Journal of Applied Mathematics and Physics, 2025, 13(11), 4002-4015
https://www.scirp.org/journal/jamp
ISSN Online: 2327-4379
ISSN Print: 2327-4352

Optimal Strategies in a Line Target Reach-Avoid Differential Game

Xinyi Zhu[1], **Yan Zhu**[1], **Fangfei Li**[1,2]

[1]School of Mathematics, East China University of Science and Technology, Shanghai, China
[2]Key Laboratory of Smart Manufacturing in Energy Chemical Process, Ministry of Education, East China University of Science and Technology, Shanghai, China
Email: zxyihemmm@163.com, zhuygraph@ecust.edu.cn, li_fangfei@163.com

How to cite this paper: Zhu, X.Y., Zhu, Y. and Li, F.F. (2025) Optimal Strategies in a Line Target Reach-Avoid Differential Game. *Journal of Applied Mathematics and Physics*, **13**, 4002-4015.
https://doi.org/10.4236/jamp.2025.1311224

Received: October 28, 2025
Accepted: November 17, 2025
Published: November 20, 2025

Abstract

This study investigates a planar differential game involving a singular attacker and defender, wherein the attacker endeavors to arrive at a specified target line while the defender seeks to intercept prior to this occurrence. The game is formulated with explicit boundary conditions, and the derivation of optimal strategies for both participants is conducted by examining the geometric characteristics of the reachable sets and terminal configurations. In contrast to traditional pursuit-evasion games, which utilize point or circular targets, the introduction of a target line presents unique geometric intricacies that complicate the determination of interception conditions. Analytical solutions for optimal controls are derived, accompanied by a lucid geometric interpretation of terminal states. The proposed formulation not only broadens the reach-avoid framework to encompass line-type targets but also offers theoretical guidance for practical defense scenarios involving extended boundaries. Ultimately, the analysis of this simplified single-attacker-single-defender scenario provides a foundational basis for future explorations into multi-agent defense systems.

Keywords

Differential Game, Pursuit-Evasion, Reach-Avoid, Optimal Control, Target Line

1. Introduction

In contemporary defense and interception frameworks [1], the attacker-defender engagement is a core issue in dynamic decision-making [2]. The attacker strives to reach a specified target area or boundary, while the defender endeavors to prevent this by intercepting the attacker beforehand. Such pursuit-evasion interactions [3] can be naturally formulated within the framework of differential games [4], which provide a rigorous mathematical foundation for deriving the optimal strategies of both players. A clear understanding of the dynamics in a simplified single-line-target scenario [5] serves as a crucial basis for extending the analysis to more complex multi-agent defense settings.

Classical pursuit-evasion and defense games have been extensively studied since Isaacs [4] introduced the theory of differential games, which laid the groundwork for analyzing optimal strategies in dynamic two-player interactions. Among these, reach-avoid games [6]-[8] represent one of the most prominent applications. Numerous studies have investigated pursuit and interception problems under various geometric configurations, such as point targets [9], circular target regions [10], and bounded domains [11]. Analytical or semi-analytical solutions have been obtained in certain cases [12]-[16], particularly when both players possess constant speeds or identical maneuvering capabilities [17]. However, when the target is represented by a line or boundary, the geometry [18] of the terminal condition becomes considerably more complex, and closed-form expressions for the optimal trajectories [19] are scarce. Moreover, most existing studies focus on instantaneous interception [20] or capture conditions [21], whereas the spatial relationship between the target line and the players' initial positions has received comparatively little attention.

Extending the approach presented in [22], this research considers a line-type target and introduces second-order dynamics for both players. Within this differential pursuit-interception framework, each player's motion is determined by acceleration control, with the attacker attempting to reach the target line and the defender working to prevent this in advance. This extension not only broadens the geometric configuration of the problem but also enhances the practical relevance of the theoretical model through refined dynamic formulations.

By incorporating second-order dynamics, an analytical framework is established based on the geometric properties of reachable sets. Under the line-target setting, we systematically investigate the optimal strategy problem by analyzing the evolution of reachable sets under acceleration constraints. New interception conditions are derived, and explicit optimal strategies for various initial configurations are obtained. Ultimately, a closed-form analytical solution, dependent on the terminal point along the target line, is provided.

Furthermore, building on the established second-order dynamics model, we provide a geometric characterization of the terminal states, highlighting how the attacker's optimal approach interacts with the defender's best response under acceleration constraints. This study extends classical pursuit-evasion games to line-shaped targets, providing a more realistic theoretical foundation for applications such as the defense of continuous perimeters or boundary lines.

This research on line-target differential games with second-order dynamics establishes an important foundation for multi-agent defense problems. Future work can build upon this framework to explore interactions among multiple attackers and defenders under coupled acceleration constraints, advancing the development of differential game theory toward more practical and complex scenarios.

