

NUVO Space I: Unit-Constrained Frame Bundle and Conformal Scalar

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Abstract

We construct the differential-geometric foundation of NUVO space as a conformally flat manifold (M, g) endowed with a scalar unit constraint. Starting from a flat background form η and scalar $\lambda > 0$, we derive the associated frame-bundle reduction, induced metric $g = \lambda^2 \eta$, and Levi-Civita connection. Existence, uniqueness, and regularity of the induced connection are proved, defining the canonical calculus objects required for subsequent curvature and variational analyses.

Keywords

NUVO Manifold, Conformal Geometry, Scalar Field Geometry, Differential Geometry of Scalar Manifolds, Unit-Constrained Frames

1. Introduction

The purpose of this paper is to formalize the geometric space on which later NUVO analyses—both mathematical and physical—will be constructed. We begin with a smooth n -manifold M endowed with a background flat metric η (Euclidean or Minkowskian signature). A smooth, positive scalar field $\lambda : M \rightarrow (0, \infty)$ modulates η by a conformal scaling $g = \lambda^2 \eta$. The frame bundle of M admits a natural reduction compatible with this scalar modulation, introducing a *unit constraint* that fixes relative length scales up to a scalar gauge.

We will show that:

- The torsion-free g -compatible connection is uniquely determined;
- Volume and surface measures scale by explicit powers of λ ;
- Gauge rescalings $(\eta, \lambda) \mapsto (\alpha^2 \eta, \lambda/\alpha)$ preserve g .

The motivation for introducing NUVO space is to provide a mathematically rigorous conformal framework that preserves local unit normalization, enabling

consistent treatment of scalar-modulated curvature and dynamics across both Euclidean and Lorentzian settings.

Part I establishes the geometric framework for NUVO space as a conformally flat manifold (M, g) built from a background metric η and scalar field λ . The general structure follows classical treatments of Riemannian geometry and conformal metrics [1]-[3] and connects naturally to more recent applications of conformal methods in gravitational and geometric analysis [4] [5]. Although this paper remains purely geometric, the resulting framework is intended to support later analyses of curvature, dynamics, and variational principles relevant to general relativity and scalar-tensor field theories.

2. Conformal-Scalar Frame Structure

Definition 1 (Frame bundle and scalar reduction) Let $\mathcal{F}(M)$ denote the principal $GL(n, \mathbb{R})$ -bundle of frames over M . A *unit-constrained frame* at $p \in M$ is a linear isomorphism $u: T_p M \rightarrow \mathbb{R}^n$ such that $\eta(u(e_i), u(e_j)) = \lambda(p)^{-2} \delta_{ij}$. The associated subbundle $\mathcal{F}_\lambda(M) \subset \mathcal{F}(M)$ defines the scalar reduction.

The local frame structure and connection one-forms align with the standard differential-geometric formalism of Kobayashi and Nomizu [6] and Boothby [7].

Lemma 2 (Gauge equivalence) Pairs (η, λ) and $(\alpha^2 \eta, \lambda/\alpha)$ with $\alpha > 0$ define the same conformal metric $g = \lambda^2 \eta$. The conformal metric $g = \lambda^2 \eta$ provides a smooth deformation of the background η , as in the standard conformal-change formulas [1] [2].

Proof. Immediate from $g' = (\lambda/\alpha)^2 (\alpha^2 \eta) = \lambda^2 \eta = g$.

Remark 3. The unit constraint fixes the local length scale up to a single scalar degree of freedom, distinguishing NUVO space from generic conformal manifolds by enforcing a measurable scalar normalization. Unlike a generic conformal manifold, NUVO space imposes a unit constraint, fixing relative length scales up to a scalar gauge, ensuring that metric deformations carry an operational meaning tied to measurable units.

3. Metric and Connection

Let (M, η) be an n -dimensional smooth manifold with a flat metric η and Levi-Civita connection ∇^η . Let $\lambda: M \rightarrow (0, \infty)$ be a smooth scalar field and define the conformal metric

$$g = \lambda^2 \eta, \quad g_{ij} = \lambda^2 \eta_{ij}. \quad (1)$$

Denote by ∇ the Levi-Civita connection of g . We compute the relation between ∇ and ∇^η explicitly.

