Scientific Research Publishing

Journal of Applied Mathematics and Physics, 2025, 13(11), 3765-3779
https://www.scirp.org/journal/jamp
ISSN Online: 2327-4379
ISSN Print: 2327-4352

Weak Pullback Attractors of Stochastic Semilinear Degenerate Equations with Memory Driven by Nonlinear Noise

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How to cite this paper: Liu, X.M. (2025) Weak Pullback Attractors of Stochastic Semilinear Degenerate Equations with Memory Driven by Nonlinear Noise. *Journal of Applied Mathematics and Physics*, **13**, 3765-3779.
https://doi.org/10.4236/jamp.2025.1311210

Received: October 13, 2025
Accepted: November 7, 2025
Published: November 10, 2025

Abstract

This paper is concerned with the long-time behavior of solutions for a class of stochastic semilinear degenerate equations with memory driven by nonlinear noise on $\mathbb{R}^n$. We primarily provide sufficient conditions for studying such problems, including the degenerate term $div\{a(x)\nabla u\}$, nonlinear drift, and diffusion terms. Based on these conditions, we establish the existence of weakly compact pullback absorbing sets and further obtain the existence and uniqueness of weak $\mathscr{D}$-pullback mean random attractors.

Keywords

Semilinear Degenerate Equation, Weak Pullback Mean Random Attractor, Nonlinear Noise, Memory

1. Introduction

In this paper, we study the asymptotic behavior of solutions for the following degenerate semilinear parabolic equations with memory driven by nonlinear multiplicative noise on unbounded domain $\mathbb{R}^n\,(n\geq 2)$:

$$\begin{aligned}&\partial_t u-div\{a(x)\nabla u\}-\int_0^\infty k(s)\Delta u(t-s)\mathrm{d}s+\lambda u+f(x,u)\\&=g(x,t)+\varepsilon\sigma(t,u)\frac{\mathrm{d}W}{\mathrm{d}t},\end{aligned}\tag{1}$$

with the initial data

$$u(x,\tau-r)=u_0(x),\ \ r\geq 0,x\in\mathbb{R}^n,\tau\in\mathbb{R},\tag{2}$$

where the variable nonnegative weighted coefficient $a(\cdot)$ denotes the diffusivity, the forcing term $g(t)=g(x,t)\in L_{loc}^2(\mathbb{R},L^2(\mathbb{R}^n))$ and the initial data $u_0\in L^2(\mathbb{R}^n)$ are given, $\lambda>0$ is a constant, $\varepsilon\in[0,\varepsilon_0]$ $(\varepsilon_0>0)$ denotes the intensity of noise, W is a two-sided U-valued cylindrical Wiener process defined on the complete filtered probability space $(\Omega,\mathscr{F},\{\mathscr{F}_t\}_{t\in\mathbb{R}},\mathbb{P})$.

This class of equations, exemplified by (1) with $a(x)=1$, models delayed reaction-diffusion processes where the reaction term depends on the historical state of the temperature field [1]-[4]. Its applicability extends to physical systems like polymers and high-viscosity liquids [5]-[7]. A critical distinction arises in the general form of (1): a spatially variable coefficient $a(x)$ enables the model to represent media with potentially "perfect" insulating properties in certain subdomains, as noted in [8].

The study of well-posedness and attractors for deterministic degenerate parabolic equations, both with and without memory terms, has been extensively conducted, see e.g., [9]-[20] and the references therein. However, research on stochastic semilinear degenerate parabolic equations is relatively scarce, see [21]-[23]. In these published works, the authors have exclusively investigated systems on bounded domains, primarily examining the existence and stability of random attractors for the corresponding stochastic systems under additive and multiplicative noise conditions. A natural question arises: can the existence of random attractors be considered for semilinear degenerate equations driven by nonlinear noise?

To the best of the author's knowledge, several studies have addressed stochastic evolution equations driven by nonlinear noise, see e.g., [24]-[29]. In these studies, the scholars mainly focus on the existence of weak mean random attractors for the corresponding stochastic systems. This is because, for stochastic evolution equations driven by nonlinear noise, we cannot use the Ornstein-Uhlenbeck transform to convert them into pathwise random differential equations, as can be done for equations driven by additive or linear multiplicative noise.

In particular, the authors in [26] investigated the existence of weak pullback random attractor for the stochastic degenerate semilinear parabolic equations driven by nonlinear noise. It is worth noting that this equation does not include a memory term. For the reaction-diffusion equation with memory term and nonlinear noise, the authors in [29] recently established the existence of weak mean random attractor for the corresponding stochastic system, though the equation in question is non-degenerate. Additionally, it should be emphasized that all the aforementioned studies are confined to bounded domains. Building on this, we may consider the existence of weak mean random attractor for the stochastic semilinear parabolic equation driven by nonlinear noise, which incorporates both degeneracy and a memory term, in an unbounded domain.

In studying the weak $\mathscr{D}$-pullback mean random attractors for problem (1)-(2), we encounter several difficulties. The primary challenges include establishing suitable conditions on the degenerate variable diffusivity $a(x)$ and the nonlinear drift and diffusion terms. Moreover, the presence of the memory term introduces additional complexity: it obstructs the direct derivation of long-time uniform estimates of the solution. Instead, the uniform boundedness of η must be inferred from that of u. To address these challenges, we fully leverage existing results on weak pullback mean random attractors for degenerate parabolic equations in unbounded domains and reaction-diffusion equations with memory driven by nonlinear noise.

The structure of this paper is as follows. In Section 2, we introduce notations, recall the abstract theory of weak mean random attractors, and establish sufficient conditions on the variable diffusivity $a(x)$ and the nonlinear drift and diffusion terms. We also address the well-posedness of problem (16)-(17). In Section 3, we derive long-time uniform estimates of solutions for (16)-(17) and prove the existence of a weak $\mathscr{D}$-pullback mean random attractor.

2. Preliminaries

We begin this section by introducing the necessary notations, proceed to recall the abstract theory of mean random dynamical systems and the existence of weak $\mathscr{D}$-pullback mean random attractors, and conclude by stating the main assumptions on the nonnegative variable diffusivity, memory kernel, as well as nonlinear drift and diffusion terms, all of which are essential for the analysis to follow.