2. Problem Description

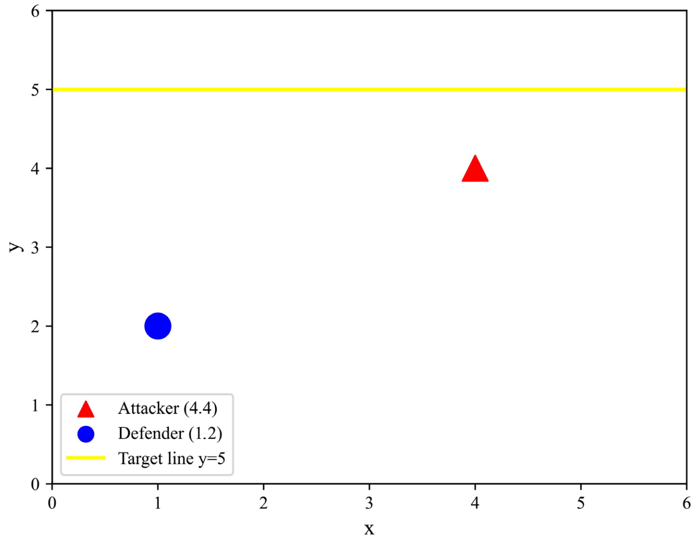

Figure 1. The initial state of the game.

The game involves two participants: an attacker and a defender (as shown in Figure 1). Positioned above both, the objective is a line defined by $y = y_s$ (*i.e.*, $y_s > y_1$ and $y_s > y_2$). The attacker has two goals: 1) arrive at the target line, 2) evade the defender. The defender has two goals: 1) catch the attacker, 2) keep the attacker away from the target line. The dynamics are as follows [13]:

$$\dot{s} = f\left(s, u_1, \theta_1, u_2, \theta_2, t\right) = \begin{bmatrix} \dot{x}_1 \\ \dot{y}_1 \\ \dot{v}_{x1} \\ \dot{v}_{y1} \\ \dot{x}_2 \\ \dot{y}_2 \\ \dot{v}_{x2} \\ \dot{v}_{y2} \end{bmatrix} = \begin{bmatrix} v_{x_1} \\ v_{y_1} \\ u_1 \cos\theta_1 \\ u_1 \sin\theta_1 \\ v_{x2} \\ v_{y2} \\ u_2 \cos\theta_2 \\ u_2 \sin\theta_2 \end{bmatrix} \tag{1}$$

in which $i = 1, 2$ represents the attacker and the defender. $\left(x_1, y_1\right)$, $\left(x_2, y_2\right)$ are the positions of the attacker and the defender. $\left(v_{x1}, v_{y1}\right)$, $\left(v_{x2}, v_{y2}\right)$ denote the velocities of the attacker and the defender, respectively. The control inputs u_i and θ_i specify the magnitude and direction of each player's self-propelled acceleration. Both players' acceleration magnitudes are assumed to be limited, *i.e.*, $u_i \in \left[0, u_{im}\right]$, which aligns with typical scenarios. Regarding the maximum accelerations u_i, we adopt the following assumption:

$$\frac{u_{2m}}{u_{1m}} = \delta > 1. \tag{2}$$

T is the terminal time when the attacker arrives at the target line. Since the boundary conditions for the attacker's and defender's victories are different, they should be discussed separately. Following [21], we can obtain the defender's minimum capture time t^* using Algorithm 1 in [21]. By comparing the relationship between t^* and T, as well as the relationships among $y_1(t^*)$, $y_1(T)$, and y_s, the result of the engagement between the attacker and defender can be determined as summarized as: if the attacker reaches the target line before being intercepted by the defender, the attacker wins; otherwise, the defender wins.

2.1. Scenario 1: Defender Wins

In [21], we can drive $\mu \to 0$ to derive the optimal control strategies for the attacker and the defender:

$$u_2^* = u_{2m}, \tag{3}$$

$$\cos\theta_2^* = \frac{2\left(x_f - x_2 - v_{2x}t^*\right)}{u_{2m}t^{*2}}, \tag{4}$$

$$\sin\theta_2^* = \frac{2\left(y_f - y_2 - v_{2y}t^*\right)}{u_{2m}t^{*2}}, \tag{5}$$

$$u_1^* = u_{1m}, \tag{6}$$

$$\cos\theta_1^* = \frac{2\left(x_f - x_1 - v_{1x}t^*\right)}{u_{1m}t^{*2}}, \tag{7}$$

$$\sin\theta_1^* = \frac{2\left(y_f - y_1 - v_{1y}t^*\right)}{u_{1m}t^{*2}}, \tag{8}$$

which is based on taking the game termination time as the cost function. $\left(x_f, y_f\right)$ is the position of the capture. The terminal time t^* can be obtained by the method mentioned in [21]. The algorithm progressively narrows the search interval through segmented solving and derivative-based validation, and ultimately determines a valid optimal capture time.

2.2. Scenario 2: Attacker Wins

The cost functions of the two players are as follows:

$$\begin{aligned} J &= \Phi\left(s_T, T\right) \\ &= \sqrt{\left(x_1(T) - x_2(T)\right)^2 + \left(y_1(T) - y_2(T)\right)^2} \\ &\quad - \left(y_s - y_1(T)\right) + \left(y_s - y_2(T)\right) \\ &= d(T) - \left(y_s - y_1(T)\right) + \left(y_s - y_2(T)\right), \end{aligned} \tag{9}$$

where T is a given terminal time of the zero-sum game,
$d(t) = \sqrt{\left(x_1(t) - x_2(t)\right)^2 + \left(y_1(t) - y_2(t)\right)^2}$ is the distance between the attacker and the defender during the game. At the terminal time $t = T$, the first term reflects the separation between the attacker and the defender, the second term de-

notes the distance between the attacker and the target line, and the third term corresponds to the distance between the defender and the target line. The attacker aims to maximize its distance from the defender at the terminal time, minimize its distance to the target line, and keep the defender as far from the target line as possible. Therefore, the attacker prefers the cost function to be as large as possible. The defender, on the other hand, has the opposite objectives: at the end of the game, it seeks to minimize the distance to the attacker, maximize the attacker's distance from the target, and minimize its own distance to the target. Therefore, the defender prefers the cost function to be as small as possible.

