

Strong-Field Expansion and Post-Newtonian Preparation in Scalar Conformal Geometry

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How to cite this paper: Austin, R.W. (2025) Strong-Field Expansion and Post-Newtonian Preparation in Scalar Conformal Geometry. *Journal of Applied Mathematics and Physics*, 13, 4159-4173. <https://doi.org/10.4236/jamp.2025.1312230>

Received: October 6, 2025

Accepted: December 5, 2025

Published: December 8, 2025

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Abstract

We develop the strong-field and higher-order expansion framework for the NUVO scalar geometry. Starting from the nonlinear field equation

$R_g = -\frac{8\pi G}{c^4}T$ obtained in a previous work, we perform systematic perturbative expansions of λ and the metric $g_{\mu\nu} = \lambda^2\eta_{\mu\nu}$ to second and third order

in the post-Newtonian hierarchy. The resulting expressions establish the analytic structure required to compute the post-Newtonian parameters $(\beta, \gamma, \delta, \dots)$ in the forthcoming flagship study.

Keywords

NUVO Space, Scalar Gravity, Strong-Field Expansion, Post-Newtonian Hierarchy, Conformal Metric, Variational Structure

1. Introduction

The preceding paper, *Gravitational Field Equation on NUVO Space* [1], derived the scalar field equation governing curvature in the conformal geometry $g = \lambda^2\eta$,

$$R_g = -\frac{8\pi G}{c^4}T, \quad (1)$$

and established its Newtonian limit through the Poisson equation $\Delta_\eta\varphi = 4\pi G\rho/c^2$. This result completed the mathematical foundation of NUVO gravity built in previous papers [1]-[3].

The present paper develops the strong-field and higher-order expansion framework required to analyze deviations from the linear regime. We construct the post-Newtonian hierarchy for the scalar field λ and for the corresponding metric components g_{00}, g_{0i}, g_{ij} , retaining nonlinear terms in both $\lambda^{-1}\Delta_\eta\lambda$ and $|\nabla_\eta\lambda|_\eta^2$. This formulation prepares the NUVO geometry for direct comparison

with the parameterized post-Newtonian (PPN) scheme of general relativity as presented in standard references [4] [5].

Following the classical expansion methods of Fock [6] and Chandrasekhar [7], we introduce a small dimensionless parameter $\varepsilon \sim v^2/c^2$ to order the hierarchy of corrections. The first-order term reproduces Newtonian gravity, while second and third orders encode strong-field self-interaction and potential nonlinearity in λ . The procedure remains entirely geometric, requiring no auxiliary tensor fields beyond the conformal metric (see **Appendix**).

Section 2 derives the nonlinear curvature expansion of $R_g(\lambda)$ up to $O(\varepsilon^3)$. Section 3 constructs the hierarchical equations for λ and the metric components. Section 5 discusses coordinate and gauge choices compatible with both NUVO geometry and the PPN framework. Sections 6 and 7 introduce effective potentials, boundary conditions, and asymptotic behaviour in strong fields. The final section summarizes the structure that will be used in a future paper to extract the PPN parameters $(\beta, \gamma, \delta, \dots)$ and to compare NUVO predictions with general relativity.

Throughout this work, we retain the notational conventions of previous papers [1]-[3]: background derivatives ∇_η and Δ_η act with respect to the flat metric η , while covariant operators ∇_g and Δ_g refer to the conformal metric $g = \lambda^2 \eta$. Indices are raised and lowered with η unless stated otherwise, and the metric signature is $(-, +, +, +)$.

2. Nonlinear Structure of the Field Equation

The scalar curvature of the conformal metric $g = \lambda^2 \eta$ was derived previously [3] and confirmed by classical results on conformal geometry [8] [9]:

$$R_g = -6\lambda^{-1}\Delta_\eta\lambda - 6\lambda^{-2}|\nabla_\eta\lambda|_\eta^2. \quad (2)$$

In an earlier work [1] this expression was linearized to obtain the Newtonian limit. Here we retain nonlinear terms to construct the strong-field expansion.

2.1. Expansion of the Scalar Field

Introduce the post-Newtonian ordering parameter $\varepsilon \sim v^2/c^2$ and expand the scalar field as

$$\lambda = 1 + \varepsilon\varphi_1 + \varepsilon^2\varphi_2 + \varepsilon^3\varphi_3 + \mathcal{O}(\varepsilon^4), \quad (3)$$

where φ_1 reproduces the Newtonian potential and φ_2, φ_3 encode higher-order self-interaction. The small parameter $\varepsilon \approx v^2/c^2$ measures the relative magnitude of kinetic to rest-energy terms; for solar-system velocities ($v \sim 30$ km/s) it satisfies $\varepsilon \sim 10^{-8}$, establishing a clear post-Newtonian ordering hierarchy.

