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Derivation of Born's Postulate and an Interpretation of Quantum Mechanics as Well as the Double Slit Experiment

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Abstract

In this paper, we give a Variational Based Proof of Born's Postulate via wave-action conservation. This aids in providing a definition of the term quanta and an interpretation of Quantum Mechanics. This in turn leads to an interpretation of the Double Slit Experiment. Other results relating Quantum Mechanics to Classical Physics are also discussed.

Keywords

Dispersive Waves, Born's Postulate, Definition of Quanta, Interpretation of Quantum Mechanics, Classical Mechanics, Wave Action, Averaged Lagrangian, Whitham's Method, Lagrangian Density

1. Introduction

This paper is a follow up on a previous paper on Deriving Results of Quantum Mechanics from Classical Physics [1]. In this paper, we give a rigorous Variational Based Proof of Born's Postulate based on Whitham's Averaged Lagrangian and the resulting conservation of wave action. This inspires a definition of quanta and a wave-action based interpretation of Quantum Mechanics as well as the Double Slit Experiment. We also remark that the correspondence between Differential operators and k, ω as evidence of the eigenfunction structure of Dispersive Waves and Quantum Mechanics. We also discuss whether the generating function—set to zero in the previous paper [1] when relating the Variational Structures of Dispersive Waves and particles—leaves the underlying physics the same. When discussing the generating function, we use the Simple Harmonic Oscillator as an example and we make some remarks regarding the quantum treatment and the classical treatment.

A Recap on Dispersive Waves and Their Relation to Particles and Other Results

This subsection summarizes definitions and results from paper [1]. First, we briefly define a Dispersive Wave. It has the form $\psi = A\mathrm{e}^{i\Theta}$, where A is the Amplitude and Θ is the phase which in turn is related to the angular frequency ω and wave number k. For the simple linear case, A, k and ω are constant and we also have

$$\Theta = kx - \omega t \tag{1}$$

As

$$-i\partial_x \psi = k\psi, i\partial_t \psi = \omega\psi \tag{2}$$

We have the correspondence

$$-i\partial_x \leftrightarrow k, i\partial_t \leftrightarrow \omega \tag{3}$$

Once ψ is plugged into the governing linear Partial Differential Equation (PDE) with constant coefficients we are lead to a Dispersion Relation relating ω and k: $\omega = \Omega(k)$. With the group velocity $v_g = \Omega'(k)$. If the group velocity is not constant the waves spread out or disperse and we have a dispersive wave. For the Extended Linear Case, *i.e.* non-uniform media or linear PDEs with non-constant coefficients the correspondence persists but the trio A, ω, k are no longer constant and are instead functions of x and t. Moreover, the dispersion relation will then have an additional dependence on x and t. Here Fourier Analysis will be of limited use and the problem may be handled by Whitham's Variational Methods [2].

While for nonlinear PDEs, we have the Nonlinear Case, the correspondence still persists and the trio still depends on x and t but this time the Dispersion relation will depend further on the Amplitude. And again, the problem may be treated via Whitham's Variational Methods [2].

Again, following paper [1], the Variational Structure of a Dispersive Wave and a Particle leads to the de Broglie and Einstein-Planck Relations ($p = \hbar k$, $E = \hbar\omega$). Basically, the phase Θ of a constant dispersive wave, with k, ω constant, may be written as

$$\Theta = kx - \Omega t \tag{4}$$

where $\Omega = \Omega(k)$ is the dispersion relation and is equal numerically to the angular frequency ω

$$\omega = \Omega(k) \tag{5}$$

For a non-constant slowly varying phase this may be generalized to (see paper [1])

$$\Theta = \int_0^t \{k\dot{x} - \Omega\} \mathrm{d}t' \tag{6}$$

where now k is varying and depends on time and the dispersion relation in addition depends on time and space. While the dot signifies taking the derivative w.r.t. time. That is,

$$\begin{aligned} x &= x(t) \\ k &= k(t) \\ \Omega &= \Omega(k,x,t) \end{aligned} \tag{7}$$

The equations of motion for the more general case where $k = k(x,t)$ have been derived in paper [1] following Whitham [2].

Note that taking partial derivatives of (6) leads to the following generalized definition of ω, k, Ω:

$$\begin{aligned} \omega &= \Omega = -\Theta_t \\ k &= \Theta_x \end{aligned} \tag{8}$$

The resemblance to the Action integral of a particle is striking

$$S = \int_0^t \{p\dot{q} - H\}\,\mathrm{d}t' = \int_0^t L\,\mathrm{d}t' \tag{9}$$

where S is the Action, p is the momentum, q the space co-ordinate, H is the Hamiltonian, L is the Lagrangian. Where,

$$\begin{aligned} q &= q(t) \\ p &= p(t) \\ H &= H(p,q,t) \end{aligned} \tag{10}$$

That is, in both cases we have

- a parameter t
- ray phase space and phase space co-ordinates (k,x) and (p,q)
- A function $\Omega = \Omega(k,x,t)$, $H = H(p,q,t)$ that is dependent on the (ray) phase space co-ordinates and the parameter t

This immediately suggests the correspondence

$$\begin{aligned} q &= x \\ p &\leftrightarrow k \\ H &\leftrightarrow \Omega = \omega \\ S &\leftrightarrow \Theta \end{aligned} \tag{11}$$