The optimal cost for each player, which defines the value of the game, can be expressed as a function

$$V = \min_{u_2,\theta_2} \max_{u_1,\theta_1} J = \max_{u_1,\theta_1} \min_{u_2,\theta_2} J \tag{10}$$

The boundary condition is:

$$\phi = y_1 - y_s = 0 \tag{11}$$

Theorem 1: If the capture cannot happen before $t = T$, the optimal control strategies for the attacker and the defender based on the cost function (9) are as follows:

$$u_1^* = u_{1m}, \tag{12}$$

$$u_2^* = u_{2m}, \tag{13}$$

$$\theta_1^* = \arctan\frac{\lambda_{y1}}{\lambda_{x1}} = \arctan\frac{\left(y_1(T) - y_2(T) + d(T)\right)(1+\eta)}{x_1(T) - x_2(T)}, \tag{14}$$

$$\theta_2^* = \arctan\frac{\lambda_{y2}}{\lambda_{x2}} = \arctan\frac{y_1(T) - y_2(T) + d(T)}{x_1(T) - x_2(T)}, \tag{15}$$

where $\left(x_1(T), y_1(T)\right)$ and $\left(x_2(T), y_2(T)\right)$ are the target points of the attacker and the target.

Proof. The Hamiltonian is

$$\begin{aligned} H = {} & \lambda_{x1} v_{x1} + \lambda_{y1} v_{y1} + \lambda_{v_{x1}} u_1 \cos\theta_1 + \lambda_{v_{y1}} u_1 \sin\theta_1 \\ & + \lambda_{x2} v_{x2} + \lambda_{y2} v_{y2} + \lambda_{v_{x2}} u_2 \cos\theta_2 + \lambda_{v_{y2}} u_2 \sin\theta_2 \end{aligned} \tag{16}$$

The costates are driven by

$$\dot{\lambda}_{x_1} = \frac{\partial H}{\partial x_1} = 0, \tag{17}$$

$$\dot{\lambda}_{y_1} = \frac{\partial H}{\partial y_1} = 0, \tag{18}$$

$$\dot{\lambda}_{v_{x_1}} = -\frac{\partial H}{\partial v_{x1}} = -\lambda_{x1}, \tag{19}$$

$$\dot{\lambda}_{v_{y1}} = -\frac{\partial H}{\partial v_{y1}} = -\lambda_{y1}, \tag{20}$$

$$\dot{\lambda}_{x_2} = \frac{\partial H}{\partial x_2} = 0, \tag{21}$$

$$\dot{\lambda}_{y_2} = \frac{\partial H}{\partial y_2} = 0, \tag{22}$$

$$\dot{\lambda}_{v_{x_2}} = -\frac{\partial H}{\partial v_{x2}} = -\lambda_{x2}, \tag{23}$$

$$\dot{\lambda}_{v_{y2}} = -\frac{\partial H}{\partial v_{y2}} = -\lambda_{y2}. \tag{24}$$

Thus, λ_{x_1}, λ_{y_1}, λ_{x_2}, λ_{y_2} are all constants. From ([23]: p. 89), one has

$$\lambda^{\top}\left(t_f\right) = \frac{\partial \Phi_d}{\partial \boldsymbol{x}_f} + \eta \frac{\partial \phi_d}{\partial \boldsymbol{x}_f}, \tag{25}$$

where η denotes an auxiliary adjoint variable, the value of which will be specified later in the analysis. Thus, considering the terminal condition:

$$\lambda_{x1}(T) = \frac{\partial \Phi}{\partial x_1} = \frac{x_1(T) - x_2(T)}{d(T)}, \tag{26}$$

$$\lambda_{y1}(T) = \frac{\partial \Phi}{\partial y_1} = \frac{y_1(T) - y_2(T)}{d(T)} + 1 + \eta, \tag{27}$$

$$\lambda_{v_{x1}}(T) = -\frac{\partial \Phi}{\partial v_{x1}} = 0, \tag{28}$$

$$\lambda_{v_{y1}}(T) = -\frac{\partial \Phi}{\partial v_{y1}} = 0, \tag{29}$$

$$\lambda_{x2}(T) = \frac{\partial \Phi}{\partial x_2} = \frac{x_2(T) - x_1(T)}{d(T)}, \tag{30}$$

$$\lambda_{y2}(T) = \frac{\partial \Phi}{\partial y_2} = \frac{y_2(T) - y_1(T)}{d(T)} - 1, \tag{31}$$

$$\lambda_{v_{x2}}(T) = -\frac{\partial \Phi}{\partial v_{x2}} = 0, \tag{32}$$

$$\lambda_{v_{y2}}(T) = -\frac{\partial \Phi}{\partial v_{y2}} = 0. \tag{33}$$

Considering (17) and (26), we can derive that

$$\lambda_{x1}(t) = \frac{\partial \Phi}{\partial x_1} = \frac{x_1(T) - x_2(T)}{d(T)} \tag{34}$$

