

Graph Games with Component-Based Control Structures

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Abstract

This paper investigates allocation rules in graph-structured cooperative games (hereinafter referred to as graph games) by integrating the notion of network control. A component-restricted game and the component control value are proposed through treating each connected component as a virtual player (termed a component player), under the stipulation that only coalitions attended by component players are eligible to obtain coalitional worth. The component control value initially assigns a Shapley payoff (Shapley value, SV) to every player, and the SV of each component player is subsequently redistributed equally among all original players belonging to its corresponding connected component. Thereby, an axiomatic characterization of this allocation rule is established, demonstrating that the component control value constitutes the unique solution satisfying component efficiency and component-restricted fairness in graph games.

Keywords

Cooperative Game, Graph Structure, Component Restriction, Allocation Rule

1. Introduction

The central issue in cooperative game theory concerns the fair and rational allocation of coalitional worth. When the worth of any coalition can be transferred freely and without loss among its members, the setting is referred to as a cooperative game with transferable utility (TU game) [1].

In a TU game, any subset of players may form a coalition and thereby obtain an associated worth. In practice, however, numerous constraints often render certain coalitions infeasible, thereby motivating the study of TU games with restricted cooperation. Myerson (1977) pioneered this line of inquiry by introducing

graph-restricted cooperative games—commonly termed graph games [2]. In his framework, only players who are connected in the underlying communication graph can form feasible coalitions; disconnected coalitions receive the sum of the worths of their respective connected components. Subsequently, the Myerson value was defined by applying the Shapley value [3] to such graph games and was axiomatically characterized through fairness and component efficiency [2]. Building upon this foundation, numerous extensions have been proposed, including models incorporating hypergraph structures [4] [5], network architectures [6] [7], union-stable systems [8] [9], and directed graph frameworks [10] [11].

Concurrently with the development of these structurally enriched models, a variety of allocation rules have been extensively investigated. Beyond the Myerson value, notable proposals grounded in the Shapley value include the two-step Shapley value [12], the Owen value [13], and the position value [14]. Moreover, integrating egalitarian principles, Yokote *et al.* (2018) introduced the r -Shapley value and provided its axiomatic foundation [15], while Li and Shan (2024) recently proposed a hybrid solution combining Shapley-based and equal-division rationales, also supported by an axiomatic characterization [16]. Nevertheless, the incorporation of control structures within graph games has received scant attention. Dietzenbacher *et al.* (2017) took a significant step by systematically introducing a broad class of network communication games [17] and analyzing their decomposition into unanimity games. Their approach embeds a network control mapping into the communication structure, yielding a TU game in which a coalition's worth equals the aggregate worth of the components it controls.

Herein, we consider a scenario wherein mutually connected players can cooperate only under the governance of a network control mechanism, which we model via connected components: each connected component acts as a virtual player embodying a control function that determines whether its constituent players may form a viable coalition. This constitutes the foundational idea of the component-restricted game.

Section 2 reviews the theoretical preliminaries of TU games and graph games, thereby establishing the basis for the component-restricted game and the component control value introduced in Section 3. In Section 3, each connected component of a graph game is treated as a control structure and introduced as a virtual player that can form coalitions with original players. Consequently, a player can obtain cooperative surplus only by allying with its corresponding connected component; otherwise, the player receives only their stand-alone worth. Based on this premise, we formulate the component-restricted game and propose the component control value by synthesizing the Shapley value [3] with the principle of equal division [18]. Section 4 introduces two novel axioms—representative fairness and component-restricted fairness—and provides an axiomatic characterization of the component control value.

2. Theoretical Foundations

A pair (N, v) denotes a cooperative game with transferable utility (hereafter, TU

game), wherein $N = \{1, \dots, n\}$ ($n \geq 2$) is the player set and $v: 2^N \rightarrow \mathbb{R}$ is the characteristic function satisfying $v(\emptyset) = 0$. Any subset $S \subseteq N$ is termed a coalition, and $v(S)$ signifies the worth of coalition S . Let \mathcal{G}^N denote the space of all TU games on player set N . Whenever no ambiguity arises, the game is identified by its characteristic function v , i.e., $v \in \mathcal{G}^N$. For every non-empty coalition $\forall S \subseteq N$, the unanimity game (N, u_S) is defined by $u_S(T) = 1$ if $S \subseteq T$, and $u_S(T) = 0$ otherwise. Furthermore, if $\forall i \in N, v(i) = 0$, then the characteristic function $v \in \mathcal{G}_0^N$ is said to be zero-normalised on player set N . For any $S \subseteq N$, $|S|$ denotes the cardinality of set S , i.e., the number of its elements.