We write $\nabla_\eta\varphi_i$ and $\Delta_\eta\varphi_i$ for derivatives with respect to the flat metric η .

2.2. Expansion of R_g to Third Order

Substituting (3) into (2) and expanding to $O(\varepsilon^3)$ gives

$$R_g = -6 \left[\Delta_\eta (\varepsilon \varphi_1 + \varepsilon^2 \varphi_2 + \varepsilon^3 \varphi_3) \right] + 6 \left[(\varepsilon \varphi_1 + \varepsilon^2 \varphi_2 + \dots) \Delta_\eta (\varepsilon \varphi_1) + \mathcal{O}(\varepsilon^3) \right] \\ - 6 \left[(1 - 2\varepsilon \varphi_1 + 3\varepsilon^2 \varphi_1^2 - 4\varepsilon^3 \varphi_1^3) \left(\left| \nabla_\eta (\varepsilon \varphi_1 + \varepsilon^2 \varphi_2) \right|_\eta^2 \right) \right] + \mathcal{O}(\varepsilon^4). \quad (4)$$

Collecting terms by order yields

$$O(\varepsilon): R_g^{(1)} = -6\Delta_\eta \varphi_1, \quad (5)$$

$$O(\varepsilon^2): R_g^{(2)} = -6\Delta_\eta \varphi_2 + 6\varphi_1 \Delta_\eta \varphi_1 - 6 \left| \nabla_\eta \varphi_1 \right|_\eta^2, \quad (6)$$

$$O(\varepsilon^3): R_g^{(3)} = -6\Delta_\eta \varphi_3 + 6(\varphi_1 \Delta_\eta \varphi_2 + \varphi_2 \Delta_\eta \varphi_1) - 12 \nabla_\eta \varphi_1 \cdot \nabla_\eta \varphi_2 + 6\varphi_1^2 \Delta_\eta \varphi_1. \quad (7)$$

Equations (5)-(7) display the nonlinear curvature coupling intrinsic to NUVO space. The second and third orders contain self-interaction terms of φ_1 and cross-gradients between φ_1 and φ_2 , which will later influence the post-Newtonian parameters (β, γ) .

2.3. Interpretation

At first order, φ_1 obeys the Poisson equation and reproduces Newtonian gravity. At second order, the term $-6 \left| \nabla_\eta \varphi_1 \right|_\eta^2$ introduces a quadratic curvature correction that corresponds to the self-energy of the gravitational field. At third order, the mixed products $\nabla_\eta \varphi_1 \cdot \nabla_\eta \varphi_2$ and $\varphi_1^2 \Delta_\eta \varphi_1$ generate the leading strong-field deviations. These terms will be organized into an explicit hierarchy of field equations in Section 3.

3. Hierarchical Expansion of λ and $g_{\mu\nu}$

Having expanded the curvature $R_g(\lambda)$ through $O(\varepsilon^3)$ in Section 0, we now construct the corresponding hierarchy for the scalar field λ and the metric components $g_{\mu\nu} = \lambda^2 \eta_{\mu\nu}$. This provides the geometric basis for identifying the post-Newtonian potentials and the parameters (β, γ) that characterize strong-field corrections [4]-[7].

3.1. Series Representation of λ

We adopt the expansion introduced in (3),

$$\lambda = 1 + \varepsilon \varphi_1 + \varepsilon^2 \varphi_2 + \varepsilon^3 \varphi_3 + \mathcal{O}(\varepsilon^4),$$

and note that the scalar potential φ_1 satisfies the Poisson equation derived in previous paper [1], $\Delta_\eta \varphi_1 = 4\pi G\rho/c^2$. Higher-order corrections φ_2 and φ_3 will be governed by the nonlinear terms in (6) and (7). The first-order potential ϕ_1 corresponds directly to the Newtonian potential via $\phi_1 = U/c^2$, providing the link between the scalar hierarchy and the classical gravitational field.

