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The Gravitational Field Equation on NUVO Space

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Abstract

We derive the gravitational field equation on NUVO space (M,g), where $g=\lambda^2\eta$ with η flat and $\lambda>0$ smooth. Starting from the scalar curvature functional and the λ-weighted calculus developed in NUVO Space I-II, we perform a constrained conformal variation to obtain the dynamical equation for λ and its coupling to matter via the trace of the energy-momentum tensor. We show that diffeomorphism invariance implies energy-momentum conservation $\nabla_g\cdot T=0$ and recover a λ-weighted continuity relation for the sinertia current $J^\mu=\lambda\rho u^\mu$. In the weak-field limit the field equation reduces to a Poisson-type equation for λ, yielding the Newtonian regime and setting the stage for a post-Newtonian (PPN) analysis in subsequent work.

Keywords

NUVO Space, Conformal Scalar Field, Weighted Geometry, Scalar Curvature, Variational Structure, Energy-Momentum Conservation, Weak-Field Limit

1. Introduction

In previous publications Part I and II [1] [2] of the NUVO Mathematical Series established the geometric and analytical foundations of *NUVO space*—a conformally flat manifold (M,g) constructed from a flat background metric η and a positive scalar field λ, such that $g=\lambda^2\eta$. The scalar field acts as a *unit constraint*, fixing local scale while preserving the global topology and causal structure of the background. The resulting space admits a complete λ-weighted calculus, including differential operators, curvature identities, and variational principles, all of which were developed rigorously in [1] [2]. Restricting gravity to a conformally flat scalar geometry isolates the essential coupling between matter and local scale. This restriction removes gauge redundancy in the full tensor formalism and focuses on how spatial variation of the scalar unit field λ alone can generate curvature. Such simplification clarifies the geometric origin of gravitational energy and provides a controlled framework for comparing scalar and tensor gravities in the weak-field regime (see **Appendix**).

The present paper derives the *gravitational field equation* on NUVO space. By restricting the Einstein-Hilbert action to the conformal class $g = \lambda^2 \eta$ and varying with respect to the single degree of freedom λ, one obtains a scalar field equation equivalent to the *trace* of Einstein's equations [3] [4]. This approach replaces the usual ten metric components by a single scalar field whose geometric behavior encapsulates curvature and energy exchange in the conformal manifold. In contrast to Brans-Dicke and other scalar-tensor theories, which introduce an independent scalar field alongside the metric, NUVO space defines the metric entirely through λ as $g = \lambda^2 \eta$. The gravitational and scalar dynamics are thus inseparable, producing a purely conformal scalar theory rather than a two-field coupling.

Coupling to matter arises naturally [5]: variation of the matter action $S_m[\psi, g]$ under conformal perturbations of g introduces the trace $T = g_{\mu\nu} T^{\mu\nu}$ of the energy-momentum tensor as the source of curvature. The resulting field equation therefore reads, in covariant form,

$$R_g = -\frac{8\pi G}{c^4} T,$$

where R_g is the scalar curvature of g expressed explicitly in terms of λ and derivatives with respect to the background metric η.

Beyond deriving this relation, we examine its conservation laws and its weak-field limit. Diffeomorphism invariance yields the usual covariant conservation $\nabla_g \cdot T = 0$, while the λ-weighted divergence identities from Part II imply an analogous continuity equation for the scalar current $J^\mu = \lambda \rho u^\mu$. In the weak-field regime $\lambda = 1 + \varepsilon\varphi$ with $\varepsilon \ll 1$, the field equation reduces to a Poisson-type equation for φ, reproducing Newtonian gravity and preparing the way for the parameterized post-Newtonian (PPN) analysis developed in the sister paper *Strong-Field Expansion and Post-Newtonian Preparation in Scalar Conformal Geometry*.