It is widely recognised that the Shapley value (SV) is the unique solution for TU games satisfying efficiency, additivity, symmetry, and the null-player property [3]. For a given $v \in \mathcal{G}^N$, its SV is defined by

$$Sh_i(v) = \sum_{S \subseteq N \setminus \{i\}} \frac{|S|!(|N| - |S| - 1)!}{|N|!} (v(S \cup \{i\}) - v(S)), \quad \forall i \in N. \quad (1)$$

An undirected graph is encoded by the pair (N, L) , where N is the set of nodes (players) and $L \subseteq \{\{i, j\} : i, j \in N, i \neq j\}$ is the edge set; an edge $\{i, j\} \in L$ signals bilateral communication between players i and j . For brevity, the graph (N, L) is subsequently identified with its edge set L , and the edge $\{i, j\}$ is written as ij . The collection of all graphs on node set N is denoted by \mathcal{L}^N . A graph $L \in \mathcal{L}^N$ is complete if every pair of distinct nodes is linked by an edge.

For any non-empty subset $S \subseteq N$, $(S, L(S))$ is the subgraph induced by S , where $L(S) = \{ij \in L : i, j \in S\}$. The set of edges incident to player i is $L_i = \{ij \in L : j \in N\}$, and $L_{-i} = L \setminus L_i$ is the set of edges not incident to i .

A sequence of distinct nodes (i_1, \dots, i_k) , $k \geq 2$, constitutes a path from i_1 to i_k in L if $i_h i_{h+1} \in L$ for every $h = 1, \dots, k-1$. If a path exists between every pair of nodes, L is connected; if, moreover, every pair is linked by exactly one path, L is a tree.

A non-empty subset $S \subseteq N$ is connected whenever its induced subgraph $L(S)$ is connected. In a graph game $L \in \mathcal{L}^N$, a maximal connected subset is termed a (connected) component: a non-empty $T \subseteq N$ is a component of L if T is connected and $T \cup \{i\}$ is disconnected for any $i \in N \setminus T$. The set of all components of L is denoted by N/L . For any $S \subseteq N$, S/L denotes the set of components of the subgraph $L(S)$; if $L = \emptyset$, then $S/L = \{\{i\} : i \in S\}$. For any $i \in N$, the component containing i is written as T_i , where $T_i \in N/L$ and $i \in T_i$.

A triple (N, v, L) specifies an undirected graph game on player set N . The space of all such games is denoted by \mathcal{G}_L^N , and a generic element is written as $(v, L) \in \mathcal{G}_L^N$. A single-valued solution is a mapping $f: \mathcal{G}_L^N \rightarrow \mathbb{R}^n$ that assigns a unique payoff vector $f(v, L)$ to every game (v, L) . The first solution of this type was introduced by [2] and is henceforth called the *Myerson value*; it is defined by

$$\mu(v, L) = Sh(v^L), \quad \forall (v, L) \in \mathcal{G}_L^N, \tag{2}$$

where v^L is the graph-restricted game given by $v^L(S) = \sum_{T \in S/L} v(T)$ for every $S \subseteq N$. The Myerson value is uniquely characterised by component efficiency and fairness.

Component Efficiency. For every $(v, L) \in \mathcal{G}_L^N$ and every component $T \in N/L$,

$$\sum_{i \in T} f_i(N, v, L) = v(T). \tag{3}$$

Fairness. For every $(v, L) \in \mathcal{G}_L^N$ and every edge $ij \in L$,

$$f_i(v, L) - f_i(v, L \setminus \{ij\}) = f_j(v, L) - f_j(v, L \setminus \{ij\}). \tag{4}$$

3. Component-Restricted Game and Component Control Value φ

Within the graph game $(N, L) \in \mathcal{G}_L^N$, it is postulated that an external force—outside the player set N —shapes the emergence of every connected component $T \in N/L$. The interpretation of a connected component $T \in N/L$ as an autonomous player—hereafter termed the *component player* $\{T\}$ —is herein motivated by the existence of an exogenous coordination protocol (e.g., corporate project charter, social-platform algorithm, or regulatory framework) that (i) endows the agents of T with complementary productive opportunities and (ii) withholds these opportunities from any coalition that omits $\{T\}$. Consequently, $\{T\}$ is modelled as the minimal governance unit embodying the synergy potential induced by the external force, and its inclusion in a coalition is necessary and sufficient to activate the worth $v(R)$ of every $R \subseteq T$. Accordingly, each component $T \in N/L$ is treated as an autonomous entity, termed a *component player* $\{T\}$, which becomes part of the game. Only coalitions that include the relevant component player can mobilise the agents located in that component; otherwise, the agents remain isolated.