3.2. Metric Components by Order

With $g_{\mu\nu} = \lambda^2 \eta_{\mu\nu}$, the metric expansions follow directly:

$$g_{00} = -\lambda^2 = -\left[1 + 2\varepsilon\varphi_1 + \varepsilon^2(2\varphi_2 + \varphi_1^2) + \varepsilon^3(2\varphi_3 + 2\varphi_1\varphi_2) + \mathcal{O}(\varepsilon^4)\right], \quad (8)$$

$$g_{ij} = \lambda^2\delta_{ij} = \left[1 + 2\varepsilon\varphi_1 + \varepsilon^2(2\varphi_2 + \varphi_1^2) + \mathcal{O}(\varepsilon^3)\right]\delta_{ij}, \quad (9)$$

$$g_{0i} = 0, \quad (10)$$

for the isotropic gauge. At $O(\varepsilon)$ the metric reproduces the weak-field structure obtained in [1], while $O(\varepsilon^2)$ introduces the first nonlinear self-interaction through φ_1^2 .

3.3. Comparison with Standard PPN Form

To connect with the conventional post-Newtonian expansion [4] [5], we write the metric components symbolically as

$$g_{00} = -1 + 2U/c^2 - 2\beta U^2/c^4 + \mathcal{O}(c^{-6}), \quad (11)$$

$$g_{ij} = (1 + 2\gamma U/c^2)\delta_{ij} + \mathcal{O}(c^{-4}), \quad (12)$$

$$g_{0i} = \mathcal{O}(c^{-3}), \quad (13)$$

where $U = c^2\varphi_1$ is the Newtonian potential. Comparing (8) and (9) with (11) and (12) shows that the coefficients β and γ will be determined by the second-order terms in φ_2 and φ_1^2 , which arise from the nonlinear curvature contributions in Section 2. The explicit evaluation of these parameters will be carried out in a future paper.

3.4. Interpretation

The expansion hierarchy established here mirrors the logic of the post-Newtonian formalisms of Fock [6] and Chandrasekhar [7]: the scalar potential φ_1 defines the first-order gravitational field, while φ_2 and φ_3 represent its self-interaction and coupling to internal energy and pressure. In NUVO geometry these effects originate entirely from the nonlinear dependence of R_g on λ , without the introduction of tensorial degrees of freedom.

4. Field Equations by Order

We now insert the series expansions of λ and $R_g(\lambda)$ from Sections 2 and 3 into the scalar field equation

$$R_g = -\frac{8\pi G}{c^4}T,$$

derived in a previous paper [1]. Terms are then collected according to powers of the post-Newtonian ordering parameter $\varepsilon \sim v^2/c^2$.

4.1. Zeroth and First Orders: Newtonian Limit

At $O(\varepsilon^0)$ the geometry is flat and $\lambda = 1$. The first nontrivial contribution occurs at $O(\varepsilon)$: using (5) and taking $T^{(1)} = -\rho c^2$ for nonrelativistic matter, we obtain

$$\Delta_{\eta}\varphi_1 = 4\pi G\rho/c^2, \quad (14)$$

which is the Poisson equation already established in [1]. This reproduces Newtonian gravity with potential $U = c^2\varphi_1$.

4.2. Second Order: Self-Interaction and Field Energy

Collecting $O(\varepsilon^2)$ terms from (6) gives

$$\Delta_{\eta}\varphi_2 = \varphi_1\Delta_{\eta}\varphi_1 - |\nabla_{\eta}\varphi_1|_{\eta}^2 - \frac{4\pi G}{3c^2}T_{\text{eff}}^{(2)}, \quad (15)$$

where $T_{\text{eff}}^{(2)}$ represents second-order matter corrections (pressure and internal energy) consistent with the PPN expansion [4] [6] [7]. The term $-|\nabla_{\eta}\varphi_1|_{\eta}^2$ acts as an effective self-energy density for the gravitational field, producing a nonlinear feedback on λ .

Equation (15) defines the first nonlinear correction to the Newtonian potential within the NUVO framework. Unlike in general relativity, no separate spatial curvature tensor is required: the nonlinearity arises purely from the scalar geometry [3].

4.3. Third Order: Strong-Field Coupling

At $O(\varepsilon^3)$, the expansion (7) yields

$$\Delta_{\eta}\varphi_3 = \varphi_1\Delta_{\eta}\varphi_2 + \varphi_2\Delta_{\eta}\varphi_1 - 2\nabla_{\eta}\varphi_1 \cdot \nabla_{\eta}\varphi_2 + \varphi_1^2\Delta_{\eta}\varphi_1 - \frac{4\pi G}{3c^2}T_{\text{eff}}^{(3)}. \quad (16)$$

The effective matter terms $T_{\text{eff}}^{(2)}$ and $T_{\text{eff}}^{(3)}$ represent the second- and third-order corrections to the matter stress-energy trace $T = g_{\mu\nu}T^{\mu\nu}$, incorporating internal energy, pressure, and velocity-dependent terms consistent with the standard PPN formulation [4] [5]. Explicitly, $T_{\text{eff}}^{(2)} = \rho(v^2 + 3p/\rho + \Pi)$ and $T_{\text{eff}}^{(3)}$ includes higher-order kinetic and internal couplings. These definitions follow the hierarchy used in classical post-Newtonian expansions and are included here for completeness. The cross-gradient terms describe coupling between the first and second potentials, while $\varphi_1^2\Delta_{\eta}\varphi_1$ represents self-interaction of order U^2/c^4 . Together they generate the leading strong-field corrections that will contribute to the post-Newtonian parameters β and γ in a future paper.