The structure of the paper is as follows. Section 2 reviews the curvature and action of the conformal metric and expresses the Einstein-Filbert functional in terms of λ. Section 3 performs the constrained variation and obtains the scalar field equation. Section 4 establishes energy-momentum and sinertia conservation. Section 5 analyzes the weak-Field limit and Newtonian regime. Section 6 summarizes well-posedness results based on the weighted Sobolev theory of Part II, and the concluding section outlines the transition to the post-Newtonian expansion. The conceptual novelty of the NUVO approach lies in the unit-constrained calculus established in Parts I-II. Here the scalar field λ defines not only the metric but also the weighting of all differential operators and conserved measures. This unified structure yields a closed variational system in which geometric and physical quantities derive self-consistently from a single scalar degree of freedom.

2. Curvature and Action in the Conformal Class

Part II derived the explicit curvature formulas for a conformal metric $g=\lambda^2\eta$ on a smooth manifold M with flat background metric η. For completeness we recall the scalar curvature expression and embed it within the variational framework required for the gravitational field equation.

2.1. Scalar Curvature of the Conformal Metric

Let $\phi=\log\lambda$ and denote by ∇_η, div_η, and Δ_η the gradient, divergence, and Laplacian with respect to η. For any dimension $n\geq 2$, the scalar curvature of $g=\lambda^2\eta$ is

$$R_g=-2(n-1)\lambda^{-1}\Delta_\eta\lambda-(n-1)(n-2)\lambda^{-2}\left|\nabla_\eta\lambda\right|_\eta^2. \tag{1}$$

Equation (1) follows from the conformal transformation formulas proved in [2]-[4]. In particular, for the physically relevant case $n=4$,

$$R_g=-6\lambda^{-1}\Delta_\eta\lambda-6\lambda^{-2}\left|\nabla_\eta\lambda\right|_\eta^2. \tag{2}$$

This curvature depends only on λ and its first two derivatives with respect to the flat background η.

***Remark* 1.** Constant λ yields $R_g=0$, confirming that the geometry is globally flat whenever the scalar field is homogeneous. Spatial or temporal variation of λ generates curvature through its gradients and Laplacian, providing the geometric origin of gravitational effects in NUVO space.

2.2. Einstein-Hilbert Action Restricted to the Conformal Class

The Einstein-Hilbert functional with matter is

$$S[g,\psi]=\frac{c^3}{16\pi G}\int_M R_g\,\mathrm{d}V_g+S_m[\psi,g], \tag{3}$$

where ψ denotes the matter fields and $\mathrm{d}V_g=\sqrt{\left|\det g\right|}\,\mathrm{d}^n x=\lambda^n\mathrm{d}V_\eta$. Substituting (1) gives an equivalent functional depending solely on λ:

$$\begin{aligned}S[\lambda,\psi]=\frac{c^3}{16\pi G}\int_M&\left[-2(n-1)\lambda^{n-1}\Delta_\eta\lambda-(n-1)(n-2)\lambda^{n-2}\left|\nabla_\eta\lambda\right|_\eta^2\right]\mathrm{d}V_\eta\\&+S_m\left[\psi,\lambda^2\eta\right].\end{aligned} \tag{4}$$

For $n=4$ the expression simplifies to

$$S[\lambda,\psi]=\frac{c^3}{16\pi G}\int_M\left[-6\lambda^3\Delta_\eta\lambda-6\lambda^2\left|\nabla_\eta\lambda\right|_\eta^2\right]\mathrm{d}V_\eta+S_m\left[\psi,\lambda^2\eta\right]. \tag{5}$$

Integration by parts (with either compact support or fixed boundary values), standard in variational formulations of geometric functionals [6], transfers derivatives from λ to the test functions, yielding the convenient gradient form

$$S_g[\lambda]=\frac{c^3}{16\pi G}\int_M 6\lambda^2\left|\nabla_\eta\lambda\right|_\eta^2\mathrm{d}V_\eta-\frac{c^3}{16\pi G}\int_{\partial M}6\lambda^3\partial_{n_\eta}\lambda\,\mathrm{d}S_\eta. \tag{6}$$

***Remark* 2.** The boundary term in (6) vanishes under compact support or asymptotically flat conditions $\lambda \to 1$ and $\nabla_\eta \lambda \to 0$ as $\|x\| \to \infty$. The remaining volume integral defines a positive-definite energy density for variations of λ and provides the starting point for the Euler-Lagrange analysis in Section 3.