Similarly,

$$\lambda_{y1}(t) = \frac{\partial \Phi}{\partial y_1} = \frac{y_1(T) - y_2(T)}{d(T)} + 1 + \eta, \tag{35}$$

$$\lambda_{x2}(t) = \frac{\partial \Phi}{\partial x_2} = \frac{x_2(T) - x_1(T)}{d(T)}, \tag{36}$$

$$\lambda_{y2}(t) = \frac{\partial \Phi}{\partial y_2} = \frac{y_2(T) - y_1(T)}{d(T)} - 1. \tag{37}$$

Considering (19) and (28), one has:

$$\lambda_{v_{x1}}(t) = \lambda_{x1}(T-t) \tag{38}$$

Similarly,

$$\lambda_{v_{y1}}(t) = \lambda_{y1}(T-t), \tag{39}$$

$$\lambda_{v_{x2}}(t) = \lambda_{x2}(T-t), \tag{40}$$

$$\lambda_{v_{y2}}(t) = \lambda_{y2}(T-t). \tag{41}$$

The terminal Hamiltonian satisfies [23]:

$$H(T) = -\frac{\partial \Phi}{\partial T} - \eta \frac{\partial \phi}{\partial T} = 0, \tag{42}$$

and $\frac{\partial H}{\partial t} = 0$, so $H(t) = 0$ for all $t \in [0, T]$. Then, we consider the optimal control inputs of the system, derived by

$$\begin{cases} \frac{\partial H}{\partial u_1} = \lambda_{v_{x1}} \cos\theta_1 + \lambda_{v_{y1}} \sin\theta_1 \\ \frac{\partial H}{\partial \theta_1} = -\lambda_{v_{x1}} u_1 \sin\theta_1 + \lambda_{v_{y1}} u_1 \cos\theta_1 = 0 \end{cases} \tag{43}$$

The optimal control actions for the attacker can be determined as follows:

$$\tan\theta_1^* = \frac{\lambda_{v_{y1}}}{\lambda_{v_{x1}}} = \frac{\lambda_{y1}(T-t)}{\lambda_{x1}(T-t)} = \frac{\lambda_{y1}}{\lambda_{x1}}, \tag{44}$$

which is a constant. Similar to [21], one has

$$u_1 = u_{1m} \cdot \frac{1}{2}\left(1 - \operatorname{sign}\left(\lambda_{v_{x1}} \cos\theta_1 + \lambda_{v_{y1}} \sin\theta_1\right)\right). \tag{45}$$

Furthermore, to minimize (16), $\frac{\partial H}{\partial u_1} < 0$ in (43), and

$$\frac{\partial^2 H}{\partial \theta_1^2} = -\lambda_{v_{x1}} u_1 \cos\theta_1 - \lambda_{v_{y1}} u_1 \sin\theta_1 > 0. \tag{46}$$

Therefore,

$$u_1 = u_{1m}. \tag{47}$$

Similarly,

$$\tan\theta_2^* = \frac{\lambda_{v_{y2}}}{\lambda_{v_{x2}}} = \frac{\lambda_{y2}(T-t)}{\lambda_{x2}(T-t)} = \frac{\lambda_{y2}}{\lambda_{x2}}, \tag{48}$$

And

$$u_2 = u_{2m}. \tag{49}$$

The velocities and trajectories of the attacker and defender can be expressed as follows:

$$v_{xi} = u_{im} \cos\theta_i^* t + v_{xi}(0), \tag{50}$$

$$v_{yi} = u_{im} \sin\theta_i^* t + v_{yi}(0), \tag{51}$$

$$x_i = \frac{1}{2}u_i \cos\theta_i^* t^2 + v_{xi}\left(0\right)t + x_i\left(0\right), \tag{52}$$

$$y_i = \frac{1}{2}u_i \sin\theta_i^* t^2 + v_{yi}\left(0\right)t + y_i\left(0\right). \tag{53}$$

The boundary conditions are:

For the Attacker (Player 1):

At $t = T$:

$$v_{x1}\left(T\right) = u_{1m}\cos\theta_1^* T + v_{x1}\left(0\right), \tag{54}$$

$$v_{y1}\left(T\right) = u_{1m}\sin\theta_1^* T + v_{y1}\left(0\right), \tag{55}$$

$$x_1\left(T\right) = \frac{1}{2}u_{1m}\cos\theta_1^* T^2 + v_{x1}\left(0\right)T + x_1\left(0\right), \tag{56}$$

$$y_1\left(T\right) = \frac{1}{2}u_{1m}\sin\theta_1^* T^2 + v_{y1}\left(0\right)T + y_1\left(0\right). \tag{57}$$

For the Defender (Player 2):

At $t = T$:

$$v_{x2}\left(T\right) = u_{2m}\cos\theta_2^* T + v_{x2}\left(0\right), \tag{58}$$

$$v_{y2}\left(T\right) = u_{2m}\sin\theta_2^* T + v_{y2}\left(0\right), \tag{59}$$

$$x_2\left(T\right) = \frac{1}{2}u_{2m}\cos\theta_2^* T^2 + v_{x2}\left(0\right)T + x_2\left(0\right), \tag{60}$$

$$y_2\left(T\right) = \frac{1}{2}u_{2m}\sin\theta_2^* T^2 + v_{y2}\left(0\right)T + y_2\left(0\right). \tag{61}$$

