar} - a_0 \quad a_N = \frac{\delta v}{\delta t} = \frac{mc^2 v\beta^2}{\hbar} - a_0\beta^3$$

so that

$$\lim_{v=0} a_N = -a_0 \quad \lim_{v=c} a_N = 0, \tag{6.15}$$

being a_N the Newtonian acceleration. The vanishing of a_N for $v \to c$ is self evident recalling (3.16).

However (6.14) waives a key point in this respect: the number n of allowed states of a gravitational quantum system, hidden in and implied by the uncertainty ranges of (6.14). To this purpose consider a spiral galaxy and introduce $\delta F_r = F_r - F_N$, being F_r and F_N the radial force components at the periphery and near the center of the galaxy. Since $F_r = \delta p_r/\delta t$, calculate first $\delta F_r = \delta(\delta p_r) = \delta(n\hbar/\delta r)$; it follows

$$\delta(\delta p_r) = \hbar\frac{\delta n}{\delta r} - n\hbar\frac{\delta(\delta r)}{\delta r^2} \Rightarrow$$
$$\frac{\delta(\delta p_r)}{\delta t} = \delta F_r = \delta\epsilon\frac{\delta n/n}{\delta r} - n\hbar\frac{\delta(\delta r)/\delta t}{\delta r^2} \quad \delta\epsilon = \frac{n\hbar}{\delta t}: \tag{6.16}$$

the radial momentum gradient along the radial distance δr from a gravity source introduces acceleration and force radial components. In general (6.16) reads

$$\delta F_r = \delta\epsilon\left(\frac{\delta n/n}{\delta r} - \frac{\delta(\delta r)}{\delta r^2}\right) \quad \delta\epsilon = \frac{n\hbar}{\delta t} \Rightarrow F_r = \delta\epsilon\frac{\delta n/n}{\delta r} \quad F_N = \delta\epsilon\frac{\delta(\delta r)}{\delta r^2}: \tag{6.17}$$

the non-Newtonian and Newtonian terms decrease according to δr^{-1} and δr^{-2}, *i.e.* at large distance from the gravity source the latter becomes negligible with respect to the former.

The notation δF_r implied by $\delta(\delta p_r/\delta t)$ is coherent with the difference of two radial forces at the right hand side of (6.16), which combine the components

$\epsilon/\delta r$ and $\epsilon\delta(\delta r)/\delta r^2$ in a unique interpolation formula governed by the Newtonian stretching rate $\delta(\delta r)/\delta t$ of the space range δr and by the quantum change rate $\delta n/\delta t$ of the number n of allowed states for the gravitational system described by δF_r. Clearly the latter is a relativistic effect of space deformation $\delta(\delta r)$ due to the interacting masses, in fact the space curvature that stretches the Euclidean δr of the special relativity, remind (4.26) and (4.29), the former is mere quantum effect. Regard thus separately these interpolation terms (6.17) of (6.16): the first addend in parenthesis dependent on the radial distance δr^{-1} is governed by $\delta n/n$, whereas the square radial distance δr^{-2} is governed by $\delta(\delta r)$. Even without explicit calculations, one infers the classical Newtonian and non-Newtonian trend at large and small δr: depending on the relative values of δn and n, intermediate behaviors are also possible without additional hypotheses and exotic assumptions about "dark matter". These considerations show that the interpolation terms have actually a profound foundation: relativistic meaning and quantum meaning.

The remainder of this section aims to emphasize the physical meaning of δn and $\delta(\delta r)$.