2.3. Physical Interpretation

In this restricted conformal class the entire gravitational dynamics are encoded in the scalar field λ. The term $\lambda^2 |\nabla_\eta \lambda|_\eta^2$ measures spatial variation of the scalar unit field, while $\Delta_\eta \lambda$ accounts for isotropic curvature. Together they form a geometrically closed system: when λ is constant the geometry is flat; when λ varies slowly the resulting curvature reproduces the Newtonian potential at leading order. These features justify treating λ as the sole dynamical variable in the subsequent field variation.

3. Constrained Conformal Variation and Field Equation

We now vary the Einstein-Hilbert functional (5) within the restricted conformal class $g = \lambda^2 \eta$, treating λ as the sole dynamical variable while holding the background η fixed. This *constrained conformal variation* yields a scalar field equation for λ that is equivalent to the trace of Einstein's equations [3].

3.1. Metric and Matter Variations

A small perturbation of λ, $\lambda \mapsto \lambda + \varepsilon h$ with compactly supported h, induces the metric variation

$$\delta g_{\mu\nu} = 2\delta\phi g_{\mu\nu}, \quad \delta\phi = \frac{h}{\lambda}. \tag{7}$$

The matter action $S_m[\psi, g]$ responds through the stress-energy tensor $T^{\mu\nu}$ [5],

$$\delta S_m = \frac{1}{2}\int_M T^{\mu\nu} \delta g_{\mu\nu} \mathrm{d}V_g = \int_M \delta\phi T \mathrm{d}V_g = \int_M h T \lambda^3 \mathrm{d}V_\eta, \tag{8}$$

where $T = g_{\mu\nu} T^{\mu\nu}$ and we have used $\mathrm{d}V_g = \lambda^4 \mathrm{d}V_\eta$ for the four-dimensional case.

3.2. Variation of the Gravitational Term

From the gradient form (6) the gravitational part of the action is

$S_g[\lambda] = \frac{c^3}{16\pi G}\int_M 6\lambda^2 |\nabla_\eta \lambda|_\eta^2 \mathrm{d}V_\eta$. Varying λ gives

$$\delta S_g = \frac{c^3}{16\pi G}\int_M \left(12\lambda \nabla_\eta \lambda \cdot \nabla_\eta h + 6h |\nabla_\eta \lambda|_\eta^2\right) \mathrm{d}V_\eta. \tag{9}$$

Integration by parts and discarding boundary terms (compact support or asymptotic flatness) lead to

$$\delta S_g = \frac{c^3}{16\pi G}\int_M h\left[-12\lambda \Delta_\eta \lambda - 6|\nabla_\eta \lambda|_\eta^2\right] \mathrm{d}V_\eta. \tag{10}$$

follows standard treatments of quasi-linear variational systems [6].

3.3. Euler-Lagrange Equation with Matter Coupling

Stationarity $\delta S_g + \delta S_m = 0$ for all smooth compactly supported h implies

$$\frac{c^3}{16\pi G}\left[-12\lambda\Delta_\eta\lambda - 6\left|\nabla_\eta\lambda\right|_\eta^2\right] + \lambda^3 T = 0 \text{ in } M. \tag{11}$$

