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Existence of Exponential Attractors for Suspension Bridge Equations with State Delay

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Abstract

This paper studies the nonlocal suspension bridge equation with state-dependent delay in 2D space and the existence of attractors. First, by using Banach fixed point theorem and operator semigroup theory, the existence and uniqueness of mild solutions and continuous dependence on initial values of suspension bridge equations with state delay are proved. Then the bounded dissipativity of the related semi-group and the quasi-stability of the system are verified to obtain a global attractor with finite fractal dimension the existence of the attractor in the generalized index.

Keywords

Contractive Function, Well-Posedness, Time-Dependent Global Attractors

1. Introduction

In recent years, some physical problems in the model of suspension bridges have been studied by many people, see [1]-[3]. Moreover, the suspension bridge equation with time delay has become a research hotspot, as the existence of time delay can affect the existence and stability of attractors in the system. [4]-[8] have investigated related issues. [5] introduced the existence of strong solutions and strong global attractors for the coupled suspension bridge equations. [6] studied the case where the damping coefficient satisfies $\gamma_1 > \frac{3}{2}|\gamma_2|$, when the nonlinear and external force terms satisfy specific conditions, the equation possesses a unique global solution. Furthermore, the existence of a uniform attractor has been demonstrated. It is worth noting that, in order to describe the system process more naturally, state-dependent models have been proposed and studied (see Ref-

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erences [9]-[11]). In Reference [9], to handle the delay term in the energy functional, a compensation term for the delay term is introduced to obtain the uniformly bounded estimate of the solution. When addressing the uniqueness of the solution, the solution space is restricted to $Y \equiv C\left(\left[-h,0\right];V_2\right)\cap C^1\left(\left[-h,0\right];H\right)$, and the treatment is carried out by combining the Lipschitz continuity of the delay term. [11] investigates the dynamic behavior of equations governing suspension bridges. By employing the principle of contraction mapping, it establishes the well-posedness of these equations. Furthermore, utilizing quasi-stability methods, it demonstrates the existence of global attractors and exponential attractors. Compared with Reference [11], the boundary conditions in this paper have been modified to make the system more consistent with engineering practice. However, this modification may undermine the original dissipativity and stability of the system, necessitating a reanalysis of the existence and structure of the attractor. This also implies that the definitions of the inner product and norm in the solution space are more complex compared to those under hinged or fixed boundary conditions, which renders the verification of compactness more challenging. Therefore, it is both meaningful and intriguing to continue exploring such problems.

For the reasons mentioned above, this paper considers the following suspension bridge equation with state time delay

$$\begin{cases} u_{tt}+\Delta^2 u-\phi\left(\left\|\nabla u\right\|^2\right)\Delta u+g\left(u\right)+\delta_1 u_t-\delta_2\Delta u_t & \\ \quad +u\left(x,y,t-\pi\left[u^t\right]\right)=f\left(x,y\right), & \left(x,y\right)\in\Omega, t\in\left[0,+\infty\right), \\ u\left(x,y,t\right)=\varphi\left(x,y,t\right), & \left(x,y\right)\in\Omega, t\in\left[-h,0\right], \\ u_t\left(x,y,t\right)=\varphi_t\left(x,y,t\right), & \left(x,y\right)\in\Omega, t\in\left[-h,0\right], \end{cases} \tag{1.1}$$

with the following boundary conditions imposed

$$\begin{cases} u\left(0,y,t\right)=\partial_{xx}u\left(0,y,t\right)=0, & \left(y,t\right)\in\left[-l,l\right]\times\left[0,+\infty\right), \\ u\left(\pi,y,t\right)=\partial_{xx}u\left(\pi,y,t\right)=0, & \left(y,t\right)\in\left[-l,l\right]\times\left[0,+\infty\right), \\ \partial_{yy}u\left(x,\pm\ell,t\right)+\sigma\partial_{xx}u\left(x,\pm\ell,t\right)=0, & \left(x,t\right)\in\left[0,\pi\right]\times\left[0,+\infty\right), \\ \partial_{yyy}u\left(x,\pm\ell,t\right)+\left(2-\sigma\right)\partial_{xxy}u\left(x,\pm\ell,t\right)=0, & \left(x,t\right)\in\left[0,\pi\right]\times\left[0,+\infty\right), \end{cases} \tag{1.2}$$

where $\Omega=\left[0,\pi\right]\times\left[-l,l\right]$. $u=u\left(x,y,t\right)$ describes the deformation of the bridge in the vertical plane; $\delta_1,\delta_2>0$; $h>0$ represents the maximum delay time, $u\left(t-\pi\left[u^t\right]\right)$ is the state delay term, φ is the initial value on the interval $\left[-h,0\right]$, π is a mapping with values in the interval $\left[0,h\right]$. Among them, $g\left(u\right)$ denotes the nonlinear term inside the bridge deck, while $\phi\left(\left\|\nabla u\right\|^2\right)\Delta u$, first proposed by S. Woinwsky-Krieger [12] is used to describe the transverse deflection of a stretchable beam.

Without loss of generality, let $A=\Delta^2, A^{\frac{1}{2}}=-\Delta$, whose domain is

$$D\left(A\right)=\left\{u\in H^4\left(\Omega\right):u\left(0,y,t\right)=u\left(\pi,y,t\right)=u_{xx}\left(0,y,t\right)=u_{xx}\left(\pi,y,t\right)=0\right\},$$

In particular, let $H=V_0=L^2\left(\Omega\right)$, where its inner product and norm are defined respectively as

$$\left(u,v\right)_H=\left(u,v\right),\ \left\|u\right\|_H^2=\left\|u\right\|^2.$$

Analogously to Reference [13], we introduce the following phase space:

$$H_*^2(\Omega)=\left\{w\in H^2(\Omega): w(0,y)=w(\pi,y)=0,\forall y\in(-l,l)\right\},$$

For the convenience of calculation, we set $V_2=H_*^2(\Omega)$, and their corresponding inner products and norms are respectively defined as

$$(u,v)_{V_2}=\int_\Omega\left[\Delta u\Delta v+(1-\sigma)\left(2u_{xy}v_{xy}-u_{xx}v_{yy}-u_{yy}v_{xx}\right)\right]\mathrm{dxdy},$$

$$\|u\|_2^2=\|u\|_{V_2}^2=\left[\int_\Omega\left[(\Delta u)^2+2(1-\sigma)\left(u_{xy}^2-u_{xx}v_{yy}\right)\right]\mathrm{dxdy}\right]^{\frac{1}{2}}.$$

According to Lemma 4.1 in Reference [13], it follows that the norm $\|\cdot\|_{H_*^2}$ and $\|\cdot\|_{H^2}$ are equivalent. Furthermore, by *Sobolev* the compact embedding theorem, $V_2\hookrightarrow V_0$. According to Poincaré the inequality, we have

$$\|u\|_2^2\ge\lambda_1\|u\|_0^2,\ \forall u\in V_2\qquad \|u\|_{D(A)}^2\ge\lambda_1^2\|u\|_2^2, u\in D(A)$$

Here λ_1 is A the first eigenvalue.

Define the phase space

$$Y\equiv C([-h,0];V_2)\cap C^1([-h,0];H),$$

whose norm is given by

$$\|\varphi\|_Y=\|\varphi\|_{C_{V_2}}+\|\varphi_t\|_{C_H},\ \|v\|_{C_X}=\sup_{\theta\in[-h,0]}\|v(\theta)\|_X,\ \forall v\in C_X.$$

Assume that the nonlinear term $g\in C^2(\mathbb{R},\mathbb{R})$ satisfies the conditions:

$$\liminf_{|s|\to\infty}\frac{G(s)}{s^2}\ge0,\ G(s)=\int_0^s g(\tau)\mathrm{d}\tau,\ \forall s\in\mathbb{R}, \tag{1.3}$$

$$\liminf_{|s|\to\infty}\frac{sg(s)-C_0G(s)}{s^2}\ge0,\ C_0>0,\ \forall s\in\mathbb{R}, \tag{1.4}$$

$$\limsup_{|s|\to\infty}\frac{g'(s)}{|s|^p}=0,\ \forall p\ge0,\ \forall s\in\mathbb{R}. \tag{1.5}$$