Definition 3.1 In a graph game $(N, L) \in \mathcal{G}_L^N$, the *component-restricted game* is the TU game $w \in \mathcal{G}^{N \cup (N/L)}$ defined by

$$w(S) = \sum_{R \in (S \setminus (N/L)) / L \cup (S \cap (N/L) T)} v(R), \quad \forall S \subseteq N \cup (N/L). \tag{5}$$

In this construction, a coalition S is composed of both original players from N and component players from N/L . The worth $w(S)$ aggregates the values of all connected components that are formed within the union of agents whose component players are present in S , and adds the stand-alone values of any remaining agents who lack their component player in S . Hence, agents can generate the cooperative worth only when their respective component player joins the coalition; otherwise, they are restricted to their individual productivities. For any graph-restricted game $v^L \in \mathcal{G}^N$ and any coalition $S \subseteq N$, one plainly has $v^L(S) = w(S \cup (N/L))$.

For notational convenience, let $S^+ = \bigcup_{T \in S \cap (N/L)} T$ and $S^- = S \setminus (N/L)$. An al-

ternative expression for the component-restricted game $w \in \mathcal{G}^{N \cup (N/L)}$ is then

$$w(S) = \sum_{R \in (S^- \cap S^+) / L(S^+)} v(R) + \sum_{i \in S^- \setminus S^+} v(i), \quad \forall S \subseteq N \cup (N/L). \quad (6)$$

We now introduce the *component control value*, abbreviated as the φ -value.

Definition 3.2 For any graph game $(v, L) \in \mathcal{G}_L^N$, the φ -value assigned to player $i \in N$ is

$$\varphi_i = Sh_i(N \cup (N/L), w) + \frac{1}{|T_i|} Sh_{T_i}(N \cup (N/L), w), \quad (7)$$

where $T_i = \{i \in T : T \in N/L\}$ is the component that contains player i .

Observe that φ_i is a linear combination of two Shapley payoffs computed in the component-restricted game w : the first term is the direct Shapley share of player i , while the second term redistributes the Shapley payoff of the component player T_i equally among all agents located in that component. Consequently, the φ -value blends marginalistic evaluation with an egalitarian disbursement internal to each component.

The equal division of $Sh_{T_i}(w)$ among $|T_i|$ members is adopted because, conditional on the activation of component T_i , no further hierarchical or productivity differential is assumed within T_i ; hence symmetry dictates an egalitarian rebate.

An example is provided below to demonstrate that the φ -value constitutes a new allocation rule distinct from existing solutions for graph games.

Example 3.1 Consider the graph game $(v, L) \in \mathcal{G}_L^N$ with player set $N = \{1, 2, 3, 4, 5, 6, 7\}$ and characteristic function $v = u_{\{1,2\}} + u_{\{4,6,7\}}$. The graph is given by $L = \{\{1, 2\}, \{1, 3\}, \{4, 5\}, \{4, 6\}, \{6, 7\}\}$. As illustrated in **Figure 1**, the graph exhibits two connected components, labelled $T_1 = \{1, 2, 3\}$ and $T_2 = \{4, 5, 6, 7\}$. Applying Definition 3.2, the Shapley payoffs allocated to the original players in the component-restricted game $w \in \mathcal{G}^{N \cup (N/L)}$ are

$\left\{ \frac{1}{3}, \frac{1}{3}, 0, \frac{1}{4}, 0, \frac{1}{4}, \frac{1}{4} \right\}$, while the Shapley payoffs assigned to the component players $\{\{T_1\}, \{T_2\}\}$ are $\left\{ \frac{1}{3}, \frac{1}{4} \right\}$. Consequently, the φ -value for the players is

$$\varphi(N, v, L) = \left\{ \frac{4}{9}, \frac{4}{9}, \frac{1}{9}, \frac{5}{16}, \frac{1}{16}, \frac{5}{16}, \frac{5}{16} \right\},$$

whereas the Myerson value of the same game is

$$\mu(N, v, L) = \left\{ \frac{1}{2}, \frac{1}{2}, 0, \frac{1}{3}, 0, \frac{1}{3}, \frac{1}{3} \right\}.$$