However this correspondence may be made more precise. The Stationary Phase Principle and Hamilton's Principle require

$$\begin{aligned} \delta\Theta &= \delta\int_0^t \{k(t)\dot{x}(t) - \Omega(k,x,t)\}\,\mathrm{d}t' = 0 \\ \delta S &= \delta\int_0^t \{p(t)\dot{q}(t) - H(p,q,t)\}\,\mathrm{d}t' = 0 \end{aligned} \tag{12}$$

Whence they are related by an Extended Canonical Transformation, see Goldstein [3]. We may expect equations of motion with the same form as we have a similar functional dependence. Without affecting the physics—as will be seen below—, we may set the generating function to zero and we are left with the scale transformations, using $\hbar$ as the scaling constant:

$$\begin{aligned} p &= \hbar k \\ q &= 1 \cdot x \end{aligned} \tag{13}$$

Leading to:

$$H = (1 \cdot \hbar)\Omega = \hbar\Omega = \hbar\omega \tag{14}$$

Finally, combining the two Equations (13), (14) and using (6), (9)… we finally have:

$$S = \hbar\Theta \tag{15}$$

And the first of (13) is the de-Broglie relation with the scaling constant $\hbar = \frac{h}{2\pi}$, h being Planck's constant. Observe that (14) is more general than the Einstein-Planck relation $E = \hbar\omega$, as in general the Hamiltonian H is not always equal to the total Energy E. Thus, a Dispersive Wave possessing an angular frequency and a wavenumber automatically possesses energy and momentum.

Combining the de Broglie relation with the first of the correspondences (3) we have:

$$-i\hbar\partial_x \leftrightarrow p \tag{16}$$

Which is the quantization rule for the momentum. Moreover, this first correspondence means that when the respective differential operator acts on the Dispersive Wave ψ it yields $k\psi$

$$-i\partial_x\psi = k\psi \tag{17}$$

Multiplying both sides with $\hbar$ we have

$$-i\hbar\partial_x\psi = p\psi \tag{18}$$

Hence, we may deduce that the momentum operator $\hat{p}$ is

$$\hat{p} = -i\hbar\partial_x \tag{19}$$

The momentum operator (19) is used when measuring the momentum.

Now, the second of the correspondences means that when the respective differential operator acts on the Dispersive Wave ψ it yields $\omega\psi$

$$i\partial_t\psi = \omega\psi \tag{20}$$

Multiplying both sides by $\hbar$ and using the generalized Einstein-Planck Relation (14) we have

$$i\hbar\partial_t\psi = H\psi \tag{21}$$

However, we may view H as the eigenvalue of an operator $\hat{H}$. That is, $\hat{H}$ may be involve x rather than be purely temporal as the LHS of (21) shows. (For, example for $H = \frac{p^2}{2m} + V(x)$, $\hat{H} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)$ see paper [1]). Thus,

$$\hat{H}\psi = H\psi \tag{22}$$

Substituting (22) in (21) we get:

$$i\hbar\partial_t\psi = \hat{H}\psi \tag{23}$$

And so, we have derived the Schrodinger equation which is an evolution equation for the wavefunction or eigenfunction ψ. But ψ is a dispersive wave.

Hence, the wavefunction is a dispersive wave. However, during measurement, we act with a differential operator on the wavefunction as is the case when using the momentum operator to measure the momentum. Observe that (18) is a differential equation so it determines the momentum p as well the eigenfunction or dependent variable ψ—which for a general case other than the momentum case may be different from the ψ determined by the Schrodinger Equation. Thus, the act of measurement leads to a new operator and so a new differential equation with a different solution. Thus, the act of measurement destroys in general the wavefunction and Schrodinger's equation only describes the time evolution of an unmeasured/unobserved/undetected wave or wavefunction. It will be shown in the next section that the wave or wavefunction collapses into a particle upon detection with a probability $A^2\mathrm{d}x = |\psi|^2\,\mathrm{d}x$ of being found in the interval $(x, x+\mathrm{d}x)$.

2. Born's Postulate and Probabilistic Behaviour

In the previous paper [1], we discussed the Intuitive and Asymptotic Approaches to the derivation of Born's Postulate. Here, we discuss the Variational Approach.

Variational Approach

In deriving the Euler-Lagrange Equations for a field, we follow Goldstein [3]. Consider the Variational Principle

$$\delta J = \delta \int_{t_1}^{t_2} \int_{x_1}^{x_2} L(\psi_x, \psi_t, \psi)\,\mathrm{d}x\mathrm{d}t = 0 \tag{24}$$

where the subscript denotes partial differentiation. If we let

$$\psi(x,t;\alpha) = \psi(x,t;0) + \alpha\zeta(x,t) \tag{25}$$

where, $\psi(x,t;0)$ is the correct wave that satisfies the variational principle, $\zeta(x,t)$ is a function that vanishes at the endpoints in x and t. Moreover, α parametrizes L the Lagrangian density and ψ. As the dependence on x and t is integrated out, J depends solely on α