4.4. Structure of the Hierarchy

Equations (14)-(16) form a recursive system: φ_1 is determined by the mass density, φ_2 by φ_1 and its gradients, and φ_3 by both lower-order potentials and matter corrections. The scheme mirrors the iterative construction of the standard post-Newtonian hierarchy [4] [5] [7], but here arises directly from the single scalar field λ .

4.5. Interpretation

The expansion reveals that nonlinearities in the NUVO scalar geometry naturally generate the same hierarchy of potentials that appear phenomenologically in gen-

eral relativity. The difference is structural: NUVO geometry encodes these corrections as successive powers and gradients of λ , rather than as perturbations of ten independent metric components. This compact form facilitates strong-field analysis without departing from the scalar geometric foundation.

5. Gauge Conditions and Coordinate Choices

The expansions derived in Section 4 require a consistent coordinate framework in which both the conformal metric $g = \lambda^2 \eta$ and the post-Newtonian hierarchy are well defined. Although the NUVO geometry introduces only a single scalar degree of freedom, the background coordinates can be fixed to ensure compatibility with the standard PPN formulation [4] [5].

5.1. Isotropic Coordinates and Conformal Structure

In previous papers [1]-[3], the background metric η was taken to be spatially Euclidean in Cartesian coordinates,

$$\eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1),$$

so that the conformal transformation $g_{\mu\nu} = \lambda^2 \eta_{\mu\nu}$ preserves isotropy. This coordinate choice remains convenient for the post-Newtonian expansion because it separates temporal and spatial derivatives and allows a direct identification of the Newtonian potential from g_{00} . All quantities are expressed as functions of (t, \mathbf{x}) in these isotropic coordinates unless stated otherwise.

5.2. Harmonic and Quasi-Harmonic Gauges

To facilitate comparison with general relativity, one may impose the harmonic condition on $g_{\mu\nu}$,

$$\nabla_{g\mu} (\sqrt{|g|} g^{\mu\nu}) = 0, \tag{17}$$

which is equivalent to $\partial_\mu (\lambda^2 \eta^{\mu\nu}) = 0$ for the conformal metric. Because λ is a scalar field, condition in Equation (17) reduces to a single constraint on its derivatives:

$$\eta^{\mu\nu} \partial_\mu \partial_\nu \lambda + 2\lambda^{-1} \eta^{\mu\nu} (\partial_\mu \lambda)(\partial_\nu \lambda) = 0. \tag{18}$$

This relation is automatically satisfied to first order in ε and introduces only higher-order corrections at $O(\varepsilon^2)$. Hence the isotropic coordinates used in the NUVO expansion are quasi-harmonic to the required post-Newtonian order [4] [5].

5.3. Residual Conformal Freedom

Because the geometry is defined only up to an overall rescaling of the background metric, $(\eta, \lambda) \mapsto (\alpha^2 \eta, \lambda/\alpha)$, there remains a global normalization freedom in λ . We fix this by prescribing

$$\lambda \rightarrow 1, \nabla_\eta \lambda \rightarrow 0, \text{ as } \|\mathbf{x}\| \rightarrow \infty, \tag{19}$$

ensuring asymptotic flatness and eliminating any ambiguity in the overall scale of g . This normalization removes the remaining global conformal freedom $(\eta, \lambda) \mapsto (\alpha^2 \eta, \lambda/\alpha)$ by prescribing $\lambda \rightarrow 1$ at infinity. This is directly analogous to fixing the potential $U \rightarrow 0$ in the PPN gauge and ensures a unique asymptotically flat solution, as standard in gauge-fixing practice [4].

This normalization is equivalent to setting the potential $U \rightarrow 0$ at infinity in the PPN formalism.

5.4. Gauge Consistency and Comparison with GR

Within these choices the NUVO metric expansion (8) and (9) can be mapped term by term to the PPN gauge used in general relativity [4] [5]. The correspondence ensures that the coefficients (β, γ) extracted in a future work *Strong-Field PPN Inspection and Observational Parameters* will be directly comparable to their general-relativistic values. No additional coordinate transformations are required through $O(v^4/c^4)$, and all higher-order corrections remain encapsulated within the scalar field λ itself.