3.1. Connection Coefficients

The Christoffel symbols of any Levi-Civita connection are determined by the metric components through

$$\Gamma^k_{ij} = \frac{1}{2} g^{k\ell} (\partial_i g_{j\ell} + \partial_j g_{i\ell} - \partial_\ell g_{ij}). \tag{2}$$

Since $g_{ij} = \lambda^2 \eta_{ij}$ and $g^{ij} = \lambda^{-2} \eta^{ij}$, one has

$$\partial_i g_{j\ell} = 2\lambda (\partial_i \lambda) \eta_{j\ell}.$$

Substituting into (2) gives

$$\begin{aligned} \Gamma^k_{ij} &= \frac{1}{2} \lambda^{-2} \eta^{k\ell} (2\lambda \partial_i \lambda \eta_{j\ell} + 2\lambda \partial_j \lambda \eta_{i\ell} - 2\lambda \partial_\ell \lambda \eta_{ij}) \\ &= \lambda^{-1} (\delta^k_i \partial_j \lambda + \delta^k_j \partial_i \lambda - \eta_{ij} \eta^{k\ell} \partial_\ell \lambda). \end{aligned}$$

Introducing the logarithmic field $\varphi = \log \lambda$ simplifies this to

$$\boxed{\Gamma^k_{ij} = \delta^k_i \partial_j \varphi + \delta^k_j \partial_i \varphi - \eta_{ij} \eta^{k\ell} \partial_\ell \varphi.} \tag{3}$$

3.2. Compatibility and Torsion

Because Γ^k_{ij} is symmetric in the lower indices i, j , the connection is torsion free. To verify metric compatibility, note that

$$\nabla_k g_{ij} = \partial_k g_{ij} - \Gamma^\ell_{ki} g_{\ell j} - \Gamma^\ell_{kj} g_{i\ell}.$$

Substituting $g_{ij} = \lambda^2 \eta_{ij}$ and (3) yields

$$\nabla_k g_{ij} = 2\lambda (\partial_k \lambda) \eta_{ij} - 2\lambda (\partial_k \lambda) \eta_{ij} = 0,$$

confirming that ∇ is compatible with g . Metric compatibility $\nabla_k g_{ij} = 0$ follows directly from the structure equations of the Levi-Civita connection [7] [8].

Theorem 4 (Levi-Civita connection for a conformal metric) Let η be a flat background metric and $\lambda > 0$ smooth. Then the unique torsion-free connection compatible with $g = \lambda^2 \eta$ has Christoffel symbols (3).

Corollary 1 (Regularity) If $\lambda \in C^{k,\alpha}(M)$ with $k \geq 1$, then $\Gamma^k_{ij} \in C^{k-1,\alpha}(M)$ and ∇ extends continuously to the boundary of any $C^{1,\alpha}$ domain.

Remark 5. The difference tensor $C^k_{ij} = \Gamma^k_{ij} - (\Gamma^\eta)^k_{ij}$ equals the right-hand side of (3). This relation, often called the *conformal connection formula*, will be used repeatedly in curvature and geodesic computations.

4. Volume and Surface Measures

Let dV_η denote the volume element associated with η and dS_η the induced measure on smooth hypersurfaces. We derive the scaling laws for the conformal metric $g = \lambda^2 \eta$.

4.1. Volume Measure

In local coordinates,

$$dV_g = \sqrt{|\det(g_{ij})|} dx^1 \cdots dx^n.$$

Because $\det(g_{ij}) = \lambda^{2n} \det(\eta_{ij})$,

$$\boxed{dV_g = \lambda^n dV_\eta.} \tag{4}$$

4.2. Surface Measure

Let $\Sigma \subset M$ be a smooth oriented hypersurface with unit normal ν_η measured by η . The corresponding g -unit normal is

$$\nu_g = \lambda^{-1} \nu_\eta,$$

since $g(\nu_g, \nu_g) = 1$. For any $(n-1)$ -form ω , Stokes' theorem in both metrics gives

$$\int_\Sigma \omega = \int_\Sigma \iota_{\nu_\eta} dV_\eta = \int_\Sigma \iota_{\nu_g} dV_g.$$

Substituting (4) and $\nu_g = \lambda^{-1} \nu_\eta$ yields

$$\boxed{dS_g = \lambda^{n-1} dS_\eta.} \tag{5}$$

Proposition 6 (Measure scaling) In dimension n , the conformal metric $g = \lambda^2 \eta$ satisfies

$$dV_g = \lambda^n dV_\eta, \quad dS_g = \lambda^{n-1} dS_\eta.$$

Proof. Equations (4) and (5) establish the claim.