Then one has:

$$\cos\theta_1^* = \frac{2\left[x_1\left(T\right) - x_1\left(0\right) - v_{x1}\left(0\right)T\right]}{u_{1m}T^2}, \tag{62}$$

$$\sin\theta_1^* = \frac{2\left[y_1\left(T\right) - y_1\left(0\right) - v_{y1}\left(0\right)T\right]}{u_{1m}T^2}, \tag{63}$$

$$\cos\theta_2^* = \frac{2\left[x_2\left(T\right) - x_2\left(0\right) - v_{x2}\left(0\right)T\right]}{u_{2m}T^2}, \tag{64}$$

$$\sin\theta_2^* = \frac{2\left[y_2\left(T\right) - y_2\left(0\right) - v_{y2}\left(0\right)T\right]}{u_{2m}T^2}. \tag{65}$$

As long as the terminal points of the attacker and the defender are known, the optimal strategies can be expressed. Based on (52) and (53), we have:

$$\left[x_i\left(t\right) - x_i\left(0\right) - v_{xi}\left(0\right)t\right]^2 + \left[y_i\left(t\right) - y_i\left(0\right) - v_{yi}\left(0\right)t\right]^2 = \frac{1}{4}u_{im}^2 t^4. \tag{66}$$

Therefore, the attacker and the defender always lie on the isochronous circles centered at their respective starting points, with radius $\frac{1}{2}u_{im}t^2$. The results indicate that while the overall structure of the optimal control strategies for attacker and defender victories is alike, the specific details vary. In the case of the defender's victory, the final state corresponds to the extreme aiming scenario described in [21]. In the case of the attacker's victory, the final state corresponds to a scenario similar to that shown in **Figure 2**.

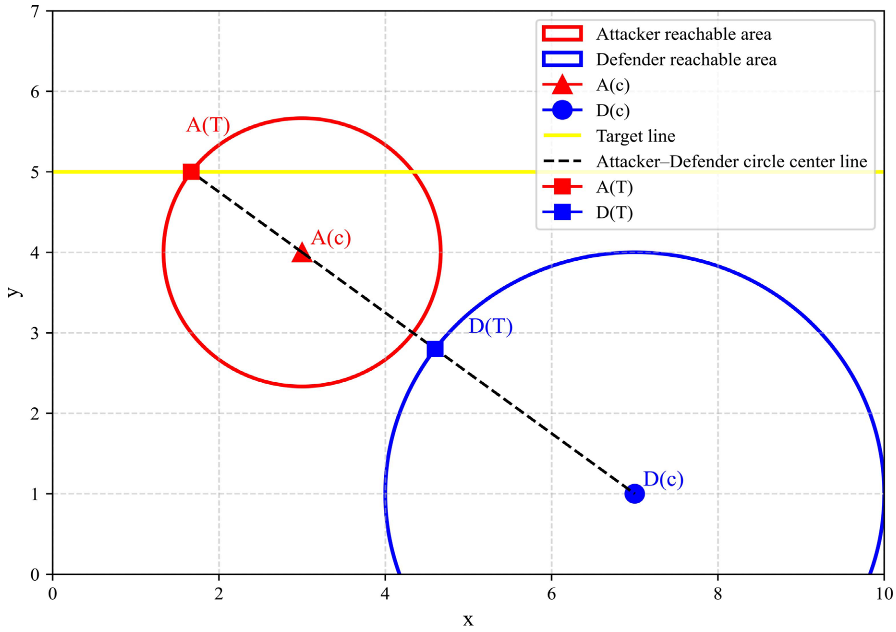

Figure 2. Selection of terminal points for the attacker and defender.

The yellow target line within the red circle denotes the region accessible to the attacker. The attacker selects the point farthest from the defender (red square) to maximize the cost function, while the defender selects the point closest to the attacker (blue square) to minimize it. $A(c)$ and $D(c)$ are the centers of the attacker's and defender's circles at the same time, while $A(T)$ and $D(T)$ are the target points chosen by the attacker and the defender, respectively.

Substituting (34)-(37), (62)-(65) into (16), we have:

$$
\begin{aligned}
H(t) = {} & \frac{x_1(T)-x_2(T)}{d(T)}\left(u_{1m}\cos\theta_1^* T + v_{x1}(0)\right) \\
& + \left(\frac{y_1(T)-y_2(T)}{d(T)} + 1 + \eta\right)\left(u_{1m}\sin\theta_1^* T + v_{y1}(0)\right) \\
& + \frac{x_2(T)-x_1(T)}{d(T)}\left(u_{2m}\cos\theta_2^* T + v_{x2}(0)\right) \\
& + \left(\frac{y_2(T)-y_1(T)}{d(T)} - 1\right)\left(u_{2m}\sin\theta_2^* T + v_{y2}(0)\right) \\
= {} & 2\frac{x_1(T)-x_2(T)}{d(T)}\left(x_1(T)-x_1(0)\right) \\
& + 2\left(\frac{y_1(T)-y_2(T)}{d(T)} + 1 + \eta\right)\left(y_1(T)-y_1(0)\right)
\end{aligned}
$$

$$
\begin{aligned}
&+2\frac{x_2(T)-x_1(T)}{d(T)}\left(x_2(T)-x_2(0)\right)\\
&+2\left(\frac{y_2(T)-y_1(T)}{d(T)}-1\right)\left(y_2(T)-y_2(0)\right)\\
=\,&0
\end{aligned}
\tag{67}
$$

