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Propagation of Emotions and Attitudes in Football

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Abstract

This article proposes a hybrid model to study the propagation of collective and individual emotions within a football team. The model combines a reaction-diffusion equation to describe the spatial dynamics of emotions on the field and a system of coupled differential equations to capture psychosocial interactions between players. We demonstrate the existence and uniqueness of the solutions using the Lax-Milgram theorem (for the partial differential equation) and the Cauchy-Lipschitz theorem (for the ordinary differential equations). This theoretical framework unifies, for the first time, the emotional and tactical aspects of football, offering perspectives for real-time analysis of team performance.

Keywords

Emotional Propagation, Reaction-Diffusion Equation, Football Tactics, Collective Behavior, Mathematical Modeling

1. Introduction

In a football match, the emotions and collective attitudes of a team evolve dynamically according to the context of the game. A team may gradually adopt a more aggressive, cautious, or euphoric behaviour in response to match events, such as a goal scored or conceded, refereeing decisions, or the reactions of the crowd. These emotional fluctuations directly influence collective performance and the tactical choices adopted in real time by the players and the coaching staff.

Recent studies have highlighted the importance of emotional processes in team sports. Anderson and May (1998) studied the dynamics of collective behaviours within sports teams and showed how complex interactions emerge among players [1]. Barros and Duchowski (2018) explored computational models of emotions and suggested that emotions influence group decision-making and performance [2]. More specifically, Tadi and Kostic (2021) proposed a mathematical model of emotional contagion applied to sports teams, demonstrating that emotions can spread like a diffusion phenomenon, thereby influencing players' cohesion and motivation [3]. While previous studies have laid the groundwork for understanding emotional dynamics in team sports, our work introduces a novel *hybrid modeling approach* that advances the field in several key ways. Below, we clarify how our model builds upon, diverges from, and extends existing frameworks, particularly the emotional contagion model proposed by Tadi and Kostic [3].

- **Integration of Spatial and Interindividual Dynamics:** Existing models, such as that of Tadi and Kostic [3], primarily focus on emotional contagion as a diffusion process among individuals, treating the team as a network of interconnected agents. While their work effectively captures the spread of emotions through social interactions, it does not explicitly account for the *spatial distribution* of emotions on the field. Our model addresses this gap by combining a *reaction-diffusion equation* for the spatial propagation of collective emotions with a *multi-agent system* for interindividual interactions. This hybrid approach allows us to study not only how emotions spread among players but also how they evolve across the physical space of the football field, providing a more comprehensive understanding of emotional dynamics in real-time game situations.
- **Unified Mathematical Framework:** Unlike prior models that often separate psychological and tactical aspects of the game, our framework unifies these dimensions within a single mathematical structure. For instance, while Anderson and May [1] and Barros and Duchowski [2] have explored collective behaviors and computational models of emotions, respectively, their approaches do not integrate spatial dynamics. Our model bridges this divide by coupling a partial differential equation (PDE) for the collective emotional field with ordinary differential equations (ODEs) for individual emotional states. This integration enables us to analyze how spatial positioning and movement influence emotional propagation, which is critical in sports like football where player positioning and tactical formations play a pivotal role.
- **Real-Time Analysis and Tactical Implications:** The model by Tadi and Kostic [3] emphasizes the contagion of emotions but does not explicitly link emotional states to tactical decision-making or real-time performance. Our approach extends this by incorporating external influences (e.g., referee decisions, crowd reactions) and exogenous factors (e.g., individual events like fouls or successful duels) into the model. This allows us to simulate how emotional fluctuations directly impact tactical choices, such as shifting from a defensive to an offensive strategy in response to a goal or a change in the emotional climate of the team.
- **Mathematical Rigor and Generalizability:** Our model leverages the Lax-Milgram theorem for the PDE and the Cauchy-Lipschitz theorem for the ODEs to ensure the existence and uniqueness of solutions, providing a robust

mathematical foundation. This rigor not only validates the model but also enhances its generalizability to other team sports and collective behavior scenarios. In contrast, many existing models rely on heuristic or simulation-based approaches, which may lack the same level of theoretical grounding.

- **Applications to Performance Analysis:** By explicitly modeling the feedback loop between individual and collective emotions, our approach offers practical applications for real-time performance analysis. For example, coaches and analysts could use this model to identify critical moments where emotional dynamics are likely to influence team performance, enabling proactive interventions such as tactical adjustments or emotional regulation strategies.

In summary, our hybrid model represents a significant step forward by integrating spatial, psychological, and tactical dimensions into a unified framework. This advancement provides a more holistic understanding of emotional propagation in football and opens new avenues for data-driven performance optimization.

However, the mathematical modelling of these phenomena in football remains underdeveloped. The diffusion of emotions and collective attitudes can be modelled using reaction-diffusion equations, which are commonly employed to describe the propagation of dynamic phenomena in various fields, including biology and neuroscience (Tuckwell, 2005) [4]. Moreover, multi-agent models have proven effective in analysing collective behaviours and tactical decision-making in game situations (Helbing, 2001) [5]. In this work, we propose a hybrid model combining a reaction-diffusion equation to capture the spatio-temporal propagation of collective emotions and a multi-agent model to represent interindividual interactions among players. This model aims to provide a better understanding of how emotions and attitudes emerge, spread, and influence the strategic decisions of a football team. Unlike existing approaches that focus either on psychological aspects or on mechanical aspects of the game, our approach makes it possible to integrate emotional and dynamic dimensions within a unified mathematical framework.

2. Mathematical Modelling of the Propagation of Emotions and Collective Attitudes

After outlining the general framework of the mathematical modeling of emotions and collective attitudes, we now introduce a rigorous formalization of the studied problem. This section presents the equations governing the dynamics of collective emotions as well as the underlying assumptions of our approach [6] [7]. We will first establish the foundations of the model before addressing, in a subsequent section, the notations and preliminary concepts necessary for its understanding [3] [8].

2.1. Mathematical Model of the Problem

The modelling of the propagation of emotions and collective attitudes in football relies on the integration of spatio-temporal and interindividual dynamics. We propose a mathematical framework that combines a reaction–diffusion equation to model the global distribution of emotions on the field and a multi-agent model to capture individual interactions between players. This framework makes it possible to study the evolution of emotional states in response to various events such as goals, refereeing decisions, and the psychological pressure exerted by the opponent.

2.1.1. Definition of the Emotion Variable

In our model, the primary variable of interest is *emotion*, denoted by $E(x,y,t)$ for the collective emotional state and $e_i(t)$ for the emotional state of each individual player i. Emotion, as a psychological construct, is inherently multi-dimensional, encompassing aspects such as valence (positive or negative affect) and arousal (intensity of emotional activation).

However, for the purposes of this study, we simplify the representation of emotion to a *scalar value*. This scalar encapsulates the overall emotional state of a player or the team, aggregating the various dimensions of emotion into a single metric. This simplification is justified by several considerations:

- **Model Tractability:** A scalar representation allows us to focus on the core dynamics of emotional propagation without the added complexity of handling multiple dimensions. This makes the model more analytically and numerically tractable.
- **Collective Behavior:** In the context of a football team, the collective emotional state $E(x,y,t)$ reflects the overall mood or emotional climate of the team, which can be approximated by a single value that captures the dominant emotional tone at a given time and location on the field.
- **Individual Emotions:** For individual players, $e_i(t)$ represents their emotional state, which is influenced by their interactions with teammates, the collective emotional field, and external events. The scalar value provides a practical way to model these influences and their effects on player behavior.