A comparison reveals that players $\{3\}$ and $\{5\}$, whose marginal contributions are nil, receive zero payoff under the Myerson value. In contrast, the φ -value equally redistributes the Shapley payoff of the respective component player among the members of each component, thereby yielding a more balanced allocation. \square

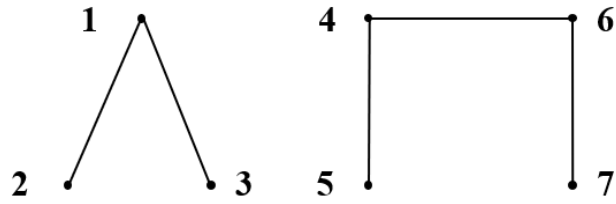


Figure 1. The graph structure (N, L) in Example 3.1.

4. Axiomatic Characterization of the φ -Value

For every connected component $T \in N/L$ of a graph game $(v, L) \in \mathcal{G}_L^N$, we introduce a *component-representative value* $c(T, v, L) \in \mathbb{R}$, abbreviated as the *representative value*, which depends on the component T , the characteristic function v , and the edge set L . The aggregate representative value is denoted by $c(N, v, L) = \sum_{T \in N/L} c(T, v, L)$.

Definition 4.1 The *representative payoff* $f^*(N, v, L) \in \mathbb{R}^n$ assigned to every player $i \in N$ in the graph game $(v, L) \in \mathcal{G}_L^N$ is defined by

$$f_i^*(N, v, L) = f_i(N, v, L) - \frac{1}{|T_i|} c(T_i, v, L). \tag{8}$$

Observe that, within any component $T \in N/L$, $T_i = T_j$ for all $i, j \in T$; hence, the representative value $c(T, v, L)$ is distributed equally among the members of T . The representative payoff $f^*(N, v, L)$ is derived from any given allocation rule $f(N, v, L)$ via Definition 4.1; consequently, f^* carries the residual variation in f after the component-wise equal rebate has been subtracted. Existing fairness axioms benchmark marginal changes against the *entire* payoff vector, thereby confounding network effects with intra-component redistribution. To disentangle these two forces, we herein introduce two novel axioms are formulated on the basis of $f^*(N, v, L)$:

Definition 4.2 A graph game $(v, L) \in \mathcal{G}_L^N$ is said to satisfy *representative fairness* if, for every component $T \in N/L$ and every edge $ij \in T$,

$$f_i^*(N, v, L) - f_i^*(N, v, L \setminus \{ij\}) = f_j^*(N, v, L) - f_j^*(N, v, L \setminus \{ij\}). \tag{9}$$

Representative fairness captures the idea that, after the removal of any edge, the two incident players experience identical gains or losses in terms of their representative payoffs. Clearly, this requirement is an edge-deletion axiom analogous to the standard fairness property; whenever the representative value $c(T, v, L) = 0$, representative fairness collapses to the latter. Consequently, representative fairness is strictly weaker than fairness, as it is formulated through the representative payment $f_i^*(N, v, L) = f_i(N, v, L) - \frac{1}{|T_i|} c(T_i, v, L)$. In other words,

the difference in payoff changes between the two players concerned depends solely on the representative values of the newly created components. Notice that if edge ij 's deletion does not split the component—*i.e.*, $T_i = T_j = T$ —representative fairness and fairness coincide.

