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Global Attractors for Nonclassical Diffusion Equations with State Delay

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Abstract

This paper first uses the Banach fixed point theorem and operator semigroup theory to prove the existence and uniqueness of mild solutions for non-classical reaction diffusion equations with state delays and their continuous dependence on the initial values. Secondly, by combining quasi-stability, the asymptotic smoothness of the relevant semigroup is obtained. Finally, the existence of a global attractor with a finite fractal dimension is obtained.

Keywords

Non-Classical Diffusion Equation, State Delay, Global Attractor

1. Introduction

This paper studies the existence of a global attractor for the nonclassical reaction-diffusion equation with state-dependent delay

$$\begin{cases} \partial_t u - \Delta\partial_t u - \Delta u + \mu u + f(u) + u\left(t - \eta\left[u^t\right]\right) = \sigma(x), & x \in \Omega, t > 0, \\ u(x,t) = 0, & x \in \partial\Omega, t \ge -h, \\ u(x,t) = \varphi(x,t), & x \in \Omega, t \in [-h,0], \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n (n \ge 3)$ is a bounded domain with smooth boundary, $\mu > 0$, $h > 0$ denotes the maximum delay time, $u^t = u(t+\theta)$ for $\theta \in [-h,0]$, and the external force term $\sigma \in L^2(\Omega)$. We assume the following conditions hold:

(H1) The nonlinear term $f \in C^2(\mathbb{R},\mathbb{R})$ satisfies the following conditions

$$\liminf_{|s|\to\infty} \frac{F(s)}{s^2} \ge 0, \ F(s) = \int_0^s f(\tau)\,\mathrm{d}\tau, \ \forall s \in \mathbb{R}; \tag{1.2}$$

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$$\liminf_{|s|\to\infty} \frac{sf(s) - C_0 F(s)}{s^2} \geq 0, \ C_0 > 0, \forall s \in \mathbb{R}; \tag{1.3}$$

$$|f'(s)| \leq C(1 + |s|^p), \ \forall 0 < p \leq 2, \forall s \in \mathbb{R}. \tag{1.4}$$

(H2) For any $\varrho > 0$, exists $C_\varrho > 0$, such that

$$\|u\|^2 \leq \varrho \left(\left\| A^{\frac{1}{2}} u \right\|^2 + \int_\Omega F(u) \mathrm{d}x \right) + C_\varrho, \ \forall u \in H_0^1(\Omega). \tag{1.5}$$

(H3) Let the mapping $\eta : \mathcal{C} \to [0, h]$ is locally Lipschitz that for any $R > 0$, there $L_R > 0$, for any $\varphi_1, \varphi_2 \in \mathcal{C}$, $\|\varphi_i\|_{\mathcal{C}} \leq R$, $i = 1, 2$, to set up

$$|\eta(\varphi_1) - \eta(\varphi_2)| \leq L_R \|\varphi_1 - \varphi_2\|_{\mathcal{C}}. \tag{1.6}$$

To obtain the compactness of the semigroup, we further assume that there exists $\epsilon > 0$ such that the delay term satisfies the subcritical local Lipschitz condition. Specifically, for any $\rho > 0$, there exists $L_\rho > 0$, such that for all $\varphi_i, i = 1, 2$ with $\|\varphi_i\|_{\mathcal{C}} \leq \rho$, the following holds

$$|\eta(\varphi_1) - \eta(\varphi_2)| \leq L_\rho \max_{\theta \in [-h, 0]} \left\| A^{\frac{1}{2} - \epsilon} (\varphi_1(\theta) - \varphi_2(\theta)) \right\|. \tag{1.7}$$

where $0 < \epsilon \leq \frac{1}{2}$. (In the subsequent energy estimation, ϵ needs to be sufficiently small to control the perturbations caused by high-order terms (such as nonlinear terms and time-delay terms), thereby ensuring that the energy decay term dominates.)