Differentiating we have:

$$\begin{gathered} L = L(\psi_x, \psi_t, \psi) \\ \mathrm{d}L = \frac{\partial L}{\partial \psi_x}\mathrm{d}\psi_x + \frac{\partial L}{\partial \psi_t}\mathrm{d}\psi_t + \frac{\partial L}{\partial \psi}\mathrm{d}\psi \\ \frac{\partial L}{\partial \alpha} = \frac{\partial L}{\partial \psi_x}\frac{\partial \psi_x}{\partial \alpha} + \frac{\partial L}{\partial \psi_t}\frac{\partial \psi_t}{\partial \alpha} + \frac{\partial L}{\partial \psi}\frac{\partial \psi}{\partial \alpha} \end{gathered} \tag{26}$$

And,

$$\frac{\mathrm{d}J}{\mathrm{d}\alpha} = \int_{t_1}^{t_2} \int_{x_1}^{x_2} \frac{\partial L}{\partial \psi_x}\frac{\partial \psi_x}{\partial \alpha} + \frac{\partial L}{\partial \psi_t}\frac{\partial \psi_t}{\partial \alpha} + \frac{\partial L}{\partial \psi}\frac{\partial \psi}{\partial \alpha}\mathrm{d}x\mathrm{d}t \tag{27}$$

Integration by parts is:

$$\int u\mathrm{d}v = uv - \int v\mathrm{d}u \tag{28}$$

Setting

$$u=\frac{\partial L}{\partial \psi_x},\mathrm{d}v=\frac{\partial \psi_x}{\partial \alpha}\mathrm{d}x \tag{29}$$

We have,

$$\int_{x_1}^{x_2}\frac{\partial L}{\partial \psi_x}\frac{\partial \psi_x}{\partial \alpha}\mathrm{d}x=\frac{\partial L}{\partial \psi_x}\left[\frac{\partial \psi}{\partial \alpha}\right]_{x_1}^{x_2}-\int_{x_1}^{x_2}\frac{\partial \psi}{\partial \alpha}\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial L}{\partial \psi_x}\right)\mathrm{d}x \tag{30}$$

But from (25) we have

$$\frac{\partial \psi}{\partial \alpha}=0+\zeta\left(x,t\right) \tag{31}$$

And so,

$$\left[\frac{\partial \psi}{\partial \alpha}\right]_{x_1}^{x_2}=0 \tag{32}$$

As $\zeta\left(x,t\right)$ vanishes at the endpoints x_2,x_1. Thus (30) becomes

$$\int_{x_1}^{x_2}\frac{\partial L}{\partial \psi_x}\frac{\partial \psi_x}{\partial \alpha}\mathrm{d}x=-\int_{x_1}^{x_2}\frac{\partial \psi}{\partial \alpha}\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial L}{\partial \psi_x}\right)\mathrm{d}x \tag{33}$$

Similarly Setting

$$u=\frac{\partial L}{\partial \psi_t},\mathrm{d}v=\frac{\partial \psi_t}{\partial \alpha}\mathrm{d}t \tag{34}$$

We have,

$$\int_{t_1}^{t_2}\frac{\partial L}{\partial \psi_t}\frac{\partial \psi_t}{\partial \alpha}\mathrm{d}x=\frac{\partial L}{\partial \psi_t}\left[\frac{\partial \psi}{\partial \alpha}\right]_{t_1}^{t_2}-\int_{t_1}^{t_2}\frac{\partial \psi}{\partial \alpha}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \psi_t}\right)\mathrm{d}t \tag{35}$$

From (31) we have,

$$\left[\frac{\partial \psi}{\partial \alpha}\right]_{t_1}^{t_2}=0 \tag{36}$$

As $\zeta\left(x,t\right)$ vanishes at the endpoints t_2,t_1. Thus (35) becomes

$$\int_{t_1}^{t_2}\frac{\partial L}{\partial \psi_t}\frac{\partial \psi_t}{\partial \alpha}\mathrm{d}x=-\int_{t_1}^{t_2}\frac{\partial \psi}{\partial \alpha}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \psi_t}\right)\mathrm{d}t \tag{37}$$

Substitute (33) and (37) in (27) to get:

$$\frac{\mathrm{d}J}{\mathrm{d}\alpha}=\int_{t_1}^{t_2}\int_{x_1}^{x_2}\left[\frac{\partial L}{\partial \psi}-\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial L}{\partial \psi_x}\right)-\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \psi}\right)\right]\frac{\partial \psi}{\partial \alpha}\mathrm{d}x\mathrm{d}t \tag{38}$$

As J is an extremum for $\psi\left(x,t;0\right)$, the derivative w.r.t. α vanishes at $\alpha=0$. Whence,

$$\int_{t_1}^{t_2}\int_{x_1}^{x_2}\left[\frac{\partial L}{\partial \psi}-\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial L}{\partial \psi_x}\right)-\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \psi}\right)\right]\left(\frac{\partial \psi}{\partial \alpha}\right)_{\alpha=0}\mathrm{d}x\mathrm{d}t=0 \tag{39}$$

Moreover, the arbitrary nature of the varied path implies the vanishing of the expression inside the square brackets

$$\frac{\mathrm{d}}{\mathrm{d}t}L_{\psi_t}+\frac{\mathrm{d}}{\mathrm{d}x}L_{\psi_x}-L_{\psi}=0 \tag{40}$$

Where again, the subscript denotes partial differentiation. That is,

$$\begin{aligned} L_{\psi_i} &= \frac{\partial L}{\partial \psi_i}; i = x, t \\ L_{\psi} &= \frac{\partial L}{\partial \psi} \end{aligned} \tag{41}$$

This is where Whitham [2] Section 11.7 appears to have made an inaccuracy. He replaced the total derivatives in (40) with partial derivatives. Thankfully, this does not affect the final result.