By adopting this scalar approach, we aim to capture the essential dynamics of emotional propagation while maintaining the simplicity and interpretability of the model. Future extensions of this work could explore the incorporation of multi-dimensional emotional states to provide a more nuanced understanding of emotional dynamics in football.

2.1.2. Modelling of Collective Emotions

We model the spatio-temporal propagation of the collective emotional state $E(x,y,t)$ on the field using a reaction–diffusion equation:

$$\frac{\partial E}{\partial t} - D_E \nabla^2 E + \lambda\left(E - \bar{E}\right) = S + I,\ \ \forall t > 0, \forall (x,y) \in \Omega. \qquad (2.1)$$

The terms of this equation represent:

- **Spatial diffusion:**: $D_E \nabla^2 E$ represents the propagation of emotions across the field.
- **Regulation**: $\lambda\left(E - \bar{E}\right)$ forces the emotional state toward an average value

$\overline{E}$.

- **Sources and influences:** S describes external disturbances (a conceded goal, a referee's decision), while I captures the psychological influence of the opponent.

2.1.3. Modelling of Individual Emotions

Each player i has an emotional state $e_i(t)$, influenced by several factors:

$$\frac{\partial e_i}{\partial t} = -\gamma e_i + \beta \sum_{j \in \mathcal{N}_i} w_{ij}\left(e_j - e_i\right) + \delta\left(E - e_i\right) + \eta F_i,\ \ \forall i \in \mathcal{P}. \tag{2.2}$$

The terms of this equation are:

- **Natural dissipation**: $-\gamma e_i$ represents a return to a neutral state.
- **Influence of teammates:** $\beta \sum_{j \in \mathcal{N}_i} w_{ij}\left(e_j - e_i\right)$ models the effect of nearby players.
- **Collective influence**: $\delta\left(E - e_i\right)$ describes the impact of the global emotional field, with E being the solution of equation (2.1).
- **Exogenous factors**: ηF_i accounts for individual events (a suffered foul, a won duel).

2.1.4. Justification of Model Parameters

The proposed model incorporates several key parameters that govern the dynamics of emotional propagation. Below, we provide a theoretical justification for each parameter and discuss how they could be estimated from empirical data.

- **Diffusion Coefficient (D_E):** The parameter D_E represents the rate at which emotions diffuse spatially across the field. This parameter is inspired by models of emotional contagion in social psychology, where emotions spread among individuals based on their proximity and interactions [3]. In the context of football, D_E captures how quickly emotional states (e.g., excitement or frustration) propagate among players as they move and interact on the field.

Empirical Estimation: D_E could be estimated by analyzing video footage of matches, tracking the spatial distribution of players, and measuring how quickly emotional responses (e.g., celebrations or reactions to a referee's decision) spread across the team.

- **Regulation Parameter (λ):** The parameter λ models the tendency of the collective emotional state to return to a baseline or equilibrium level $\overline{E}$. This aligns with psychological theories of emotional regulation, where individuals and groups naturally tend to stabilize their emotional states over time [2].

Empirical Estimation: λ could be estimated by observing how long it takes for a team's emotional state to return to baseline after a significant event, such as scoring or conceding a goal. Longitudinal data on team performance and emotional responses could provide insights into this parameter.

- **Influence of Teammates (β):** The parameter β quantifies the strength of emotional influence among teammates. It reflects the degree to which a player's emotional state is affected by the emotional states of their nearby teammates. This is grounded in theories of social influence and emotional contagion, where individuals in close proximity tend to synchronize their emotional states [5].

Empirical Estimation: β could be estimated by analyzing interactions among players during matches, such as their reactions to each other's actions (e.g., passing, tackling, or celebrating). Surveys or interviews with players could also provide qualitative insights into the perceived influence of teammates on their emotional states.

- **Collective Influence (δ):** The parameter δ represents the impact of the collective emotional field $E(x,y,t)$ on individual players' emotional states. This parameter captures how the overall emotional climate of the team influences each player, aligning with theories of group dynamics and collective behavior [1].

Empirical Estimation: δ could be estimated by comparing individual emotional responses to the overall team mood during matches. For example, if a team's collective emotional state is highly positive, individual players might exhibit elevated emotional states, and vice versa.

- **Exogenous Factors (η):** The parameter η accounts for the influence of external events, such as referee decisions or crowd reactions, on individual players' emotional states. This parameter is rooted in theories of environmental psychology, where external stimuli can significantly impact emotional responses [4].

Empirical Estimation: η could be estimated by analyzing how players react to specific external events, such as referee decisions or crowd noise. Data from matches, including player reactions and changes in performance, could be used to quantify this influence.

By grounding these parameters in psychological and sports science theories, we ensure that our model captures realistic dynamics of emotional propagation in football. Future work could involve refining these estimates using empirical data from real matches, thereby enhancing the model's predictive accuracy and applicability.

2.1.5. Boundary Conditions and Complete Modeling

The boundaries of the field influence the system through a Robin condition:

$$\frac{\partial E}{\partial n} + \alpha_E E = g_E, \quad \forall (x,y) \in \partial\Omega. \tag{2.3}$$

The complete model can then be written in compact form as:

$$\begin{cases} \dfrac{\partial E}{\partial t} - D_E \nabla^2 E + \lambda\left(E - \bar{E}\right) = S + I, & \forall t > 0, \forall (x,y) \in \Omega, \\ \dfrac{\partial e_i}{\partial t} = -\gamma e_i + \beta \sum_{j\in\mathcal{N}_i} w_{ij}\left(e_j - e_i\right) + \delta\left(E - e_i\right) + \eta F_i, & \forall i \in \mathcal{P}, \\ \dfrac{\partial E}{\partial n} + \alpha_E E = g_E, & \forall (x,y) \in \partial\Omega. \end{cases} \tag{2.4}$$

This hybrid model links the global diffusion of emotions and the interindividual influences among players. It helps understand how a team transitions from one emotional state to another (calm $\rightarrow$ frustration $\rightarrow$ aggressiveness) in response to match events.

2.2. Preliminaries and Notations

Before establishing the existence and uniqueness of solutions to our problem, we introduce the functional spaces and notations that will be used later. We define the Hilbert spaces in which we will formulate our diffusion equation for the collective emotion, as well as the system of equations describing the evolution of individual emotions.

The assumptions on the system parameters are as follows:

Hypothesis 2.1 *We assume that*:

1) *The model coefficients satisfy*:

$$D > 0,\ \alpha > 0,\ \gamma > 0,\ \delta > 0.$$

2) *The feedback function* F_i *is continuous in* t.

Assumption (2.1) allows for several types of boundary conditions, including Dirichlet, Neumann, or mixed ones.

We define the main Hilbert space in which the collective emotion evolves as:

$$\mathbb{H} := L^2(\Omega, \mathbb{R}), \tag{2.5}$$

where Ω represents the spatial domain of the football field. An element v of $\mathbb{H}$ thus corresponds to a spatial distribution of the collective emotion.

We define the inner product in $\mathbb{H}$ as:

$$\langle v, u \rangle_{\mathbb{H}} := \int_\Omega v(x, y) u(x, y) \mathrm{d}x\mathrm{d}y, \tag{2.6}$$

and the associated norm:

$$\|v\|_{\mathbb{H}} = \left(\int_\Omega v^2(x, y) \mathrm{d}x\mathrm{d}y \right)^{\frac{1}{2}}. \tag{2.7}$$

We then define the Sobolev spaces:

$$\mathbb{H}^m := W^{m,2}(\Omega, \mathbb{R}),\ m = 1, 2. \tag{2.8}$$

With these notations, we introduce the gradient and Laplacian operators:

$$\nabla v := \left(\frac{\partial v}{\partial x}, \frac{\partial v}{\partial y} \right),\ \forall v \in \mathbb{H}^1, \tag{2.9}$$

$$\Delta v := \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2},\ \forall v \in \mathbb{H}^2. \tag{2.10}$$

Finally, we define the norm of the space $\mathbb{H}^1$ as follows:

$$\|v\|_{\mathbb{H}^1} := \left(\|v\|_{\mathbb{H}}^2 + \|\nabla v\|_{\mathbb{H}}^2 \right)^{\frac{1}{2}},\ \forall v \in \mathbb{H}^1. \tag{2.11}$$

These definitions and assumptions will allow us to establish the existence and uniqueness results in the next section.