According to assumption (H1) and Poincaré's inequalities, there exist constants $K_1, K_2 > 0$ such that

$$\int_\Omega F(u) \mathrm{d}x + \frac{1}{8} \|\nabla u\|^2 \geq -K_1, \ \forall u \in H_0^1(\Omega), \tag{1.8}$$

$$(f(u), u) - C_0 \int_\Omega F(u) \mathrm{d}x + \frac{1}{8} \|\nabla u\|^2 \geq -K_2, \ \forall u \in H_0^1(\Omega). \tag{1.9}$$

In recent years, the long-time behavior of solutions to non-classical diffusion equations without delay has been extensively studied. For instance, in [1], the authors proved the existence of global attractors in $H_0^1(\Omega)$ under the condition that the nonlinear term satisfies subcritical growth. In [2], the compactness of the semigroup $S(t)$ in $H_0^1(\Omega)$ is obtained via the method of asymptotic a priori estimates, thereby proving the existence of a global attractor for a class of nonclassical diffusion equations. In [3], the authors employed a novel asymptotic contraction semigroup method to prove the existence of a global attractor in $H^1(\mathbb{R}^n)$ for the nonclassical diffusion equation, where the nonlinear term satisfies a polynomial growth condition of arbitrary order and the external force term g only belongs to $H^{-1}(\mathbb{R}^n)$. For research on attractors with delays, please refer to [4]-[9] and other related works. Among them, [4] investigates the existence of a pullback attractor for the delayed nonclassical diffusion equation, where the nonlinear term exhibits both critical growth and arbitrary-order growth. In Reference [9], the authors prove the existence and uniqueness of strong solutions, as well as the existence of a pullback attractor, for the classical reaction-diffusion equation with state-dependent delays by using the standard Fadeo-Galerkin approximation method.

In this paper, we focus on the existence of global attractors for autonomous non-classical diffusion with state-dependent delays. Compared with constant delays or time-dependent delays, state-dependent delays bring new difficulties in analysis, including adaptability and corresponding prior estimates. Therefore, the results for systems with state-dependent delays are not as rich as those for other types of delay differential equations. For problem (1.1), due to the existence of state time delay, we need to select an appropriate phase space. Moreover, when proving the prior estimation, an additional term is required in the energy functional as compensation for the time delay term. Finally, the existence of the global attractor of the system is proved by using the quasi-stable method.

The structure of this article is as follows. In Section 2, we define some abstract results regarding the global attractor. In Section 3, we conduct a prior estimation and establish the conformity of the problem (1.1). In Section 4, we prove the existence of the global attractor of (1.1).

2. Preliminary Knowledge

Let $H = L^2(\Omega)$, where its norm and inner product are denoted by $\|\cdot\|$ and $(\cdot,\cdot)$. Let $A = -\Delta$, then A is positive definite operator with a discrete spectrum on H. For any $\alpha \in \mathbb{R}$, define the space $H^\alpha = D\left(A^{\frac{\alpha}{2}}\right)$, whose inner product and norm are given by

$$(u,v)_\alpha = \left(A^{\frac{\alpha}{2}}u, A^{\frac{\alpha}{2}}v\right);\ \|u\|_\alpha = \left\|A^{\frac{\alpha}{2}}u\right\|.$$

In particular, we denote $H^0 = H = L^2(\Omega)$, $H^1 = D\left(A^{\frac{1}{2}}\right) = H_0^1(\Omega)$, and $H^2 = D(A) = H^2(\Omega) \cap H_0^1(\Omega)$. Additionally, $|\cdot|$ denotes the absolute value. Let $C_\alpha = \left([-h,0]; H^\alpha\right)$, and define the following norm

$$|\varphi|_{\mathcal{C}} = \sup_{\theta\in[-h,0]} \|\varphi(\theta)\|,\ \forall \varphi \in \mathcal{C}, \mathcal{C} = \mathcal{C}_0;$$

$$|\varphi|_{\mathcal{C}_1} = \sup_{\theta\in[-h,0]} \left(\|\varphi(\theta)\| + \left\|A^{\frac{1}{2}}\varphi(\theta)\right\|\right) \sim \sup_{\theta\in[-h,0]} \left\|A^{\frac{1}{2}}\varphi(\theta)\right\|,\ \forall \varphi \in \mathcal{C}_1.$$

Theorem 2.1. [10] If the dynamical system $(S(t), \mathcal{H})$ is quasi-stable on every bounded positively invariant set $B \subseteq \mathcal{H}$, then $(S(t), \mathcal{H})$ is asymptotically smooth.

Theorem 2.2. [10] A dissipative dynamical system $(S(t), \mathcal{C}_1)$ possesses a compact global attractor if and only if it is asymptotically smooth.