For a slowly changing Dispersive Wave in 1D,

$$\begin{aligned} \psi(x,t) &= A(x,t)\mathrm{e}^{i\theta(x,t)} \\ \omega &= -\theta_t \\ k &= \theta_x \end{aligned} \tag{42}$$

Where the angular frequency is ω, wave number is k and Amplitude is A. See also the definition in the paper [1] as well as Whitham Chapter 11 [2] and Tracy Appendix B [4]. Furthermore, for continuous θ we have

$$\theta_{xt} = \theta_{tx} \tag{43}$$

Using the second and third of (42) this means,

$$\frac{\partial k}{\partial t} + \frac{\partial \omega}{\partial x} = 0 \tag{44}$$

But the group velocity is

$$v_g = \frac{\partial \omega}{\partial k} \tag{45}$$

Whence we may rewrite (44) as:

$$\frac{\partial k}{\partial t} + \frac{\partial \omega}{\partial k}\frac{\partial k}{\partial x} = 0 \tag{46}$$

Or,

$$\frac{\partial k}{\partial t} + v_g \frac{\partial k}{\partial x} = 0 \tag{47}$$

Now if we substitute (42) into the Lagrangian Density. Whitham [2], unlike Tracy *et al.* [4] mentions that derivatives of these parameters are neglected (Though Tracy *et al.* in [4] does emphasize that the Dispersive Wave or the Eikonal representation is slowly varying). We average the Lagrangian Density L over one period and we get the Averaged Lagrangian Density $\mathscr{L}$. Observe that Averaging when dealing with waves is very natural and reminds us of Averaging in Quantum Mechanics when calculating the Expectation values, see the paper [1] and the references therein.

And at this stage, Whitham [2] proposes the Averaged Variational Principle:

$$\delta \iint \mathscr{L}(-\theta_t, \theta_x, A)\,\mathrm{d}t\mathrm{d}x \tag{48}$$

As derivatives of A are not present, by analogy with (40), it's variation gives:

$$\mathfrak{L}_A = 0 \tag{49}$$

And as θ is not present, again by the same analogy, it's variation gives:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathfrak{L}_{\theta_t} + \frac{\mathrm{d}}{\mathrm{d}x}\mathfrak{L}_{\theta_x} = 0 \tag{50}$$

Again, this is different from the equation derived by Whitham in Section 11.7 [2], where he replaced the total derivatives by partial derivatives.

Equation (49) is a relation between A, ω, k; whence it is a dispersion relation. For the Linear Extended Case, L is quadratic in ψ and it's derivatives and so $\mathfrak{L}$ takes the form:

$$\mathfrak{L} = G(\omega, k) A^2 \tag{51}$$

where $G(\omega, k)$ is an equation in ω and k *i.e.* it's an implicit dispersion relation. And so from (49) we have,

$$G(\omega, k) = 0 \tag{52}$$

Applying Noether's Theorem to (50) by considering what happens to the Averaged Lagrangian due to variations in the phase, see [4], we get the wave-action conservation in 1D. (This is a generalization of the conservation of classical energy—which in turn is proportional to the Intensity or the square of the Amplitude. This means that the derivation given in this subsection of the Born's Postulate is more general than the intuitive argument given in paper [1] that assumed that the classical energy is conserved.)

$$\frac{\partial}{\partial t}\left(G_\omega A^2\right) - \frac{\partial}{\partial x}\left(G_k A^2\right) = 0 \tag{53}$$

Which is Equation (11.85) in Whitham [2], so we are back on track and the effect of the inaccuracy is gone. Equation (52) can be solved to yield the explicit dispersion relation

$$\omega = \Omega(k) \tag{54}$$

Thus,

$$G(\Omega(k), k) = 0 \tag{55}$$

And taking the total derivative

$$\mathrm{d}G = \frac{\partial G}{\partial \Omega}\mathrm{d}\Omega + \frac{\partial G}{\partial k}\mathrm{d}k = 0 \tag{56}$$

or,

$$G_\omega \frac{\mathrm{d}\Omega}{\mathrm{d}k} + G_k = 0 \tag{57}$$

or the group velocity is given by

$$v_g = \frac{\mathrm{d}\Omega}{\mathrm{d}k} = -\frac{G_k}{G_\omega} \tag{58}$$

In other words, (53) can be written as

$$\frac{\partial}{\partial t}\left(G_\omega A^2\right) + \frac{\partial}{\partial x}\left(G_\omega v_g A^2\right) = 0 \tag{59}$$

or,

$$\begin{aligned}0&=A^2\frac{\partial}{\partial t}G_\omega+G_\omega\frac{\partial}{\partial t}A^2+G_\omega\frac{\partial}{\partial x}\left(v_gA^2\right)+v_gA^2\frac{\partial}{\partial x}G_\omega\\&=A^2\left(\frac{\partial}{\partial t}G_\omega+v_g\frac{\partial}{\partial x}G_\omega\right)+G_\omega\left(\frac{\partial}{\partial t}A^2+\frac{\partial}{\partial x}\left(v_gA^2\right)\right)\end{aligned}\tag{60}$$