3. Existence and Uniqueness of Solutions

Regarding the existence and uniqueness of the solution to problem (2.4), we will proceed in two parts, since problem (2.4) is a mixed problem composed of two independent equations: a PDE (collective emotional state equation) and an ODE (individual emotional evolution equation). For the collective emotional state equation, we will use the Lax-Milgram theorem to prove the existence and uniqueness of the solution, and for the individual emotion evolution equation, we will use the Cauchy-Lipschitz theorem.

3.1. Existence and Uniqueness of the Collective Emotion Problem (2.1)

Proving the existence and uniqueness of the collective emotion problem amounts to proving the existence and uniqueness of the following problem:

$$\begin{cases} \dfrac{\partial E}{\partial t} - D_E \nabla^2 E + \lambda\left(E - \bar{E}\right) = S + I, & \forall t > 0, \forall (x, y) \in \Omega, \\ \dfrac{\partial E}{\partial n} + \alpha_E E = g_E, & \forall (x, y) \in \partial\Omega. \end{cases} \tag{3.1}$$

We use a variational approach and the Lax-Milgram theorem [9]. To this end, we consider the following stationary version of problem (3.1):

$$\begin{cases} -D_E \nabla^2 E + \lambda\left(E - \bar{E}\right) = S + I, & \forall t > 0, \forall (x, y) \in \Omega, \\ \dfrac{\partial E}{\partial n} + \alpha_E E = g_E, & \forall (x, y) \in \partial\Omega. \end{cases} \tag{3.2}$$

In numerical simulations, time is generally known and discretized at each iteration. Each iteration represents a discrete time interval, and at each time step, variable values are computed. In this case, solving problem (3.2) numerically corresponds to a specific instance of the time-evolution problem at a given time. Thus, solving problem (3.2) for each time t in the numerical simulation provides an approximate solution of the time-dependent problem (3.1) over the full duration of the simulation. Therefore, in this numerical context, the existence and uniqueness of problem (3.2) for each time t can be considered as ensuring the existence and uniqueness of the time-dependent problem (3.1) at each discrete time step of the simulation.

3.1.1. Variational Formulation of Problem (3.2)

Consider the reaction-diffusion equation

$$\frac{\partial E}{\partial t} - D_E \nabla^2 E + \lambda\left(E - \bar{E}\right) = S + I, \ \forall t > 0, \ \forall (x, y) \in \Omega, \tag{3.3}$$

with the Robin boundary condition

$$\frac{\partial E}{\partial n} + \alpha_E E = g_E, \ \forall (x, y) \in \partial\Omega. \tag{3.4}$$

To derive the variational formulation, we multiply equation (3.3) by a test function $v \in H^1(\Omega)$ and integrate over the domain Ω. The diffusion term is treated by integration by parts [9]. Indeed, we have

$$-D_E \int_\Omega \Delta E v \mathrm{d}x\mathrm{d}y = D_E \int_\Omega \nabla E \cdot \nabla v \mathrm{d}x\mathrm{d}y - D_E \int_{\partial\Omega} \frac{\partial E}{\partial n} v \mathrm{d}\sigma.$$

Using the Robin condition (3.4), written as

$$\frac{\partial E}{\partial n} = g_E - \alpha_E E,$$

we obtain

$$-D_E \int_\Omega \Delta E v \mathrm{d}x\mathrm{d}y = D_E \int_\Omega \nabla E \cdot \nabla v \mathrm{d}x\mathrm{d}y - D_E \int_{\partial\Omega} (g_E - \alpha_E E) v \mathrm{d}\sigma.$$

Collecting the terms, the variational formulation consists in finding $E \in H^1(\Omega)$ such that

$$a(E, v) = L(v), \ \forall v \in H^1(\Omega), \tag{3.5}$$

where the bilinear form $a(\cdot,\cdot)$ and the linear form $L(\cdot)$ are defined as follows:

$$a(E, v) := D_E \int_\Omega \nabla E \cdot \nabla v \mathrm{d}x\mathrm{d}y + \lambda \int_\Omega E v \mathrm{d}x\mathrm{d}y + D_E \alpha_E \int_{\partial\Omega} E v \mathrm{d}\sigma, \tag{3.6}$$

and

$$L(v) := \lambda \int_\Omega \bar{E} v \mathrm{d}x\mathrm{d}y + \int_\Omega (S + I) v \mathrm{d}x\mathrm{d}y + D_E \int_{\partial\Omega} g_E v \mathrm{d}\sigma. \tag{3.7}$$

In the following lemma, we show the correspondence between the solutions of the variational problem (3.5) and those of (3.2) (or equivalently (??)).

Lemma 3.1 *The following properties are equivalent*:

i) *There exists* $E \in H^2(\Omega)$ *such that* $-D_E \nabla^2 E + \lambda (E - \bar{E}) = S + I$.

ii) *There exists* $E \in H^1(\Omega)$ *such that* $a(E, v) = L(v)$, $\forall v \in H^1(\Omega)$.

Proof 3.2 *The implication* i) $\Rightarrow$ ii) *is a consequence of the derivation of the variational formulation. Thus, we only need to prove that* ii) *implies* i). *Suppose that* ii) *holds. Let* $E \in H^1(\Omega)$ *such that*

$$a(E, v) = L(v), \ \forall v \in H^1(\Omega).$$

Since $E \in \mathbb{H}^1$ and satisfies the boundary conditions, it suffices to show that $E' \in W^{1,2}(\Omega, \mathbb{R})$. Let $C_c^\infty(\Omega, \mathbb{R})$ denote the space of smooth functions with compact support in Ω. Define, for each $\varphi \in C_c^\infty(\Omega, \mathbb{R})$, the function

$$\tilde{v}_\varphi = \begin{pmatrix} 0 \\ \vdots \\ \varphi \\ \vdots \\ 0 \end{pmatrix}$$

with φ in the k th position and all other components being zero. Clearly, $\tilde{v}_\varphi \in \mathbb{V}$ for all $\varphi \in C_c^\infty(\Omega, \mathbb{R})$ and

$$a(E, \tilde{v}_\varphi) = L(\tilde{v}_\varphi),$$

which is equivalent to:

$$D_E \int_\Omega \nabla E \cdot \nabla \varphi \mathrm{d}x\mathrm{d}y + \lambda \int_\Omega E\varphi \mathrm{d}x\mathrm{d}y + D_E \alpha_E \int_{\partial\Omega} E\varphi \mathrm{d}\sigma$$
$$= \lambda \int_\Omega \bar{E}\varphi \mathrm{d}x\mathrm{d}y + \int_\Omega (S+I)\varphi \mathrm{d}x\mathrm{d}y + D_E \int_{\partial\Omega} g_E \varphi \mathrm{d}\sigma$$

Therefore, for all $\varphi \in C_c^\infty(\Omega, \mathbb{R})$, we have

$$\int_\Omega D_E \nabla E \cdot \nabla \varphi \mathrm{d}x = \int_\Omega \left(-\lambda E + \lambda \bar{E} + (S+I)\right)\varphi(x)\mathrm{d}x\mathrm{d}y$$
$$+ \int_{\partial\Omega} \left(-D_E \alpha_E E + D_E g_E\right)\varphi \mathrm{d}\sigma$$