By solving (67), one has:

$$\eta=-\frac{\Lambda}{d(T)\left(y_1(T)-y_1(0)\right)}, \tag{68}$$

where

$$
\begin{aligned}
\Lambda=&\left(x_1(T)-x_2(T)\right)\left(\left[x_1(T)-x_1(0)\right]-\left[x_2(T)-x_2(0)\right]\right)\\
&+\left(y_1(T)-y_2(T)+d(T)\right)\left(\left[y_1(T)-y_1(0)\right]-\left[y_2(T)-y_2(0)\right]\right).
\end{aligned}
\tag{69}
$$

Therefore, we can rewrite θ_1^* and θ_2^* as

$$\theta_1^*=\arctan\frac{\lambda_{y1}}{\lambda_{x1}}=\arctan\frac{\left(y_1(T)-y_2(T)+d(T)\right)(1+\eta)}{x_1(T)-x_2(T)}, \tag{70}$$

$$\theta_2^*=\arctan\frac{\lambda_{y2}}{\lambda_{x2}}=\arctan\frac{y_1(T)-y_2(T)+d(T)}{x_1(T)-x_2(T)}, \tag{71}$$

where η is shown as (68). Since we have derived θ_1^*, we can substitute θ_1^* into (57) to solve it for the terminal time T. Up to this point, all unknowns have been solved, and the proof is thereby complete.

□

3. Simulation

We conducted simulations for both scenarios—attacker victory and defender victory—and the results are shown below.

3.1. Attacker-Win Scenario

To validate the effectiveness of the proposed method, we conducted a simulation for the attacker victory scenario. The attacker starts at $p_1(0)=(3.0,4.0)$ and the defender at $p_2(0)=(7.0,1.0)$, with initial velocities $v_1(0)=(-0.5,0.2)$ and $v_2(0)=(-0.6,0.3)$, and $u_{1m}=0.343$, $u_{2m}=1.553$. The attacker aims for a reachable point on the yellow horizontal target line $y=5$, selecting the point farthest from the defender to maximize the cost function, while the defender chooses the point most aligned toward the attacker to minimize it. The resulting trajectories, shown in Figure 3, illustrate that the attacker reaches the target line (red point) successfully, while the defender (blue trajectory) cannot intercept. The attacker's final position $\left(x_1(T),y_1(T)\right)\approx(1.667,5.0)$ exactly matches the target, and the defender reaches the analytically chosen interception point. This simulation confirms the attacker's optimal strategy and the defender's trajectory adjustment based on the corrected target selection.

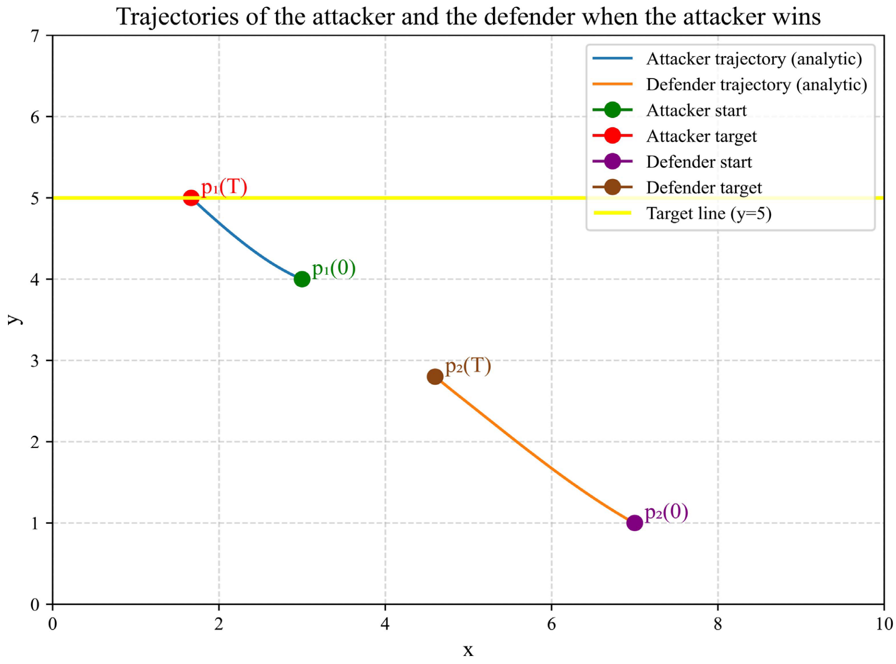

Figure 3. The simulation of attacker win-scenario.