and

$$\begin{aligned}\frac{\partial G_\omega}{\partial t}+v_g\frac{\partial G_\omega}{\partial x}&=\frac{\partial G_\omega}{\partial k}\frac{\partial k}{\partial t}+v_g\frac{\partial G_\omega}{\partial k}\frac{\partial k}{\partial x}\\&=\frac{\partial G_\omega}{\partial k}\left(\frac{\partial k}{\partial t}+v_g\frac{\partial k}{\partial x}\right)\\&=0\end{aligned}\tag{61}$$

where at the last equality we used (47). Substituting (61) in (60) we get,

$$\frac{\partial}{\partial t}A^2+\frac{\partial}{\partial x}\left(v_gA^2\right)=0\tag{62}$$

Comparing this to the continuity equation in fluid mechanics in 1D

$$\frac{\partial}{\partial t}\rho+\frac{\partial}{\partial x}(\rho v)=0\tag{63}$$

where ρ is the linear density, ρv is the flux, v is the velocity of the mass element $\rho\mathrm{d}x$. The comparison leads to the fact that v_g is the speed of the intensity A^2 or more precisely $A^2\mathrm{d}x$. in accordance with Witham's [2] discussion of the dual role of the group velocity—the other role that it is the velocity of the wave number. The mass m is conserved with

$$m=\int\rho\mathrm{d}x\tag{64}$$

we see that ρ corresponds to A^2 and the quantity

$$\int A^2\mathrm{d}x=const\left(conserved\right)\tag{65}$$

But from (42),

$$A^2=|\psi|^2\tag{66}$$

And we end up with

$$\int|\psi|^2\,\mathrm{d}x=const\left(conserved\right)\tag{67}$$

But v_g is also the speed of the particle (by the identification of the Hamiltonian Structures in Dispersive waves and particle mechanics see paper [1]). Thus $A^2\mathrm{d}x$ moves at the velocity of the particle and may be identified with the particle. The fact that $\int A^2\mathrm{d}x=const$ means that upon normalization it may be set to 1 and so may identified with the total probability of finding a particle. Now $A^2\mathrm{d}x$ is identified with detecting the particle in the interval $(x,x+\mathrm{d}x)$ with $A^2\mathrm{d}x$ being a part of unity as we are dealing with an indivisible particle. That is, we are excluding the cases where the particle splits or spontaneously decays. Thus, we can't have a part of a particle but we can have the probability of finding a particle. Thus, the conservation of Whitham's wave action leads to conservation of probability.

3. Definition of a Quanta and an Interpretation of Quantum Mechanics

A quanta is defined here as the identification of a dispersive wave $\psi = \mathrm{e}^{i\Theta}$ and a particle by their respective Variational Structures. It is created and propagates as a wave and is detected(measured) like a particle. After being detected or measured it moves, acts and remains as a particle. Also, the wavefunction is identified with a dispersive wave as was mentioned in the introduction.

The Wave aspect means that the quanta possesses an angular frequency ω and wavenumber k. And so automatically possesses Energy and Momentum as shown in the introduction.

As waves may interfere and exist in superposition, the quanta may interfere and exist as a superposition of states (before measurement or detection). This is in accordance with the Copenhagen Interpretation. Furthermore, this Unification of Waves and Particles applies to all scales. It's only wave-particle duality (where the particle and associated wave coincide… rather than just having the Intensity and particle coincide) that works on a small scale see [1].

This superposition and interference are of the core principles of Quantum Computing permitting the performance of many calculations in parallel see [5].

As mentioned in the Introduction, the act of measurement leads to a new operator and so a new differential equation with a different solution. Thus, the act of measurement destroys in general the wavefunction and Schrodinger's equation only describes the time evolution of an unmeasured/unobserved/undetected wave or wavefunction up until detection. However, during detection or measurement at an interval $(x, x+\mathrm{d}x)$ the wave transforms, actually collapses, into a particle in accordance with the square of the Amplitude $A^2\mathrm{d}x = |\psi|^2\,\mathrm{d}x$ as in the Born's Postulate just proven. The Born's Postulate describes the bridge between wave state and particle state of a quanta. After detection the wave character is forgotten and the quanta behaves as a particle and never returns to the wave state.

The wavefunction or wave character is not observed directly. What is observed directly is the collapse of the wavefunction into a particle after observation or detection. If many quanta are detected the resulting detection may start to reveal the wave character in accordance with $A^2\mathrm{d}x = |\psi|^2\,\mathrm{d}x$ as happens at the detection screen of the Double Slit Experiment (with no detectors at the slits).