By identifying both sides of the equality, we get:

$$\int_\Omega D_E \nabla E \cdot \nabla \varphi(x,y)\mathrm{d}x = \int_\Omega \left(-\lambda E + \lambda \bar{E} + (S+I)\right)\varphi(x,y)\mathrm{d}x\mathrm{d}y,$$

which implies that $E' \in W^{1,2}(\Omega, \mathbb{R})$ with

$$D_E (E')' = -\lambda(E + \bar{E}) + (S+I), \ \forall (x,y) \in \Omega.$$

We finally obtain

$$-D_E \nabla^2 E + \lambda(E - \bar{E}) = S + I.$$

3.1.2. Verification of the Hypotheses of the Lax-Milgram Theorem

To apply the Lax-Milgram theorem, it is necessary to show that:

1) The bilinear form $a(\cdot,\cdot)$ is continuous on $H^1(\Omega)$.
2) The bilinear form $a(\cdot,\cdot)$ is coercive on $H^1(\Omega)$.
3) The linear form $L(\cdot)$ is continuous on $H^1(\Omega)$.

Continuity of $a(\cdot,\cdot)$:

For all $u, v \in H^1(\Omega)$, using the Cauchy–Schwarz inequality and the trace theorem (which allows one to control the $L^2(\partial\Omega)$ norm by the $H^1(\Omega)$ norm), we have:

$$|a(u,v)| \le D_E \|\nabla u\|_{L^2(\Omega)} \|\nabla v\|_{L^2(\Omega)} + \lambda \|u\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} + D_E \alpha_E \|u\|_{L^2(\partial\Omega)} \|v\|_{L^2(\partial\Omega)}.$$

Hence, there exists a constant $C > 0$ such that:

$$|a(u,v)| \le C \|u\|_{H^1(\Omega)} \|v\|_{H^1(\Omega)}.$$

Coercivity of $a(\cdot,\cdot)$:

For all $u \in H^1(\Omega)$, we have:

$$a(u,u) = D_E \|\nabla u\|_{L^2(\Omega)}^2 + \lambda \|u\|_{L^2(\Omega)}^2 + D_E \alpha_E \|u\|_{L^2(\partial\Omega)}^2.$$

Since $D_E > 0$, $\lambda > 0$, and $\alpha_E > 0$, and using the Poincaré and trace inequalities, there exists a constant $c > 0$ such that:

$$a(u,u) \ge c \|u\|_{H^1(\Omega)}^2.$$

Continuity of $L(\cdot)$:

Assume that $\bar{E}$, $S+I \in L^2(\Omega)$ and that $g_E \in L^2(\partial\Omega)$. For all $v \in H^1(\Omega)$, we have:

$$\left|L(v)\right| \le \lambda \|\bar{E}\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} + \|S+I\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} + D_E \|g_E\|_{L^2(\partial\Omega)} \|v\|_{L^2(\partial\Omega)}.$$

According to the trace theorem, there exists a constant C_{trace} such that:

$$\|v\|_{L^2(\partial\Omega)} \le C_{\text{trace}} \|v\|_{H^1(\Omega)}.$$

Hence, there exists a constant C' such that:

$$\left|L(v)\right| \le C' \|v\|_{H^1(\Omega)}.$$

3.1.3. Application of the Lax-Milgram Theorem

The Lax-Milgram theorem ensures that, given the continuity and coercivity of $a(\cdot,\cdot)$ and the continuity of $L(\cdot)$, there exists a unique $E \in H^1(\Omega)$ such that:

$$a(E,v) = L(v), \ \forall v \in H^1(\Omega).$$

This solution E is the weak solution of the diffusion equation (3.3) with the Robin boundary condition (3.4).

In the case of a time-dependent (parabolic) equation, a time discretization (for instance, by the implicit Euler method) is applied so that at each time step, an elliptic problem of the above form is solved. The uniqueness and existence of the solution at each time, combined with an induction argument, allow for the construction of the global-in-time solution.

3.1.4. Conclusion

Under the assumptions:

$$D_E > 0, \ \lambda > 0, \ \alpha_E > 0,$$

and with appropriate regularity conditions on $\bar{E}$, $S+I$, and g_E, the Lax-Milgram theorem guarantees the existence and uniqueness of the weak solution of the collective emotion diffusion problem (3.3) with Robin condition (3.4). Consequently, by Lemma (3.1), the existence and uniqueness of the solution to problem (3.2) are also ensured.

3.2. Existence and Uniqueness of Individual Emotions

The evolution equation of individual emotions e_i is an ordinary differential equation given by:

$$\frac{\mathrm{d}e_i}{\mathrm{d}t} = -\gamma e_i + \beta \sum_{j \in \mathcal{N}_i} w_{ij}\left(e_j - e_i\right) + \delta\left(E - e_i\right) + \eta F_i. \tag{3.8}$$

We rewrite this equation in matrix form:

$$\frac{\mathrm{d}\boldsymbol{e}}{\mathrm{d}t} = A\boldsymbol{e} + b(t), \tag{3.9}$$

where $\boldsymbol{e} = \left(e_1, \cdots, e_N\right)^\top$, and A is a matrix defined by:

$$A_{ii} = -\gamma - \beta \sum_{j \in \mathcal{N}_i} w_{ij} - \delta, \ A_{ij} = \beta w_{ij}, \ i \neq j, \tag{3.10}$$

and $b(t)$ is given by:

$$b_i(t) = \delta E + \eta F_i. \tag{3.11}$$

3.2.1. Application of the Cauchy-Lipschitz Theorem

Lemma 3.3 *Assuming that hypothesis* (2.1) *holds, then* $f(t, e_i) = A\boldsymbol{e} + b(t)$ *is linear in* e_i*, hence Lipschitz continuous with respect to* e_i*. Therefore, by the Cauchy–Lipschitz theorem, problem* (3.8) *admits a unique solution.*

Proof 3.4 *According to subsection* (3.1) *and assumption* (2.1), $E(x, y, t)$ *and* F_i *are sufficiently regular* (*continuous in* t), *hence* $b(t)$ *is also continuous. Moreover, by construction,* A *is a constant-coefficient matrix, so* $f(t, e_i) = A\boldsymbol{e} + b(t)$ *is linear in* e_i *and therefore Lipschitz continuous with respect to* e_i.

3.2.2. Conclusion

We have shown that:

- The collective emotion equation $E(x, y, t)$ admits a unique solution by the Lax-Milgram theorem.
- The individual emotion equations $e_i(t)$ admit a unique solution by the Cauchy-Lipschitz theorem.

Thus, the proposed hybrid model is well-posed and guarantees the existence and uniqueness of the solutions under reasonable assumptions.

4. Simulation and Numerical Results

In this section, we present the numerical implementation of the model describing the propagation of emotions and attitudes on the field. The simulation is based first on a diffusion equation representing the collective emotional state, then on a multi-agent model describing the individual evolution of interacting players, and finally on a coupling between these two equations.

For each simulation, we first detail the parameters and initial conditions used, then analyze the results obtained. The objective is to observe how emotions spread within the team in response to external stimuli and interactions among players. Finally, we discuss the implications of the results and their interpretation in a tactical and behavioral context.

4.1. Numerical Methods

To numerically solve the hybrid model, we employ a combination of the *finite element method* (FEM) for the partial differential equation (PDE) governing the collective emotional state $E(x, y, t)$, and the *Crank-Nicolson scheme* for the system of ordinary differential equations (ODEs) describing the evolution of individual emotions $e_i(t)$.