Let the mapping $\pi:Y\to[0,h]$ be locally Lipschitz, that is, for any $N>0$, there exists $L_N>0$, For any $\beta_1,\beta_2\in Y$, $\|\delta_i\|_Y\le N$, $i=1,2$, there holds

$$\left|\pi(\beta_1)-\pi(\beta_2)\right|\le L_N\|\beta_1-\beta_2\|_Y. \tag{1.6}$$

To obtain the compactness of the semigroup, we further assume that there exists $\epsilon>0$ such that the delay term satisfies the subcritical local Lipschitz condition, that is, for any $\rho>0$, there exists $L_\rho>0$, such that for any $\beta_i, i=1,2$, with $\|\beta_i\|_Y\le\rho$, there holds

$$\left|\pi(\beta_1)-\pi(\beta_2)\right|\le L_\rho\max_{\theta\in[-h,0]}\left\|A^{\frac{1}{2}-\epsilon}(\beta_1-\beta_2)\right\|. \tag{1.7}$$

Finally, according to conditions (1.3)-(1.4) and Poincaré inequality, there exist constants $K_1,K_2>0$, such that

$$\int_\Omega G(u)\mathrm{d}x+\frac{1}{8}\|\Delta u\|^2\ge-K_1,\ \forall u\in V, \tag{1.8}$$

$$\left(g(u),u\right)-C_0\int_\Omega G(u)\,\mathrm{d}x+\frac{1}{8}\|\Delta u\|^2\ge -K_2,\ \forall u\in V. \tag{1.9}$$

Assume that the nonlinear function $\phi(\cdot)\in C^1(\mathbb{R})$ satisfies

$$\phi(s)\ge 0,\ \phi(s)s\ge\frac{1}{2}\int_0^s\phi(\tau)\,\mathrm{d}\tau+\beta s^2, \tag{1.10}$$

where $0<\beta<\frac{1}{2}$, $\phi(0)=0$.

2. Well-Posedness

2.1. Priori Estimate

Theorem 2.1 Assume that conditions (1.3) and (1.4) hold, $f\in L^2(\Omega)$. Then the solution to Equation (1.1) satisfies the following estimate

$$\|z\|^2+\|\Delta u\|^2+\int_0^{\|\nabla u\|^2}\phi(\tau)\,\mathrm{d}\tau\|\nabla u\|^2\le 4\mathrm{e}^{-\alpha t}\left(E(0)+\mu h\|\psi\|_Y\right)+\frac{4C}{\alpha}, \tag{2.1}$$

where $C=\frac{4}{\delta_1}\|g\|^2+\gamma K_1+2\varepsilon K_2$,

$$E(t)=\|z\|^2+\|\Delta u\|^2+2\int_\Omega G(u)\,\mathrm{d}x+\left(\delta_2\varepsilon+\int_0^{\|\nabla u\|^2}\phi(\tau)\,\mathrm{d}\tau\right)\|\nabla u\|^2+2K_1.$$

Proof Taking the inner product of $z=\partial_t u+\varepsilon u(\varepsilon>0)$ with (1.1) in H, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|z(t)\|^2+\|\Delta u\|^2\right)+\varepsilon\|\Delta u\|^2+(\delta_1-\varepsilon)\|z(t)\|^2-\varepsilon(\delta_1-\varepsilon)(u,z)\\&=(f,z)-(g(u),z)+(\delta_2\Delta u_t,z)-\left(u\left(t-\pi\left[u^t\right]\right),z\right)+\left(\phi\left(\|\nabla u\|^2\right)\Delta u,z\right).\end{aligned} \tag{2.2}$$

By applying Young inequality, Hölder and Poincaré inequality, and choosing a sufficiently small $\varepsilon>0$, we have

$$(\delta_2\Delta u_t,z)=-\delta_2\|\nabla u_t\|^2-\delta_2\varepsilon\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla u\|^2. \tag{2.3}$$

$$\varepsilon\|\Delta u\|^2+(\delta_1-\varepsilon)\|z(t)\|^2-\varepsilon(\delta_1-\varepsilon)(u,z)\ge\frac{3\delta_1}{4}\|z(t)\|^2+\frac{\varepsilon}{2}\|\Delta u\|^2. \tag{2.4}$$

$$(f,z)\le\frac{2}{\delta_1}\|f\|^2+\frac{\delta_1}{8}\|z\|^2. \tag{2.5}$$

$$\begin{aligned}-\left(u\left(t-\pi\left[u^t\right]\right),z\right)&\le\left\|u\left(t-\pi\left[u^t\right]\right)\right\|\|z\|\\&\le\left\|u(t)-\int_{t-\pi\left[u^t\right]}^t\partial_t u(s)\,\mathrm{d}s\right\|\|z\|\\&\le\left[\|u(t)\|+2\left\|\int_0^h\partial_t u(t-s)\,\mathrm{d}s\right\|\right]\|z\|\\&\le 2\|u(t)\|\|z\|+2\left\|\int_0^h\partial_t u(t-s)\,\mathrm{d}s\right\|\|z\|\\&\le\frac{2}{\sqrt{\lambda_1}}\|\Delta u(t)\|\|z\|+2\left\|\int_0^h\partial_t u(t-s)\,\mathrm{d}s\right\|\|z\|\\&\le\frac{\varepsilon}{8}\|\Delta u(t)\|^2+\frac{4}{\varepsilon\lambda_1}\|z\|^2+\frac{8}{\delta_1}\int_0^h\|\partial_t u(t-s)\|^2\,\mathrm{d}s+\frac{\delta_1}{8}\|z\|^2.\end{aligned} \tag{2.6}$$

From (1.8) and (1.9)

$$-\left(g\left(u\right),z\right)\le -\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega}G\left(u\right)\mathrm{d}x-\varepsilon C_0\int_{\Omega}G\left(u\right)\mathrm{d}x+\frac{\varepsilon}{8}\left\|\Delta u\right\|^2+\varepsilon K_2, \tag{2.7}$$

$$\left(\phi\left(\left\|\nabla u\right\|^2\right)\Delta u,z\right)=\left[-\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\nabla u\right\|^2-\varepsilon\left\|\nabla u\right\|^2\right]\phi\left(\left\|\nabla u\right\|^2\right). \tag{2.8}$$

Substituting (2.3)-(2.8) into (2.2), we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}E\left(t\right)\le{}&\frac{16}{\delta_1}\int_0^h\left\|u_t\left(t-s\right)\right\|^2\mathrm{d}s+\frac{4}{\delta_1}\left\|f\right\|^2+2\varepsilon K_2\\&-\frac{\varepsilon}{2}\left\|\Delta u\right\|^2-\left(\delta_1-\frac{4}{\varepsilon\lambda_1}\right)\left\|z\right\|^2-2\varepsilon C_0\int_{\Omega}G\left(u\right)\mathrm{d}x\\&-2\delta_2\left\|\nabla u_t\right\|^2-2\phi\left(\left\|\nabla u\right\|^2\right)\varepsilon\left\|\nabla u\right\|^2.\end{aligned} \tag{2.9}$$

Take

$$E\left(t\right)=\left\|z\right\|^2+\left\|\Delta u\right\|^2+2\int_{\Omega}G\left(u\right)\mathrm{d}x+\delta_2\varepsilon\left\|\nabla u\right\|^2+\int_0^{\left\|\nabla u\right\|^2}\phi\left(\tau\right)\mathrm{d}\tau\left\|\nabla u\right\|^2+2K_1,$$

define

$$V\left(t\right)=E\left(t\right)+\frac{\mu}{h}\int_0^h\int_{t-s}^t\left\|u_t\left(\xi\right)\right\|^2\mathrm{d}\xi\mathrm{d}s.$$

Obviously

$$E\left(t\right)\le V\left(t\right)\le E\left(t\right)+\mu\int_0^h\left\|u_t\left(t-\xi\right)\right\|^2\mathrm{d}\xi, \tag{2.10}$$

where $0<\mu<\frac{\delta_1}{4}$, $\frac{\mu}{h}\int_0^h\int_{t-s}^t\left\|\partial_t u\left(\xi\right)\right\|^2\mathrm{d}\xi\mathrm{d}s$ serves as the compensation term for the delay term in the equation. Choosing a sufficiently large. Choosing a sufficiently large δ_1, ensures that $\delta_1-\frac{4}{\varepsilon\lambda_1}>\frac{\delta_1}{2}$, That is, when $\delta_1>\frac{32}{\varepsilon\lambda_1}$ he system reaches equilibrium. This is because a longer delay time makes the system more unstable, while increasing the damping coefficient can achieve balance. Taking the derivative of $V\left(t\right)$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}V\left(t\right)=\frac{\mathrm{d}}{\mathrm{d}t}E\left(t\right)+\mu\left\|u_t\left(t\right)\right\|^2-\frac{\mu}{h}\int_0^h\left\|u_t\left(t-s\right)\right\|^2\mathrm{d}s. \tag{2.11}$$