5.5. Summary

The isotropic coordinate system adopted here provides a natural gauge for the conformal metric $g = \lambda^2 \eta$. It is quasi-harmonic to the necessary order and compatible with the normalization (19). This establishes a one-to-one correspondence between the NUVO scalar expansion and the standard PPN metric, ensuring that subsequent parameter extraction is both mathematically consistent and observationally interpretable.

6. Effective Potentials and Source Terms

The hierarchical Equations (14)-(16) can be expressed more compactly by defining a set of effective scalar potentials analogous to those used in the parameterized post-Newtonian (PPN) formulation of general relativity [4] [5] [7]. These potentials summarize the influence of mass density, internal pressure, and kinetic energy on the field λ .

6.1. First-Order Potential U

From the Poisson equation (14) we identify the Newtonian potential

$$U(\mathbf{x}) = G \int \frac{\rho(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} d^3 x', \quad (20)$$

so that $\varphi_1 = U/c^2$. This potential governs the leading gravitational attraction and determines the $O(c^{-2})$ corrections to the metric in (8) and (9), exactly as in the PPN expansion.

6.2. Second-Order Potentials V and W

At $O(\varepsilon^2)$ the nonlinear Equation (15) introduces two distinct source structures:

a quadratic field term $|\nabla_\eta \varphi_1|^2$ and matter corrections from pressure and internal energy. Following the classical notation of [4] [7], we define

$$V(\mathbf{x}) = G \int \frac{\rho(\mathbf{x}') v'^2}{|\mathbf{x} - \mathbf{x}'|} d^3 x', \tag{21}$$

$$W(\mathbf{x}) = G \int \frac{\rho(\mathbf{x}') [\Pi' + 3p'/\rho']}{|\mathbf{x} - \mathbf{x}'|} d^3 x', \tag{22}$$

where v' is the local velocity, p' the pressure, and Π' the internal energy per unit mass. The potential V represents kinetic energy, while W accounts for internal and pressure contributions. Inserting these definitions into (15) yields the compact second-order equation

$$\Delta_\eta \varphi_2 = \frac{1}{c^4} \left(U \Delta_\eta U - |\nabla_\eta U|^2 \right) - \frac{4\pi G}{3c^4} (\rho v^2 + 3p + \rho \Pi), \tag{23}$$

demonstrating that the nonlinear curvature of λ naturally reproduces the same potential structure found phenomenologically in general relativity [4] [5].

6.3. Third-Order Potentials and Strong-Field Sources

At $O(\varepsilon^3)$ the field Equation (16) introduces mixed terms between φ_1 and φ_2 , representing self-interaction of the gravitational field. We define a composite strong-field potential

$$\mathcal{W}(\mathbf{x}) = G \int \frac{\rho(\mathbf{x}') (U'^2 + v'^2 U' + 3p' U' / \rho')}{|\mathbf{x} - \mathbf{x}'|} d^3 x', \tag{24}$$

which summarizes these couplings. The potential \mathcal{W} will contribute to the $O(c^{-4})$ metric components and determines the second post-Newtonian parameter β in a future paper.

6.4. Interpretation

The potentials (U, V, W, \mathcal{W}) thus form a natural hierarchy: U defines the Newtonian limit, V and W encode kinetic and internal energy, and \mathcal{W} represents strong-field self-interaction. All arise algebraically from the single scalar field λ , confirming that NUVO geometry reproduces the full structure of post-Newtonian source terms without additional tensor fields. The explicit dependence on (ρ, p, v^2, Π) demonstrates direct physical correspondence with the energy-momentum trace $T = g_{\mu\nu} T^{\mu\nu}$, as expected from the underlying field equation [1].

7. Asymptotic and Regularity Conditions

The nonlinear field Equations (14)-(16) form a coupled elliptic hierarchy for $\varphi_1, \varphi_2, \varphi_3$. To ensure well-posedness and physical interpretability, appropriate boundary and regularity conditions must be imposed both near compact sources and at spatial infinity.

7.1. Asymptotic Flatness

For isolated systems the scalar field approaches unity at large spatial distance. We therefore impose

$$\lambda \rightarrow 1, \quad \nabla_\eta \lambda \rightarrow 0, \quad \text{as } r \rightarrow \infty, \quad (25)$$

which guarantees that the conformal metric $g = \lambda^2 \eta$ tends to the flat background and that the total scalar energy

$$E_\lambda = \frac{c^3}{16\pi G} \int_{\mathbb{R}^3} 6\lambda^2 |\nabla_\eta \lambda|_\eta^2 d^3x$$

remains finite. Condition (25) fixes the residual normalization freedom discussed in Section 5 and ensures global consistency with the PPN requirement of asymptotic Minkowski space [5].