2.1. Notations

Let C be a generic constant, which is allowed to vary and may depend on the context, even within a single line. Denote $\mathbb{R}^+ = [0,\infty)$ and $\mathbb{R}^\tau = [\tau,\infty)$. Let

$$|u|_p = \left(\int_{\mathbb{R}^n} |u(x)|^p \,\mathrm{d}x\right)^{\frac{1}{p}}$$

be the norm of $L^p(\mathbb{R}^n)$ with $2 \le p < \infty$. We denote the norms of $L^1(\mathbb{R}^n)$ and $L^\infty(\mathbb{R}^n)$ by $\|\cdot\|_{L^1}$ and $\|\cdot\|_{L^\infty}$, respectively. Moreover, let $H = L^2(\mathbb{R}^n)$ and $H^1(\mathbb{R}^n)$ be endowed with the following inner products:

$$(u,v) = \int_{\mathbb{R}^n} u(x)v(x)\,\mathrm{d}x, \ \forall u,v \in L^2(\mathbb{R}^n),$$

$$\langle u,v\rangle_{H^1(\mathbb{R}^n)} = \int_{\mathbb{R}^n} u(x)v(x)\,\mathrm{d}x + \int_{\mathbb{R}^n} \nabla u(x)\nabla v(x)\,\mathrm{d}x, \ \forall u,v \in H^1(\mathbb{R}^n).$$

To formulate our problem, we work in the Hilbert space $\mathcal{H}^1(\mathbb{R}^n,a)$ equipped with the norm

$$\|u\|^2_{\mathcal{H}^1(\mathbb{R}^n,a)} := \int_{\mathbb{R}^n} |u(x)|^2 \,\mathrm{d}x + \int_{\mathbb{R}^n} a(x)|\nabla u(x)|^2 \,\mathrm{d}x.$$

Denote the weight spaces $\mathcal{V}_0 = L^2_\mu(\mathbb{R}^+; L^2(\mathbb{R}^n))$ and $\mathcal{V}_1 = L^2_\mu(\mathbb{R}^+; H^1(\mathbb{R}^n))$, as well as their inner products and norms are defined as

$$\langle \xi,\eta\rangle_{\mathcal{V}_0} = \int_0^\infty \mu(s)(\xi,\eta)\,\mathrm{d}s, \ \|\eta^t\|^2_{\mu,0} = \int_0^\infty \mu(s)|\eta^t(s)|_2^2\,\mathrm{d}s,$$

and

$$\langle \xi,\eta\rangle_{\mathcal{V}_1} = \int_0^\infty \mu(s)\langle \psi,\eta\rangle_{H^1(\mathbb{R}^n)}\,\mathrm{ds},\ \left\|\eta^t\right\|_{\mu,1}^2 = \int_0^\infty \mu(s)\left(\left|\eta^t(s)\right|_2^2 + \left|\nabla\eta^t(s)\right|_2^2\right)\mathrm{ds},$$

respectively. With the help of the aforementioned notations, the phase space of the problem (16)-(17) can be represented as

$$\mathcal{M} := H \times \mathcal{V}_1,$$

it is endowed the following norm

$$\|\cdot\|_{\mathcal{M}}^2 = |\cdot|_2^2 + \|\cdot\|_{\mu,1}^2.$$

2.2. Abstract Theory on Weak Pullback Attractors

This subsection recalls the basic theory of mean random dynamical systems. We consider a complete filtered probability space $\left(\Omega,\mathscr{F},\{\mathscr{F}_t\}_{t\in\mathbb{R}},\mathbb{P}\right)$, where $\{\mathscr{F}_t\}_{t\in\mathbb{R}}$ is an increasing, right continuous the family of sub-σ-algebras of $\mathcal{F}$ that contains all $\mathbb{P}$-null sets. We then introduce the necessary definitions and lemmas concerning the existence of weak $\mathscr{D}$-pullback mean random attractors, following the framework in [27] [28] [30].

Let $(X,\|\cdot\|_X)$ be a complete and separable metric space with Borel sigma-algebra, and denote by $L^p(\Omega,\mathscr{F};X)$ (for $p>1$) the Banach space of all equivalence classes of Bochner integrable functions $\phi:\Omega\to X$ satisfying

$$\|\phi\|_{L^p(\Omega,\mathscr{F};X)} = \left(\int_\Omega \|\phi\|_X^p\right)^{\frac{1}{p}} < \infty.$$

The space $L^p(\Omega,\mathscr{F}_\tau;X)$ is defined similarly for any $\tau\in\mathbb{R}$. Furthermore, for any given $t\in\mathbb{R}$ and $p>1$, the space $L^p(\Omega,\mathscr{F}_\tau;X)$ is subspace of $L^p(\Omega,\mathscr{F};X)$; precisely, it comprises all strongly $\mathscr{F}_t$-measurable functions ϕ in $L^p(\Omega,\mathscr{F};X)$

Definition 1. Let X,Y be two metric spaces, $\pi\mapsto D(\pi)$ be a set-valued mapping of the family of sets consisting of all nonempty bounded subsets form X to Y. Then a family of sets $\mathcal{D}$ is called universal set of Y, if

$$\mathscr{D} = \Big\{\mathcal{D} = \big\{D(\pi)\subset Y : D(\pi)\neq\varnothing \text{ is bounded}, \pi\in X\big\}: \\ \mathcal{D} \text{ satisfies some conditions}\Big\}.$$

Following Definition 1, we work with a universal set in $L^p(\Omega;X)$ defined by

$$\mathscr{D} = \Big\{\mathcal{D} = \big\{D(\tau)\subset L^p(\Omega;X) : D(\pi)\neq\varnothing \text{ bounded}, \tau\in\mathbb{R}\big\}: \\ \mathcal{D} \text{ satisfies some conditions}\Big\}. \tag{3}$$

Definition 2. For $\mathcal{D} = \{D(\tau)\}_{\tau\in\mathbb{R}} \in \mathscr{D}$ and $\tilde{\mathcal{D}} = \{\tilde{D}(\tau)\}_{\tau\in\mathbb{R}}$, if

$$\tilde{D}(\tau) \subseteq D(\omega),\ \forall\tau\in\mathbb{R}$$

implies $\tilde{\mathcal{D}}\in\mathscr{D}$, then $\mathscr{D}$ is called inclusion-closed.