Dividing by λ^3 and using the operator relation (for $n=4$)

$$\Delta_g f = \lambda^{-2}\left(\Delta_\eta f + 2\nabla_\eta\phi\cdot\nabla_\eta f\right), \quad \phi = \log\lambda,$$

one recovers the covariant expression

$$R_g = -\frac{8\pi G}{c^4}T. \tag{12}$$

Hence the constrained conformal variation yields precisely the *trace equation* of general relativity, expressed entirely in terms of the scalar field λ and background derivatives. In explicit η-coordinates, equation (12) is equivalent to

$$-6\lambda^{-1}\Delta_\eta\lambda - 6\lambda^{-2}\left|\nabla_\eta\lambda\right|_\eta^2 = -\frac{8\pi G}{c^4}T. \tag{13}$$

***Remark* 3. (Gauge and normalization)** The rescaling $(\eta,\lambda)\mapsto\left(\alpha^2\eta,\lambda/\alpha\right)$ leaves g invariant. A global normalization may therefore be fixed either by $\lambda\to 1$ at spatial infinity or by specifying the λ-average on a chosen Cauchy slice.

3.4. Interpretation

Equation (12) is the gravitational field equation of NUVO space. The geometry is entirely controlled by the scalar field λ: regions of constant λ are flat, while gradients and Laplacians of λ generate curvature. The coupling to the matter trace T links the conformal geometry directly to local energy density. Because the field equation couples only to the trace $T = g_{\mu\nu}T^{\mu\nu}$, sources with vanishing trace—such as pure electromagnetic fields—do not directly generate curvature in λ. Their gravitational influence would appear only indirectly through interactions with matter or through higher-order scalar couplings introduced in extended versions of the framework. This scalar-geometric field equation provides the mathematical bridge between the differential geometry of Parts I-II and the physical regime analyzed in Sections 4 and 5.

4. Conservation Laws

Conservation principles follow from the diffeomorphism invariance of the total action $S[g,\psi] = S_g[g] + S_m[\psi,g]$ and from the λ-weighted calculus developed in Part II [2]. We collect the two complementary formulations that arise on NUVO space.

4.1. Covariant Conservation of Stress-Energy

Under an infinitesimal diffeomorphism generated by a compactly supported vector field X, the metric varies by $\delta g = \mathcal{L}_X g$. Invariance of the total action implies [3]

$$\nabla_{g\mu} T^{\mu\nu} = 0, \tag{14}$$

the standard covariant conservation of the energy-momentum tensor. Because the geometry of NUVO space is entirely determined by λ, Equation (14) remains valid without modification: matter follows the Levi-Civita connection of $g = \lambda^2 \eta$, and any additional coupling enters only through the trace term T in the scalar field equation.

***Remark* 4.** Equation (14) guarantees that the matter source in (12) is self-consistent. Taking the divergence of the field equation and using Bianchi identities for g recovers (14) automatically, ensuring no further constraints on λ are introduced.

4.2. λ-Weighted Continuity for Scalar Flux

Independently of (14), Part II established that the divergence of any vector field X with respect to g can be expressed in background form as

$$\operatorname{div}_g X = \lambda^{-n} \operatorname{div}_\eta \left(\lambda^n X \right), \quad n = \dim M.$$

For $n = 4$ this yields the λ-weighted continuity law [2]. Let ρ denote a scalar density and u^μ a g-normalized velocity field satisfying $g_{\mu\nu} u^\mu u^\nu = -1$. Define the *sinertia current*

$$J^\mu = \lambda \rho u^\mu. \tag{15}$$

Then

$$\operatorname{div}_g J = 0 \iff \lambda^{-4} \operatorname{div}_\eta \left(\lambda^5 \rho u \right) = 0, \tag{16}$$

expressing conservation of the scalar-weighted flux through any closed hypersurface. The sinertia current $J^\mu = \lambda \rho u^\mu$ therefore represents conservation of the scalar-weighted flux rather than ordinary mass flux. Its divergence expresses how the conformal measure λ modulates inertial content in curved regions. This conservation law complements the covariant energy-momentum conservation $\nabla_g \cdot T = 0$ by tracking the exchange between matter and the scalar geometric background.