Definition 4.3 A graph game $(v, L) \in \mathcal{G}_L^N$ is said to satisfy *component-restricted fairness* if, for every player $i \in N$ whose departure from his component $T_i \in N/L$ creates the new collection T_i/L_{-i} ,

$$f_i^*(N, v, L) - f_i^*(N, v, L_{-i}) = c(N, v, L) - c(N, v, L_{-i}). \tag{10}$$

Here, $L_{-i} = L \setminus L_i$ is the set of edges not incident to player i . In fact, after player i leaves component T_i , one has

$c(N, v, L) - c(N, v, L_{-i}) = c(T_i, v, L) - \sum_{Z \in T_i/L_{-i}} c(Z, v, L_{-i})$, where $\sum_{Z \in T_i/L_{-i}} c(Z, v, L_{-i})$ is the aggregate representative value of the newly formed components. Component-restricted fairness can therefore be rewritten as

$$f_i^*(N, v, L) - f_i^*(N, v, L_{-i}) = c(T_i, v, L) - \sum_{Z \in T_i/L_{-i}} c(Z, v, L_{-i}).$$

Once player i severs all links with his component, he becomes isolated in the graph game $(v, L_{-i}) \in \mathcal{G}_L^N$. As illustrated in **Figure 2**, player $\{4\}$'s withdrawal breaks edges $\{45\}$ and $\{46\}$, turning $\{4\}$ into an isolated node and splitting the former component $\{4, 5, 6, 7\}$ into $\{4, 5, \{6, 7\}\}$.

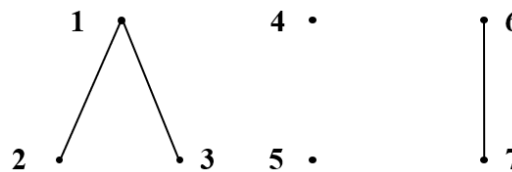


Figure 2. The graph structure after player $\{4\}$ disconnects from the component.

Treating a component as an integrated entity, the departure of any insider restructures the component itself; component-restricted fairness therefore requires that the representative payoff change of the departing player equals the change in his component's representative value.

Lemma 4.1 For any graph game $(v, L) \in \mathcal{G}_L^N$, if the payoff vector $f(N, v, L)$ satisfies component efficiency, then $f_i^*(N, v, L_{-i}) = v(i)$.

Proof. By component efficiency, once player $i \in N$ leaves his component $T_i \in N/L$ and becomes isolated, he earns only his stand-alone worth. \square

Definition 4.4 For any component-restricted game $w \in \mathcal{G}^{N \cup (N/L)}$ and any edge $e \in L$, the game after e 's removal is $w' \in \mathcal{G}^{N \cup (N/(L \setminus e))}$ defined by

$$w'(S) = \sum_{R \in (S \setminus (N/(L \setminus e))) / L(\cup_{T \in S \cap (N/(L \setminus e))} T)} v(R), \quad \forall S \subseteq N \cup (N/(L \setminus e)). \tag{11}$$

Definition 4.5 For any component-restricted game $w \in \mathcal{G}^{N \cup (N/L)}$, its sub-game $w_T \in \mathcal{G}^{T \cup \{T\}}$ is given by

$$w_T(S) = \sum_{R \in S/L(S \cap \{T\})} v(R), \quad \forall S \subseteq T \cup \{T\}. \tag{12}$$

Definition 4.6 For any sub-game $w_T \in \mathcal{G}^{T \cup \{T\}}$ of a component-restricted game,

the game obtained after player $i \in T$ leaves component T is $w_T^{-\{i\}} \in \mathcal{G}^{T \cup (T/L_i)}$ defined by

$$w_T^{-\{i\}}(S) = \sum_{R \in (S \setminus (T/L_i)) / L(\cup_{T \in S \cap (T/L_i)} T)} v(R), \quad \forall S \subseteq T \cup (T/L_i). \tag{13}$$

Lemma 4.2 In any graph game $(N, v, L) \in \mathcal{G}_L^N$, the φ -value satisfies component efficiency, representative fairness, and component-restricted fairness.

Proof. Component efficiency is verified first. Let $N/L = \{T_1, \dots, T_m\}$ be the partition of N into components. For every coalition $S \subseteq N \cup (N/L)$,

$$w(S) = \sum_{j=1}^m w(S \cap (T_j \cup \{T_j\})).$$

Define $w_j(S) = w(S \cap (T_j \cup \{T_j\}))$ for $1 \leq j \leq m$; then $w(S) = \sum_{j=1}^m w_j(S)$. Owing to the additivity of the Shapley value, for every $i \in N$,

$$\begin{aligned} \varphi_i(N, v, L) &= Sh_i(N \cup (N/L), w) + \frac{1}{|T_i|} Sh_{T_i}(N \cup (N/L), w) \\ &= \sum_{j=1}^m Sh_i(N \cup (N/L), w_j) + \frac{1}{|T_i|} \sum_{j=1}^m Sh_{T_i}(N \cup (N/L), w_j) \end{aligned}$$