4. Double Slit Experiment

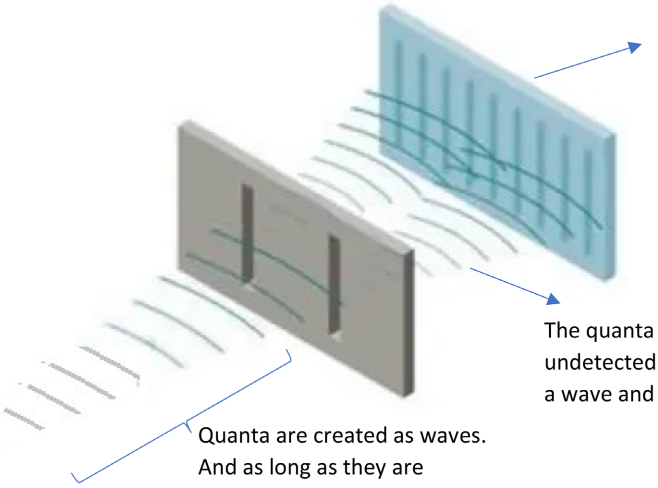
This wave-action based interpretation that a quanta is created and propagates as a wave and is detected as a particle helps to explain the double slit experiment. When passing through the double slits (without being detected) the wave splits and then we have an interference of the resulting waves. However, at the detection screen, they appear as blobs because quanta are detected as particles... But for many blobs the quanta form a wave-interference pattern… Thanks to the probabilistic nature of the link between a particle and a wave inherent in Born's Postulate, many particles at the detection screen will reproduce the wave pattern.

Moreover, if the quanta are detected before entering the double slits, then they collapse to being particles and form on the detection screen the pattern of being particles passing through the slits *i.e.* two bands.

What if the quanta are fired one by one with the detectors at the double slits removed? Then **Figure 1** still applies. The quanta travels as a wave to the double slits and splits over there as waves are expected to. Then the two waves interfere. However, when the quanta reaches the detection screen it is detected as a single particle marking a single blob anywhere the probability waves deems permissible *i.e.* the detection happens in the region of the interference bands (more than two bands are created).

What if the quanta are fired one by one but this time with the detectors at the double slits in place? Then **Figure 2** applies. The quanta travels as a wave to the detectors and at the detectors collapse into a particle and remains as a particle afterwards as it passes through either of the slits. When the particle reaches the detection screen it is detected as a single particle marking a single blob anywhere permissible *i.e.* the region of the two bands expected of particles.

All this was inferred from the Variational Structure, Whitham's Averaged Variational Principle and the resulting wave action conservation… without the need to consider the underlying Schrodinger Equation as is the treatment in say Liboff [6]. It was only assumed that the governing PDE is linear. This shows the power of Whitham's methods in understanding waves and understanding quantum mechanics.

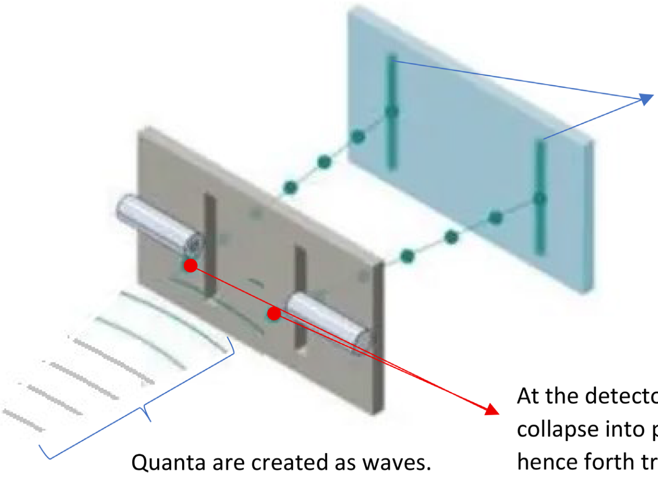


A single quanta is detected as a particle or a blob. Thus, at the detection screen it appears as a blob. For many blobs they form a wave-interference pattern (more than two bands). This is due to the probabilistic nature of the link A^2dx between a wave and a particle.

The quanta travel as a wave and as they are undetected at the double-slits, they pass through as a wave and split and the resulting waves interfere.

Quanta are created as waves. And as long as they are undetected, they travel and behave as waves

Figure 1. Wave-Action based interpretation applied to Double-Slit Experiment with no detectors before the slits. When quanta are passing through the double slits, without being detected, they travel as waves. They split at the double slits and interfere as waves. However, at the detection screen, they appear as blobs because they are detected as particles… But for many blobs they form a wave-interference pattern. Thanks to the probabilistic nature of the link between a particle and a wave inherent in Born's Postulate, many particles at the detection screen will reproduce the wave pattern.



On the detection screen, the already collapsed quanta are particles and are detected as particles. The pattern created is that of two bands.

At the detectors, the quanta collapse into particles and hence forth travel and behave as particles

Quanta are created as waves. And as long as they are undetected, they travel and behave as waves

Figure 2. Wave-Action based interpretation applied to Double-Slit Experiment with detectors before the slits. If quanta are detected before entering the double slits then they collapse to being particles and form on the detection screen the pattern of particles passing through the slits.

It would be interesting to see the kind of physics that would arise had we considered a nonlinear PDE and the associated more general Nonlinear Case of the Dispersive Wave (where now the dispersion relation is a function of the Amplitude as well as the wave number and time and space) rather than just the Extended Linear Case (where the Dispersion Relation is a function of the wave number and time and space), for detailed definitions of the cases see paper [1]. In the Nonlinear Case, $\mathcal{L}$ is no longer proportional to A^2 though the wave action conservation still applies see Whitham [2].