***Remark* 5.** Equation (16) reduces to the ordinary mass-continuity equation $\operatorname{div}_\eta (\rho u) = 0$ when $\lambda \equiv 1$. Spatial variation of λ therefore represents a modulation of the local inertial measure, consistent with the geometric interpretation of λ as a unit constraint field.

4.3. Integral Form

For any compact domain $\Omega \subset M$ with smooth boundary $\partial\Omega$ and outward g-unit normal n, integration of (16) gives the flux identity

$$\int_\Omega \operatorname{div}_g J \, \mathrm{d}V_g = \int_{\partial\Omega} g(J, n) \, \mathrm{d}S_g = 0, \tag{17}$$

showing that the total sinertia crossing any closed surface vanishes. Equation (17) will provide the conserved quantity required for the weak-field and post-Newtonian analyses in Section 5.

4.4. Summary

Two conservation statements therefore coexist on NUVO space:

(i) The *covariant conservation law* $\nabla_g \cdot T = 0$, arising from diffeomorphism invariance of the matter action.

(ii) The λ *-weighted continuity law* $\mathrm{div}_g J = 0$, encoding conservation of scalar flux (sinertia) in the conformal geometry.

Together these form the complete conservation structure of NUVO space: the first governs local energy-momentum exchange within matter, and the second governs the geometric balance of the scalar field itself. Both remain consistent with the field equation (12) and reduce to their classical counterparts when λ is constant.

5. Weak-Field Limit and Newtonian Regime

We now examine the leading-order behavior of the field equation (12) in regions where departures from flatness are small [3]. This limit identifies the Newtonian potential and provides the bridge to the post-Newtonian analysis of the sister paper *Strong-Field Expansion and Post-Newtonian Preparation in Scalar Conformal Geometry*.

5.1. Linearization of the Scalar Field

Let the scalar field be written as

$$\lambda = 1 + \varepsilon\varphi, \quad 0 < \varepsilon \ll 1, \tag{18}$$

with φ smooth and dimensionless. Substituting (18) into the curvature expression (2) and retaining terms through first order in ε gives

$$R_g = -6\varepsilon\Delta_\eta\varphi + \mathcal{O}\left(\varepsilon^2\right). \tag{19}$$

To this order $\left|\nabla_\eta\lambda\right|^2$ is already $\mathcal{O}\left(\varepsilon^2\right)$ and may be neglected.

5.2. Matter Source and Field Equation

For nonrelativistic matter, the dominant component of the stress-energy tensor is $T^{00} \simeq \rho c^2$, so that the trace is $T \simeq -\rho c^2$. Inserting (19) into the field equation (12) yields

$$\Delta_\eta\varphi = 4\pi G\rho/c^2. \tag{20}$$

Equation (20) is precisely the Poisson equation [4] for the Newtonian potential when we identify

$$U = c^2\varphi, \tag{21}$$

so that $\Delta_\eta U = 4\pi G\rho$.

5.3. Metric Components in the Weak Field

In this limit the metric $g_{\mu\nu} = \lambda^2 \eta_{\mu\nu}$ takes the approximate form

$$g_{00} \simeq -\lambda^2 = -(1+2\varepsilon\varphi) = -\left(1+2U/c^2\right), \tag{22}$$

$$g_{ij} \simeq (1+2\varepsilon\varphi)\delta_{ij} = \left(1+2U/c^2\right)\delta_{ij}, \;\; g_{0i} = 0. \tag{23}$$

The geodesic equation then reduces, at leading order, to Newton's second law $\ddot{x}^i = -\partial_i U$, confirming the consistency of the scalar field geometry with the classical gravitational limit.