To explain why n changes, consider first that the gravitational interaction propagates at rate v_r' between two masses δr apart m and m'; thus (6.16) reads

$$\delta F_r = \frac{n\hbar\delta r}{\delta t\delta r^2}\left(\frac{\delta n}{n}-\frac{\delta(\delta r)}{\delta r}\right)=\frac{n\hbar v_r''}{\delta r^2}\quad v_r''=v_r'\left(\frac{\delta n}{n}-\frac{\delta(\delta r)}{\delta r}\right)\quad v_r'=\frac{\delta r}{\delta t}, \tag{6.18}$$

i.e. the radial distance range δr between the masses is defined by the propagation rate itself of the gravitational interaction. So, δF_r related to v_r'' requires anyway $|v_r''|\le c$. Therefore put $v_r'=c$ and write

$$\delta F_r=\frac{n\hbar v_r''}{\delta r^2}\quad v_r''=c\left(\frac{\delta n}{n}-\frac{\delta(\delta r)}{\delta r}\right)\Rightarrow\left|\frac{\delta n}{n}-\frac{\delta(\delta r)}{\delta r}\right|\le 1, \tag{6.19}$$

with gravitational interaction field propagating at rate c as already found in (DVQ).

-On the one hand put in (6.18) $n\hbar v_r''=\pm Gm'm$ by dimensional reasons, with G proportionality constant, and $\delta r/\delta t=c$ to intend that the distance δr between the interacting masses is compliant itself with the propagation rate c of the gravitational interaction. Thus δF_r fulfills the Newtonian form

$$\delta F_r=\pm\frac{Gm'm}{\delta r^2}\quad v_r''=\pm\frac{Gm'm}{n\hbar}\quad a_r=\frac{Gm}{\delta r^2}, \tag{6.20}$$

where however two different terms are hidden in and summarized by v_r''. So, if $m'm$ is large enough so that $Gm'm/n\hbar>c$ for a given n, then n must increase to $n+n'$ in order that $Gm'm/(n+n')\hbar<c$ is still compatible with a new $|v_r'''|<c$. Simply rewriting (6.14) as (6.20), the uncertainty introduces the quantization and removes the idea of instantaneous action at a distance once having introduced space ranges through which the gravitational interaction propagates at radial rate c. Even without higher order potential terms of (4.22) or (2.27), the Newton law is approximate but not conceptually wrong like the deterministic clas-

sical form $Gm'm/r^2$.

-On the other hand, an analogous conclusion holds also for the potential energy. Indeed (6.18) yields

$$c\left(\frac{\delta n}{n}-\frac{\delta(\delta r)}{\delta r}\right)=\pm\frac{Gm'm}{n\hbar}$$
$$\Rightarrow\left(\frac{\delta n}{n}-\frac{\delta(\delta r)}{\delta r}\right)\delta\epsilon=\pm\frac{m'mG}{\delta\ell}\quad n\hbar c=\delta\epsilon\delta\ell\quad\delta\ell=c\delta t. \tag{6.21}$$

To understand the physical meaning of δn remind (4.4) to (4.7) and note that $\delta \boldsymbol{p}/\delta t=\boldsymbol{F}$ whereas $\delta\mathbf{M}/\delta t=\boldsymbol{\mathcal{T}}$, being $\boldsymbol{\mathcal{T}}$ torque. The former has been concerned in (6.16) and implies F_N of (FFF), the latter explains what has to do F_r with the rotation of galaxies; the key point is δn. Is interesting in particular (4.4), because the quantization of M_u implies $\delta\mathrm{M}_u=\hbar\delta n$ introduced in (FFF); it suggests that $\hbar\delta n/\delta t$ is thinkable as torque acting on a rotating quantized system. Since $\delta v_r=\delta(\delta r/\delta t)=\delta(\delta r)/\delta t$ during a fixed time lapse δt, (6.16) yields

$$\delta\frac{\delta p_r}{\delta t}=\frac{\mathcal{T}}{\delta r}-\frac{n\hbar}{\delta t}\frac{\delta(\delta r)}{\delta r^2}\quad\mathcal{T}_u=\frac{\delta(n\hbar)}{\delta t}. \tag{6.22}$$