Definition 3. A family $\Psi = \{\Psi(t,\tau): t\in\mathbb{R}^+, \tau\in\mathbb{R}\}$ of mappings is said to be a mean random dynamical system on $L^p(\Omega,\mathscr{F};X)$ over $\left(\Omega,\mathscr{F},\{\mathscr{F}_t\}_{t\in\mathbb{R}},\mathbb{P}\right)$

if for $\tau \in \mathbb{R}$,

1) $\Psi(0,\tau) = id$ (Identity operator on $L^p(\Omega, \mathcal{F}_\tau; X)$);
2) $\varphi(t+s,\tau) = \varphi(t,\tau+s) \circ \varphi(s,\tau)$ (cocycle property), $\forall s,t \in \mathbb{R}^+$;
3) $\varphi(t,\tau): L^p(\Omega, \mathcal{F}_\tau; X) \mapsto L^p(\Omega, \mathcal{F}_t; X)$.

As shown in [24] [27] [28], the mean random dynamical system Ψ from Definition 3 corresponds to a non-autonomous deterministic dynamical system on $L^p(\Omega, \mathcal{F}; X)$ over $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in\mathbb{R}}, \mathbb{P}\right)$.

Definition 4. A family $K = \{K(\tau): \tau \in \mathbb{R}\} \in \mathcal{D}$ is said to be a $\mathcal{D}$-pullback absorbing set for Ψ over $L^p(\Omega, \mathcal{F}; X)$ if for any $\tau \in \mathbb{R}$ and $\mathcal{D} \in \mathcal{D}$, there exists $T := T(\tau, \mathcal{D})$ such that for all $t \geq T$

$$\varphi(t, \tau - t) D(\tau - t) \subseteq K(\tau),$$

where T is called the absorption time. In particular, the family $K = \{K(\tau): \tau \in \mathbb{R}\}$ is called a weakly $\mathcal{D}$-pullback absorbing set for Ψ over $L^p(\Omega, \mathcal{F}; X)$, if $K(\tau) \neq \varnothing$ is weakly compact subset of $L^p(\Omega, \mathcal{F}_\tau; X)$ for any $\tau \in \mathbb{R}$.

Definition 5. A family $K = \{K(\tau): \tau \in \mathbb{R}\} \in \mathcal{D}$ is said to be a $\mathcal{D}$-pullback weakly attracting set for Ψ in $L^p(\Omega, \mathcal{F}; X)$ over $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in\mathbb{R}}, \mathbb{P}\right)$ if for any $\tau \in \mathbb{R}$, $\mathcal{D} \in \mathcal{D}$ and each weak neighborhood $\mathcal{N}^w(K(\tau))$ of $K(\tau)$ in $L^p(\Omega, \mathcal{F}_\tau; X)$, there exists $T := T\left(\tau, \mathcal{D}, \mathcal{N}^w(K(\tau))\right)$ such that

$$\varphi(t, \tau - t) D(\tau - t) \subseteq \mathcal{N}^w(K(\tau))$$

holds true for all $t \geq T$, where T is called the attracting time.

In particular, the family $K = \{K(\tau): \tau \in \mathbb{R}\}$ is said to be a weakly $\mathcal{D}$-pullback weakly compact and weakly attracting set for Ψ over $L^p(\Omega, \mathcal{F}; X)$, if $K(\tau) \neq \varnothing$ is weakly compact subset of $L^p(\Omega, \mathcal{F}_\tau; X)$ for any $\tau \in \mathbb{R}$.

Definition 6. Let $\mathcal{A} = \{\mathcal{A}(\tau)\}_{\tau\in\mathbb{R}} \in \mathcal{D}$. Then $\mathcal{A}$ is said to be a $\mathcal{D}$-pullback mean random attractor of Ψ in $L^p(\Omega, \mathcal{F}; X)$ over $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in\mathbb{R}}, \mathbb{P}\right)$ if he following conditions hold:

(i) $\mathcal{A}(\tau)$ is a weakly compact subset of $L^p(\Omega, \mathcal{F}_\tau; X)$ for every $\tau \in \mathbb{R}$;

(ii) $\{\mathcal{A}(\tau)\}_{\tau\in\mathbb{R}}$ is a $\mathcal{D}$-pullback weakly attracting set for Ψ;

(iii) $\mathcal{A}$ is the minimal element of $\mathcal{D}$ satisfying (i) and (ii), that is to say, if $\mathcal{B} = \{\mathcal{B}(\tau)\}_{\tau\in\mathbb{R}} \in \mathcal{D}$ is a $\mathcal{D}$-pullback weakly compact and weakly attracting set for Ψ, then $\mathcal{A}(\tau) \in \mathcal{B}(\tau)$ for any $\tau \in \mathbb{R}$.

The following theorem establishes the existence and uniqueness of a weak $\mathcal{D}$-pullback mean random attractor for the dynamical system Ψ on $L^p(\Omega, \mathcal{F}; X)$ over $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in\mathbb{R}}, \mathbb{P}\right)$.

Theorem 7. Assume that X is a reflexive Banach space and $p \in (0, \infty)$. Let $\mathcal{D}$ be an inclusion-closed collection of some families of nonempty bounded subsets of $L^p(\Omega, \mathcal{F}; X)$ as given by (3) and Ψ be a mean random dynamical system on $L^p(\Omega,\mathscr{F};X)$ over $(\Omega,\mathscr{F},\{\mathscr{F}_t\}_{t\in\mathbb{R}},\mathbb{P})$. If Ψ has a weakly compact $\mathscr{D}$-pullback absorbing set $K\in\mathscr{D}$ on $L^p(\Omega,\mathscr{F};X)$ over $(\Omega,\mathscr{F},\{\mathscr{F}_t\}_{t\in\mathbb{R}},\mathbb{P})$, then Ψ possesses a unique weak $\mathscr{D}$-pullback mean random attractors $\mathscr{A}\in\mathscr{D}$ on $L^p(\Omega,\mathscr{F};X)$ over $(\Omega,\mathscr{F},\{\mathscr{F}_t\}_{t\in\mathbb{R}},\mathbb{P})$ which is given by the set as follows:

$$\mathscr{A}(\tau)=\bigcap_{s\geq 0}\overline{\bigcup_{t\geq s}\Psi(t,\tau-t)K(\tau-t)}^{w},$$

where the closure represents the weak topology of $L^p(\Omega,\mathscr{F}_\tau;X)$.

2.3. Some Assumptions and Precise Model

To establish the well-posedness and investigate the asymptotic behavior of solutions to Equation (1) subject to the initial condition (2), we make the following assumptions on the nonnegative variable diffusivity $a(\cdot)$, the nonlinearity f, and the memory $k(s)$.