3. Well-Posedness of Solutions

3.1. Prior Estimation

To establish the well-posedness of solutions to Problem (1.1), we first make the following a priori estimates.

Lemma 3.1. Suppose that conditions (H1)-(H3) hold and $\sigma \in L^2(\Omega)$. Then the solution to Problem (1.1) satisfies the following estimate

$$\|\nabla u\|^2 + \|u\|^2 \le 4e^{-\gamma t}\left(E(0) + \alpha h\|\varphi\|_{C_1}^2\right) + \frac{4C}{\gamma}, \tag{3.1}$$

where $C = \frac{2}{\mu}\|\sigma\|^2 + 2\gamma K_1 + 2\varepsilon K_2 + \frac{4}{\mu}C_\varrho$.

Proof. For any $\forall \varepsilon > 0$, take the inner product of $z(t) = \partial_t u(t) + \varepsilon u(t)$ with Equation (1.1) in $L^2(\Omega)$, we then have

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left[(1+\varepsilon)\|\nabla u\|^2 + (\mu+\varepsilon)\|u\|^2\right] + \varepsilon\|\nabla u\|^2 + \mu\varepsilon\|u\|^2 + \|\partial_t u\|^2 + \|\nabla\partial_t u\|^2 \\ &= -\left(f(u), z\right) - \left(u(t-\eta[u]), z\right) + (\sigma, z).\end{aligned} \tag{3.2}$$

From (1.8) and (1.9), we have

$$-\left(f(u), z\right) \le -\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega F(u)\mathrm{d}x - \varepsilon C_0\int_\Omega F(u)\mathrm{d}x + \frac{\varepsilon}{8}\|\nabla u\|^2 + \varepsilon K_2. \tag{3.3}$$

Furthermore, using the inequality Hölder' and Young's inequality, we obtain

$$\begin{aligned}&\left(u(t-\eta[u]), z\right) + (\sigma, z) \\ &\le \frac{1}{\mu}\left\|u(t-\eta[u])\right\|^2 + \frac{1}{\mu}\|\sigma\|^2 + \frac{\mu}{2}\|\partial_t u\|^2 + \frac{\mu\varepsilon}{2}\|u\|^2.\end{aligned} \tag{3.4}$$

Substitute (3.3) and (3.4) into (3.2), apply condition (1.5), and set $\varrho = \frac{\varepsilon\mu}{8}$, we then have

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left[(1+\varepsilon)\|\nabla u\|^2 + (\mu+\varepsilon)\|u\|^2 + 2\int_\Omega F(u)\mathrm{d}x\right] + \frac{5\varepsilon}{8}\|\nabla u\|^2 \\ &+ \frac{\mu\varepsilon}{2}\|u\|^2 + \left(1-\frac{\mu}{2}\right)\|\partial_t u\|^2 + \|\nabla\partial_t u\|^2 + \varepsilon\left(C_0 - \frac{1}{4}\right)\int_\Omega F(u)dx \\ &\le \frac{2h}{\mu}\int_0^h\|\partial_t u(t-\xi)\|^2\mathrm{d}\xi + \frac{1}{\mu}\|\sigma\|^2 + \varepsilon K_2 + \frac{2}{\mu}C_\varrho.\end{aligned} \tag{3.5}$$

Let

$$E(t) = (1+\varepsilon)\|\nabla u\|^2 + (\mu+\varepsilon)\|u\|^2 + 2\int_\Omega F(u)\mathrm{d}x + 2K_1 \ge 0,$$

Define

$$V(t) = E(t) + \frac{\alpha}{h}\int_0^h\int_{t-s}^t\|\partial_t u(\xi)\|^2\mathrm{d}\xi\mathrm{d}s,$$

It is obvious that

$$E(t) \le V(t) \le E(t) + \alpha\int_0^h\|\partial_t u(t-\xi)\|^2\mathrm{d}\xi, \tag{3.6}$$

where $\alpha > 0$, and the term $\frac{\alpha}{h}\int_0^h\int_{t-s}^t\|\partial_t u(\xi)\|^2\mathrm{d}\xi\mathrm{d}s$ serves as a compensation term for the time-delay term in the equation. Specifically, a longer delay time tends to make the system more unstable, while increasing the coefficient of the damping term can balance this instability. Therefore, in the process of proof, the delay time is restricted by the damping coefficient μ.