***Remark* 6.** Equation (20) verifies that the λ-field reproduces the Newtonian potential without the introduction of additional parameters or functions. The next corrections, of order $\mathcal{O}\left(U^2/c^4\right)$, generate the post-Newtonian terms that will be developed in the sister paper *Strong-Field Expansion and Post-Newtonian Preparation in Scalar Conformal Geometry.*

5.4. Boundary Conditions

For isolated sources it is natural to impose

$$\lambda \to 1,\;\; \nabla_\eta \lambda \to 0,\;\; \text{as } \|x\| \to \infty, \tag{24}$$

ensuring asymptotic flatness and finiteness of the total scalar energy $E = \frac{c^3}{16\pi G}\int 6\lambda^2 \left|\nabla_\eta \lambda\right|_\eta^2 \mathrm{d}V_\eta$. With these boundary conditions the Newtonian potential U is uniquely determined by the mass density ρ via (20).

5.5. Trace of the Stress-Energy Tensor and Boundary Normalization

For a perfect fluid with rest-mass density ρ, pressure p, 4-velocity u^μ (normalized by $g_{\mu\nu}u^\mu u^\nu = -1$), and specific internal energy Π, the stress-energy tensor is

$$T^{\mu\nu} = \left(\rho c^2 + \rho\Pi + p\right)u^\mu u^\nu + p g^{\mu\nu}.$$

Its trace with respect to g is

$$T = g_{\mu\nu}T^{\mu\nu} = \left(\rho c^2 + \rho\Pi + p\right)g_{\mu\nu}u^\mu u^\nu + 4p = -\rho c^2 - \rho\Pi + 3p. \tag{25}$$

In the nonrelativistic regime ($p \ll \rho c^2$, $\Pi \ll c^2$) we have $T \simeq -\rho c^2$, recovering the source used in the linearized analysis of §0. The next corrections ($-\rho\Pi + 3p$) enter at $O\left(v^4/c^4\right)$ and feed the second-order scalar hierarchy used in the post-Newtonian expansion of the sister paper *Strong-Field Expansion and Post-Newtonian Preparation in Scalar Conformal Geometry.*

Boundary data and finite scalar energy. Asymptotic flatness (cf. (24)) fixes the residual conformal normalization and ensures finiteness of the scalar energy

$$E_\lambda = \frac{c^3}{16\pi G}\int 6\lambda^2 \left|\nabla_\eta \lambda\right|_\eta^2 \mathrm{d}V_\eta < \infty,$$

since $\lambda \to 1$ and $\nabla_\eta \lambda \to 0$ as $r \to \infty$. With compactly supported matter, the multipole falloff implies $\lambda(r)=1+\frac{GM}{c^2r}+\mathcal{O}(r^{-2})$, so E_λ converges and the Newtonian potential $U=c^2(\lambda-1)$ is uniquely determined by ρ.

5.6. Summary of the Weak-Field Structure

At first order in ε the NUVO gravitational field obeys:

$$R_g=-6\Delta_\eta\varphi,\ \Delta_\eta\varphi=4\pi G\rho/c^2,\ g_{00}=-1-2U/c^2.$$

These relations demonstrate that NUVO space recovers the Newtonian limit exactly and provides a direct geometric path to the higher-order post-Newtonian expansion.

6. Well-Posedness in the Geometric Class

The scalar field equation obtained in Section 3,

$$-6\lambda^{-1}\Delta_\eta\lambda-6\lambda^{-2}\left|\nabla_\eta\lambda\right|_\eta^2=-\frac{8\pi G}{c^4}T, \tag{26}$$

is a quasi-linear second-order partial differential equation on the flat background η. We summarize the analytic properties that follow from the weighted functional framework developed in Part II [2].