(A_1) The weight function $a(x)$ is nonnegative and locally integrable on $\mathbb{R}^n$. Furthermore, it satisfies the following conditions: there exists $0<\alpha<2$, such that for all $z\in\mathbb{R}^n$,

$$\liminf_{x\to z}|x-z|^{-\alpha}a(x)>0, \tag{4}$$

and is bounded on the annular domains $\{x:k\leq|x|\leq\sqrt{2}k\}$ for all sufficiently large k.

(A_2) The memory kernel $k(s)$ is a nonnegative integrable function of total mass $\int_0^\infty k(s)\mathrm{ds}=1$. Let $\mu(s)=-k'(s)$, and we suppose that

$$\mu\in C^1(\mathbb{R}^+)\cap L^1(\mathbb{R}^+),\ \mu(s)\geq 0,\ \mu'(s)\leq 0,\ \forall s\in\mathbb{R}^+, \tag{5}$$

and there exists a constant $\delta>0$, such that

$$\mu'(s)+\delta\mu(s)\leq 0,\ \forall s\in\mathbb{R}^+. \tag{6}$$

Combining (5) and (6) yields

$$\mu(\infty)=\lim_{s\to\infty}\mu(s)=0. \tag{7}$$

For simplicity, we suppress non-essential constants by setting

$$\int_0^\infty\mu(s)\mathrm{ds}=1. \tag{8}$$

(A_3) The nonlinearity $f\in C^1(\mathbb{R}^n\times\mathbb{R},\mathbb{R})$ fulfills $f(x,0)=0$, along with the dissipation condition

$$\frac{\partial}{\partial s}f(x,s)\geq-\phi_1(x), \tag{9}$$

and the arbitrary order polynomial growth restriction

$$f(s)s\geq\alpha_1|s|^p-\varphi_1(x)\ \text{and}\ |f(s)|\leq\alpha_2|s|^{p-1}+\varphi_2(x),\ p\geq 2, \tag{10}$$

where $\alpha_i\,(i=1,2)$ are the positive constants, $\phi_1(x)\in L^\infty(\mathbb{R}^n)$ with $\phi_1(x)>0$ $\varphi_1\in L^1(\mathbb{R}^n)$, $\varphi_2\in L^{\frac{p}{p-1}}(\mathbb{R}^n)$ are the nonnegative functions. Furthermore, we assume that $f(x,s)$ is locally Lipschitz continuous with s, more precisely, for any compact subset $I \subseteq \mathbb{R}$, there exists $\mathscr{L}_I > 0$ such that

$$\left|f(x,s_1)-f(x,s_2)\right| \le \mathscr{L}_I \left|s_1-s_2\right|,\ \forall x\in\mathbb{R}^n,\ s_1,s_2\in I, \tag{11}$$

(A_4) Let $\sigma:\mathbb{R}\times H \to L_2(U,H)$ satisfy that there exists the constants $\beta_i(i=1,2)>0$ such that for all $s,s_1,s_1\in H$ and $t\in\mathbb{R}$, there are

$$\left\|\sigma(t,s_1)-\sigma(t,s_2)\right\|_{L_2(U,H)}^2 \le \beta_1\left|s_1-s_2\right|_2^2,$$

and

$$\left\|\sigma(t,s)\right\|_{L_2(U,H)}^2 \le \beta_2\left|s\right|_2^2 + h(t),$$

where $h(t)\in L_{loc}^1(\mathbb{R})$.

In particular, we assume that

$$\int_\tau^{\tau+T}\left(\left|g(s)\right|_2^2+\left|h(s)\right|\right)\mathrm{d}s<\infty \tag{12}$$

Remark. We provide an example that satisfies condition (A1). For $a(x)$, it can take the following form:

$$a(x)=|x|^\beta \text{ for } \beta\in[0,2),$$

It is not difficult to verify that $a(x)$ satisfies all the conditions of (A1).

Following Dafermos [31], we characterize the past history of u by introducing a new variable η^t, defined as

$$\eta^t=\eta^t(x,s):=\int_0^s u(x,t-\xi)\mathrm{d}\xi,\ \forall s\in\mathbb{R}^+. \tag{13}$$

Setting $\eta_t^t=\frac{\partial}{\partial t}\eta^t$, $\eta_s^t=\frac{\partial}{\partial s}\eta^t$, then we easily obtain

$$\eta_t^t=-\eta_s^t+u. \tag{14}$$

The historical variable $u_0(\cdot,\tau-r)$ of u satisfies the integrability condition

$$\int_0^\infty \mathrm{e}^{-\kappa t}\left\|u_0(\tau-r)\right\|_{H^1(\mathbb{R}^n)}^2\mathrm{d}s\le\Re, \tag{15}$$

where $\Re>0$ is a constant and $\kappa\le\delta$, with δ given in (6).

Therefore, the original problem (1)-(2) can be recast as follows:

$$\begin{cases}\partial_t u-div\{a(x)\nabla u\}-\int_0^\infty\mu(s)\Delta\eta^t(s)\mathrm{d}s+\lambda u+f(x,u)=g(x,t)+\varepsilon\sigma(t,u)\frac{\mathrm{d}W}{\mathrm{d}t},\\ \eta_t^t=-\eta_s^t+u,\end{cases} \tag{16}$$

with the initial data

$$u(x,\tau)=u_0(x),\ \eta^0(x,s)=\int_0^s u_0(x,\tau-\xi)\mathrm{d}\xi. \tag{17}$$

From (15), we deduce the following estimate

$$\int_0^\infty\mu(s)\left\|\eta^0(s)\right\|_{H^1(\mathbb{R}^n)}^2\mathrm{d}s\le\Re.$$

2.4. Well-Posedness and Mean Random Dynamical System

The objective of this subsection is to demonstrate that problem (16)-(17) generates a mean random dynamical system. To this end, we first introduce the solution concept and establish the well-posedness of the problem.