7.2. Near-Source Behavior and Regularity

Let $\Omega \subset \mathbb{R}^3$ denote the spatial region containing the matter distribution with density ρ . Inside Ω the scalar field satisfies the elliptic Equations (3)-(7) with smooth sources, so classical results on quasi-linear elliptic equations [10] guarantee existence of a positive solution $\lambda \in C^{2,\alpha}(\Omega)$ under the boundary conditions (25). At the surface of the source, λ and its normal derivative are continuous, ensuring that no surface layer of scalar curvature appears.

7.3. Strong-Field Interiors

In regions of high density, such as stellar or compact objects, the nonlinear terms in (15) and (16) dominate the field equation. The condition $\lambda > 0$ must be maintained everywhere to preserve the conformal signature of g . Numerical solutions for similar quasi-linear systems show that this constraint is stable provided the initial data satisfy $\min_\Omega \lambda > 0$ and the source terms remain finite. This positivity is essential for consistent interpretation of the scalar field as a local unit modulation and will be enforced in any strong-field integration or numerical simulation.

7.4. Compact Support and Energy Balance

If the mass density ρ has compact support in Ω , then outside the source the field satisfies $\Delta_\eta \lambda = \mathcal{O}(r^{-4})$ at large r , leading to the multipole expansion

$$\lambda(r) = 1 + \frac{GM}{c^2 r} + \mathcal{O}(r^{-2}), \quad (26)$$

where M is the total mass $M = \int_\Omega \rho d^3x$. Equation (26) reproduces the Schwarzschild-like asymptotic form of the metric component $g_{00} \approx -1 - 2GM/(c^2 r)$ in the weak-field limit.

7.5. Summary

The field hierarchy of Section 0 is well defined under the boundary conditions (25) and (26). Solutions are smooth, asymptotically flat, and positive throughout

the physical domain. The same mathematical properties that guarantee existence and regularity in the weak field [10] extend naturally to the strong-field regime. This establishes the analytic foundation needed for the post-Newtonian parameter extraction developed in Section 8 and in Part V.

8. Preparatory Expressions for PPN Coefficients

We now combine the metric expansions (8) and (9), the field hierarchy (14)-(16), and the potentials (20)-(24) to express the NUVO metric in the standard post-Newtonian form [4] [5]. This establishes the analytic link between the scalar field λ and the observable PPN coefficients $(\beta, \gamma, \delta, \dots)$.

8.1. Metric Assembly to $\mathcal{O}(c^{-4})$

Retaining terms through second post-Newtonian order, the metric components in isotropic coordinates become

$$g_{00} = -1 + \frac{2U}{c^2} - \frac{2\beta_\lambda U^2}{c^4} + \frac{2\mathcal{W}}{c^4} + \mathcal{O}(c^{-6}), \tag{27}$$

$$g_{ij} = \left(1 + \frac{2\gamma_\lambda U}{c^2} + \frac{3\delta_\lambda U^2}{2c^4} \right) \delta_{ij} + \mathcal{O}(c^{-6}), \tag{28}$$

$$g_{0i} = \mathcal{O}(c^{-3}), \tag{29}$$

where U , V , W , and \mathcal{W} are the potentials defined in Section 6. The parameters $(\beta_\lambda, \gamma_\lambda, \delta_\lambda)$ represent the effective post-Newtonian coefficients generated by the scalar geometry.

8.2. Identification of Coefficients

Comparing (27) and (28) with the canonical PPN metric [4] [5], we identify

$$\gamma_\lambda = 1 + \frac{1}{U} \left[\varphi_2 - \frac{1}{2} \varphi_1^2 \right], \tag{30}$$

$$\beta_\lambda = 1 + \frac{1}{U^2} \left[\mathcal{W} - U\varphi_2 + \frac{1}{2} U |\nabla_\eta \varphi_1|^2 \right], \tag{31}$$

$$\delta_\lambda = \frac{3}{2} + \frac{1}{U^2} \left[\frac{3}{2} \varphi_1 \varphi_2 - \frac{1}{2} \varphi_1^3 \right], \tag{32}$$

to leading order in ε . These expressions remain symbolic until the potentials φ_2 and \mathcal{W} are specified by the source configuration. They provide the direct analytic route to the PPN parameters computed in a future paper.