In general the torque expresses the ability of a force to impart a rotation around an axis. Given a two body system, the uncertainty evidences that the radial component of a gravitational gravitational force acting on a mass determines properties like radial a_r and F_N components, whereas the change δn of the number n of allowed states of the system accounts for their orbiting behavior. However this latter is triggered by large orbiting systems like a huge galaxy, where reasonably peripheral δr imply the δr^{-2} Newtonian term negligible with respect to δr^{-1} uncertainty driven term: in other words the masses and their distance determine whether the attractive force implies implies mere radial effect or also combined δn driven orbiting motion to fulfill $|v_r''|\le c$ in (6.19). A few considerations highlight further these notes paying attention to the specific system of a spiral galaxy, whose behavior is controlled not only by the relative values of radial ranges δr^{-1} and δr^{-2} but also by the finite n and δn. In fact the near field behavior of (6.21) is reasonably expected for stars with small average distance from the galaxy rotation center; the far field behavior prevails instead for stars very distant from the galaxy center. Anyway, the expected whole behavior of a large gravitational system, *i.e.* with variable number of allowed quantum states, cannot be in principle merely Newtonian. In practice, in order for the addend $\delta(\delta r)/\delta r$ to be negligible with respect to the non-Newtonian term $\delta n/n$, the galaxy must be huge and massive; these conditions allow large δr and δn. Thus small galaxies should not exhibit the MOND behavior.

At this point, once having explained the physical meaning of the terms r^{-1} and r^{-2}, here merely replaced by the respective uncertainty ranges (6.17), the well known (2.35) does not need further comments. It is worth emphasizing that according to (6.20), putting $\delta r=r_u$ and $m'=M_u$, universe radius and mass, one finds

$$a_r = \frac{M_u G}{r_u^2} = 1.9\times10^{-10}\,\frac{\text{m}}{\text{s}^2}, \tag{6.23}$$

which agrees reasonably with $a_0 = F_{Pl}/M_u = 2\times10^{-10}\ \text{m/s}^2$ and $c^2/r_u = 2\times10^{-10}\ \text{m/s}^2$; also, as concerns (2.35) note that $(M_u G a_0)^{1/4} = c$ so that $v_1 = c\zeta^{-1/4}\ \text{m/s}$. These results support the value of M_u and suggests that the galaxies rotate consistently with centripetal acceleration whose order of magnitude is definable through parameters at the universe scale; in other words, this centripetal acceleration is not mere local feature but appears to have more general meaning. It appears that ρ_{de} and η_{de} calculated with $M_{de}c^2/V_u$ of (6.3) fit reasonably the literature values of ρ_{vac} and η_{vac}. Moreover note the dimensional relationship $G\times \textit{energy density} = \textit{acceleration}^2$; indeed

$$\sqrt{\eta_{de} G} = 2\times10^{-10}\ \text{m/s}^2 . \tag{6.24}$$

in agreement with (6.23). The previous (2.34) had classical meaning; here the addends (6.17) and (6.22) have quantum meaning, the value of a_0 is calculated in (5.12) and (6.23) in the cosmological frame of parameters of the universe.

7. Discussion and Conclusions

One of main tasks of this paper was to regard the cosmological parameters not separately, *i.e.* as individual quantities that concur to calculate something, but as values collectively connected themselves by a rational link. Whit this strategy, any calculation confirms other calculations if the input data are a self-consistent basis to describe physical effects apparently independent. Thanks to this basic idea, it has been possible to ascertain by direct calculation that the total visible mass M_{ob} extrapolated to the whole universe volume, beyond our boundary of sight due to the finite light speed propagation, is enough to include the so called "dark matter" as mere completion of the visible m_{ob}. Moreover, once having calculated η_{vac} in a previous paper, the numerical calculations hitherto carried out bring to the conclusion the connection between dark energy and quantum vacuum energy. Eventually, replacing systematically ∂ with δ not only aims to overcome the deterministic approach based on any physical variable, but also to bypass the idea of immediate cause-effect and instantaneous long range effect. The fact a finite time range is necessary for a finite change of velocity of the particle subjected to a force, implies in a natural way to define the distance δr via the propagation rate of any perturbation.