If $i \notin T_k \cup \{T_k\}$ (hence $i \neq k$), then $w_k(S \cup i) - w_k(S) = 0$, implying $Sh_i(N \cup (N/L), w_k) = 0$. Consequently, for each fixed component $T_k \cup \{T_k\}$,

$$\begin{aligned} \sum_{i \in N \cup (N/L)} Sh_i(N \cup (N/L), w_k) &= \sum_{i \in (T_k \cup \{T_k\})} Sh_i(N \cup (N/L), w_k) \\ &= \sum_{i \in T_k} Sh_i(N \cup (N/L), w_k) + Sh_{T_k}(N \cup (N/L), w_k) \\ &= w_j(T_k) = w(T_k) = v(T_k) \end{aligned}$$

Therefore, for every component T_k ,

$$\begin{aligned} \sum_{i \in T_k} \varphi_i(N, v, L) &= \sum_{i \in T_k} Sh_i(N \cup (N/L), w) + Sh_{T_k}(N \cup (N/L), w) \\ &= \sum_{i \in T_k} \sum_{j=1}^m Sh_i(N \cup (N/L), w_j) + \sum_{j=1}^m Sh_{T_k}(N \cup (N/L), w_j) \\ &= \sum_{i \in T_k} Sh_i(N \cup (N/L), w_k) + \sum_{j=1}^m Sh_{T_k}(N \cup (N/L), w_k) \\ &= w_j(T_k) = w(T_k) = v(T_k) \end{aligned}$$

establishing component efficiency.

To establish representative fairness, set $c(T, v, L) = Sh_T(N \cup (N/L), w)$, so that the representative payoff becomes $f_i^*(N, v, L) = Sh_i(N \cup (N/L), w)$. It suffices to show that for every component $T \in N/L$ and every edge $ij \in T$,

$$\begin{aligned} &Sh_i(N \cup (N/L), w) - Sh_i(N \cup (N/L), w') \\ &= Sh_j(N \cup (N/L), w) - Sh_j(N \cup (N/L), w'), \end{aligned}$$

where w' is the game obtained after deleting edge $e = ij$. By additivity and sym-

metry of the Shapley value, one only needs

$$\begin{aligned} & w(S \cup i) - w(S) - (w'(S \cup i) - w'(S)) \\ & = w(S \cup j) - w(S) - (w'(S \cup j) - w'(S)) \end{aligned}$$

for all $S \subseteq N \cup (N/L)$ with $i, j \notin S$. Because $L(S \cup i) = (L \setminus e)(S \cup i)$, the definition of w yields $w(S \cup i) - w'(S \cup i) = 0$; an identical argument gives $w(S \cup j) - w'(S \cup j) = 0$, proving representative fairness.

Finally, component-restricted fairness is verified component-wise. Fix $c(T, v, L) = Sh_T(N \cup (N/L), w)$; then

$\sum_{Z \in T/L_i} c(Z, v, L_{-i}) = \sum_{Z \in T/L_i} Sh_Z(N \cup (N/L_{-i}), w)$ and the representative payoff is $f_i^*(N, v, L) = Sh_i(N \cup (N/L), w)$. Let

$$\begin{aligned} V_1 &= Sh_i(T \cup \{T\}, w_T) - Sh_i(T \cup (T/L_{-i}), w_T^{-\{i\}}), \\ V_2 &= Sh_T(T \cup \{T\}, w_T) - \sum_{Z \in T/L_{-i}} Sh_Z(T \cup (T/L_{-i}), w_T^{-\{i\}}). \end{aligned}$$

Additivity and symmetry of the Shapley value reduce the claim to $V_1 = V_2$ for all $S \subseteq T \cup \{T\}$ with $i, \{T\} \notin S$. Direct computation gives

$$V_1 = w_T(S \cup i) - w_T(S) - (w_T^{-\{i\}}(S \cup i) - w_T^{-\{i\}}(S)) = v(i) - v(i) = 0.$$

Similarly,

$$V_2 = w_T(S \cup \{T\}) - w_T(S) - \sum_{Z \in T/L_{-i}} (w_T^{-\{i\}}(S \cup \{Z\}) - w_T^{-\{i\}}(S)) = 0.$$