Definition 8. Let $\tau \in \mathbb{R}$ and $z_0 = (u_0, \eta^0) \in \mathcal{M}$. An $\mathcal{M}$-valued $\mathscr{F}$-adapted stochastic process $\{z(t)\}_{t\in\mathbb{R}^\tau}$ is called a solution of the equation (16) with initial data (17), if

$u \in C(\mathbb{R}^\tau; H) \cap L^2((\tau,\infty); \mathcal{H}^1(\mathbb{R}^n, a)) \cap L^p((\tau,\infty); L^p(\mathbb{R}^n))$, $\eta^t \in C(\mathbb{R}^\tau; \mathcal{V}_1)$,

and for every $t > \tau$, $(\zeta,\varphi) \in (\mathcal{H}^1(\mathbb{R}^n, a) \cap L^p(\mathbb{R}^n)) \times \mathcal{V}_1$ and $\mathbb{P}$-a.s.

$$\begin{cases} (u(t),\zeta) + \int_\tau^t (a(x)\nabla u, \nabla\zeta)\mathrm{d}r + \int_\tau^t \langle \eta^r, \zeta\rangle_{\mathcal{V}_1} \mathrm{d}r + \lambda \int_\tau^t (u,\zeta)\mathrm{d}s \\ \quad + \int_\tau^t \langle f(x,u),\zeta\rangle \mathrm{d}r = (u_0,\zeta) + \int_\tau^t (g(r),\zeta)\mathrm{d}r + \varepsilon \int_\tau^t (\zeta, \sigma(r,u)\mathrm{d}W(r)), \\ \int_\tau^t \langle \eta_t^r + \eta_s^r, \varphi\rangle_{\mathcal{V}_1} \mathrm{d}r = \int_\tau^t \langle u, \varphi\rangle_{\mathcal{V}_1} \mathrm{d}r. \end{cases}$$

Using the argument of [32] (Theorem 1), we can obtain the following theorem. For the reader's convenience, we state only the final result.

Theorem 9. Suppose (A_1) - (A_5) hold, and let $\tau \in \mathbb{R}$, $z_0 = (u_0, \eta^0) \in L^2(\Omega, \mathscr{F}_\tau, \mathbb{P}; \mathcal{M})$. The problem (16)-(17) possesses a unique solution $z(t) = (u(t), \eta^t)$ under the sense of Definition 8. In addition, for any $T > 0$, it holds

$$\mathbb{E}\left(\sup_{t\in[\tau,\tau+T]} \|z(t)\|_{\mathcal{M}}^2 \right) < \infty, \tag{18}$$

which implies that $z \in C([\tau, \tau+T], \mathcal{M})$ $\mathbb{P}$-a.s.

An application of Lebesgue's dominated convergence theorem to (18) shows that $z(t) \in C([\tau,\tau+T], L^2(\Omega,\mathcal{M}))$.

Now, we define the mapping Ψ by

$$\begin{aligned} &\Psi : \mathbb{R}^+ \times \mathbb{R} \times L^2(\Omega,\mathcal{M}) \mapsto L^2(\Omega,\mathcal{M}), \\ &(t,\tau,z_0) \mapsto \Psi(t,\tau) z_0 := z(t+\tau,\tau,z_0), \end{aligned} \tag{19}$$

where z is the solution of the problem (16)-(17) with initial value $z_0 \in L^2(\Omega, \mathscr{F}_\tau, \mathcal{M})$.

Proof. We mainly prove the estimate (19).

Using Ito's formula to the process $|u(t)|_2^2 + \|\nabla\eta^t\|_{\mu,0}^2$, we can obtain from (16) that

$$\begin{aligned} &\mathrm{d}\left(|u(t)|_2^2 + \|\nabla\eta^t\|_{\mu,0}^2 \right) + 2\lambda |u(t)|_2^2 \mathrm{d}t \\ &+ \left(2\int_{\mathbb{R}^n} a(x) |\nabla u(t)|^2 \mathrm{d}x + \delta \|\nabla\eta^t\|_{\mu,0}^2 + 2\int_{\mathbb{R}^n} f(x,u(t)) u \mathrm{d}x \right) \mathrm{d}t \\ &= 2(g(t),u(t))\mathrm{d}t + \varepsilon^2 \|\sigma(t,u(t))\|_{L_2(U,H)}^2 \mathrm{d}t + 2\varepsilon (u(t), \sigma(t,u(t))\mathrm{d}W(t)), \end{aligned} \tag{20}$$

and we can also obtain

$$\mathrm{d}\left\|\eta^t\right\|_{\mu,0}^2+\frac{\delta}{2}\left\|\eta^t\right\|_{\mu,0}^2\mathrm{d}t\le\frac{2}{\delta}\left|u(t)\right|_2^2\mathrm{d}t. \tag{21}$$

By Hölder's inequality, Young's inequality, (A_3) and (A_4) we have

$$\begin{aligned}&\mathrm{d}\left(\left|u(t)\right|_2^2+\left\|\nabla\eta^t\right\|_{\mu,0}^2+\left\|\eta^t\right\|_{\mu,0}^2\right)\le\left(2\beta_1\left\|\varphi_1\right\|_{L^1}+\left|g(t)\right|_2^2+\varepsilon_0^2\left|h(t)\right|\right)\mathrm{d}t\\&+C\left(\left|u(t)\right|_2^2+\left\|\nabla\eta^t\right\|_{\mu,0}^2+\left\|\eta^t\right\|_{\mu,0}^2\right)\mathrm{d}t+2\varepsilon\left(u(t),\sigma(t,u(t))\mathrm{d}W(t)\right).\end{aligned} \tag{22}$$

Taking supremum and expectation of (22), we deduce

$$\begin{aligned}&\mathbb{E}\left(\sup_{s\in[\tau,t]}\left(\left|u(s)\right|_2^2+\left\|\nabla\eta^s\right\|_{\mu,0}^2+\left\|\eta^s\right\|_{\mu,0}^2\right)\right)\\&\le\mathbb{E}\left(\left|u_0\right|_2^2+\left\|\nabla\eta^0\right\|_{\mu,0}^2+\left\|\eta^0\right\|_{\mu,0}^2\right)+\int_\tau^t\left(2\beta_1\left\|\varphi_1\right\|_{L^1}+\left|g(s)\right|_2^2+\varepsilon_0^2\left|h(s)\right|\right)\mathrm{d}s\\&+C\int_\tau^t\mathbb{E}\left(\left|u(s)\right|_2^2+\left\|\nabla\eta^s\right\|_{\mu,0}^2+\left\|\eta^s\right\|_{\mu,0}^2\right)\mathrm{d}s\\&+2\varepsilon\mathbb{E}\left(\sup_{r\in[\tau,t]}\left|\int_\tau^r\left(u(s),\sigma(s,u(s))\mathrm{d}W(s)\right)\right|\right).\end{aligned} \tag{23}$$