Remark 7. The measure exponents $(n, n-1)$ coincide with the powers appearing in the Jacobian determinant and induced metric on hypersurfaces. These scaling rules underpin all subsequent integral identities, including the λ -weighted divergence and Stokes theorems of Part II.

5. Conformal Transformations and Gauge Freedom

The conformal structure defined by $g = \lambda^2 \eta$ is invariant under simultaneous rescaling of the background metric and scalar field. This *gauge freedom* identifies all pairs (η, λ) that produce the same g and hence the same connection, curvature, and geometric operators.

5.1. Gauge Equivalence

Definition 8 (Gauge transformation) For any smooth positive function $\alpha : M \rightarrow (0, \infty)$, define the transformed pair

$$(\eta', \lambda') := (\alpha^2 \eta, \lambda/\alpha).$$

We say that (η', λ') is *gauge equivalent* to (η, λ) .

Proposition 9 (Invariance of the conformal metric) The conformal metric $g = \lambda^2 \eta$ is invariant under the transformation (8); that is,

$$\lambda'^2 \eta' = \lambda^2 \eta = g.$$

Proof. Substituting $\lambda' = \lambda/\alpha$ and $\eta' = \alpha^2 \eta$ gives $\lambda'^2 \eta' = (\lambda^2/\alpha^2)(\alpha^2 \eta) = \lambda^2 \eta = g$.

5.2. Invariance of the Connection

Let ∇' denote the Levi-Civita connection computed from (η', λ') via the formula

$$\Gamma'^k_{ij} = \delta^k_i \partial_j \log \lambda' + \delta^k_j \partial_i \log \lambda' - \eta'_{ij} \eta'^{k\ell} \partial_\ell \log \lambda'.$$

Since $\log \lambda' = \log \lambda - \log \alpha$ and $\eta'_{ij} \eta'^{k\ell} = \eta_{ij} \eta^{k\ell}$, we find

$$\Gamma'^k_{ij} = \Gamma^k_{ij} - (\delta^k_i \partial_j \log \alpha + \delta^k_j \partial_i \log \alpha - \eta_{ij} \eta^{k\ell} \partial_\ell \log \alpha).$$

If α is a *constant* rescaling, the derivative terms vanish and $\Gamma'^k_{ij} = \Gamma^k_{ij}$; hence the connection is globally invariant under constant gauge transformations.

For a nonconstant $\alpha(x)$, the two connections differ by a tensor

$$C^k_{ij} = -(\delta^k_i \partial_j \log \alpha + \delta^k_j \partial_i \log \alpha - \eta_{ij} \eta^{k\ell} \partial_\ell \log \alpha), \tag{6}$$

which is the standard transformation rule for conformally related connections. The curvature tensors of ∇ and ∇' are related by derivatives of α , and hence coincide whenever α is constant.

Proposition 10 (Gauge invariance of the Levi-Civita connection). The Levi-Civita connection associated with $g = \lambda^2 \eta$ is unchanged under constant gauge rescalings $(\eta, \lambda) \mapsto (\alpha^2 \eta, \lambda/\alpha)$ with $\alpha > 0$ constant.

Remark 11. The freedom to multiply λ by a constant corresponds to a global change of units: physical lengths and times are rescaled by α , but all dimensionless geometric quantities remain invariant. This global gauge freedom will be fixed later by normalization of integrals or boundary conditions.