However, it seems that the Extended Linear Case is sufficient for Quantum Mechanics as the Hamiltonian is only needed to be a function of p,q,t (momentum, space co-ordinate and time). That is, we have the correspondence (see [1]):

$$\begin{array}{c} H\left(p,q,t\right) \\ \updownarrow \ \updownarrow \updownarrow \updownarrow \\ \Omega\left(k,x,t\right) \end{array} \tag{68}$$

Rather than having the correspondence

$$\begin{array}{c} H\left(?,\ \ p,q,t\right) \\ \updownarrow \ \ \updownarrow \ \updownarrow \updownarrow \updownarrow \\ \Omega\left(A^2,k,x,t\right) \end{array} \tag{69}$$

It was to ensure the application of superposition that Schrodinger proposed a Linear Differential Equation. However, in Quantum Field Theory, we may have nonlinear equations but even then, we may still have periodic solutions and the wave-action conservation still applies see Whitham [3]… However, it's unclear whether the Born's Postulate would follow. This is because linearity was assumed when suggesting the form of $\mathcal{L}$ in Equation (51).

Finally, it's worth mentioning that in [4], the wave action density was related to the particle(photon) number density $n\hbar$.

5. Remark on Correspondence between Differential Operators and Wave Parameters

The correspondence between Differential Operators with k and ω (see paper [1]),

$$-i\partial_x \leftrightarrow k, i\partial_t \leftrightarrow \omega \tag{70}$$

is evidence of the eigen structure of Dispersive Waves as well as Quantum Mechanics. This is because in order for the differential operator to yield the Wave parameter (as an eigenvalue) it needs to act on an eigenfunction/wavefunction.

$$\begin{gathered} -i\partial_x \leftrightarrow k \Leftrightarrow -i\partial_x \psi = k\psi \\ i\partial_t \leftrightarrow \omega \Leftrightarrow i\partial_t \psi = \omega\psi \end{gathered} \tag{71}$$

This shows that the Dispersive Waves possess an eigen structure. Moreover, this eigen-formulation is inherited by Quantum Mechanics (with the aid of the de Broglie and Einstein-Planck Relations). This eigen-formulation then leads to the Measurement postulate, where the act of measurement changes the state and the wavefunction/eigenfunction as discussed in the paper [1].

6. On the Generating Function

In the paper [1], an Extended Canonical Transformation was found to relate a Dispersive Wave's Stationary Principle and a particle's Hamilton's Principle. By Setting the Generating Function to zero, the Hamiltonian structures of the waves and particles had been related by a scaling transformation (a subset of Extended Canonical Transformation). The de Broglie and Einstein-Planck relations were then derived. It was then remarked that it would be interesting to see the kind of physics that would arise had the generating function was not set to zero.

Here, it is argued that the Generating Function leads to a geometric transformation that might simplify the equations involved yet leaves the underlying physics the same—a mere change of variables. By geometric, we don't mean a mere coordinate transformation that leaves the momenta untouched… momenta and coordinates may be mixed together. By geometric we mean a transformation that does not change the physics. For example, in Goldstein [3] when considering the Simple Harmonic Oscillator (SHO) with the Hamiltonian

$$H = \frac{1}{2m}p^2 + \frac{1}{2}kq^2 \tag{72}$$

where m is the mass of the block and k is the spring constant. It was proposed to find a Canonical Transformation that leaves the Hamiltonian as cyclic … that would greatly simplify the mathematics and make the physics more transparent. Transforming from $(p,q) \to (P,Q)$ a generating function of the first kind was proposed:

$$F_1 = \frac{1}{2}m\omega q^2 \cot Q \tag{73}$$

With

$$p = \frac{\partial F_1}{\partial q} = m\omega q \cot Q \qquad P = \frac{\partial F_1}{\partial Q} = \frac{m\omega q^2}{2\sin^2 Q} \tag{74}$$

Solving for p and q in terms of (P,Q) we have (observe that here the old co-ordinate is written in terms of the new co-ordinate as well as the new momentum so this is not a mere co-ordinate transform there is mixing as was noted earlier),

$$q = \sqrt{\frac{2P}{m\omega}} \sin Q \qquad p = \sqrt{2Pm\omega} \cos Q \tag{75}$$

And the Hamiltonian (which is conserved and is identical to the total energy E) is found to be

$$H = E = \omega P \tag{76}$$

And now we may solve for Q using

$$\dot{Q} = \frac{\partial H}{\partial P} = \omega \tag{77}$$

or

$$Q = \omega t + \alpha \tag{78}$$

where α is a constant. And we may now write p and q in terms of *t*:

$$q = \sqrt{\frac{2E}{m\omega^2}} \sin(\omega t + \alpha) \qquad p = \sqrt{2mE} \cos(\omega t + \alpha) \tag{79}$$

And we went a full circle from $(p,q) \to (P,Q) \to (p,q)$. We have solved for p and q in terms of t using the simplifying Generating Function. But the physics is the same, only more transparent… all what we have done is a change of variables. The Generating function allows us to transform the co-ordinates and momenta but the underlying physics remains the same. This is because the physics lies in the Hamiltonian Equations

$$\dot{q} = \frac{\partial H}{\partial p}, \dot{p} = -\frac{\partial H}{\partial q} \qquad \dot{Q} = \frac{\partial H}{\partial P}, \dot{P} = -\frac{\partial H}{\partial Q} \tag{80}$$

which are left unchanged by the Generating Function. Thus, there was no physics left out when we set the generating function to zero in the previous paper [1]. The generating function generates a canonical transformation which preserves the form of Hamilton's equations, which contain the physics.