3.2. Defender-Win Scenario

In this simulation, the attacker and defender are initially positioned as $\boldsymbol{p}_1(0)=(3,4)$ and $\boldsymbol{p}_2(0)=(7,1)$, with initial velocities $\boldsymbol{v}_1(0)=(-0.5,0.2)$ and $\boldsymbol{v}_2(0)=(-0.6,0.3)$, respectively. For the defender-winning case, the internal tangency condition is considered, where the defender's reachable set fully contains that of the attacker. The control magnitudes used are $u_1 \approx 0.708$ for the attacker and $u_2 \approx 3.069$ for the defender, ensuring both can reach the shared internal tangent point $\boldsymbol{p}(T) \approx (0.86, 5.24)$ at the terminal time.

The simulation results are shown in **Figure 4**, where the red and blue trajectories represent the attacker and defender, respectively. The yellow line denotes the target boundary. As observed, both players reach the terminal point simultaneously, and the defender successfully intercepts the attacker before it crosses the boundary, indicating a defender-winning scenario.

4. Conclusions

This paper investigated a planar differential game between one attacker and one defender with a target line. The attacker sought to reach the target line, whereas the defender endeavored to intercept beforehand. Through an analysis of the geometric features of the reachable sets and terminal conditions for both success scenarios, explicit optimal strategies were obtained for various initial configurations. Compared with classical point-target scenarios, the line-type target introduces distinctive geometric complexities, leading to more intricate interception conditions and richer strategic interactions.

The proposed analysis provides a clear geometric interpretation of how the interaction between the attacker's choice of its target point and the defender's

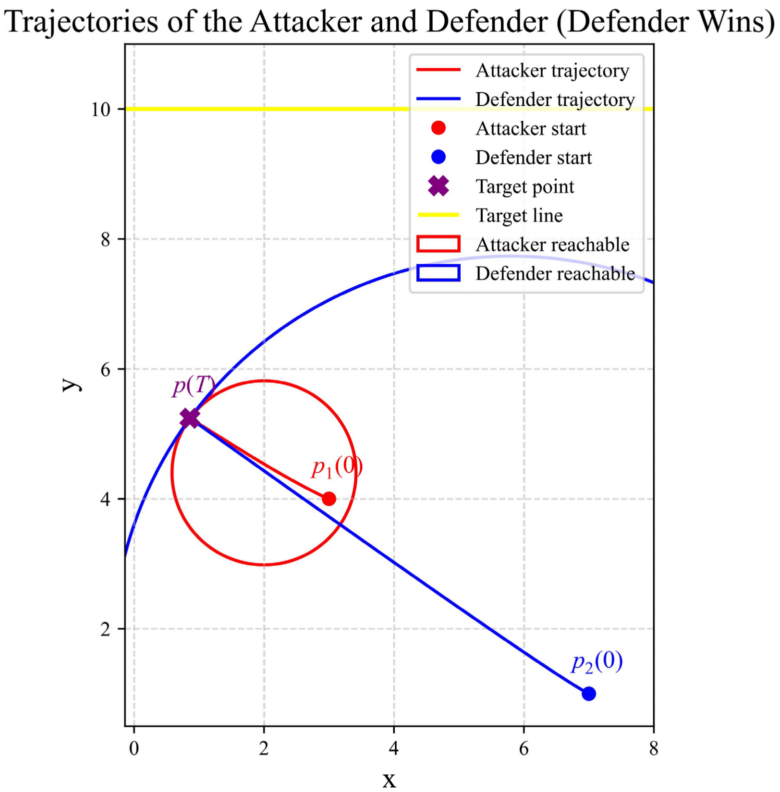

Figure 4. The simulation of defender win-scenario.

response determines the final outcome. The obtained results not only generalize traditional pursuit-evasion and reach-avoid formulations to the case of line targets, but also offer practical guidance for designing defense strategies in applications involving extended boundaries or perimeters.

The framework developed in this study establishes a theoretical foundation for future work on multi-agent defense systems, where multiple attackers and defenders interact simultaneously under coupled dynamics and strategic constraints. On one hand, isochrones characterize the reachable sets of the defender and the target, offering a universal metric for evaluating local time advantage in decentralized decision-making. On the other hand, the method reduces computational complexity through piecewise solving and derivative-based validation, thereby enabling a scalable computational framework for designing real-time multi-agent coordination strategies.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

References

[1] Starý, V. and Křivánek, V. (2024) Integrating Mechatronics Principles in Modern Air Defence Systems. 2024 21*st International Conference on Mechatronics—Mechatronika* (*ME*), Brno, 4-6 December 2024, 1-6. https://doi.org/10.1109/me61309.2024.10789743

[2] Bhatt, M., Jia, Y. and Mehr, N. (2025) Strategic Decision-Making in Multiagent Domains: A Weighted Constrained Potential Dynamic Game Approach. *IEEE Transactions on Robotics*, **41**, 2749-2764. https://doi.org/10.1109/tro.2025.3552325

[3] Li, S., Wang, C., Sun, J., Zhang, S. and Xie, G. (2025) Distributed Task Allocation with Minimum Makespan for Heterogeneous Multiplayer Pursuit-Evasion Games. *IEEE Transactions on Automatic Control*, **70**, 2827-2842. https://doi.org/10.1109/tac.2024.3485433

[4] Isaacs, R. (1999) Differential Games: A Mathematical Theory with Applications to Warfare and Pursuit, Control and Optimization. Courier Corporation.