Differentiating $V(t)$ from t, we obtain

$$\frac{d}{dt}V(t)=\frac{d}{dt}E(t)+\alpha\|\partial_t u(t)\|^2-\frac{\alpha}{h}\int_0^h\|\partial_t u(t-s)\|^2\,ds. \tag{3.7}$$

Combining (3.6) and (3.7), we obtain

$$\frac{d}{dt}V(t)+\frac{5\varepsilon}{4}\|\nabla u\|^2+\mu\varepsilon\|u\|^2+\left[2\left(1-\frac{\mu}{2}\right)-\alpha\right]\|\partial_t u\|^2+2\varepsilon\left(C_0-\frac{1}{4}\right)\int_\Omega F(u)\,dx$$
$$\le\left(\frac{4h}{\mu}-\frac{\alpha}{h}\right)\int_0^h\|\partial_t u(t-\xi)\|^2\,d\xi+\frac{2}{\mu}\|\sigma\|^2+2\varepsilon K_2+\frac{4}{\mu}C_\varrho.$$

choose $\varepsilon>0$ sufficiently small such that $\frac{1}{2}<\mu<\frac{3}{2}$, and set $\alpha=\frac{\mu}{8}$, and $C_0>\frac{1}{4}$, which ensures $\mu\varepsilon>0$, $2\left(1-\frac{\mu}{2}\right)-\alpha>0$, and $C_0-\frac{1}{4}>0$. Let $\gamma=\min\left\{\frac{5\varepsilon}{4},\mu\varepsilon,2\left(1-\frac{\mu}{2}\right)-\alpha,2\varepsilon\left(C_0-\frac{1}{4}\right)\right\}$. Therefore

$$\frac{d}{dt}V(t)+\gamma V(t)\le C.$$

Applying Gronwall's lemma, we have

$$V(t)\le V(0)e^{-\gamma t}+\frac{C}{\gamma}\left(1-e^{-\gamma t}\right). \tag{3.8}$$

According to the definition of $E(t)$ as well as (3.7) and (3.8), we obtain

$$\|\nabla u\|^2+\|u\|^2\le 4e^{-\gamma t}\left(E(0)+\alpha h\|\varphi\|_{C_1}^2\right)+\frac{4C}{\gamma},$$

where $C=\frac{2}{\mu}\|\sigma\|^2+2\gamma K_1+2\varepsilon K_2+\frac{4}{\mu}C_\varrho$.

3.2. Existence and Uniqueness

Multiply both sides of Equation (1.1) by $(I-\Delta)^{-1}$, we then obtain

$$\partial_t u(t)+(\mu-\Delta)(I-\Delta)^{-1}u(t)=(I-\Delta)^{-1}\left(-f(u(t))-u\left(t-\eta\left[u^t\right]\right)+\sigma(x)\right).$$

Let $A=-\Delta$, $\mathcal{L}=(\mu-\Delta)(I-\Delta)^{-1}$. Next, we present the corresponding estimates.

Lemma 3.2.

$$(\mathcal{L}u,u)\ge 0,\ \forall u\in C([-h,T];H).$$

Proof. Since H is a separable Hilbert space and $A=-\Delta$ is a positive definite operator with discrete spectrum on H, there exists a sequence of orthogonal bases $\{e_k\}_{k=1}^\infty\subset H$ such that

$$Ae_k=\lambda_k e_k,\ 0<\lambda_1\le\lambda_2\le\cdots,\ \lim_{k\to\infty}\lambda_k=\infty,$$

By spectral theory, we have

$$\mathcal{L}e_k=(\mu+\lambda_k)(1+\lambda_k)^{-1}e_k,\ \inf_{\lambda>0}(\mu+\lambda)(1+\lambda)^{-1}=0.$$

Thus

$$\left(\mathcal{L}u,u\right)\geq 0,\ \ \forall u\in C\left(\left[-h,T\right];H\right).$$

In particular, it follows from [11] that $\left\{\mathrm{e}^{-\mathcal{L}t}\right\}_{t\geq 0}$ is a C_0-semigroup generated by the operator $\mathcal{L}$ in the space H.

Lemma 3.3. There exists a positive constant c such that

(1) $\left\|A^{-\alpha}\left(I-\mathrm{e}^{-\mathcal{L}t}\right)\right\|\leq ct,\forall t\geq 0$;

(2) $\left\|A^{-\alpha}\mathrm{e}^{-\mathcal{L}t}\right\|\leq c\mathrm{e}^{-\lambda_1(1+\lambda_1)^{-1}t},\forall t\geq 0$, where λ_1 is the first eigenvalue of $-\Delta$.