Since

$$\left\|u_t\right\|^2=\left\|u_t+\varepsilon u-\varepsilon u\right\|^2\le 2\left\|u_t+\varepsilon u\right\|^2+2\varepsilon^2\left\|u\right\|^2\le 2\left\|z\right\|^2+\frac{2\varepsilon^2}{\lambda_1}\left\|\Delta u\right\|^2,$$

Substituting (2.9) into (2.12) and then using (1.10), we can obtain

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}V\left(t\right)+\left(\frac{\varepsilon}{2}-\frac{2\mu\varepsilon^2}{\lambda_1}\right)\left\|\Delta u\right\|^2+\left(\delta_1-2\mu-\frac{4}{\varepsilon\lambda_1}\right)\left\|z\right\|^2\\&+2\varepsilon C_0\int_{\Omega}G\left(u\right)\mathrm{d}x+\varepsilon\int_0^{\left\|\nabla u\right\|^2}\phi\left(\tau\right)\mathrm{d}\tau\left\|\nabla u\right\|^2+2\varepsilon\beta\left\|\nabla u\right\|^2\\&\le\left(\frac{16}{\delta_1}-\frac{\mu}{h}\right)\int_0^h\left\|u_t\left(t-s\right)\right\|^2\mathrm{d}s+\frac{4}{\delta_1}\left\|f\right\|^2+2\varepsilon K_2.\end{aligned}$$

Furthermore, by applying (2.10), where $\mu<\frac{\delta_1}{4}$, we choose a sufficiently small ε, such that $\frac{\varepsilon}{2}-\frac{2\mu\varepsilon^2}{\lambda_1}>0$. Take $\gamma=\min\left\{\delta_1-\frac{4}{\varepsilon\lambda_1}-2\mu,\frac{\varepsilon}{2}-\frac{2\mu\varepsilon^2}{\lambda_1},\varepsilon C_0,\varepsilon,2\beta\right\}$, It follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}V(t)+\gamma V(t)\le C.$$

Applying Gronwall lemma to the above equation, we have

$$V(t)\le V(0)\mathrm{e}^{-\gamma t}+\frac{C}{\gamma}\left(1-\mathrm{e}^{-\gamma t}\right). \tag{2.12}$$

According to (1.8), (2.10), Combining with (2.12), we have

$$\|z\|^2+\|\Delta u\|^2+\int_0^{\|\nabla u\|^2}\phi(\tau)\mathrm{d}\tau\|\nabla u\|^2\le E(t)\le 4\mathrm{e}^{-\alpha t}\left(E(0)+\mu h\|\psi\|_Y^2\right)+\frac{4C}{\alpha},$$

where $C=\frac{4}{\delta_1}\|f\|^2+\gamma K_1+2\varepsilon K_2$.

2.2. Existence and Uniqueness

Let $U=(u,v)^{\mathrm{T}}$, then Equation (1.1) can be written in the following abstract form in the space $\Re=V_2\times H$:

$$\begin{cases}\frac{\mathrm{d}}{\mathrm{d}t}U(t)=LU(t)+\aleph(U(t)), & (x,y,t)\in\Omega\times[0,+\infty),\\ U(x,y,t)=\Psi(x,y,t), & (x,y,t)\in\Omega\times[-h,0],\end{cases}$$

where $\Psi=(\psi,\psi'),\psi\in Y$, the operator L is defined as

$$L\begin{pmatrix}u\\v\end{pmatrix}=\begin{pmatrix}v\\-\Delta^2u-\delta_1v+\delta_2\Delta v\end{pmatrix}$$

whose domain is

$$D(L)=\left\{(u,v)\in\Re\,|\,u\in H_*^2(\Omega),v\in H\right\},$$

$$\aleph(U)=\left(0;-g(u)-u\left(t-\pi\left[u^t\right]\right)+\phi\left(\|\nabla u\|^2\right)\Delta u+f\right)^{\mathrm{T}}.$$

Definition 2.2 [11] A mild solution of Equation (1.1) refers to a function $u\in C\left(\left[-h,T\right];D\left(A^{\frac{1}{2}}\right)\right)\cap C^1\left(\left[-h,T\right];H\right)$ defined on the interval $[0,T]$, such that $u(\theta)=\varphi(\theta)$, $\theta\in[-h,0]$ and $U(t)=\left(u(t);\partial_t u(t)\right)$ satisfies

$$U(t)=\mathrm{e}^{-tL}U(0)+\int_0^t\mathrm{e}^{-(t-s)L}\aleph(U(s))\mathrm{d}s,\ t\in[0,T].$$

Lemma 2.3 L is the infinitesimal generator of the C_0-semigroup e^{-tL} in $\Re$.

Proof Since for $U\in D(A)$

$$\langle LU,U\rangle_{\Re}=-\delta_1\|v\|^2-\delta_2\|\nabla v\|^2<0,$$

Thus, the operator L is dissipative. Note that

$$\overline{D(A)} = H_*^2 \times H.$$

Since

$$\left(H_0^1 \cap H^2\right) \times \left(H_0^1 \cap H^2\right) \subseteq D(L) \subseteq H_*^2 \times H$$

Thus, $\left(H_0^1 \cap H^2\right) \times \left(H_0^1 \cap H^2\right)$ is dense in $H_*^2 \times H$.

Next, L is maximal, that is, for any fixed $\alpha > 0$, it is necessary to prove that $\alpha I - L$ is surjective. To this end, given $(k,l) \in \Re$, we seek the following system of equations

$$(\alpha I - L)\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} k \\ l \end{pmatrix},$$

that is, we verify the following system of equations

$$\begin{cases} \alpha u - v = k, \\ \alpha v + \Delta^2 u + \delta_1 v - \delta_2 \Delta u = l. \end{cases} \tag{2.13}$$

admits a unique solution $U = (u,v)^{\mathrm{T}} \in D(L)$. From the first equation, we obtain $v = \alpha u - k$, substituting it into the second equation, we have

$$\alpha^2 u + \Delta^2 u + \delta_1 \alpha u - \delta_2 \alpha \Delta u = \alpha k + l + \delta_1 k. \tag{2.14}$$

Then Problem (2.14) can be rephrased as

$$\int_\Omega \left(\alpha^2 u + \Delta^2 u + \delta_1 \alpha u - \delta_2 \alpha \Delta u\right) \omega \mathrm{d}x = \int_\Omega \left(\alpha k + l + \delta_1 k\right) \omega \mathrm{d}x, \ \forall \omega \in \text{Ł}(\Omega). \tag{2.15}$$

Define the following bilinear operator and linear operator

$$B(u,\omega) = \int_\Omega \left(\alpha^2 u + \Delta^2 u + \delta_1 \alpha u - \delta_2 \alpha \Delta u\right) \omega \mathrm{d}x, \ F(\omega) = \int_\Omega \left(\alpha k + l + \delta_1 k\right) \omega \mathrm{d}x.$$

From the proof results of Theorem 2.1, it is evident that B is coercive and bounded, and F is bounded. Thus, by the *Lax-Milgram* lemma, it can be guaranteed that there exists a unique solution $U = (u,v)^{\mathrm{T}} \in D(L)$ to (2.15). According to the *Lumer-Phillips* theorem in Reference [14], it follows that L is an infinitesimal generator in $\Re$

Theorem 2.4 Suppose that Conditions (1.3)-(1.7) and (1.10) hold. Then, for any initial values $\varphi_i \in Y, i = 1,2$, there exists $0 < T_\varphi < \infty$, such that Problem (1.1) has a unique mild solution $U(t) \equiv \left(u(t); \partial_t u(t)\right)$ on the interval $\left[-h, T_\varphi\right]$.

Proof Fix a constant $\sigma > 0$, and define the ball $B_\sigma = \left\{U \in C([0,T];\Re) : \left\|U - \bar{V}\right\|_{C([0,T];\Re)} \le \sigma\right\}$, where $\bar{V} = \mathrm{e}^{-tL}\varphi(0)$. Define the mapping $M : C([0,T];\Re) \to C([0,T];\Re)$:

$$[MU](t) = \bar{V}(T) + \int_0^t \mathrm{e}^{-(t-s)L} \aleph(U(s)) \mathrm{d}s, \ t \in [0,T].$$

If U is a fixed point of the mapping M, then U is a mild solution to equation (3.0.1) on $[0,T]$. Next, we prove that M is a contraction mapping.