- **Finite Element Method (FEM) for the PDE:** The PDE in our model is a reaction-diffusion equation, which describes the spatio-temporal propagation of the collective emotional state across the football field. FEM is particularly well-suited for this task due to its ability to handle complex geometries and boundary conditions, such as the irregular shape of a football field or localized emotional disturbances. By discretizing the spatial domain into smaller elements, FEM allows us to approximate the solution $E(x, y, t)$ with high accuracy, even in the presence of heterogeneous emotional dynamics.

- **Crank-Nicolson Scheme for the ODEs:** The system of ODEs, which models the evolution of individual emotions $e_i(t)$, is solved using the Crank-Nicolson scheme. This implicit method is chosen for its stability and second-order accuracy in time, which are critical for capturing the potentially rapid changes in players' emotional states. The Crank-Nicolson scheme ensures that the numerical solution remains stable even for larger time steps, while preserving the physical meaning of the emotional interactions among players.

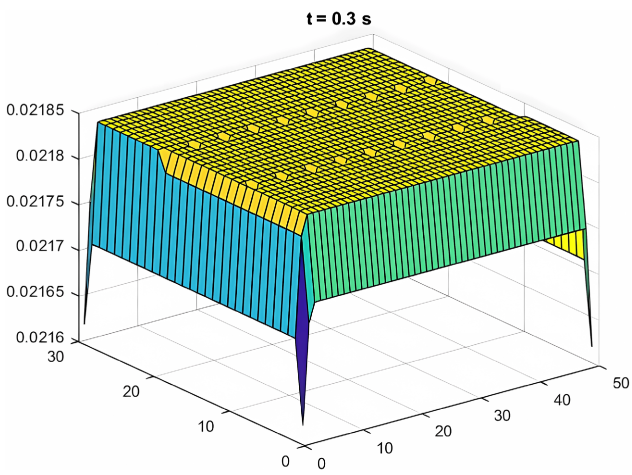
The coupling between the PDE and ODE systems is handled by iteratively updating the collective emotional field $E(x, y, t)$ and the individual emotional states $e_i(t)$ at each time step. This approach ensures that the feedback loop between individual and collective emotions is accurately captured, providing a realistic simulation of emotional dynamics on the field.

By using these numerical methods, we ensure both the stability and accuracy of our simulations, allowing us to explore the complex interplay between individual and collective emotions in a football team.

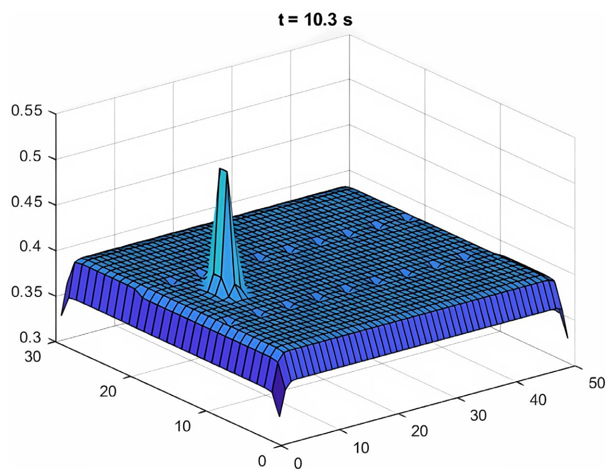
4.2. Results and Interpretation of the Collective Emotion Simulation

After setting up the numerical simulation of the collective emotion diffusion model, we analyze here the results obtained. We first present the different configurations observed during the simulation and the emotional dynamics that result from them. This analysis then allows us to interpret the influence of model parameters and draw conclusions about the collective behavior of the players.

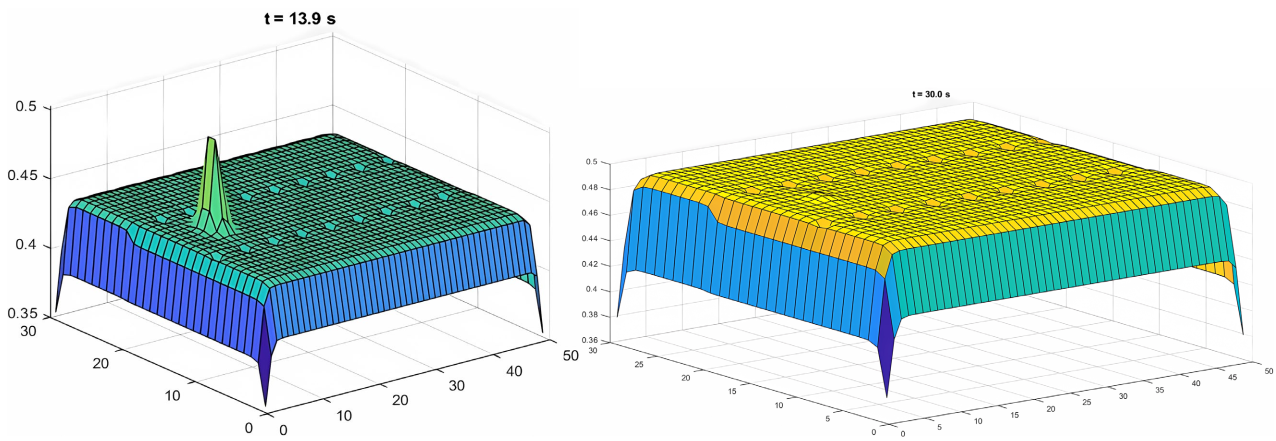
4.2.1. Results



(a) Graph 1



(b) Graph 2



(c) Graph 3 (d) Graph 4

Figure 1. Set of four graphs illustrating collective emotions.

4.2.2. Interpretation of the Results

As illustrated in Figure 1, the simulation highlights the spatio-temporal evolution of collective emotions within the team. The formation of emotional clusters observed in Figures 1(c)-(d) reflects the impact of significant game events.

General Dynamics The simulation highlights the spatio-temporal evolution of collective emotions within the team. Analysis of the results reveals several significant trends:

- **Homogeneous initialization:** At the beginning of the simulation, the collective emotional state is relatively neutral and evenly distributed over the field. This corresponds to a team in an initial emotional equilibrium, without strong external influences.
- **Propagation and diffusion:** Over time, zones of more intense emotions appear and spread across the field. This phenomenon agrees with the diffusion equation integrated into the model, representing the transmission of emotions among nearby players.
- **Regulation influence:** The regulation parameter λ prevents excessive amplification of emotions and promotes a progressive stabilization of emotional states. This mechanism avoids uncontrolled dynamics and contributes to the emotional resilience of the team.

Spatial Analysis The evolution of emotions across the field exhibits distinct patterns reflecting the collective dynamics of players:

- **Formation of emotional clusters:** Zones of stronger emotions emerge, corresponding to significant game events (e.g., a conceded goal, a foul, or a missed opportunity).

Anisotropic diffusion: The propagation of emotions does not occur uniformly on the field. This anisotropy may be due to the team's structure, player positions, or differentiated interactions.

- **Emotional persistence:** Some emotional zones persist longer before dissipating, suggesting that emotions do not vanish instantly but leave a lasting imprint on the team.

Temporal Evolution The temporal dynamics highlight several phases in the evolution of collective emotions:

- **Activation phase:** Initially, emotions evolve slowly before interactions and diffusion accelerate the collective dynamics.
- **Fluctuations and adjustments:** Gradual adjustments occur due to individual and collective interactions. As the simulation progresses, the dynamics tend toward a more homogeneous emotional state.