Proof. (1) Since $A^{-\alpha}$ is compact on $L^2\left(\Omega\right)$, there exists a constant $c>0$ such that

$$I-\mathrm{e}^{-\mathcal{L}t}\leq ct,\ \ \forall t\geq 0.$$

The assertion that $\mathrm{e}^{-\mathcal{L}t}$ is bounded for $t\geq 1$ holds true;

for $t\in\left[0,1\right]$, the spectrum of $I-\mathrm{e}^{-\mathcal{L}t}$ is $1-\mathrm{e}^{-\lambda(1+\lambda)^{-1}t}$, and

$$I-\mathrm{e}^{-\mathcal{L}t}\leq 1-\mathrm{e}^{-c\lambda_1(1+\lambda_1)^{-1}t}\leq c\lambda_1\left(1+\lambda_1\right)^{-1}t\leq ct.$$

(2) Similar to the proof of (1), we only need to prove that

$$\mathrm{e}^{-\mathcal{L}t}\leq c\mathrm{e}^{-\lambda_1(1+\lambda_1)^{-1}t},\ \ \forall t\geq 0.$$

In fact, by spectral theory and the property that $\left(\mathcal{L}u,u\right)\geq\lambda_1\left(u,u\right)$, for all $u\in L^2\left(\Omega\right)$, it is easy to see that $\mathcal{L}$ is bounded for $t\geq 0$.

The following is a mild solution to (1.1).

Definition 3.4. A mild solution to Equation (1.1) is a function $u\in C\left(\left[-h,T\right];H\right)$, defined in the interval $\left[-h,T\right]$ such that $u\left(\theta\right)=\varphi\left(\theta\right)$, $\theta\in\left[-h,0\right]$ and satisfies

$$u\left(t\right)=\mathrm{e}^{-\mathcal{L}t}\varphi\left(\theta\right)+\int_0^T\mathrm{e}^{-\mathcal{L}(t-s)}F\left(u^s\right)\mathrm{d}s,\ \ t\in\left[-h,T\right],$$

where $F\left(u^s\right)=\left(I-\Delta\right)^{-1}\left(-f\left(u\right)-u\left(t-\eta\left[u^t\right]\right)+\sigma\left(x\right)\right)$, and $\mathcal{L}$ is the infinitesimal generator of the C_0-semigroup $\left\{\mathrm{e}^{-\mathcal{L}t}\right\}_{t\geq 0}$ in H.

Theorem 3.5. Suppose that conditions (H1)-(H3) hold. Then, for any initial values (H1)-(H3), there exists $T_\varphi>0$ such that Equation (1.1) has a unique mild solution $u\left(t\right)$ on the interval $\left[-h,T_\varphi\right]$.

Proof. For a fixed $\omega>0$, define the ball:

$$B_\omega=\left\{u\in C\left(\left[-h,T\right];H\right):\left\|u\right\|_{C([-h,T];H)}\leq\omega\right\},$$

where $v=\mathrm{e}^{-\mathcal{L}t}\varphi\left(0\right)$.

Define the mapping $K:C\left(\left[-h,T\right];H\right)\rightarrow C\left(\left[-h,T\right];H\right)$ as follows

$$\left[Ku\right]\left(t\right)=\begin{cases}\varphi\left(t\right),\ \ t\in\left[-h,0\right],\\ v\left(t\right)+\int_0^t\mathrm{e}^{-\mathcal{L}(t-s)}F\left(u^s\right)\mathrm{d}s,\ \ t\in\left[0,T\right].\end{cases}$$

If u is a fixed point of the mapping K, then u is a mild solution to Equation (1.1) in $\left[0,T\right]$. Now we prove that K is a contraction mapping.