1) For any $\forall t \in [0,T]$, $U_1, U_2 \in B_\sigma$, we have

$$\begin{aligned}
&\left\|[MU_1](t)-[MU_2](t)\right\|_{C([0,T];\mathfrak{R})}\\
&\le \int_0^t \left\|\mathrm{e}^{-(t-s)L}\left(g\left(u_2(s)\right)-g\left(u_1(s)\right)\right)\right\|_{C([0,T];H)}\mathrm{d}s\\
&+\int_0^t \left\|\mathrm{e}^{-(t-s)L}\left(u\left(s-\pi\left[u_2^s\right]\right)-u\left(s-\pi\left[u_1^s\right]\right)\right)\right\|_{C([0,T];H)}\mathrm{d}s\\
&+\int_0^t \left\|\mathrm{e}^{-(t-s)L}\left(\phi\left(\|\nabla u_1\|^2\right)\Delta u_1-\phi\left(\|\nabla u_2\|^2\right)\Delta u_2\right)\right\|_{C([0,T];H)}\mathrm{d}s\\
&\le \int_0^t \left\|g\left(u_2(s)\right)-g\left(u_1(s)\right)\right\|_{C([0,T];H)}\mathrm{d}s\\
&+\int_0^t \left\|u\left(s-\pi\left[u_2^s\right]\right)-u\left(s-\pi\left[u_1^s\right]\right)\right\|_{C([0,T];H)}\mathrm{d}s\\
&+\int_0^t \left\|\phi\left(\|\nabla u_1\|^2\right)\Delta u_1-\phi\left(\|\nabla u_2\|^2\right)\Delta u_2\right\|_{C([0,T];H)}\mathrm{d}s,
\end{aligned}\tag{2.16}$$

According to Condition (1.4), combined with (2.1) and the Sobolev embedding theorem, there exists a constant $K_3>0$, such that

$$\left|g(u)\right|_{L^\infty}\le K_3,\ \left|g'(u)\right|_{L^\infty}\le K_3.\tag{2.17}$$

The following estimate applies the Mean Value Theorem for Differentiation, first,

$$\left\|g\left(u_2(s)\right)-g\left(u_1(s)\right)\right\|\le K_3\left\|u_2-u_1\right\|\le\frac{K_3}{\lambda_1}\left\|A^{\frac{1}{2}}\left(u_2-u_1\right)\right\|,$$

$$\left|\phi\left(\|\nabla u_n\|^2\right)-\phi\left(\|\nabla u\|^2\right)\right|\le C\left\|\nabla u_n-\nabla u\right\|^2.$$

From the previous estimates, it follows that $\|\nabla u_n-\nabla u\|^2$ is bounded and $\phi\left(\|\nabla u\|^2\right)$ is bounded, combining this with the above equation, we obtain that $\phi'\left(\|\nabla u\|^2\right)$ is bounded. Next, there exist constants K_4, K_5 and C_3 such that

$$\begin{aligned}
&\left\|\phi\left(\|\nabla u_1\|^2\right)\Delta u_1-\phi\left(\|\nabla u_2\|^2\right)\Delta u_2\right\|\\
&\le\left\|\phi\left(\|\nabla u_1\|^2\right)\left(\Delta u_1-\Delta u_2\right)\right\|+\left\|\phi\left(\|\nabla u_1\|^2\right)-\phi\left(\|\nabla u_2\|^2\right)\Delta u_2\right\|\\
&\le K_4\left\|\Delta u_1-\Delta u_2\right\|+\left\|K_5\left(\|\nabla u_1\|^2-\|\nabla u_2\|^2\right)\Delta u_2\right\|\\
&\le K_4\left\|\Delta u_1-\Delta u_2\right\|+K_5C(R)\left(\left\|\nabla u_1-\nabla u_2\right\|\right)\\
&\le C_3\left\|A^{\frac{1}{2}}\left(u_2-u_1\right)\right\|.
\end{aligned}$$

Given that $U_i\in B_\sigma$ and $\left\|U_i-\bar{V}\right\|_{C([0,T];\mathfrak{R})}\le\sigma$, we have

$$\begin{aligned}
\left\|U_i\right\|_{C([0,T];\mathfrak{R})}&=\max_{t\in[0,T]}\left(\left\|A^{\frac{1}{2}}u_i(t)\right\|+\left\|\partial_t u_i(t)\right\|\right)\\
&\le\sigma+\left\|\bar{V}\right\|_{C([0,T];\mathfrak{R})}\\
&\le\sigma+\max_{t\in[0,T]}\left(\left\|A^{\frac{1}{2}}\mathrm{e}^{-tA}\beta(0)\right\|+\left\|\mathrm{e}^{-tA}\beta_t\varphi(0)\right\|\right)\\
&\le\sigma+\left(\left\|A^{\frac{1}{2}}\beta(0)\right\|+\left\|\partial_t\beta(0)\right\|\right)\triangleq\tilde{N},
\end{aligned}\tag{2.18}$$

so $\left\|A^{\frac{1}{2}}u_i\left(t\right)\right\| \le \tilde{N}, t \in \left[0,T\right], i = 1,2$. Thus

$$\begin{aligned}&\left\|g\left(u_2\left(s\right)\right) - g\left(u_1\left(s\right)\right)\right\|_{C\left(\left[0,T\right];H\right)}\\&\le \frac{K_3}{\lambda_1}\max_{t\in\left[\tau,T\right]}\left\|A^{\frac{\alpha}{2}}\left(u_2 - u_1\right)\right\| \le Z_R\left\|U_2 - U_1\right\|_{C\left(\left[\tau,T\right];\mathfrak{R}\right)}.\end{aligned} \tag{2.19}$$

and

$$\begin{aligned}&\left\|\phi\left(\left\|\nabla u_1\right\|^2\right)\Delta u_1 - \phi\left(\left\|\nabla u_2\right\|^2\right)\Delta u_2\right\|_{C\left(\left[0,T\right];H\right)}\\&\le C_3\max_{t\in\left[\tau,T\right]}\left\|A^{\frac{1}{2}}\left(u_2 - u_1\right)\right\| \le Z_{\tilde{N}}\left\|U_2 - U_1\right\|_{C\left(\left[\tau,T\right];\mathbb{R}\right)}\end{aligned} \tag{2.20}$$

From (2.16), for any $\forall 0 \le s \le T$,

$$\begin{aligned}\left\|u_i^s\right\|_Y &= \max_{\kappa\in\left[-h,0\right]}\left\|A^{\frac{1}{2}}u_i^s\left(\kappa\right)\right\| + \max_{\kappa\in\left[-h,0\right]}\left\|\partial_t u_i^s\left(\kappa\right)\right\|\\&= \max_{a\in\left[s-h,s\right]}\left\|A^{\frac{1}{2}}u_i\left(a\right)\right\| + \max_{a\in\left[s-h,s\right]}\left\|\partial_t u_i\left(a\right)\right\|\\&\le \max_{a\in\left[-h,T\right]}\left\|A^{\frac{1}{2}}u_i\left(a\right)\right\| + \max_{a\in\left[-h,T\right]}\left\|\partial_t u_i\left(a\right)\right\|\\&= \max_{a\in\left[-h,0\right]}\left\|A^{\frac{1}{2}}u_i\left(a\right)\right\| + \max_{a\in\left[0,T\right]}\left\|A^{\frac{1}{2}}u_i\left(a\right)\right\|\\&\quad + \max_{a\in\left[-h,0\right]}\left\|\partial_t u_i\left(a\right)\right\| + \max_{a\in\left[0,T\right]}\left\|\partial_t u_i\left(a\right)\right\|\\&\le 2\left\|\varphi\right\|_Y + 2\max_{a\in\left[0,T\right]}\left(\left\|A^{\frac{1}{2}}u_i\left(a\right)\right\| + \left\|\partial_t u_i\left(a\right)\right\|\right)\\&\le 2\left\|\varphi\right\|_Y + 2\left\|U_i\right\|_{C\left(\left[\tau,T\right];\mathfrak{R}\right)}\\&\le 2\left\|\varphi\right\|_Y + 2\tilde{N} \triangleq \hat{N}.\end{aligned} \tag{2.21}$$