Comparison with Real Scenarios The interpretation of results allows analogies with real football situations:

- A team leading the score may display a more stable and uniform emotional dynamic.
- A struggling team may exhibit more chaotic emotional fluctuations with significant variations among players.
- A disruptive event (e.g., a red card) may trigger a wave of emotions progressively spreading throughout the team.

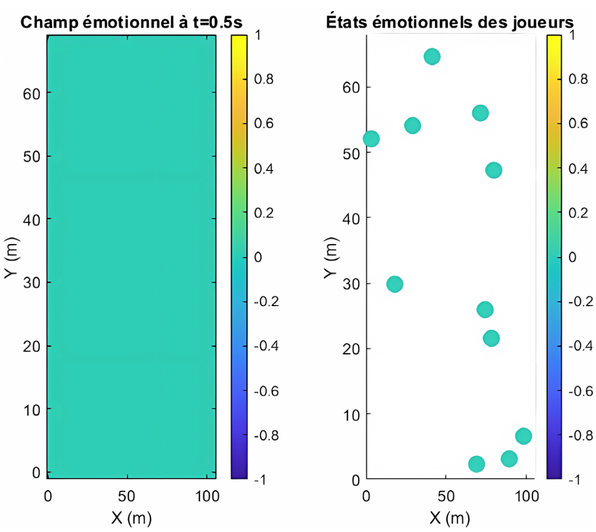
4.2.3. Conclusion

This simulation highlights the emerging collective behavior induced by emotional diffusion and player interactions. Individual emotions gradually influence the entire team, confirming the central role of emotional dynamics in collective play. These results offer promising perspectives for modeling real match situations, such as the psychological impact of a goal or stress management during defensive phases.

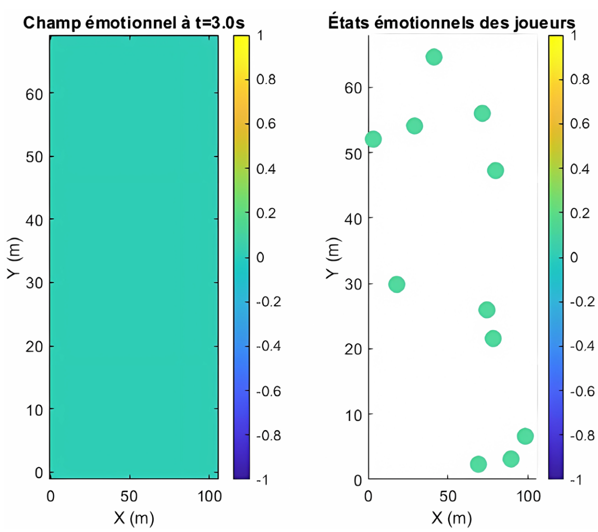
4.3. Analysis of Results: Simulation of Individual Emotions

In this section, we analyze the results obtained from the simulation of the individual emotion equation. The goal of this simulation is to study how players' emotions evolve as a function of their interactions, external stimuli, and internal regulation parameters.

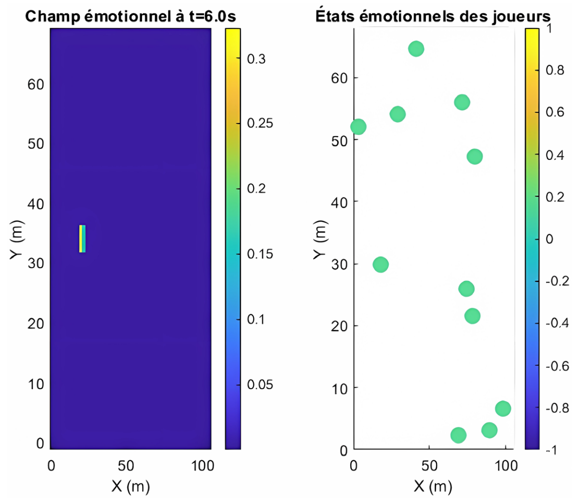
Results



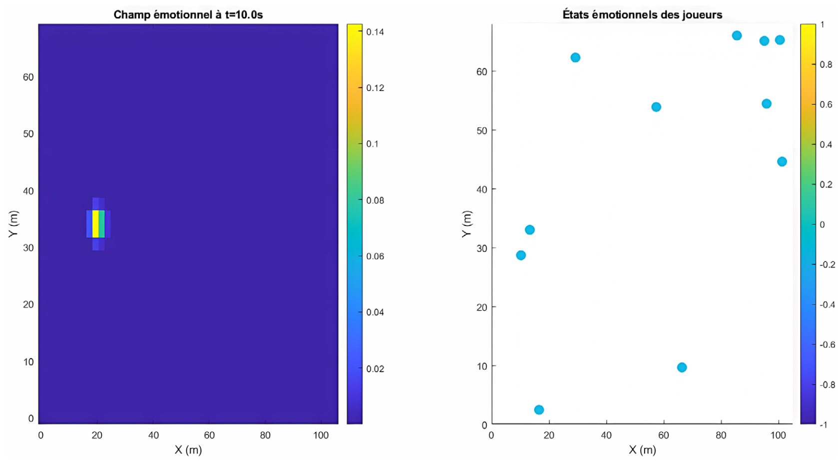
(a) Graph 5



(b) Graph 6



(c) Graph 7



(d) Graph 8

Figure 2. Set of four graphs illustrating individual emotions.

4.4. Interpretation of the Simulation

The evolution of individual emotional states shown in Figure 2 reveals significant heterogeneity among players.In particular, the oscillatory behaviors observed in Figure 2(a), Figure 2(b) indicate high sensitivity to external stimuli.

Dynamics of Individual Emotions Analysis of the obtained curves highlights significant variations in players' emotional states. Several behaviors can be identified:

- Some players show strong emotional oscillations, indicating high sensitivity to interactions and external stimuli.
- Others exhibit more gradual evolution and stabilize rapidly, suggesting greater resistance to external influence.

Propagation and Mutual Influence Individual emotions are not isolated but are influenced by other players' emotional states. Specifically:

- When certain players experience a sudden increase in their emotional level (positive or negative), this dynamic gradually propagates to neighboring players.
- Local interactions play a central role in this propagation, reflecting social

influence phenomena similar to those observed in collective psychology.

Impact of External Stimuli The influence of external events is particularly evident in certain simulation phases. When an external stimulus is introduced, it triggers an immediate response that spreads differently depending on several factors:

- The player's proximity to the stimulus source.
- The intrinsic sensitivity of the player (possibly modeled by a sensitivity parameter in the equation).
- The player's previous emotional state, which affects their reaction intensity.

Regulation and Stabilization Mechanisms Long-term observation of individual emotions reveals self-regulation mechanisms specific to each player:

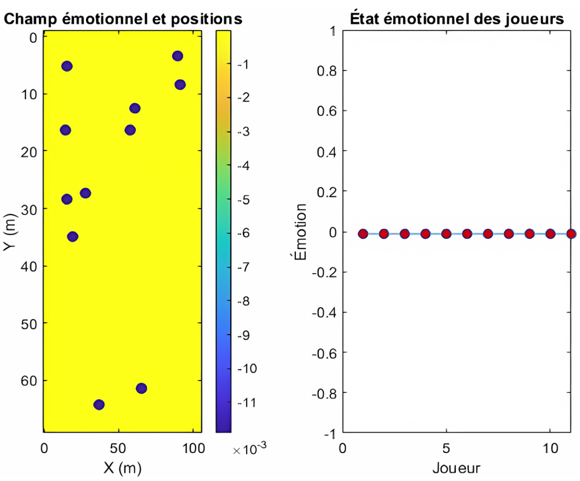
- Some players gradually return to a more stable emotional state after an initial disturbance.
- The relaxation effect is modulated by social interactions: a player surrounded by highly reactive teammates will take longer to regain stability.
- Conclusion
- These observations confirm the importance of emotional diffusion and coupling between players in shaping the evolution of individual emotions on the field.