As concerns ordinary matter existing in the universe, note interesting information coming from the wavelike and corpuscular definitions of p ; the formalism is driven by (3.12), (3.30) and (3.31). By differentiating (3.20) one finds

$$\delta p = -\frac{h}{\lambda^2}\delta\lambda = -\frac{hc^2}{\lambda^2 c^2}\delta\lambda = -\frac{h\nu}{c^2}\nu\delta\lambda = -\frac{h\nu}{c^2}\delta v \quad \delta v = \nu\delta\lambda.$$

Now rewrite this result via corpuscular formalism; the wavelike to corpuscular correspondence reads

$$\delta\left(\frac{h}{\lambda}\right)=-\frac{h\nu}{c^2}\delta v \Leftrightarrow \delta\left(\frac{mv}{\beta}\right)=-\frac{\epsilon}{c^2\beta}\delta v, \qquad (7.1)$$

being ϵ/β the amount of energy corresponding to $h\nu$. Rewrite the right hand side of (7.1) as follows

$$\frac{mv_2}{\beta_2}-\frac{mv_1}{\beta_1}=\frac{m}{\beta}v_1-\frac{m}{\beta}v_2 \Rightarrow \frac{mv_1}{\beta_1}=\frac{\epsilon/\beta}{c^2}v_2 \quad \frac{mv_2}{\beta_2}=\frac{\epsilon/\beta}{c^2}v_1.$$

Multiply side by side these two equations; then

$$m^2\frac{v_1v_2}{\beta_1\beta_2}=\frac{\epsilon^2}{c^4\beta^2}v_1v_2 \Rightarrow \frac{\left(mc^2\right)^2}{\epsilon^2}=\frac{\beta_1\beta_2}{\beta^2} \Rightarrow mc^2=\pm\frac{\epsilon}{\beta}\sqrt{\beta_1\beta_2}=\pm h\nu\sqrt{\beta_1\beta_2}.$$

The states of negative energy are well known since Dirac. The negative quantum energy, *i.e.* less than that of the quantum vacuum, is also known in the field theory; it is allowed to occur for a certain time and space ranges [11]. Once more, see e.g. (2.45), dimensional mass and material mass of (7.1) are self-consistent. In fact m results here extracted from immaterial momentum h/λ through the corpuscular momentum mv/β via $h\nu \leftrightarrow mc^2/\beta$: *i.e.* a massive particle corresponds to wavelike energy. The well known concept *mass*↔ *energy* still holds in the form *mass*↔ *quantum wave energy*; the last equation is in fact an energy balance. Further work on this subtle topic is in progress, in particular as concerns the physical meaning of $\beta_{eff}=\sqrt{\beta_1\beta_2}$. Express the effective energy and momentum at the macroscopic and Planck scales as $\epsilon_{eff}\ell_{eff}\nu_{eff}/c=\epsilon_{Pl}$ and $m_{eff}\nu_{eff}\ell_{eff}/c=p_{Pl}$, functions of $\nu_{eff}\ell_{eff}$ only, plus the obvious condition $m_{eff}c^2=h\nu_{eff}$. Test a set of trial values ℓ_{eff} to identify $\ell_{eff}\nu_{eff}$ fulfilling both equations, and select the one with ν_{eff} satisfying also the given condition. It yields ν_{eff} = 3 × 10^{25} s^{-1} and m_{eff} = 2.2 × 10^{25} Kg: in fact $h\nu_{eff}=m_{eff}c^2$ = 125 GeV implies $\epsilon_{eff}=a_0 m_{eff} r_u$ and $\epsilon_{eff}=m_{eff}\Lambda_t/\Lambda$ too, which link Planck and cosmological scales.

It is worth noticing that the definition of (3.1) is in fact included in a more general context, e.g. noting that

$$c^2-\frac{c^2}{\beta}=-\frac{1}{2}v^2-\frac{3}{8}\frac{v^4}{c^2}+\cdots \Rightarrow \frac{\varphi}{c^2}=1-\frac{1}{\beta} \Rightarrow \frac{\delta\varphi}{c^2}=\left(1-\frac{1}{\beta}\right);$$

it could provide in principle a more fundamental basis to extend further the ideas proposed in this paper.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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