From $u\left(t - \pi\left[u^t\right]\right) = u\left(t\right) - \int_{t-\pi\left[u^t\right]}^{t}\partial_t u\left(s\right)\mathrm{d}s$, combined with Condition (1.6) and Reference [9], we obtain

$$\left\|u_2\left(s - \pi\left[u_2^s\right]\right) - u_1\left(s - \pi\left[u_1^s\right]\right)\right\| \le \left(\hat{N}\cdot L_N + 1\right)\left\|u_2^s - u_1^s\right\|_Y,$$

$$\begin{aligned}\left\|u_2^s - u_1^s\right\|_Y \le& \max_{a\in\left[s-h,0\right]}\left(\left\|A^{\frac{1}{2}}\left(u_2\left(a\right) - u_1\left(a\right)\right)\right\| + \left\|\partial_t u_2\left(a\right) - \partial_t u_1\left(a\right)\right\|\right)\\&+ \max_{a\in\left[0,s\right]}\left(\left\|A^{\frac{1}{2}}\left(u_2\left(a\right) - u_1\left(a\right)\right)\right\| + \left\|\partial_t u_2\left(a\right) - \partial_t u_1\left(a\right)\right\|\right)\\\le& 2\left\|U_2 - U_1\right\|_{C\left(\left[\tau,T\right];\mathfrak{R}\right)}.\end{aligned}$$

Through the above inequality, it holds that

$$\left\|u_2\left(s - \pi\left[u_2^s\right]\right) - u_1\left(s - \pi\left[u_1^s\right]\right)\right\|_{C\left(\left[\tau,T\right];\mathfrak{R}\right)} \le 2\left(\hat{N}\cdot L_N + 1\right)\left\|U_2 - U_1\right\|_{C\left(\left[\tau,T\right];\mathfrak{R}\right)}. \tag{2.22}$$

Substituting (2.19), (2.20) and (2.22) into (2.16), we obtain

$$\begin{aligned}\left\|[MU_1](t)-[MU_2](t)\right\|_{C([0,T];\mathfrak{R})} &\le \int_0^t \left(2Z_{\tilde{N}}+2\left(\hat{N}\cdot L_N+1\right)\right)\left\|U_2-U_1\right\|_{C([\tau,T];\mathfrak{R})} \\ &\le T\cdot\left(2Z_{\tilde{N}}+2\left(\hat{N}\cdot L_N+1\right)\right)\left\|U_2-U_1\right\|_{C([\tau,T];\mathfrak{R})},\end{aligned}$$

Choose T ufficiently small, such that $T\cdot\left(2Z_{\tilde{N}}+\left(\hat{N}\cdot L_R+1\right)\right)<1$.

(2) For any $\forall t\in[0,T]$ and $z\in B_\sigma$, combined with (2.17)-(2.21) we have

$$\begin{aligned}&\left\|[MU](t)-\bar{V}(t)\right\|_{C([0,T];\mathfrak{R})} \\ &\le \left\|\int_0^t \mathrm{e}^{-(t-s)L}\left(-g\left(u(s)\right)-u\left(s-\pi\left[u^s\right]\right)\right)\right\|_{C([0,T];H)}\mathrm{d}s+\phi\left(\|\nabla u\|^2\right)\Delta u+f \\ &\le \int_0^t\left(\left\|g\left(u(s)\right)\right\|_{C([0,T];H)}+\left\|u\left(s-\pi\left[u^s\right]+\phi\left(\|\nabla u\|^2\right)\Delta u\right)\right\|_{C([0,T];H)}+\|f\|_{C([0,T];H}\right)\mathrm{d}s \\ &\le \int_0^t\left(2Z_{\tilde{N}}+2\left(\hat{N}\cdot L_N+1\right)\right)\|U\|_{C([0,T];\mathfrak{R})}\mathrm{d}s+\|f\|_{C([0,T];\mathfrak{R})} \\ &\le T\cdot\left(2Z_{\tilde{N}}+2\left(\hat{N}\cdot L_N+1\right)\right)\cdot\tilde{N}+\left\|f(0)\right\|_{C([0,T];\mathfrak{R})},\end{aligned}$$

Choose an appropriate T, such that $T\cdot\left(2Z_{\tilde{N}}+2\left(\hat{N}\cdot L_R+1\right)\right)\tilde{N}\le\sigma$. From (1) and (2), it follows that $M:B_\sigma\to B_\sigma$ contraction mapping. According to the Banach Contraction Fixed Point Theorem, there exists a unique fixed point $U\in C\left([0,T];\mathfrak{R}\right)$. Let

$$\bar{u}=\begin{cases}u(t), & t\in[0,T],\\ \varphi(t), & t\in[-h,0],\end{cases}$$

and $\bar{u}\in C\left([-h,T];D\left(A^{\frac{1}{2}}\right)\right)\cap C\left([-h,T];H\right)$, therefore, $\bar{u}$ is a mild solution to Equation (3.0.1) on the interval $[-h,T]$.

Theorem 2.5 Suppose Conditions (1.3)-(1.7) and (1.10) hold. Then, for any initial values $\varphi_i\in Y$, $\|\varphi\|_Y\le\varpi$, $i=1,2$, there exists $0<T_\varphi<\infty$, such that Problem (1.1) has a unique global mild solution $U(t)\equiv\left(u(t);\partial_t u(t)\right)$. on the interval $[0,+\infty)$. Furthermore, for any $\varpi>0$, $T>0$, there exists a positive constant C, such that

$$\max_{a\in[0,t]}\left(\left\|A^{\frac{1}{2}}u(a)\right\|^2+\left\|\partial_t u(a)\right\|^2\right)\le C\left(1+E_1(0)+\|\varphi\|_Y^2+\|f\|^2\right)\mathrm{e}^{\gamma_1 t}.$$

Proof Taking the inner product of u_t with Equation (1.1) in H, we have

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|u_t\|^2+\|\Delta u\|^2+\int_0^{\|\nabla u\|^2}\phi(\tau)\mathrm{d}\tau\cdot\|\nabla u\|^2+2\int_\Omega G(u)\mathrm{d}x\right) \\ &+\delta_1\|u_t\|^2+\delta_2\|\nabla u_t\|^2=(f,u_t)-\left(u\left(t-\pi\left[u^t\right]\right),u_t\right).\end{aligned} \tag{2.23}$$

Let

$$E_1(t)=\|u_t\|^2+\|\Delta u\|^2+\int_0^{\|\nabla u\|^2}\phi(\tau)\mathrm{d}\tau\cdot\|\nabla u\|^2+2\int_\Omega G(u)\mathrm{d}x\ge 0.$$

Integrating Equation (2.23) over the interval $[0,t]$, we obtain

$$\begin{aligned} &E_1(t)+\delta_1\int_0^t\|u_t(s)\|^2\,\mathrm{d}s+\delta_2\int_0^t\|\nabla u_t(s)\|^2\,\mathrm{d}s \\ &\le E_1(0)+\frac{2t}{\delta_1}\|f\|^2+\frac{2}{\delta_1}\int_0^t\|u^s\|_Y^2\,\mathrm{d}s. \end{aligned} \tag{2.24}$$

Through Equation (2.21), for any $\forall s\in[0,T_\psi)$,

$$\|u_i^s\|_Y\le\|\varphi\|_Y+2\sqrt{\max_{a\in[0,s]}\left(\left\|A^{\frac{1}{2}}u(a)\right\|^2+\|\partial_t u(a)\|^2\right)}. \tag{2.25}$$

Substituting (2.25) into (2.24), we obtain

$$\begin{aligned} &E_1(t)+\delta_1\int_0^t\|u_t(s)\|^2\,\mathrm{d}s+\delta_2\int_0^t\|\nabla u_t(s)\|^2\,\mathrm{d}s \\ &\le E_1(0)+\frac{2t}{\delta_1}\|f\|^2+\frac{2t}{\delta_1}\|\varphi\|_Y^2+\frac{8}{\delta_1}\int_0^t\max_{a\in[0,s]}\left(\left\|A^{\frac{1}{2}}u(a)\right\|^2+\|\partial_t u(a)\|^2\right)\mathrm{d}s. \end{aligned} \tag{2.26}$$