For the last term on the right-hand side of (23), by (A_4) and the Burkholder-Davis-Gundy (BDG) inequality we have

$$\begin{aligned}&2\varepsilon\mathbb{E}\left(\sup_{r\in[\tau,t]}\left|\int_\tau^r\left(u(s),\sigma(s,u(s))\mathrm{d}W(s)\right)\right|\right)\\&\le2\varepsilon_0\mathbb{E}\left(\sup_{s\in[\tau,t]}\left|u(s)\right|_2\left(\int_\tau^t\left\|\sigma(s,u(s))\right\|_{L_2(U,H)}^2\mathrm{d}s\right)^{1/2}\right)\\&\le\frac{1}{2}\mathbb{E}\left(\sup_{s\in[\tau,t]}\left|u(s)\right|_2^2\right)+2\varepsilon_0^2\int_\tau^t\left(\beta_2\left|u(s)\right|_2^2+\left|h(s)\right|\right)\mathrm{d}s.\end{aligned} \tag{24}$$

Combining with (23) and (24), we get

$$\begin{aligned}&\mathbb{E}\left(\sup_{s\in[\tau,t]}\left\|z(s)\right\|_{\mathcal{M}}^2\right)\\&\le2\mathbb{E}\left(\left\|z_0\right\|_{\mathcal{M}}^2\right)+\int_\tau^t\left(4\beta_1\left\|\varphi_1\right\|_{L^1}+2\left|g(s)\right|_2^2+6\varepsilon_0^2\left|h(s)\right|\right)\mathrm{d}s\\&+C\int_\tau^t\mathbb{E}\left(\sup_{s\in[\tau,r]}\left\|z(s)\right\|_{\mathcal{M}}^2\right)\mathrm{d}r.\end{aligned} \tag{25}$$

Using Gronwall's lemma yields that for all $t\in[\tau,\tau+T]$,

$$\mathbb{E}\left(\sup_{s\in[\tau,t]}\left\|z(s)\right\|_{\mathcal{M}}^2\right)\le\tilde{C}\mathrm{e}^{CT},$$

where $\tilde{C}=2\mathbb{E}\left(\left\|z_0\right\|_{\mathcal{M}}^2\right)+\int_\tau^{\tau+T}\left(4\beta_1\left\|\varphi_1\right\|_{L^1}+2\left|g(s)\right|_2^2+6\varepsilon_0^2\left|h(s)\right|\right)\mathrm{d}s$, and by (12) we know that $\tilde{C}$ is well-defined. This proof is finished. □

By the uniqueness of solutions, it follows that $\Psi=\left\{\Psi(t,\tau):t\in\mathbb{R}^+,\tau\in\mathbb{R}\right\}$ defines a mean random dynamical system on $L^2(\Omega,\mathscr{F};\mathcal{M})$ over $(\Omega,\mathscr{F},\{\mathscr{F}_t\}_{t\in\mathbb{R}},\mathbb{P})$. Suppose $\mathcal{D}=\{D(\tau)\subseteq L^2(\Omega,\mathscr{F}_\tau;\mathcal{M}):\tau\in\mathbb{R}\}$ is a family of nonempty bounded sets such that

$$\lim_{\tau\to-\infty} e^{\gamma\tau}\|D(\tau)\|^2_{L^2(\Omega,\mathscr{F}_\tau;\mathcal{M})}=0, \tag{26}$$

where γ shall be given later, and $\|D(\tau)\|_{L^2(\Omega,\mathscr{F}_\tau;\mathcal{M})}=\sup_{u\in D(\tau)}\|u\|_{L^2(\Omega,\mathscr{F}_\tau;\mathcal{M})}$.

Furthermore, we use $\mathscr{D}$ to denote the collection of all families of nonempty bounded sets satisfying (26), that is,

$$\mathscr{D}=\left\{\mathcal{D}=\left\{D(\tau)\subseteq L^2(\Omega,\mathscr{F}_\tau;\mathcal{M}):D(\tau)\neq\varnothing \text{ bounded},\tau\in\mathbb{R}\right\}:D \text{ fulfills } (26)\right\}$$

3. Weak $\mathscr{D}$-Pullback Mean Random Attractors

In this section, we shall investigate the existence and uniqueness of weak $\mathscr{D}$-pullback mean random attractors for problem (16)-(17). For this purpose, we assume that

$$\int_{-\infty}^{\tau} e^{\gamma r}\left(|g(r)|_2^2+|h(r)|\right)dr<\infty,\ \ \forall\tau\in\mathbb{R}. \tag{27}$$

Hereafter, unless otherwise specified, the solution of problem (16)-(17) is denoted by $z(t)=(u(t),\eta^t)$. In what follows, we first establish uniform priori estimates of solutions of the problem (16)-(17).

Lemma 10. Let (A_1)-(A_4) and (27) hold. Then, there exists $\varepsilon_0>0$ such that for all $\varepsilon\in(0,\varepsilon_0]$, as well as for any $\tau\in\mathbb{R}$ and $\mathcal{D}=\{D(t)\}_{t\in\mathbb{R}}\in\mathscr{D}$, there exists $T=T(\tau,\mathcal{D})>0$ such that for all $t\geq T$, the following estimate

$$\mathbb{E}\left(\|z(\tau,\tau-t,z_0)\|^2_{\mathcal{M}}\right)\leq\mathcal{Q}+\mathcal{Q}e^{-\kappa\tau}\int_{-\infty}^{\tau}e^{\kappa r}\left(|g(r)|_2^2+|h(r)|\right)dr$$

holds for any $z_0\in D(\tau-t)$, where $\mathcal{Q}=\mathcal{Q}\left(\delta,\lambda,\beta_1,\|\varphi_1\|_{L^1}\right)>0$ is a constant which is not related to τ and $\mathcal{D}$.

Proof. Applying Ito's formula to the process $|u(t)|_2^2$, then by the first equation of (16) we can obtain that

$$\begin{aligned}&\frac{d}{dt}\mathbb{E}\left(|u(t)|_2^2+\|\nabla\eta^t\|^2_{\mu,0}\right)\\&+\mathbb{E}\left(2\int_{\mathbb{R}^n}a(x)|\nabla u(t)|^2dx+\delta\|\nabla\eta^t\|^2_{\mu,0}+2\int_{\mathbb{R}^n}f(x,u(t))u\,dx\right)\\&=-2\lambda\mathbb{E}\left(|u(t)|_2^2\right)+2\mathbb{E}\left((g(t),u(t))\right)+\varepsilon^2\mathbb{E}\left(\|\sigma(t,u(t))\|^2_{L_2(U,H)}\right).\end{aligned} \tag{28}$$

Now, we deal with the equality (28). By Hölder's and Young's inequalities we have

$$2\mathbb{E}\left((g,u(t))\right)\leq\frac{\lambda}{2}\mathbb{E}\left(|u(t)|_2^2\right)+\frac{2}{\lambda}|g(t)|_2^2. \tag{29}$$