In conclusion, this simulation highlights the complexity of emotional dynamics within a team. The balance between individual regulation and collective influence generates emergent behaviors that may be decisive for team performance. These results confirm the importance of considering emotional factors in strategic game analysis and open perspectives for tactical adjustments based on players' emotional states.

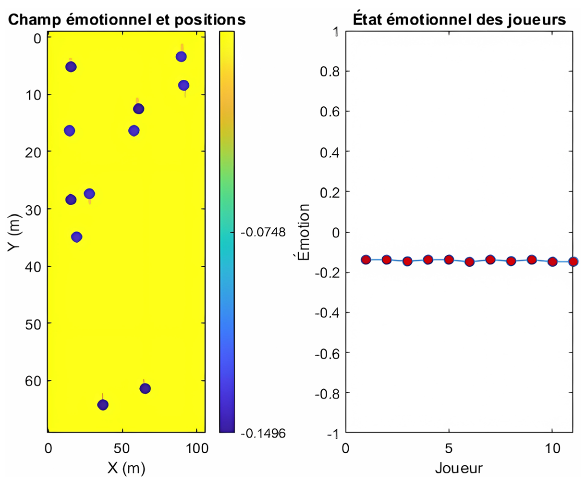
4.5. Analysis of the Coupled Simulation Results

In this final simulation, we examine the coupling between the equations governing individual and collective emotions. The goal is to evaluate how individual behaviors influence the collective emotional dynamics and, conversely, how the global emotional state affects individual players.

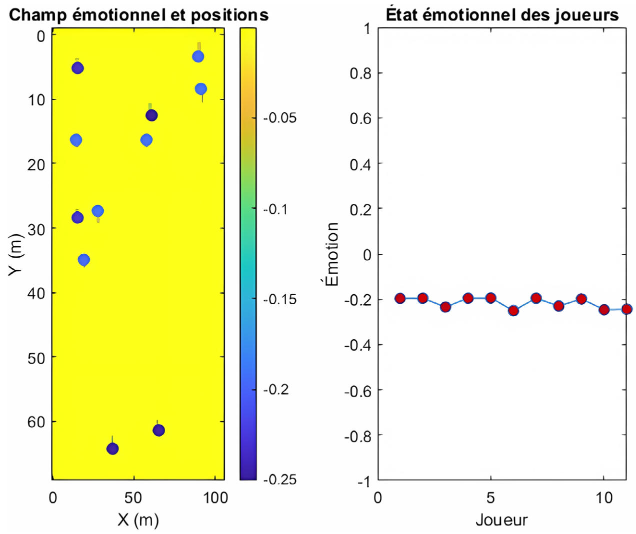
Results



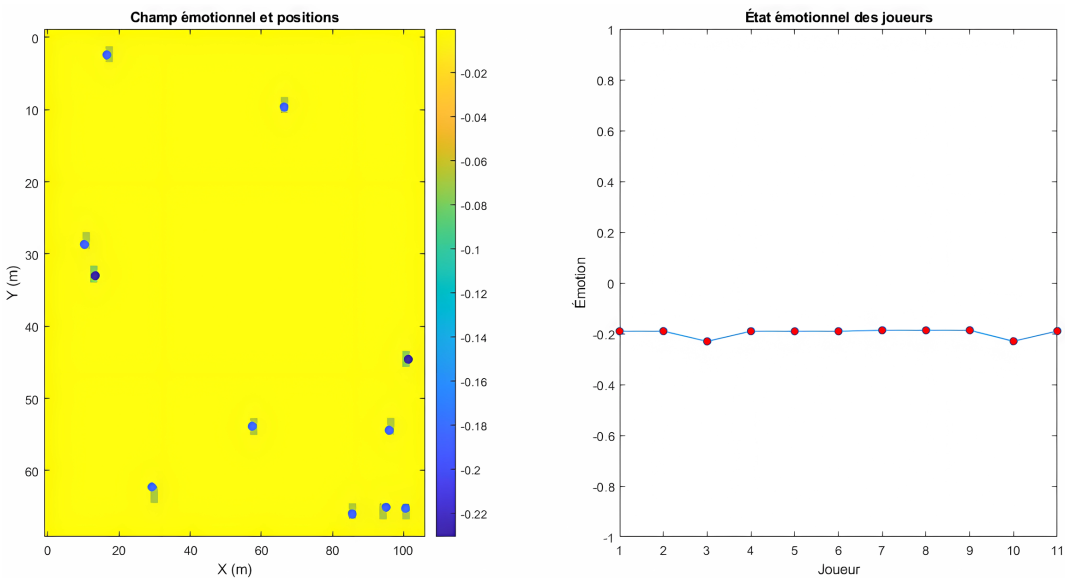
(a) Graph 9



(b) Graph 10



(c) Graph 11



(d) Graph 12

Figure 3. Set of four graphs illustrating the coupling between collective and individual emotions.

4.6. Interpretation of the Simulation

The coupled dynamics between individual and collective emotions are illustrated in Figure 3. As shown in Figures 3(c)-(d), emotional amplification and stabilization phases emerge from the bidirectional coupling.

Dynamics of the Coupled System The simulation reveals several noteworthy phenomena:

- **Bidirectional propagation:** Unlike the previous simulations, we observe a joint evolution of individual and collective emotions. A localized excitation in one player can gradually influence teammates, while a globally tense or calm environment directly affects each individual.
- **Emergence of spatio-temporal patterns:** Interaction between players and the

collective field generates regular structures such as zones of high emotional intensity that form and dissipate dynamically over time.

- **Amplification and stabilization effects:** Certain moments show rapid amplification of individual emotions, likely due to a positive feedback loop between players and the collective field. Conversely, other periods show stabilization, suggesting that the system possesses intrinsic regulation mechanisms.

Interpretation and Strategic Impact These results provide a deeper understanding of the role of interactions in emotional dynamics on the field:

- **Stress and motivation management:** Intense emotional disturbances (e.g., a conceded goal or player error) do not remain isolated but spread throughout the team. Appropriate strategies—such as effective communication or tactical adjustments—can help limit these fluctuations and stabilize collective performance.
- **Leadership effect:** Certain players exert a stronger influence on the team's emotional state, acting as regulators or catalysts of collective energy. Identifying these natural leaders and understanding their roles could improve emotional management during matches.
- **Detection of critical phases:** By observing patterns of emotional amplification, it becomes possible to predict moments when the team risks losing control (e.g., moral collapse or disorganization) and to anticipate corrective interventions.

Conclusion

The coupled system simulation enhances our understanding of team emotional dynamics in action. It reveals complex interactions between individual emotions and collective feelings, offering valuable insights for sports performance analysis. The results could be used to design targeted intervention strategies aimed at improving team cohesion and resilience to emotional fluctuations during matches.

4.7. Linking Simulation Dynamics to Model Parameters

The simulation results provide valuable insights into how specific model parameters influence the emotional dynamics of the team. Below, we discuss the impact of key parameters on the observed patterns, such as the formation of emotional clusters and the team's emotional resilience.