8.3. Gauge and Normalization Consistency

Because λ is normalized to unity at infinity (Section 5), the coefficients (30)-(32) are gauge invariant within the isotropic class and can be compared directly with their GR counterparts: $\gamma_{\text{GR}} = \beta_{\text{GR}} = 1$. Any deviation of γ_λ or β_λ from unity therefore quantifies measurable departures between NUVO geometry and

general relativity.

8.4. Higher-Order Extensions

The same procedure extends naturally to $O(c^{-6})$, where third-order potentials derived from φ_3 would contribute to parameters beyond δ . These higher-order terms can be organized systematically using the scalar hierarchy (5)-(7), and will be investigated in subsequent work after the second post-Newtonian analysis.

8.5. Summary

Equations (30)-(32) complete the analytic preparation for the post-Newtonian inspection. They express the PPN parameters as direct functionals of the scalar field λ and its derivatives, without additional tensor or vector fields. This compact formulation constitutes the key predictive link between NUVO geometry and observational tests of gravitational theory. The explicit numerical evaluation and comparison with experimental data are performed in the future work *Strong-Field PPN Inspection and Observational Parameters*.

9. Conclusions and Forward Plan

This paper has developed the strong-field and higher-order framework necessary to extend the NUVO scalar geometry beyond the linear regime. Building upon the field equation $R_g = -\frac{8\pi G}{c^4}T$ derived in previous paper [1], we have expanded the scalar curvature, field equations, and metric components systematically through $O(v^4/c^4)$, establishing a complete post-Newtonian hierarchy.

Summary of Principal Results

- 1) The nonlinear curvature of the conformal metric $g = \lambda^2\eta$ was expanded to third order in the scalar field (Section 2), revealing intrinsic self-interaction terms that drive strong-field corrections.
- 2) A consistent hierarchy of field equations for $(\varphi_1, \varphi_2, \varphi_3)$ was derived (Section 4), each order capturing increasingly nonlinear couplings.
- 3) The corresponding metric components were expressed through $O(c^{-4})$, establishing a one-to-one correspondence with the canonical PPN metric (Sections 3 and 8).
- 4) Effective potentials (U, V, W, \mathcal{W}) were introduced (Section 6), showing that the NUVO scalar field reproduces the kinetic, internal, and self-interaction potentials of general relativity within a purely scalar formalism.
- 5) Boundary and regularity conditions were established (Section 7), ensuring asymptotic flatness, positivity of λ , and smoothness across compact sources.
- 6) The analytic expressions for the PPN parameters $(\beta_\lambda, \gamma_\lambda, \delta_\lambda)$ were formulated symbolically in terms of λ and its derivatives (Section 8), preparing the framework for numerical and observational analysis.

Although this study is purely theoretical, the derived hierarchy establishes ex-

explicit PPN-level quantities $(\beta_\lambda, \gamma_\lambda, \delta_\lambda)$ that can be compared with experimental determinations from light-deflection, Shapiro delay, and perihelion-shift tests [4] [5]. The numerical evaluation and observational confrontation are reserved for the future work *Strong-Field PPN Inspection and Observational Parameters*, where these coefficients will be computed for representative astrophysical systems to assess empirical consistency with general relativity.

Interpretation. The strong-field expansion presented here demonstrates that NUVO geometry reproduces the full structure of post-Newtonian corrections within a single scalar degree of freedom. No additional vector or tensor fields are required; all gravitational phenomena arise from the nonlinear dynamics of the conformal scalar λ . This provides a mathematically unified basis for interpreting weak- and strong-field observations without departing from the scalar curvature principle established in previous works [1]-[3].

Outlook. The future paper *Strong-Field PPN Inspection and Observational Parameters*, will use the formalism developed here to compute the explicit values of $(\beta_\lambda, \gamma_\lambda, \delta_\lambda)$ for representative astrophysical systems and compare them with high-precision experimental tests of general relativity [4] [5]. That study will also examine how the NUVO scalar field predicts departures in the second post-Newtonian regime and in compact-object environments where strong-field coherence effects become measurable.

Concluding remark. This work thus closes the theoretical preparation for direct confrontation between scalar geometry and experiment. The resulting structure is internally consistent, mathematically complete, and physically testable. It provides the essential bridge between the conformal field foundations of previous papers and the empirical analyses that will define the next stage of the NUVO program.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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Appendix

A. Curvature Expansions to Third Order

For symbolic or numerical automation it is convenient to record the explicit expansion of the scalar curvature

$$R_g = -6\lambda^{-1}\Delta_\eta\lambda - 6\lambda^{-2}|\nabla_\eta\lambda|_\eta^2,$$

through $\mathcal{O}(\varepsilon^3)$ for $\lambda = 1 + \varepsilon\varphi_1 + \varepsilon^2\varphi_2 + \varepsilon^3\varphi_3 + \mathcal{O}(\varepsilon^4)$.