[5] Fu, H. and Liu, H.H. (2023) Defending a Target Area with a Slower Defender. *IEEE Control Systems Letters*, **7**, 661-666. https://doi.org/10.1109/lcsys.2022.3215944

[6] Selvakumar, J. and Bakolas, E. (2021) Feedback Strategies for a Reach-Avoid Game with a Single Evader and Multiple Pursuers. *IEEE Transactions on Cybernetics*, **51**, 696-707. https://doi.org/10.1109/tcyb.2019.2914869

[7] Yan, R., Deng, R., Lai, H., Zhang, W., Shi, Z. and Zhong, Y. (2024) Homicidal Chauffeur Reach-Avoid Games via Guaranteed Winning Strategies. *IEEE Transactions on Automatic Control*, **69**, 2367-2382. https://doi.org/10.1109/tac.2023.3329693

[8] Yan, R., Mi, S., Duan, X., Chen, J. and Ji, X. (2025) Pursuit Winning Strategies for Reach-Avoid Games with Polygonal Obstacles. *IEEE Transactions on Automatic Control*, **70**, 814-829. https://doi.org/10.1109/tac.2024.3438806

[9] Li, G., Wang, J., Liu, F. and Deng, F. (2025) Target-Attackers-Defenders Linear-Quadratic Exponential Stochastic Differential Games with Distributed Control. *IEEE Transactions on Cybernetics*, **55**, 574-587. https://doi.org/10.1109/tcyb.2024.3508694

[10] Von Moll, A., Pachter, M., Shishika, D. and Fuchs, Z. (2023) Circular Target Defense Differential Games. *IEEE Transactions on Automatic Control*, **68**, 4065-4078.

[11] Lai, H., Yan, R., Zhang, W., Shi, Z. and Zhong, Y. (2021) Reach-Avoid Differential Games via Finite-Time Heading Tracking. 2021 60*th IEEE Conference on Decision and Control* (*CDC*), Austin, 14-17 December 2021, 1656-1662. https://doi.org/10.1109/cdc45484.2021.9683266

[12] Deng, R., Shi, Z. and Zhong, Y. (2024) Reach-Avoid Games with Two Cooperative Attackers: Value Function and Singular Surfaces. *IEEE Transactions on Aerospace and Electronic Systems*, **60**, 558-573. https://doi.org/10.1109/taes.2023.3326780

[13] Zou, R., Shi, Z. and Zhong, Y. (2022) Reach-Avoid Game with Circular Goal Region. 2022 41*st Chinese Control Conference* (*CCC*), Hefei, 25-27 July 2022, 6870-6875. https://doi.org/10.23919/ccc55666.2022.9902598

[14] Yan, R., Shi, Z. and Zhong, Y. (2020) Task Assignment for Multiplayer Reach-Avoid Games in Convex Domains via Analytical Barriers. *IEEE Transactions on Robotics*, **36**, 107-124. https://doi.org/10.1109/tro.2019.2935345

[15] Deng, R., Zhang, W., Yan, R., Shi, Z. and Zhong, Y. (2024) Multiple-Pursuer Single-Evader Reach-Avoid Games in Constant Flow Fields. *IEEE Transactions on Automatic Control*, **69**, 1789-1795. https://doi.org/10.1109/tac.2023.3329537

[16] Garcia, E., Casbeer, D.W. and Pachter, M. (2020) Optimal Strategies of the Differential Game in a Circular Region. *IEEE Control Systems Letters*, **4**, 492-497. https://doi.org/10.1109/lcsys.2019.2963173

[17] Yan, R., Shi, Z. and Zhong, Y. (2019) Reach-Avoid Games with Two Defenders and One Attacker: An Analytical Approach. *IEEE Transactions on Cybernetics*, **49**, 1035-1046. https://doi.org/10.1109/tcyb.2018.2794769

[18] Long, Y., Han, L., Dong, F., Hu, Q. and Li, Q. (2024) Optimal Assignment for Multi-

player Target Defense Differential Games via Analytical Geometric Approach. 2024 *American Control Conference* (*ACC*), Toronto, 10-12 July 2024, 3637-3642. https://doi.org/10.23919/acc60939.2024.10644596

[19] Huang, Y., Hu, Y., Yuan, X. and Schmeink, A. (2025) Analytical Optimal Joint Resource Allocation and Continuous Trajectory Design for UAV-Assisted Covert Communications. *IEEE Transactions on Wireless Communications*, **24**, 213-227. https://doi.org/10.1109/twc.2024.3490898

[20] Oshman, Y. and Arad, D. (2006) Differential-Game-Based Guidance Law Using Target Orientation Observations. *IEEE Transactions on Aerospace and Electronic Systems*, **42**, 316-326. https://doi.org/10.1109/taes.2006.1603425

[21] Li, S., Wang, C. and Xie, G. (2024) Optimal Strategies for Pursuit-Evasion Differential Games of Players with Damped Double Integrator Dynamics. *IEEE Transactions on Automatic Control*, **69**, 5278-5293. https://doi.org/10.1109/tac.2023.3346815

[22] Singh, S.K. and Reddy, P.V. (2023) Dynamic Network Analysis of a Target Defense Differential Game with Limited Observations. *IEEE Transactions on Control of Network Systems*, **10**, 308-320. https://doi.org/10.1109/tcns.2022.3203358

[23] Bryson, A.E. (2018) Applied Optimal Control: Optimization, Estimation and Control. Routledge.