By (10) in (A_3) we have

$$2\mathbb{E}\left(\int_{\mathbb{R}^n} f\left(x,u(t)\right)u(t)\,\mathrm{d}x\right) \ge 2\alpha_1\mathbb{E}\left(\left|u(t)\right|_p^p\right) - 2\beta_1\left\|\varphi_1\right\|_{L^1}. \tag{30}$$

From (A_4), we can deduce

$$\varepsilon^2\mathbb{E}\left(\left\|\sigma\left(t,u(t)\right)\right\|_{L_2(U,H)}^2\right) \le \varepsilon^2\beta_2\mathbb{E}\left(\left|u(t)\right|_2^2\right) + \varepsilon^2\left|h(t)\right|. \tag{31}$$

Substituting (29)-(31) into (28), we obtain

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}\left(\left|u(t)\right|_2^2 + \left\|\nabla\eta^t\right\|_{\mu,0}^2\right) + \mathbb{E}\left(2\int_{\mathbb{R}^n} a(x)\left|\nabla u(t)\right|^2\mathrm{d}x + \delta\left\|\nabla\eta^t\right\|_{\mu,0}^2 + \left|u(t)\right|_p^p\right)\\ &\le \left(-2\lambda + \varepsilon^2\beta_2\right)\mathbb{E}\left(\left|u(t)\right|_2^2\right) + \frac{2}{\lambda}\left|g(t)\right|_2^2 + 2\beta_1\left\|\varphi_1\right\|_{L^1}.\end{aligned} \tag{32}$$

Let $0 < \varepsilon < \varepsilon_0 = \min\left\{1, \sqrt{\dfrac{\lambda}{2\beta_2}}\right\}$, then we get from (32) that for any $r \ge \tau - t$,

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}\left(\left|u(r,\tau-t,u_0)\right|_2^2 + \left\|\nabla\eta^r\right\|_{\mu,0}^2\right)\\ &+\mathbb{E}\left(2\int_{\mathbb{R}^n} a(x)\left|\nabla u(r,\tau-t,u_0)\right|^2\mathrm{d}x + \delta\left\|\nabla\eta^r\right\|_{\mu,0}^2 + \left|u(r,\tau-t,u_0)\right|_p^p\right)\\ &\le -\lambda\mathbb{E}\left(\left|u(r,\tau-t,u_0)\right|_2^2\right) + \frac{2}{\lambda}\left|g(t)\right|_2^2 + 2\beta_1\left\|\varphi_1\right\|_{L^1} + \left|h(t)\right|.\end{aligned} \tag{33}$$

Taking $\kappa = \min\left\{\lambda, \dfrac{\delta}{2}\right\}$ in (32), one has

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}\left(\left|u(r,\tau-t,u_0)\right|_2^2 + \left\|\nabla\eta^r\right\|_{\mu,0}^2\right) + \kappa\mathbb{E}\left(\left|u(r,\tau-t,u_0)\right|_2^2 + \left\|\nabla\eta^r\right\|_{\mu,0}^2\right)\\ &\le \frac{2}{\lambda}\left|g(t)\right|_2^2 + 2\beta_1\left\|\varphi_1\right\|_{L^1} + \left|h(t)\right|.\end{aligned} \tag{34}$$

Multiplying (34) by $\mathrm{e}^{\kappa r}$, and then integrating it from $\tau - t$ $(t > 0)$ to τ, it yields

$$\begin{aligned}&\mathbb{E}\left(\left|u(\tau,\tau-t,u_0)\right|_2^2 + \left\|\nabla\eta^\tau\right\|_{\mu,0}^2\right) \le \mathrm{e}^{-\kappa\tau}\mathrm{e}^{\kappa(\tau-t)}\mathbb{E}\left(\left|u_0\right|_2^2 + \left\|\nabla\eta^0\right\|_{\mu,0}^2\right)\\ &+\mathrm{e}^{-\kappa\tau}\int_{\tau-t}^{\tau}\mathrm{e}^{\kappa r}\left(\frac{2}{\lambda}\left|g(r)\right|_2^2 + \left|h(r)\right|\right)\mathrm{d}r + 2\beta_1\left\|\varphi_1\right\|_{L^1}.\end{aligned} \tag{35}$$

Moreover, using η^t to multiply the second equation of (16) in $\mathcal{V}_0$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}\left(\left\|\eta^t\right\|_{\mu,0}^2\right) + \delta\mathbb{E}\left(\left\|\eta^t\right\|_{\mu,0}^2\right) \le 2\mathbb{E}\left(\int_0^\infty \mu(s)\left(u(t),\eta^t\right)\mathrm{d}s\right). \tag{36}$$

By Hölder's inequality, Young's inequality and (8), it is easy to get that

$$2\mathbb{E}\left(\int_0^\infty \mu(s)\left(u(t),\eta^t\right)\mathrm{d}s\right) \le \frac{2}{\delta}\mathbb{E}\left(\left|u(t)\right|_2^2\right) + \frac{\delta}{2}\mathbb{E}\left(\left\|\eta^t(s)\right\|_{\mu,0}^2\right),$$

which together with (36) can infer

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}\left(\left\|\eta^\tau\right\|_{\mu,0}^2\right) + \kappa\mathbb{E}\left(\left\|\eta^\tau\right\|_{\mu,0}^2\right) \le \frac{2}{\delta}\Big[\mathrm{e}^{-\kappa t}\mathbb{E}\left(\left|u_0\right|_2^2 + \left\|\nabla\eta^0\right\|_{\mu,0}^2\right)\\ &+\mathrm{e}^{-\kappa\tau}\int_{\tau-t}^{\tau}\mathrm{e}^{\kappa r}\left(\frac{2}{\lambda}\left|g(r)\right|_2^2 + \left|h(r)\right|\right)\mathrm{d}r + 2\beta_1\left\|\varphi_1\right\|_{L^1}\Big],\end{aligned} \tag{37}$$

where we used (35). Applying Gronwall's inequality on $(\tau - t, \tau)$ to (37), we have

$$\mathbb{E}\left(\left\|\eta^{\tau}\right\|_{\mu,0}^{2}\right)\leq Ce^{-\kappa\tau}\mathbb{E}\left(\left\|z_0\right\|_{\mathcal{M}}^{2}\right)+Ce^{-\kappa\tau}\int_{\tau-t}^{\tau}e^{\kappa r}\left(\left|g(r)\right|_2^2+\left|h(r)\right|\right)\mathrm{d}r+C\left\|\varphi_1\right\|_{L^1}. \quad (38)$$