6.1. Elliptic Character and Weak Formulation

For spacelike slices or static sources, the principal part of (26) is the Laplacian $\Delta_\eta\lambda$ multiplied by a positive coefficient λ^{-1}, so the equation is elliptic wherever $\lambda>0$. Introducing the weighted Sobolev space $W_\lambda^{1,2}(M)$ defined by

$$\|u\|_{W_\lambda^{1,2}}^2=\int_M\left(\left|\nabla_\eta u\right|_\eta^2+|u|^2\right)\lambda^4\mathrm{d}V_\eta,$$

we obtain the weak formulation: find $\lambda>0$ such that for all test functions $h\in W_\lambda^{1,2}(M)$,

$$\frac{c^3}{16\pi G}\int_M\left(12\lambda\nabla_\eta\lambda\cdot\nabla_\eta h+6h\left|\nabla_\eta\lambda\right|_\eta^2\right)\mathrm{d}V_\eta=\int_M h\lambda^3T\mathrm{d}V_\eta. \tag{27}$$

6.2. Existence and Regularity

The coercivity and monotonicity properties of the quadratic form on the left-hand side of (27) allow application of the standard results of Gilbarg-Trudinger and Zeidler [7] [8] for quasi-linear elliptic equations. If the source satisfies $T\in L_{\mathrm{loc}}^2(M)$ and boundary data $\lambda|_{\partial M}=\lambda_0>0$ are prescribed, then there exists a weak solution $\lambda\in W_\lambda^{1,2}(M)$ with $\lambda>0$ almost everywhere. Moreover, if T and ∂M are $C^{k,\alpha}$, elliptic regularity implies $\lambda\in C^{k+1,\alpha}(M)$ [6] [7].

***Remark* 7.** For isolated sources the boundary conditions (24) ensure asymptotic flatness and decay of $\nabla_\eta\lambda$, so that the weak solution obtained above extends smoothly to spatial infinity.

6.3. Time-Dependent Extensions

In dynamical situations the full metric $g=\lambda^2\eta$ induces a hyperbolic-elliptic system when λ depends on time. Local well-posedness in this case follows from the same energy estimates applied to the covariant wave operator $\Box_g\phi=\lambda^{-2}\left(\Box_\eta\phi+2\nabla_\eta\ln(\lambda)\cdot\nabla_\eta\phi\right)$, where $\Box_\eta$ is the flat d'Alembertian. The static analysis above thus provides the spatial foundation for the more general time-dependent theory.

6.4. Summary

The gravitational field Equation (26) is therefore well posed within the λ-weighted geometric class:

- For $\lambda>0$ the equation is elliptic on each spacelike slice and admits weak solutions in $W_\lambda^{1,2}(M)$.
- Regularity and uniqueness follow from standard elliptic theory under physically reasonable boundary conditions.
- Time-dependent generalizations inherit local well-posedness from the corresponding hyperbolic system.

These results complete the analytic closure of the scalar field equation derived in Section 3 and establish the mathematical consistency of NUVO gravity within the conformal geometric framework.

7. Discussion and Outlook

The developments in this paper complete the mathematical construction of the gravitational field equation on NUVO space. Starting from the conformal geometry $g=\lambda^2\eta$ defined in Parts I and II, we derived the scalar curvature functional [3] [4], performed the constrained conformal variation, and obtained the trace equation [2]

$$R_g=-\frac{8\pi G}{c^4}T,$$

which governs the dynamics of the scalar field λ in the presence of matter. The weak-field limit reproduces the Poisson equation and the classical Newtonian potential, establishing the empirical consistency of the formalism at leading order.

Summary of principal results.

1) The Einstein-Hilbert action restricted to the conformal class $g=\lambda^2\eta$ yields, upon variation in λ, a single scalar equation equivalent to the trace of Einstein's equations.

2) Diffeomorphism invariance ensures covariant conservation of the stress-energy tensor, while the λ-weighted divergence structure introduces an additional conserved current $\mathrm{div}_g J=0$ representing scalar flux (sinertia) conservation.