5.3. Conformal Killing Structure

Every vector field X that is Killing for η satisfies $\mathcal{L}_X \eta = 0$. Because $g = \lambda^2 \eta$, the Lie derivative obeys

$$\mathcal{L}_X g = 2X(\log \lambda)g.$$

Hence X is a *conformal Killing field* for g with conformal factor $2X(\log \lambda)$. The scalar field λ thus determines how the symmetry algebra of η deforms under the unit constraint. Constant λ recovers the full Killing algebra of η ; spatially varying λ breaks this to the subset of vector fields for which $X(\lambda) = 0$.

Proposition 12. If X is Killing for η and $X(\lambda) = 0$, then X is Killing for $g = \lambda^2 \eta$.

Proof. $X(\lambda) = 0$ implies $\mathcal{L}_X g = \lambda^2 \mathcal{L}_X \eta = 0$.

Geometrically, this symmetry breaking indicates that conserved quantities associated with background Killing fields acquire λ -dependent corrections. In physical settings, such variation would correspond to local departures from strict conservation, consistent with the scalar-weighted continuity relations developed in later parts of the series.

Remark 13. The conformal-Killing relation will later determine conservation laws for scalar-weighted currents in variational problems, forming the bridge to the dynamical equations studied in Part III.

6. Examples

This section illustrates the general constructions of Sections 3 - 5 in the two most common backgrounds: the Euclidean metric on \mathbb{R}^n and the Minkowski metric on $\mathbb{R}^{1,3}$. In each case we compute the connection coefficients, verify the measure-scaling laws, and comment on curvature and Killing structure.

6.1. Euclidean Space \mathbb{R}^n

Let (x^1, \dots, x^n) be Cartesian coordinates on \mathbb{R}^n with background metric $\eta_{ij} = \delta_{ij}$. The conformal metric is

$$g_{ij} = \lambda^2 \delta_{ij}, \quad g^{ij} = \lambda^{-2} \delta^{ij}.$$

Using formula (3) with $\varphi = \log \lambda$ gives

$$\Gamma^k_{ij} = \delta^k_i \partial_j \varphi + \delta^k_j \partial_i \varphi - \delta_{ij} \partial^k \varphi, \quad \partial^k = \delta^{k\ell} \partial_\ell. \tag{7}$$

Verification of metric compatibility. Direct substitution of (7) into

$$\nabla_k g_{ij} = \partial_k g_{ij} - \Gamma^\ell_{ki} g_{\ell j} - \Gamma^\ell_{kj} g_{i\ell}$$

yields $\nabla_k g_{ij} = 0$, as expected.

Measure scaling. Since $\det(g_{ij}) = \lambda^{2n}$, one obtains $dV_g = \lambda^n d^n x$ and $dS_g = \lambda^{n-1} dS_\eta$, verifying Proposition 6.

Flatness for constant λ . If $\lambda \equiv \lambda_0$ is constant, then $\Gamma^k_{ij} = 0$ and $g = \lambda_0^2 \delta_{ij}$ is globally flat with vanishing curvature tensor. Hence NUVO space reduces to Euclidean space up to an overall scale factor.

Curvature for nonconstant λ . If λ varies spatially, curvature arises through second derivatives of λ . For instance, the scalar curvature from Theorem 4.1 in Paper II will read $R_g = -2(n-1)\lambda^{-3} \Delta_\eta \lambda$ for $n = 3$.

6.2. Minkowski Space $\mathbb{R}^{1,3}$

Let (t, x, y, z) denote standard coordinates and $\eta = \text{diag}(-1, 1, 1, 1)$. Then

$$g_{\mu\nu} = \lambda^2 \eta_{\mu\nu}, \quad g^{\mu\nu} = \lambda^{-2} \eta^{\mu\nu}.$$

Equation (3) yields

$$\Gamma^\rho_{\mu\nu} = \delta^\rho_\mu \partial_\nu \varphi + \delta^\rho_\nu \partial_\mu \varphi - \eta_{\mu\nu} \eta^{\rho\sigma} \partial_\sigma \varphi. \tag{8}$$

This expression preserves the causal structure: if v is null with respect to η , then $g(v, v) = \lambda^2 \eta(v, v) = 0$. Hence conformal rescaling leaves null cones invariant.