- **Diffusion Coefficient (D_E):** The diffusion coefficient D_E plays a critical role in determining how quickly emotions spread across the field. Our simulations reveal that:
  - A **higher** D_E leads to faster and more widespread diffusion of emotions, resulting in the rapid formation of large emotional clusters. This can be interpreted as a team where emotions, such as excitement or frustration, spread quickly among players, potentially leading to synchronized emotional responses (e.g., collective motivation or demoralization).
  - Conversely, a **lower** D_E restricts emotional propagation, causing emotions

to remain localized around their source. This may reflect a team with weaker emotional contagion, where individual emotional states are less influenced by teammates, leading to more isolated emotional responses.

Implications: Teams with high D_E may experience stronger collective emotional responses to events like scoring a goal or conceding a penalty, while teams with low D_E may show more individualized emotional reactions.

- **Regulation Parameter (λ):** The regulation parameter λ governs the team's ability to return to an emotional baseline after a disturbance. Our results indicate that:
  - A **higher** λ enhances emotional resilience, as the team's collective emotional state quickly stabilizes after perturbations. This suggests a team that is better at managing emotional fluctuations, maintaining focus, and avoiding prolonged periods of negative emotions (e.g., frustration or anxiety).
  - A **lower** λ results in slower emotional recovery, with emotions persisting for longer periods. This may reflect a team that struggles to regulate its emotional state, making it more vulnerable to prolonged emotional disturbances, such as lingering frustration after a controversial referee decision.

Implications: Teams with high λ are likely to exhibit greater emotional stability during high-pressure moments, while those with low λ may require additional support (e.g., coaching interventions) to regain emotional equilibrium.

- **Influence of Teammates (β):** The parameter β, which quantifies the strength of emotional influence among teammates, affects the synchronization of emotional states within the team:
  - A **higher** β fosters stronger emotional alignment among players, leading to cohesive emotional responses. This can be beneficial for team unity but may also amplify negative emotions if not managed properly.
  - A **lower** β reduces the emotional influence among players, allowing for more independent emotional responses. This may be advantageous in situations where individual focus is required, but it could also lead to a lack of emotional cohesion.

Implications: Teams with high β may benefit from strong leadership to channel collective emotions positively, while teams with low β might need strategies to enhance emotional connectivity among players.

- **Collective Influence (δ):** The parameter δ captures the impact of the collective emotional field on individual players. Our simulations show that:
  - A **higher** δ amplifies the influence of the team's overall emotional state on individual players, leading to stronger alignment between individual and collective emotions. This can enhance team cohesion but may also make players more susceptible to collective emotional swings.
  - A **lower** δ reduces this influence, allowing players to maintain more independent emotional states. This may help players stay focused on

individual tasks but could also lead to a disconnect between individual and team emotions.

Implications: Teams with high δ may require strategies to manage collective emotions effectively, while teams with low δ might focus on fostering a stronger connection between individual and team emotions.

By analyzing how these parameters shape emotional dynamics, we gain a deeper understanding of the factors that contribute to team resilience, cohesion, and performance. These insights can inform targeted interventions, such as adjusting team communication strategies or implementing emotional regulation techniques, to optimize team performance in real-world scenarios.

5. Conclusions

In this paper, we proposed an innovative mathematical model to analyze the propagation of emotions and collective attitudes in a football team depending on the match context. By combining a reaction–diffusion equation to capture the spatio-temporal dynamics of collective emotions with a multi-agent model representing individual interactions among players, we built a hybrid framework providing a deeper understanding of emotional dynamics on the field.

Our approach relies on solving a mixed system integrating a partial differential equation (PDE) for the collective emotional state and a set of ordinary differential equations (ODEs) for the evolution of individual emotions. We demonstrated the existence and uniqueness of solutions using the Lax–Milgram theorem for the PDE and the Cauchy–Lipschitz theorem for the ODEs. The numerical approximation was carried out using the finite element method for the PDE and a Crank–Nicolson scheme for the ODEs, ensuring both stability and accuracy.

Simulation results show that collective emotions evolve coherently with match events, significantly influencing player attitudes. We observed emotional contagion and internal regulation phenomena, illustrating how a team can transition from calmness to heightened aggression in response to external stimuli. These findings confirm the hypothesis that emotions and psychological environment play a crucial role in team performance and tactical decision-making.

Empirical Validation and Data-Driven Perspectives

To further validate and refine our hybrid model, real-world data can be leveraged to estimate key parameters and test the model's predictive accuracy. Below, we outline specific types of empirical data that could be used to ground the model in observable phenomena, enhancing its applicability to real-world scenarios.

- **Player Tracking Data:** Modern football analytics relies heavily on *player tracking data*, which captures the spatial positions and movements of all players on the field at high temporal resolution. This data can be used to:
  - Estimate the *interaction weights* w_{ij} by analyzing the frequency and duration of proximity between players. For example, players who frequently interact or occupy similar spatial regions may have stronger emotional

influence on each other, reflected by higher w_{ij} values.

- Validate the spatial diffusion of emotions by observing how emotional responses (e.g., celebrations or frustrations) propagate across the field following key events, such as goals or referee decisions.

Example: Tracking data from leagues like the English Premier League or UEFA Champions League could be analyzed to quantify how quickly and widely emotions spread among players after significant in-game events.

- **Biometric Data:** Wearable sensors and biometric monitoring devices can provide real-time measurements of players' physiological states, such as heart rate variability, skin conductance, and cortisol levels. These metrics are strongly correlated with emotional states, such as stress, excitement, or frustration. Biometric data can be used to:
  - Approximate individual emotional states $e_i(t)$ by mapping physiological responses to emotional valence and arousal.
  - Validate the model's predictions of emotional fluctuations by comparing simulated emotional states with observed physiological changes during matches.

Example: Biometric data collected during training sessions or competitive matches could be used to calibrate the model's parameters, such as the regulation parameter λ or the influence of exogenous factors η.

- **Video and Behavioral Analysis:** Video footage of matches, combined with manual or automated behavioral coding, can provide insights into players' emotional expressions and interactions. For instance:
  - Facial expression analysis and body language cues (e.g., gestures, postures) can be used to infer emotional states and validate the model's predictions of individual and collective emotions.
  - Observations of team cohesion, such as synchronized celebrations or coordinated defensive reactions, can help assess the model's ability to capture emergent emotional dynamics.

Example: Video analysis tools, such as those used in sports psychology research, could be employed to track emotional contagion in real-time and compare it with the model's simulations.

- **Survey and Self-Report Data:** Post-match surveys or real-time self-reports from players can provide subjective assessments of emotional states, complementing objective data sources. This data can be used to:
  - Validate the model's predictions of individual emotional responses to specific events, such as scoring a goal or receiving a yellow card.
  - Refine parameters related to exogenous influences η by correlating self-reported emotional states with in-game events.

Example: Players could use mobile apps or wearable devices to log their emotional states at key moments during a match, providing a direct comparison with the model's outputs.

- **Tactical and Performance Data:** Data on tactical decisions (e.g., formation changes, substitutions) and performance metrics (e.g., pass completion rates, shot accuracy) can be linked to emotional dynamics. For example:

- Changes in tactical behavior following emotional events (e.g., shifting to a more defensive formation after conceding a goal) can be analyzed to validate the model's predictions of emotional influence on decision-making.
- Performance fluctuations correlated with emotional states can provide insights into the model's ability to capture the impact of emotions on team effectiveness.

Example: Performance analytics platforms, such as Opta or Wyscout, could be used to correlate emotional simulations with on-field performance outcomes.

By integrating these diverse data sources, our model can be empirically validated and refined, enhancing its utility for real-world applications. This data-driven approach not only strengthens the model's credibility but also opens new avenues for applying emotional analytics to improve team performance, inform coaching strategies, and optimize player well-being.

Thus, this modeling approach represents a step forward in studying collective behavior in sports, highlighting the interplay between emotions, game dynamics, and strategic decision-making.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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