3) In the weak-field regime $\lambda=1+\varepsilon\varphi$ the field equation reduces to $\Delta_\eta\varphi=4\pi G\rho/c^2$, identifying $U=c^2\varphi$ as the Newtonian potential and confirming $g_{00}\simeq-1-2U/c^2$.

4) The resulting quasi-linear elliptic equation for λ is well posed in the weighted Sobolev spaces established in Part II and admits smooth solutions under standard boundary conditions.

Interpretation. Within the restricted conformal class, all gravitational effects arise from spatial and temporal variations of the scalar unit field λ. Constant λ corresponds to flat space, while gradients and Laplacians of λ generate curvature and govern energy exchange through the trace T of the matter tensor. The geometry therefore encodes gravity entirely as a modulation of the local scalar scale, without introducing additional tensor degrees of freedom. Because the metric of NUVO space is everywhere conformally flat, the theory propagates only a single scalar degree of freedom. Tensorial perturbations and the two polarization states detected by LIGO/Virgo are therefore absent within this restricted model. Such effects would arise only in extensions that include additional geometric structures—for example, coupling of λ to power connections or higher-order conformal operators—beyond the scalar sector treated here.

Mapping to classical GR structures. For accessibility, we summarize the correspondence between standard GR elements and the NUVO conformal scalar geometry:

GR quantity	**NUVO counterpart**	**Comment**
Metric $g_{\mu\nu}$	$\lambda^2\eta_{\mu\nu}$	Conformal scalar metric
Einstein eqs. $G_{\mu\nu}=8\pi GT_{\mu\nu}/c^4$	$R_g=-8\pi GT/c^4$	Trace equation in conformal class
Conservation $\nabla_g\cdot T=0$	Same	From diffeo invariance (see §4)
Newtonian potential U	$c^2\varphi_1$	From $\lambda=1+\varepsilon\varphi_1$
PPN coefficients (β,γ)	From φ_2, gradients of φ_1	

This table emphasizes that observational content (Newtonian and post-Newtonian) follows directly from the single scalar λ.

Outlook. The results presented here conclude the mathematical foundation of NUVO gravity. The next work in this series, *Strong-Field Preparation for Post-Newtonian Analysis*, extends the scalar field equation into the nonlinear regime, develops the hierarchy of higher-order corrections, and establishes the framework for extracting post-Newtonian parameters $(\beta,\gamma,\cdots)$. That study will form the bridge to the forthcoming flagship paper devoted to the full strong-field PPN inspection and empirical comparison with general relativity.

Concluding remark. NUVO space thus provides a mathematically rigorous, self-contained platform for studying gravitation through a single scalar degree of freedom. Its internal consistency, correct Newtonian limit, and compatibility with standard conservation laws establish a firm foundation for the physical extensions developed in the later parts of the series.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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Appendix: Illustrative Example: Static Spherical Source (Weak Field)

Consider a static, spherically symmetric source of total mass M and radius R. In the weak field, we write $\lambda = 1 + \varepsilon\varphi(r)$ with $\varepsilon \ll 1$ and $r = \|\boldsymbol{x}\|$. Giving

$$\Delta_\eta \varphi = 4\pi G\rho / c^2 .$$

Outside the source ($r > R$), $\rho = 0$ so $\Delta_\eta \varphi = 0$ and the decaying solution is $\varphi(r) = A/r$. Matching to the total mass yields $A = GM/c^2$, hence

$$\lambda(r) = 1 + \frac{GM}{c^2 r} + \mathcal{O}\left(\frac{1}{r^2}\right), \quad g_{00} = -\lambda^2 \simeq -\left(1 + \frac{2GM}{c^2 r}\right),$$

reproducing the Newtonian limit and the standard redshift/time-dilation to leading order. Interior solutions ($r < R$) satisfy $\Delta_\eta \varphi = 4\pi G\rho / c^2$ with regularity at $r = 0$ and continuity of λ and $\partial_r \lambda$ at $r = R$; the explicit form depends on the density profile $\rho(r)$.