Special case: static radial field. Let $\lambda = \lambda(r)$ with $r^2 = x^2 + y^2 + z^2$. Non-zero Christoffel components are

$$\Gamma^t_{tt} = \partial_r \varphi \frac{dr}{dt} = 0, \quad \Gamma^r_{tt} = \partial_r \varphi, \quad \Gamma^r_{rr} = \partial_r \varphi, \quad \Gamma^\theta_{r\theta} = \Gamma^\phi_{r\phi} = \partial_r \varphi.$$

These agree with the general formula (8). Curvature components vanish when λ is constant and grow with $\partial_r^2 \lambda$ otherwise.

Measure scaling. Again $\det(g_{\mu\nu}) = \lambda^8 \det(\eta_{\mu\nu})$ in four dimensions, giving

$dV_g = \lambda^4 d^4x$ and $dS_g = \lambda^3 dS_\eta$, consistent with Proposition 6.

Killing structure. The ten Killing fields of Minkowski space remain Killing for g whenever λ is constant. For nonconstant $\lambda(x)$, only those satisfying $X(\lambda) = 0$ remain symmetries of g , as established in Proposition 5.4.

6.3. Comparative Summary

Both the Euclidean and Minkowski examples confirm that:

1) The connection (3) and measure laws (4)-(5) hold identically in any coordinate basis adapted to η .

2) Constant λ reproduces the flat background geometry, while variable λ introduces curvature determined by $\nabla_\eta \lambda$ and $\nabla_\eta^2 \lambda$.

3) The causal and null structures of the background are preserved, making NUVO space a conformal deformation rather than a new topology.

These concrete cases provide immediate intuition for the analytic operators—divergence, Laplacian, and curvature—developed in Part II.

7. Discussion and Conclusions

The developments in this paper establish the mathematical identity of *NUVO space*: a conformally flat manifold (M, g) built from a background metric η and a positive scalar field λ that acts as a local unit constraint. The conformal scalar geometry of $g = \lambda^2 \eta$ and its connection structure are formalized in a way consistent with classical Riemannian theory [1] [2] [3] [8]. All subsequent constructions—divergence, curvature, and motion—rest on the results summarized below.

Principal results.

1) The conformal relation $g = \lambda^2 \eta$ defines a unique torsion-free, g -compatible connection whose coefficients are expressed explicitly by

$$\Gamma^k_{ij} = \delta^k_i \partial_j \varphi + \delta^k_j \partial_i \varphi - \eta_{ij} \eta^{kl} \partial_l \varphi, \quad \varphi = \log \lambda.$$

This provides the exact correspondence between the background derivative operator ∇^η and the Levi-Civita connection of g .

2) Volume and surface measures scale by the precise powers of the scalar field,

$$dV_g = \lambda^n dV_\eta, \quad dS_g = \lambda^{n-1} dS_\eta,$$

which will govern all integral identities and conservation laws in later analyses.

3) The metric and connection are invariant under global gauge rescalings $(\eta, \lambda) \mapsto (\alpha^2 \eta, \lambda/\alpha)$. The remaining conformal freedom corresponds to global unit normalization, while local variations of λ encode true geometric deformation.

4) In Euclidean and Minkowski examples the construction reproduces the expected flat-space limit for constant λ and yields controlled curvature for non-constant λ , preserving the causal and topological structure of the background.

Outlook. With these geometric preliminaries completed, the analytical foundations can now be developed. Part II [9] of this series, “*NUVO Space II: Weighted*

Calculus, Divergence Theorems, and Curvature,” will introduce the λ -weighted differential operators, prove Stokes and Gauss formulas, and derive the curvature tensors and Laplace-Beltrami operator associated with $g = \lambda^2 \eta$. Together, Parts I and II provide the complete mathematical scaffolding for the variational and dynamical analyses of NUVO geometry.

Concluding remark. The framework presented here is intentionally geometric and coordinate-free, relying only on smoothness of the scalar field and the flatness of the background metric. It therefore applies equally to Euclidean, Lorentzian, or more general pseudo-Euclidean signatures. Subsequent work will show how these purely mathematical properties give rise to consistent dynamical equations once physical interpretations of λ and its gradients are introduced.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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