It's worth noting that in phase space (p,q) (79) traces an ellipse. With an area (for $m \neq 1$ which more general than what was done in the paper [1]):

$$Area = \frac{2\pi E}{\omega} = 2\pi\hbar = h \tag{81}$$

Whence h is still a multiple of a Poincare Invariant even for $m \neq 1$.

It is noteworthy that the Energy E equals the Hamiltonian H and so corresponds to the Explicit Dispersion Relation $\Omega(k,x,t)$ (see paper [1]) which in turn is equal to the angular frequency ω (actually it's equal up to a constant again see paper [1]). Thus, the Energy in this classical example is proportional to the angular frequency as in the quantum case. Prior to the application of the correspondence, the form of the first of (1.79) suggests that the Energy is proportional to the Intensity (Amplitude Squared) as in the classical case. This is another result from (large scale) Classical Physics that can be used to explain results of Quantum Mechanics.

7. Comparison with Other Interpretations of Quantum Mechanics

The Interpretation given in this work is in line with the Copenhagen Interpretation [7]. In that the probability of detection of the particle is derived from the wavefunction via Born's Postulate. The principle of complementarity—that certain pair properties that cannot be observed or measured simultaneously—follows from the eigen structure of Quantum Mechanics and the fact that the measurement is encoded mathematically via an eigen operator… and that two operators might not commute.

The Objectivity of Quantum Mechanics follows from the fact that no reference to the state of the observer is made. While the superposition aspect follows from the fact that the wavefunction is actually a wave and waves may superpose. The difficulty of the fact that measurement is described using classical physics rather than Quantum Mechanics might have been circumvented in this work by noting that the Quantum Mechanics here as well as the Born's Postulate were derived from Classical Physics—this is the advantage of the current work. Thus, there is no jump from the Classical to the Quantum. Detection is a classical process in accordance with the wave action conservation. A wave is a wave and a particle is a particle both described in this work using classical physics—the former using Whitham's Methods and the latter using Classical Mechanics. The troublesome transition from wave to particle occurs only during detection... and that too is in accordance with the Classical Treatment of Waves by Whitham. As noted in paper [1], Whitham's Method provides a framework for understanding Classical Waves as well as Quantum Mechanics and it's Wavefunction.

Detection and wavefunction collapse arise naturally from the framework of Whitham's Method of Dispersive Waves and that the wave function is a dispersive wave (that also resulted in the derivation of Schrodinger Equation and the de Broglie and Einstein Planck Relations) and so is not ad hoc. In fact, Born's Postulate has been derived—as noted earlier. Thus, there is no need to for a Many-

Worlds Interpretation [8] that avoids the wavefunction collapse and speaks of many universes or worlds existing simultaneously due to superposition. A wavefunction is a wave and so it may be a superposition of states. However, at detection, the wavefunction collapses into one of the possible states.

There is no need either of a Hidden Variables Theory [9]. The Uncertainty Principle for example emerges naturally from the de Broglie Relation as well as the Fourier Relation $\Delta k \Delta x = \frac{1}{2}$ and there is nothing probabilistic or indeterminate about the Fourier Relation. The Born's Postulate derived here arises from the mapping of waves and classical particle through the marriage of Dispersive Waves and Classical Mechanics… Through the matching of the existing (Variational) structures found in the Stationary Principle and the Hamilton's Principle. No new structures were proposed.

That is, no new physical structure e.g. strings, was proposed to explain Quantum Mechanics. Neither was a new mathematical structure proposed e.g. twistors [10] to explain Quantum Mechanics. The structures involved were individually naturally occurring in deterministic frameworks. Waves are deterministic and evolve according to a differential equation. Similarly, particles are deterministic and evolve according to a differential equation. It was only when they were mapped into each other and linked did Probabilistic Behavior Arise. This is because a wave has a large wavefront and is extended… while a point particle is localized in space. Mapping the two requires us to take a portion of the wavefront or intensity and map it to a point in accordance with the proven Born Postulate. No need to suppose hidden variables to explain the probabilistic correspondence between a wave's intensity and a particle's location—it's a result of the unification of waves and particles. There is no need to suppose branching of time during measurement either… the collapse of the wave into a point particle during measurement is due to the natural correspondence between waves and particles.

8. Conclusion

In this paper, we gave a rigorous Variational Based Proof of Born's Postulate based on Whitham's Averaged Lagrangian and the resulting conservation of wave action. This aided in giving a definition of Quanta and a wave-action based interpretation of Quantum Mechanics that was later applied to the double slit experiment. Comparisons to other Interpretations of Quantum Mechanics were made. We also remarked that the correspondence between Differential operators and k, ω as evidence of eigen structure of Dispersive Waves and Quantum Mechanics—the latter inheriting the structure from the former, in our treatment. We also showed that setting the generating function to zero leaves the underlying physics the same.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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