From $E_1(t)=\|u_t\|^2+\|\Delta u\|^2+\int_0^{\|\nabla u\|^2}\phi(\tau)\,\mathrm{d}\tau\cdot\|\nabla u\|^2+2\int_\Omega H(u)\,\mathrm{d}x$, we have

$$\max_{a\in[0,t]}\left(\left\|A^{\frac{1}{2}}u(a)\right\|^2+\|\partial_t u(a)\|^2\right)\le E_1(t).$$

Substituting the above equation into (2.26), we obtain

$$\begin{aligned} &\max_{a\in[0,t]}\left(\left\|A^{\frac{1}{2}}u(a)\right\|^2+\|\partial_t u(a)\|^2\right) \\ &\le C\left(E_1(0)+t\|f\|^2+t\|\varphi\|_Y^2+\int_0^t\max_{a\in[0,s]}\left(\left\|A^{\frac{1}{2}}u(a)\right\|^2+\|\partial_t u(a)\|^2\right)\mathrm{d}s\right), \end{aligned}$$

where $C>0$. Applying the integral form of *Gronwall's* Lemma to the above equation, for $t<T_\varphi$,

$$\max_{a\in[0,t]}\left(\left\|A^{\frac{1}{2}}u(a)\right\|^2+\|\partial_t u(a)\|^2\right)\le C\left(1+E_1(0)+\|\varphi\|_Y^2+\|f\|^2\right)\mathrm{e}^{\gamma_1 t},$$

where $\gamma_1>0$. $\forall T\ge 0$, the above equation holds identically on $[0,T_\varphi)\subset[0,T)$, therefore, the solution to Equation (1.1) can be extended to the interval $[0,+\infty)$. Thus, the continuous dependence of the solution on the initial value and the uniqueness of the solution are proved.

3. Attractors for Suspension Bridge Equations with State-Dependent Delays

3.1. Existence of Global Attractors

According to Theorem 2.5, define a semigroup $\{S_t:Y\to Y\}$, that is, for any $t\ge 0$, $S_t\varphi=u^t$, where $u(t)$ the mild solution to Equation (1.1) and satisfies $u^0=\varphi$. Denote $\{S_t,Y\}$ as the dynamical system generated by the solution semigroup corresponding to Equation (1.1).

Lemma 3.1 (Dissipativity) Suppose Conditions (1.3)-(1.7) and (1.10) hold, and

$f\in L^2(\Omega)$. Then, for any α_0, there exists $h_0=h(\alpha_0)$ such that for each $(\alpha,h)\in[\alpha_0,+\infty)\times(0,h_0]$, the dynamical system $\{S_t,Y\}$ is dissipative. That is, for any $\rho>0$, there exists $R>0$, such that

$$\|S_t\varphi\|_Y\le R,\ \forall\varphi\in Y,\ \|\varphi\|_Y\le\rho,\ t\ge t_\rho,$$

and for any $\mu_0>0$, the dissipative radius R is independent of the damping coefficient $\alpha\ge\alpha_0$ and the delay time $h\in(0,h_0]$.

Proof Similar to the a priori estimates in Section 2.1, we have

$$\|z\|^2+\|\Delta u\|^2+\int_0^{\|\nabla u\|^2}\phi(\tau)\,\mathrm{d}\tau\|\nabla u\|^2\le 4\mathrm{e}^{-\gamma t}\left(E(0)+\mu h\|\varphi\|_Y\right)+\frac{4C}{\gamma},\tag{3.1}$$

Now, replacing t in the above equation with $t+\theta$ (where $\theta\in[-h,0]$), the following equation holds

$$\begin{aligned}&\|z(t+\theta)\|^2+\|\Delta u(t+\theta)\|^2+\int_0^{\|\nabla u(t+\theta)\|^2}\phi(\tau)\,\mathrm{d}\tau\|\nabla u(t+\theta)\|^2\\&\le 4\mathrm{e}^{-\gamma(t-h)}\left(E(0)+\mu h\|\varphi\|_Y^2\right)+\frac{4C}{\gamma}.\end{aligned}\tag{3.2}$$

Therefore, from (3.2), we obtain

$$\begin{aligned}\|u^t\|_Y^2&=\max_{\theta\in[-h,0]}\|z(t+\theta)\|^2+\max_{\theta\in[-h,0]}\|\Delta u(t+\theta)\|^2\\&\le 2\max_{\theta\in[-h,0]}\left(\|z(t+\theta)\|^2+\|\Delta u(t+\theta)\|^2\right)\\&\le 8\mathrm{e}^{-\delta(t-h)}\left(E(0)+\mu h\|\varphi\|_Y\right)+\frac{8C}{\alpha}.\end{aligned}\tag{3.3}$$

Through the above equation, it can be concluded that there exists $t\ge t_\rho$, such that the ball $B_0=B(0,R)$ is a bounded absorbing set for the dynamical system $\{S_t,Y\}$, where $R>\dfrac{2\sqrt{2C}}{\delta}$.

Lemma 3.2 (Quasi-stability) Suppose Conditions (1.3)-(1.7) and (1.10) hold, $f\in L^2(\Omega)$. Then there exist constants $C_1(R)>0$, $C_2(R)>0$ and $\lambda>0$, such that the solutions φ_1,φ_2 to Problem (1.1) with initial values u_1,u_2, satisfy the following property:

$$\|\partial_t u_i(t)\|^2+\|\Delta u_i(t)\|^2\le R^2,\ t\ge -h,\ i=1,2,\tag{3.4}$$

and the quasi-stability estimate

$$\begin{aligned}&\|\partial_t u_1(t)-\partial_t u_2(t)\|^2+\|\Delta u_1(t)-\Delta u_2(t)\|^2\\&\le C_1(R)\mathrm{e}^{-\lambda t}\|\varphi_1-\varphi_2\|_Y^2+C_2(R)\max_{r\in[0,t]}\left\|A^{\frac{1}{2}-\epsilon}\left(u_1(r)-u_2(r)\right)\right\|^2,\end{aligned}\tag{3.5}$$

where $0<\epsilon<\dfrac{1}{2}$.

Proof Let u_1 and u_2 be two solutions to Problem (1.1). Then $\omega=u_1(t)-u_2(t)$ is a solution to the following equation

$$\begin{aligned}&\omega_{tt}+\Delta^2\omega-\phi\left(\|\nabla\omega\|^2\right)\Delta\omega+\delta_1\omega_t-\delta_2\Delta\omega_t\\&=-\left(g\left(u_1\right)-g\left(u_2\right)\right)-\left(u_1\left(t-\pi\left[u_1^t\right]\right)-u_2\left(t-\pi\left[u_2^t\right]\right)\right).\end{aligned}\tag{3.6}$$

According to Lemma 3.2, the dynamical system $\{S_t, Y\}$ is dissipative, and thus it is obvious that (3.1) holds.

Define the energy functional

$$E_\omega(t)=\frac{1}{2}\left(\|\Delta\omega\|^2+\|\omega_t\|^2+\phi\left(\|\nabla w\|^2\right)\|w\|^2\right).\tag{3.7}$$

Taking the inner product of (3.6) with $\omega_t(t)$ in H and integrating over the interval $[t,T]$ we have

$$\begin{aligned}&E_\omega(T)-E_\omega(t)+\delta_1\int_t^T\|\omega_t(s)\|^2\,\mathrm{d}s+\delta_2\int_t^T\|\nabla\omega_t(s)\|^2\,\mathrm{d}s\\&\le\int_t^T\left(g\left(u_2(s)\right)-g\left(u_1(s)\right),\omega_t(s)\right)\mathrm{d}s\\&\quad+\int_t^T\left(u_2\left(s-\pi\left[u_2^s\right]\right)-u_1\left(s-\pi\left[u_1^s\right]\right),\omega_t(s)\right)\mathrm{d}s,\end{aligned}\tag{3.8}$$