Hence $V_1 = V_2 = 0$, and component-restricted fairness follows. □

Example 4.1 Consider again the graph game $(v, L) \in \mathcal{G}_L^N$ of Example 3.1. If edge $\{1, 2\}$ is removed, the remaining graph is $L = \{\{4, 5\}, \{4, 6\}, \{6, 7\}\}$. The resulting allocation is

$$\varphi(N, v, L \setminus \{1, 2\}) = \left\{ 0, 0, 0, \frac{5}{16}, \frac{1}{16}, \frac{5}{16}, \frac{5}{16} \right\},$$

and the representative payoffs of players $\{1\}$ and $\{2\}$ become

$f_1^*(N, v, L \setminus \{1, 2\}) = 0$ and $f_2^*(N, v, L \setminus \{1, 2\}) = 0$. Using the values reported in Example 3.1, $f_1^*(N, v, L) = \frac{1}{3}$ and $f_2^*(N, v, L) = \frac{1}{3}$; hence

$$f_1^*(N, v, L) - f_1^*(N, v, L \setminus \{1, 2\}) = f_2^*(N, v, L) - f_2^*(N, v, L \setminus \{1, 2\}) = \frac{1}{3},$$

confirming representative fairness for the deletion of edge $\{1, 2\}$.

Likewise, after player $\{4\}$ severs all links within his component (as illustrated in **Figure 2**), the allocation becomes

$$\varphi(N, v, L_{-\{4\}}) = \left\{ \frac{4}{9}, \frac{4}{9}, \frac{1}{9}, 0, 0, 0, 0 \right\},$$

with $f_{\{4\}}^*(N, v, L_{-\{4\}}) = 0$ and $\sum_{Z \in T/L_{-\{4\}}} c(Z, v, L_{-\{4\}}) = 0$. From Example 3.1 we

have $f_{\{4\}}^*(N, v, L) = \frac{1}{4}$ and $c(T, v, L) = \frac{1}{4}$; thus

$$f_{\{4\}}^*(N, v, L) - f_{\{4\}}^*(N, v, L_{-\{4\}}) = c(T, v, L) - \sum_{Z \in T/L_{-\{4\}}} c(Z, v, L_{-Z}) = \frac{1}{4},$$

verifying component-restricted fairness for player $\{4\}$'s departure. □

Lemma 4.3 For every graph game $(N, v, L) \in \mathcal{G}_L^N$ and every component $T \in N/L$, there exists at most one payoff vector $f(N, v, L) \in \mathbb{R}^n$ and at most one component-representative value $c(T, v, L) \in \mathbb{R}$ that satisfy component efficiency and component-restricted fairness.

Proof. Uniqueness is established by contradiction. Suppose two distinct allocation rules with associated representative values, namely $f(N, v, L), c(T, v, L)$ and $g(N, v, L), d(T, v, L)$, both satisfy the two axioms. Summing the identity obtained from component-restricted fairness over all players in a component $T \in N/L$ and invoking Lemma 4.1 yields

$$\begin{aligned} \sum_{i \in T} f_i^*(N, v, L) - \sum_{i \in T} v(i) &= |T|c(T, v, L) + \sum_{i \in T} \sum_{Z \in T/L_{-i}} c(T, v, L_{-i}), \\ \sum_{i \in T} g_i^*(N, v, L) - \sum_{i \in T} v(i) &= |T|d(T, v, L) + \sum_{i \in T} \sum_{Z \in T/L_{-i}} d(T, v, L_{-i}). \end{aligned}$$

Component efficiency implies

$$\begin{aligned} \sum_{i \in T} f_i(N, v, L) &= \sum_{i \in T} f_i^*(N, v, L) + c(T, v, L) = v(T), \\ \sum_{i \in T} g_i(N, v, L) &= \sum_{i \in T} g_i^*(N, v, L) + d(T, v, L) = v(T). \end{aligned}$$

Consequently,

$$\begin{aligned} &(|T|+1)(c(T, v, L) - d(T, v, L)) \\ &= \sum_{i \in T} \sum_{Z \in T/L_{-i}} c(T, v, L_{-i}) - \sum_{i \in T} \sum_{Z \in T/L_{-i}} d(T, v, L_{-i}). \end{aligned}$$

An inductive argument on $|T|$ now gives $c(T, v, L) = d(T, v, L)$ for every component T ; details are identical to those in the proof of Lemma 4.3 and are omitted here.