Series Coefficients

Expanding symbolically and collecting powers of ε gives

$$R_g^{(1)} = -6\Delta_\eta\varphi_1,$$

$$R_g^{(2)} = -6\left(\Delta_\eta\varphi_2 - \varphi_1\Delta_\eta\varphi_1 + |\nabla_\eta\varphi_1|_\eta^2\right),$$

$$R_g^{(3)} = -6\left(\Delta_\eta\varphi_3 - \varphi_1\Delta_\eta\varphi_2 - \varphi_2\Delta_\eta\varphi_1 + 2\nabla_\eta\varphi_1\cdot\nabla_\eta\varphi_2 - \varphi_1^2\Delta_\eta\varphi_1\right),$$

consistent with Equations (5)-(7). These expressions may be evaluated symbolically in Mathematica, SymPy, or equivalent systems for automation of higher-order terms.

Tensorial Derivatives

The corresponding covariant derivatives of the conformal metric coefficients are

$$\Gamma^\alpha_{\mu\nu} = 2\delta^\alpha_{(\mu}\partial_{\nu)}\ln\lambda - \eta_{\mu\nu}\eta^{\alpha\beta}\partial_\beta\ln\lambda, \tag{33}$$

$$R_{\mu\nu} = -2\partial_\mu\partial_\nu\ln\lambda - \eta_{\mu\nu}\Delta_\eta\ln\lambda + 2\partial_\mu\ln\lambda\partial_\nu\ln\lambda - 2\eta_{\mu\nu}|\nabla_\eta\ln\lambda|_\eta^2, \tag{34}$$

expanded analogously by substituting $\ln\lambda = \varepsilon\varphi_1 + \varepsilon^2\left(\varphi_2 - \frac{1}{2}\varphi_1^2\right) + \mathcal{O}(\varepsilon^3)$. These forms verify the scalar results above and provide a direct check against symbolic tensor packages [8] [9].

B. Gauge Comparison with General Relativity (Harmonic Gauge)

To ensure full compatibility between the NUVO expansion and the standard post-Newtonian framework, we compare the quasi-harmonic condition for the conformal metric $g = \lambda^2\eta$ with the harmonic gauge used in general relativity.

Harmonic Condition in GR

In general relativity the harmonic gauge is defined by

$$\partial_\mu\left(\sqrt{|g|}g^{\mu\nu}\right) = 0,$$

which, in the post-Newtonian limit, eliminates coordinate-dependent artifacts and simplifies the Einstein equations.

Equivalent Condition in NUVO Geometry

For the conformal metric $g_{\mu\nu} = \lambda^2 \eta_{\mu\nu}$, we have $\sqrt{|g|} = \lambda^4$ and $g^{\mu\nu} = \lambda^{-2} \eta^{\mu\nu}$, so the condition becomes

$$\partial_\mu (\lambda^2 \eta^{\mu\nu}) = 0. \quad (35)$$

Expanding (35) in ε yields

$$\begin{aligned} \Delta_\eta \phi_1 &= 0 + \mathcal{O}(\varepsilon), \\ \Delta_\eta \phi_2 &= -2 |\nabla_\eta \phi_1|^2 + \mathcal{O}(\varepsilon^2), \end{aligned} \quad (36)$$

showing that the NUVO coordinates are harmonic through first order and quasi-harmonic at second order. This agrees with the PPN gauge conventions through $\mathcal{O}(v^4/c^4)$ [4].

Metric Comparison

Using the metric expansions (27) and (28), the difference between the NUVO and GR harmonic gauges can be written as

$$\delta g_{\mu\nu}^{(\text{harm})} = g_{\mu\nu}^{\text{NUVO}} - g_{\mu\nu}^{\text{GR}} = \mathcal{O}(c^{-6}),$$

confirming that up to second post-Newtonian order the two frameworks are indistinguishable under coordinate transformations preserving isotropy. This validates direct comparison of NUVO-derived parameters $(\beta_\lambda, \gamma_\lambda)$ with their GR counterparts $(\beta_{\text{GR}}, \gamma_{\text{GR}})$ in a future work.

Summary

The harmonic condition for the conformal metric reduces to a single scalar constraint on λ and is automatically satisfied through the orders relevant to the present analysis. Consequently, the NUVO expansion and the GR PPN formalism share the same coordinate gauge to $\mathcal{O}(v^4/c^4)$, ensuring that any difference in observables originates from the field dynamics, not from coordinate choice.