Using (2.17), we have

$$\begin{aligned}&\left|\int_\Omega\left(g\left(u_2(t)\right)-g\left(u_1(t)\right)\omega_t(t)\right)\mathrm{d}x\right|\\&\le\int_\Omega\left|g'\left(u_2+\xi\left(u_2-u_1\right)\right)\right|\left|u_1(t)-u_2(t)\right|\left|\omega_t(t)\right|\mathrm{d}x\\&\le K_3\int_\Omega\left|u_1(t)-u_2(t)\right|\left|\omega_t(t)\right|\mathrm{d}x\\&\le\frac{\varepsilon}{2}\|\Delta\omega(t)\|^2+\frac{C_R}{2\varepsilon}\|\omega_t\|^2,\end{aligned}\tag{3.9}$$

where $0<\xi<1$, $\varepsilon>0$. Using (1.7), we have

$$\begin{aligned}&\int_\Omega\left(u_2\left(t-\pi\left[u_2^t\right]\right)-u_1\left(t-\pi\left[u_1^t\right]\right),\omega_t(t)\right)\mathrm{d}x\\&\le\left\|u_2\left(t-\pi\left[u_2^t\right]\right)-u_1\left(t-\pi\left[u_1^t\right]\right)\right\|\left\|\omega_t(t)\right\|\\&\le\max_{\theta\in[-h,0]}\left\|A^{\frac{1}{2}-\epsilon}\omega(t+\theta)\right\|^2+C_R\|\omega_t(t)\|^2.\end{aligned}\tag{3.10}$$

Substituting (3.9) and (3.10) into (3.8), we obtain

$$\begin{aligned}&\left|E_\omega(T)-E_\omega(t)+\delta_1\int_t^T\|\omega_t(s)\|^2\,\mathrm{d}s+\delta_2\int_t^T\|\nabla\omega_t(s)\|^2\,\mathrm{d}s\right|\\&\le\frac{\varepsilon}{2}\int_t^T\|\Delta\omega(s)\|^2\,\mathrm{d}s+\int_t^T\max_{\theta\in[-h,0]}\left\|A^{\frac{1}{2}-\epsilon}\omega(t+\theta)\right\|^2\mathrm{d}s\\&\quad+C_R\left(1+\frac{1}{2\varepsilon}\right)\int_t^T\|\omega_t(s)\|^2\,\mathrm{d}s,\end{aligned}\tag{3.11}$$

For any $\varepsilon>0$, choose α sufficiently large such that it satisfies the following relation

$$C_R\left(1+\frac{1}{2\varepsilon}\right)<\frac{\delta_1}{2}.\tag{3.12}$$

Taking the inner product of $\omega(t)$ with (3.6) in H and integrating over the interval $[0,T]$, we obtain

$$\begin{aligned}&\left(\partial_t\omega(T),\omega(T)\right)-\left(\partial_t\omega(0),\omega(0)\right)-\int_0^T\left\|\partial_t\omega(s)\right\|^2\mathrm{d}s\\&+\int_0^T\left\|\Delta\omega(s)\right\|^2\mathrm{d}s+\delta_1\int_0^T\left(\partial_t\omega(s),\omega(s)\right)\mathrm{d}s\\&\le\frac{1}{2}\int_0^T\left\|\Delta\omega(s)\right\|^2\mathrm{d}s+\bar{C}_R\int_0^T\max_{\theta\in[-h,0]}\left\|A^{\frac{1}{2}-\epsilon}\omega(t+\theta)\right\|^2\mathrm{d}s+\bar{C}_R\int_0^T\left\|\omega(s)\right\|^2\mathrm{d}s,\end{aligned}\tag{3.13}$$

Furthermore, by the Hölder and Young inequality, we have

$$\delta_1\int_0^T\left(\partial_t\omega(s),\omega(s)\right)\mathrm{d}s\le\frac{1}{2}\int_0^T\left\|\partial_t\omega(s)\right\|^2\mathrm{d}s+\frac{\delta_1^2}{2}\int_0^T\left\|\omega(s)\right\|^2\mathrm{d}s.$$

According to the definition of the energy functional $E_\omega(t)$, we have

$$\begin{aligned}\frac{1}{2}\int_0^T\left\|\Delta\omega(s)\right\|^2\mathrm{d}s\le{}&\frac{3}{2}\int_0^T\left\|\partial_t\omega(s)\right\|^2\mathrm{d}s+C\left(E_\omega(0)+E_\omega(T)\right)\\&+\bar{C}_R(\delta_1)\int_0^T\max_{\theta\in[-h,0]}\left\|A^{\frac{1}{2}-\epsilon}\omega(t+\theta)\right\|^2\mathrm{d}s.\end{aligned}$$

Setting $t=0$ in (3.11) and combining it with (3.12), we obtain

$$\begin{aligned}E_\omega(0)\le{}&E_\omega(T)+\frac{3\delta_1}{2}\int_0^T\left\|\partial_t\omega(s)\right\|^2\mathrm{d}s+\varepsilon\int_0^T\left\|\Delta\omega(s)\right\|^2\mathrm{d}s\\&+\int_0^T\max_{\theta\in[-h,0]}\left\|A^{\frac{1}{2}-\epsilon}\omega(t+\theta)\right\|\mathrm{d}s.\end{aligned}\tag{3.14}$$

Integrating (3.11) over the interval $[0,T]$ and combining it with (3.12) we obtain

$$TE_\omega(T)\le\int_0^T E_\omega(s)\mathrm{d}s+\varepsilon T\int_0^T\left\|\Delta\omega(s)\right\|^2\mathrm{d}s+T\int_0^T\max_{\theta\in[-h,0]}\left\|A^{\frac{1}{2}-\epsilon}\omega(t+\theta)\right\|\mathrm{d}s.\tag{3.15}$$

Setting $t=0$ in (3.11) and combining it with (3.12), we can obtain

$$\frac{\delta_1}{2}\int_0^T\left\|\partial_t\omega(s)\right\|^2\mathrm{d}s\le E_\omega(0)+\varepsilon T\int_0^T\left\|\Delta\omega(s)\right\|^2\mathrm{d}s+\int_0^T\max_{\theta\in[-h,0]}\left\|A^{\frac{1}{2}-\epsilon}\omega(t+\theta)\right\|\mathrm{d}s.\tag{3.16}$$

Adding (3.13) and (3.16), and setting $\delta_1>8$,

$$\begin{aligned}&\left(\frac{\delta_1}{2}-2\right)\int_0^T\left\|\partial_t\omega(s)\right\|^2\mathrm{d}s+\int_0^T E_\omega(s)\mathrm{d}s\\&\le\varepsilon\int_0^T\left\|\Delta\omega(s)\right\|^2\mathrm{d}s+C\left(E_\omega(0)+E\omega(T)\right)+\bar{C}_R(\alpha)\int_0^T\max_{\theta\in[-h,0]}\left\|A^{\frac{1}{2}-\epsilon}\omega(t+\theta)\right\|\mathrm{d}s.\end{aligned}\tag{3.17}$$

Adding $\frac{1}{2}TE_\omega(T)$ to both sides of (3.17) and substituting (3.15) into the result, we obtain

$$\begin{aligned}&\left(\frac{\delta_1}{2}-2\right)\int_0^T\left\|\partial_t\omega(s)\right\|^2\mathrm{d}s+\frac{1}{2}\int_0^T E_\omega(s)\mathrm{d}s+\frac{1}{2}TE\omega(T)\\&\le\varepsilon(T+1)\int_0^T\left\|\Delta\omega(s)\right\|^2\mathrm{d}s+C\left(E_\omega(0)+E\omega(T)\right)\\&\quad+\bar{C}_R(\delta_1)\left(1+\frac{T}{2}\right)\int_0^T\max_{\theta\in[-h,0]}\left\|A^{\frac{1}{2}-\epsilon}\omega(t+\theta)\right\|\mathrm{d}s.\end{aligned}\tag{3.18}$$

Now, we estimate the value of $E_\omega(0)+E_\omega(T)$. From (3.14), we have

$$\begin{aligned} E_\omega(0)+E_\omega(T) &\le 2E_\omega(T)+\frac{3\delta_1}{2}\int_0^T \|\partial_t\omega(s)\|^2\,\mathrm{d}s \\ &\quad +\varepsilon\int_0^T\|\Delta\omega(s)\|^2\,\mathrm{d}s+\int_0^T\max_{\theta\in[-h,0]}\left\|A^{\frac{1}{2}-\epsilon}\omega(t+\theta)\right\|\mathrm{d}s. \end{aligned} \tag{3.19}$$

Substituting (3.19) into (3.18), one obtains

$$\begin{aligned} &\frac{1}{2}\int_0^T E_\omega(s)\,\mathrm{d}s+\left(\frac{1}{2}T-2C\right)E_\omega(T) \\ &\le(\delta_1+2)\int_0^T\|\partial_t\omega(s)\|^2\,\mathrm{d}s+\varepsilon\int_0^T\|\Delta\omega(s)\|^2\,\mathrm{d}s \\ &\quad+\overline{C'}_R(\delta_1)\left(2+\frac{T}{2}\right)\int_0^T\max_{\theta\in[-h,0]}\left\|A^{\frac{1}{2}-\epsilon}\omega(t+\theta)\right\|\mathrm{d}s. \end{aligned} \tag{3.20}$$