Base step: For any graph game $(N, v, L) \in \mathcal{G}_L^N$ and any component $T \in N/L$ with $|T| = 1$, the component is an isolated node and $L(T) = \emptyset$. Component efficiency immediately gives $c(T, v, L) = d(T, v, L) = 0$.

For $|T| = 2$, the component contains exactly one edge, $|L(T)| = 1$. After any player leaves, the remaining agent becomes isolated. The base case $|T| = 1$ implies $\sum_{i \in T} \sum_{Z \in T/L_{-i}} c(T, v, L_{-i}) = \sum_{i \in T} \sum_{Z \in T/L_{-i}} d(T, v, L_{-i}) = 0$. Component efficiency and component-restricted fairness then force $c(T, v, L) = d(T, v, L)$.

When $|T| = 3$, two possible edge counts arise: a tree ($|L(T)| = 2$) or a complete triangle ($|L(T)| = 3$). If $|L(T)| = 2$, a player with $|L_i| = 1$ becomes isolated after departure, while the remaining two form a component of size two; a player with $|L_i| = 2$ isolates everyone. If $|L(T)| = 3$, any single departure leaves a component

of size two. Hence only components of sizes 1 or 2 can emerge, so the induction hypothesis yields $\sum_{i \in T} \sum_{Z \in T/L_{-i}} c(T, v, L_{-i}) = \sum_{i \in T} \sum_{Z \in T/L_{-i}} d(T, v, L_{-i})$. Component efficiency and component-restricted fairness again imply $c(T, v, L) = d(T, v, L)$ for every $|T| = 3$ and $|L(T)| \in [2, 3]$.

Induction hypothesis: For any $(N, v, L) \in \mathcal{G}_L^N$ and any component $T \in N/L$ with $|T| = k \geq 3$ and $|L(T)| \in [k-1, k(k-1)/2]$, we have $c(T, v, L) = d(T, v, L)$.

Induction step: Consider $|T| = k+1$ and $|L(T)| \in [k, k(k+1)/2]$. After any player $i \in T$ leaves, the emerging component T' satisfies $|T'| \in \{1, 2, \dots, k\}$. The base and induction hypotheses give

$$\sum_{i \in T} \sum_{Z \in T/L_{-i}} c(T, v, L_{-i}) = \sum_{i \in T} \sum_{Z \in T/L_{-i}} d(T, v, L_{-i}).$$

Component efficiency and component-restricted fairness then yield $c(T, v, L) = d(T, v, L)$ for all such T .

Consequently, for every $(N, v, L) \in \mathcal{G}_L^N$ and every component $T \in N/L$,

$$f_i(N, v, L) = \frac{|T_i|+1}{|T_i|} c(T, v, L) - \sum_{Z \in T/L_{-i}} c(Z, v, L_{-i}) + v(i),$$

and an identical expression holds for $g_i(N, v, L)$ with d in place of c . Thus $f(N, v, L) = g(N, v, L)$ and $c(T, v, L) = d(T, v, L)$, contradicting the assumption of distinct solutions. uniqueness follows. \square

Theorem 4.1 In every graph game $(N, v, L) \in \mathcal{G}_L^N$, the φ -value is the unique solution satisfying component efficiency and component-restricted fairness; the component-representative value $c(T, v, L) = Sh_T(N \cup (N/L), w)$ is the unique mapping that satisfies the same two axioms.

Proof. Immediate from Lemmas 4.2 and 4.3. \square

5. Conclusion and Suggestions

By embedding network-control ideas into cooperative games on graphs, we have introduced the component-restricted game and the component control value, together with two novel axioms—representative fairness and component-restricted fairness. It has been shown that the component control value is the unique allocation rule satisfying component efficiency and component-restricted fairness. The proposed framework is readily applicable to revenue sharing within project teams, cost allocation of airport runways, and other group-decision contexts. As an illustration, consider revenue sharing in open-source software foundations: each repository's maintainers and contributors form a connected component, while the foundation's charter (the external force) requires that any official grant application must be submitted by the repository as a whole. The component player $\{T\}$ therefore corresponds to the "project entity" recognised by the foundation, and φ_i quantifies how the foundation's lump-sum grant is split between the coders, thereby providing a normative benchmark for transparent disbursement. Future work will extend the analysis to multiplex network games and provide an axio-

matic characterization of the component control value in that richer environment.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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