Combining with (35) and (37), we obtain

$$\begin{aligned}&\mathbb{E}\left(\left|u(\tau,\tau-t,u_0)\right|_2^2+\left\|\eta^{\tau}\right\|_{\mu,1}^{2}\right)\\&\leq Ce^{-\kappa\tau}e^{\kappa(\tau-t)}\mathbb{E}\left(\left\|z_0\right\|_{\mathcal{M}}^{2}\right)+Ce^{-\kappa\tau}\int_{\tau-t}^{\tau}e^{\kappa r}\left(\left|g(r)\right|_2^2+\left|h(r)\right|\right)\mathrm{d}r+C\left\|\varphi_1\right\|_{L^1}.\end{aligned}$$

Thanks to $z_0\in D(\tau-t)$ and $\mathcal{D}=\{D(t)\}_{t\in\mathbb{R}}\in\mathscr{D}$, thereby we have

$$\lim_{t\to\infty}e^{-\kappa\tau}e^{\kappa(\tau-t)}\mathbb{E}\left(\left\|z_0\right\|_{\mathcal{M}}^{2}\right)\leq\lim_{t\to\infty}e^{-\kappa\tau}e^{\kappa(\tau-t)}\left\|D(\tau-t)\right\|_{L^2(\Omega,\mathscr{F}_{\tau-t};\mathcal{M})}^{2}=0.$$

So, there exists $T:=T(\tau,\mathcal{D})$ such that

$$e^{-\kappa\tau}e^{\kappa(\tau-t)}\mathbb{E}\left(\left\|z_0\right\|_{\mathcal{M}}^{2}\right)\leq 1,\ \forall t\geq T.$$

In conclusion, by choosing appropriate $\mathcal{Q}$ we can conclude the desired conclusion. The proof is completed. □

According to Lemma 10, we can give the existence of weakly compact $\mathscr{D}$-pullback bounded absorbing set.

Lemma 11. Let (A_1)-(A_5) and (27) hold. Then, there exists $\varepsilon_0>0$ such that for all $\varepsilon\in(0,\varepsilon_0]$, the mean random dynamical system Ψ associated with the problem (16)-(17) possesses a weakly compact $\mathscr{D}$-pullback bounded absorbing set $\mathcal{K}=\{\mathcal{K}(\tau):\tau\in\mathbb{R}\}\in\mathscr{D}$, where, for any $\tau\in\mathbb{R}$, $\mathcal{K}(\tau)$ is given by

$$\mathcal{K}(\tau)=\left\{u\in L^2(\Omega,\mathscr{F}_\tau;\mathcal{M}):\mathbb{E}\left(|z|_2^2\right)\leq\mathcal{Q}+\mathcal{Q}e^{-\kappa\tau}\int_{-\infty}^{\tau}e^{\kappa r}\left(\left|g(r)\right|_2^2+\left|h(r)\right|\right)\mathrm{d}r\right\},$$

where $\mathcal{Q}$ can be found in Lemma 10.

Proof. First, from (27), it is straightforward to verify that the integral $\int_{-\infty}^{\tau}e^{\kappa r}\left(\left|g(r)\right|_2^2+\left|h(r)\right|\right)\mathrm{d}r$ is well-defined. Moreover, since $\mathcal{K}(\tau)$ is a bounded, closed, and convex subset of the reflexive Banach space $L^2(\Omega,\mathscr{F}_\tau;\mathcal{M})$, it follows that $\mathcal{K}(\tau)$ is weakly compact in this space. In particular, by Lemma 10, for any $\tau\in\mathbb{R}$ and any family $\mathcal{D}=\{D(t)\}_{t\in\mathbb{R}}$, there exists $T=T(\tau,\mathcal{D})>0$ such that

$$\Psi(t,\tau-t,D(\tau-t))=u(\tau,\tau-t,D(\tau-t))\subseteq K(\tau)$$

holds for any $t\geq T$ and $\varepsilon\in(0,\varepsilon_0]$. Lastly, it remains to prove that $\mathcal{K}\in\mathscr{D}$, *i.e.*, that $\mathcal{K}$ satisfies condition (26). From (27), we obtain

$$\begin{aligned}0&\leq\lim_{\tau\to-\infty}e^{\kappa\tau}\left\|\mathcal{K}(\tau)\right\|_{L^2(\Omega,\mathscr{F}_\tau;\mathcal{M})}^{2}\\&\leq\lim_{\tau\to-\infty}e^{\kappa\tau}\mathcal{Q}+\lim_{\tau\to-\infty}\left(e^{\kappa\tau}\mathcal{Q}e^{-\kappa\tau}\int_{-\infty}^{\tau}e^{\kappa r}\left(\left|g(r)\right|_2^2+\left|h(r)\right|\right)\mathrm{d}r\right)=0,\end{aligned}$$

from which we can obtain

$$\lim_{\tau\to-\infty}e^{\kappa\tau}\left\|\mathcal{K}(\tau)\right\|_{L^2(\Omega,\mathscr{F}_\tau;\mathcal{M})}^{2}=0.$$

This completes the proof. □

Combining Theorem 7 and Lemma 11, we now establish the following theorem on the existence and uniqueness of the weak $\mathscr{D}$-pullback mean random attractor for the mean random dynamical system Ψ.

Theorem 12. Let assumptions (A_1)-(A_4) and (27) hold. Then the mean random dynamical system $\Psi = \{\Psi(t,\tau): t \in \mathbb{R}^+, \tau \in \mathbb{R}\}$ generated by problem (16)-(17) has a unique weak $\mathscr{D}$-pullback mean random attractors $\mathscr{A} = \{\mathscr{A}(\tau)\}_{\tau\in\mathbb{R}}$ belonging to $\mathscr{D}$ on $L^2(\Omega,\mathscr{F};\mathcal{M})$ over $(\Omega,\mathscr{F},\{\mathscr{F}_t\}_{t\in\mathbb{R}},\mathbb{P})$, and the attractors $\mathscr{A}$ can be given, for every $\tau \in \mathbb{R}$, $\mathscr{A}(\tau)$ is given as follows:

$$\mathscr{A}(\tau) = \bigcap_{s\geq 0} \overline{\bigcup_{t\geq s} \Psi(t,\tau-t)\mathcal{K}(\tau-t)}^{w},$$

where the closure is taken as weak topology of $L^2(\Omega,\mathscr{F}_\tau;\mathcal{M})$.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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