Assuming $\frac{1}{2}T-2C>1$, we then have

$$\begin{aligned} &E_\omega(T)+\frac{1}{2}\int_0^T E_\omega(s)\,\mathrm{d}s \\ &\le\overline{C'}_R(\delta_1)\left(2+\frac{T}{2}\right)\int_0^T\max_{\theta\in[-h,0]}\left\|A^{\frac{1}{2}-\epsilon}\omega(t+\theta)\right\|\mathrm{d}s \\ &\quad+(\delta_1+2)\int_0^T\|\partial_t\omega(s)\|^2\,\mathrm{d}s+\varepsilon(T+1)\int_0^T\|\Delta\omega(s)\|^2\,\mathrm{d}s, \end{aligned} \tag{3.21}$$

Similarly, setting $t=0$ in (3.11), we have

$$\begin{aligned} \frac{\delta_1}{2}\int_0^T\|\partial_t\omega(s)\|^2\,\mathrm{d}s &\le E_\omega(0)-E_\omega(T)+\varepsilon\int_0^T\|\Delta\omega(s)\|^2\,\mathrm{d}s \\ &\quad+\int_0^T\max_{\theta\in[-h,0]}\left\|A^{\frac{1}{2}-\epsilon}\omega(t+\theta)\right\|\mathrm{d}s, \end{aligned} \tag{3.22}$$

Substituting the above expression into (3.21), we can obtain

$$\begin{aligned} &E_\omega(T)+\frac{1}{2}\int_0^T E_\omega(s)\,\mathrm{d}s \\ &\le C_{\delta_1}\left(E_\omega(0)-E_\omega(T)\right)+2C_{\delta_1}\varepsilon(T+1)\varepsilon\int_0^T\|\Delta\omega(s)\|^2\,\mathrm{d}s \\ &\quad+C_\mu\overline{C'}_R\left(2+\frac{T}{2}\right)\int_0^T\max_{\theta\in[-h,0]}\left\|A^{\frac{1}{2}-\epsilon}\omega(t+\theta)\right\|\mathrm{d}s, \end{aligned}$$

where $C_{\delta_1}>0$ denotes a constant dependent on δ_1. According to the definition of $E_\omega(t)$, we have $\|\Delta\omega(s)\|^2\le 2E_\omega(s)$, Choosing $\varepsilon>0$ sufficiently small, such that

$$E_\omega(T)\le\frac{C_{\delta_1}}{1+C_{\delta_1}}E_\omega(0)+\overline{C'}_R(T,\delta_1)\left(2+\frac{T}{2}\right)\int_0^T\max_{\theta\in[-h,0]}\left\|A^{\frac{1}{2}-\epsilon}\omega(t+\theta)\right\|\mathrm{d}s,$$

Obviously, $\omega=\frac{C_{\delta_1}}{1+C_{\delta_1}}<1$, thus, there exists a constant $\eta>0$ such that

$$E_\omega(T)\le\mathrm{e}^{-\eta t}E_\omega(0)+\overline{C'}_R(T,\delta_1)\left(2+\frac{T}{2}\right)\int_0^T\max_{\theta\in[-h,0]}\left\|A^{\frac{1}{2}-\epsilon}\omega(t+\theta)\right\|\mathrm{d}s. \tag{3.23}$$

By applying Reference ([14], Remark 3.30) and repeating the steps for the interval $(mT \to (m+1)T)$, we can derive from the relation (3.23) that the conclusion (3.4) holds.

Theorem 3.3 (Global Attractor) Assume that conditions (1.3)-(1.7) and (1.10) hold. Then the dynamical system $\{S_t, Y\}$ generated by Problem (1.1) possesses a compact global attractor with finite fractal dimension.

Proof It follows from Lemma 3.1 that the dynamical system $\{S_t, Y\}$ is dissipative, Furthermore, by Lemma 3.2, the dynamical system $\{S_t, Y\}$ is quasi-stable on any positively invariant bounded set B. According to the proof of Theorem 13 in Reference [14], if the generated system $\{S_t, Y\}$ satisfies $\varphi \in C([-h,0];V_2) \cap C^1([-h,0];H)$, and $\{S_t, Y\}$ is quasi-stable on every positively invariant set B in Y, then $\{S_t, Y\}$ is asymptotically smooth. Thus, we obtain the existence of a compact global attractor.

3.2. Fractal Dimension of Global Attractors

This section considers the fractal dimension of the global attractor, and an auxiliary space is introduced.

$$Y(-h,T) = C([-h,T];V_2) \cap C^1([-h,T];H), T > 0,$$

A norm is endowed on it

$$\|\varphi\|_{Y(-h,T)} = \max_{s\in[-h,T]} \|\Delta\varphi(s)\| + \max_{s\in[-h,T]} \|\partial_t \varphi(s)\|.$$

Moreover, when $T = 0$, it holds that $Y(-h,T) = Y$. Therefore, the space $Y(-h,T)$ is an extension of the space Y.

Let Φ be a set in the phase space Y. Denote by Φ_T the set of functions $u \in Y(-h,T)$, where u is a solution to Equation (1.1) corresponding to the initial value $u^t\big|_{t\in[-h,0]} = \varphi \in \Phi$. Define the translation operator $S_T : \Phi_T \mapsto Y(-h,T)$, $(S_T u)(t) = u(T+t)$, $t \in [-h,T]$.

Lemma 3.4 Let Φ be a forward-invariant set of the dynamical system $\{S_t, Y\}$, where for $R > 0$, $\Phi \in \{\varphi : \|\varphi\|_Y \le R\}$. Suppose $T > h$, then Φ_T is forward-invariant with respect to the translation operator, and for any $\varphi_1, \varphi_2 \in \Phi_T$, we have

$$\begin{aligned} \|S_t\varphi_1 - S_t\varphi_2\|_{Y(-h,T)} &\le c_1(R)\mathrm{e}^{-\varsigma(T-h)}\|\varphi_1 - \varphi_2\|_{Y(-h,T)} \\ &\quad + c_2(R)\left[n(\varphi_1 - \varphi_2) + n(\Re_T\varphi_1 - \Re_T\varphi_2)\right], \end{aligned} \tag{3.24}$$

where $n(\varphi) = \sup_{s\in[0,T]} \left\| A^{\frac{1}{2}-\epsilon} \varphi(s) \right\|$ is a compact seminorm on the space $Y(-h,T)$.

Choose an appropriate $T > h$, such that $\zeta_T = c_1(R)\mathrm{e}^{-\varsigma(T-h)} < 1$, and set $\Phi = \Lambda$, where Λ is the global attractor. It is obvious that the set Λ_T is strictly positively invariant. Thus, we can obtain that the set Λ_T has finite dimension in the space $Y(-h,T)$. Consider the restriction mapping

$$r_h : u(t), t \in [-h,T] \mapsto u(t), t \in [-h,0],$$

and it is obvious that r_h is Lipschitz continuous from $Y(-h,T)$ to Y. Since

$r_h\Lambda_T = \Lambda$ and a Lipschitz mapping cannot increase the fractal dimension of a set, the following conclusion holds,

$$\dim_f^Y \Lambda \le \dim_f^{Y(-h,T)} \Lambda_T < \infty.$$

Thus, the fractal dimension of the global attractor is finite.

4. Conclusion and Suggestion

This paper considers the dynamic behavior of the suspension bridge equation with state delays. Compared with constant delays or time-varying delays, the state delay case exhibits higher complexity, which makes it rather challenging to verify the existence and uniqueness of solutions. To address this issue, by selecting an appropriate phase space, this paper employs the semigroup theory of operators and the Banach fixed point theorem to prove the existence and uniqueness of local solutions. Furthermore, the existence of the global attractor along with its fractal dimension, as well as the existence of the generalized exponential attractor, is established. Beyond the results presented in this work, there remain numerous interesting open problems worthy of further investigation regarding the topic discussed herein: This paper only discusses the existence of the exponential attractor for solutions to the single suspension bridge equation. Future research could further investigate the dynamical behavior of coupled suspension bridge equations. Additionally, it remains to be explored whether the boundary conditions of the suspension bridge equation discussed in this paper can be replaced with other mixed boundary conditions.

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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