(I) For any $t\in[0,T]$ and $u_1,u_2\in B_\omega$, we have

$$\begin{aligned}
&\left\|[Ku_1](t)-[Ku_2](t)\right\|_{C([0,T];H)}\\
&\le\int_0^t\left\|e^{-\mathcal{L}(t-s)}(I-\Delta)^{-1}\left(f(u_2(s))-f(u_1(s))\right)\right\|_{C([0,T];H)}ds\\
&\quad+\int_0^t\left\|e^{-\mathcal{L}(t-s)}(I-\Delta)^{-1}\left(u_2\left(s-\eta\left[u_2^s\right]\right)-u_1\left(s-\eta\left[u_1^s\right]\right)\right)\right\|_{C([0,T];H)}ds\\
&\le\int_0^t\left\|f(u_2(s))-f(u_1(s))\right\|_{C([0,T];H)}ds\\
&\quad+\int_0^t\left\|u_2\left(s-\eta\left[u_2^s\right]\right)-u_1\left(s-\eta\left[u_1^s\right]\right)\right\|_{C([0,T];H)}ds.
\end{aligned}\tag{3.9}$$

By condition (1.4), we have

$$\begin{aligned}
\int_\Omega\left|f(u_2)-f(u_1)\right|dx&=\int_\Omega\left|f'\left(u_1+\theta(u_2-u_1)\right)(u_2-u_1)\right|dx\\
&\le C\int_\Omega\left(1+|u_1|^2+|u_2|^2\right)|u_2-u_1|dx\\
&\le C\left(1+\|u_1\|_{L^4}^2+\|u_2\|_{L^4}^2\right)\|u_2-u_1\|\\
&\le C\|u_2-u_1\|,
\end{aligned}\tag{3.10}$$

Thus

$$\left\|f(u_2)-f(u_1)\right\|_{C([0,T];H)}\le C\max_{t\in[0,T]}\|u_2-u_1\|\le M_{\hat{R}}\|u_2-u_1\|_{C([0,T];H)}.\tag{3.11}$$

Since $u\left(t-\eta\left[u^t\right]\right)=u(t)-\int_{t-\eta\left[u^t\right]}^t\partial_t u(s)ds$, by using condition (1.6) and the above inequality

$$\begin{aligned}
&\left\|u_2\left(s-\eta\left[u_2^s\right]\right)-u_1\left(s-\eta\left[u_1^s\right]\right)\right\|\\
&\le\left\|u_2\left(s-\eta\left[u_2^s\right]\right)-u_2\left(s-\eta\left[u_1^s\right]\right)\right\|+\left\|u_2\left(s-\eta\left[u_1^s\right]\right)-u_1\left(s-\eta\left[u_1^s\right]\right)\right\|\\
&=\left\|u_2(s)-\int_{s-\eta\left[u_2^s\right]}^s\partial_t u_2(r)dr-u_2(s)+\int_{s-\eta\left[u_1^s\right]}^s\partial_t u_2(r)dr\right\|\\
&\quad+\max_{\theta\in[-h,0]}\left\|u_2(s+\theta)-u_1(s+\theta)\right\|\\
&\le\left|\int_{s-\eta\left[u_1^s\right]}^{s-\eta\left[u_2^s\right]}\left|\partial_t u_2(r)\right|dr\right|+\left\|u_2^s-u_1^s\right\|_{\mathcal{C}}\\
&\le\hat{R}\left|\eta\left[u_1^s\right]-\eta\left[u_2^s\right]\right|+\left\|u_2^s-u_1^s\right\|_{\mathcal{C}}\\
&\le\left(\hat{R}\cdot L_R+1\right)\left\|u_2^s-u_1^s\right\|_{\mathcal{C}},
\end{aligned}\tag{3.12}$$

However

$$\begin{aligned}
\left\|u_2^s-u_1^s\right\|_{\mathcal{C}}&\le\max_{r\in[s-h,0]}\left\|u_2(r)-u_1(r)\right\|+\max_{r\in[0,s]}\left\|u_2(r)-u_1(r)\right\|\\
&\le\|u_2-u_1\|_{C([0,T];H)},
\end{aligned}$$

Thus, we have

$$\begin{aligned}
&\left\|u_2\left(s-\eta\left[u_2^s\right]\right)-u_1\left(s-\eta\left[u_1^s\right]\right)\right\|_{C([0,T];H)}\\
&\le\left(\hat{R}\cdot L_R+1\right)\|u_2-u_1\|_{C([0,T];H)}.
\end{aligned}\tag{3.13}$$

Substituting (3.11) and (3.13) into (3.9), we obtain

$$\begin{aligned}&\left\|[Ku_1](t)-[Ku_2](t)\right\|_{C([0,T];H)}\\&\le\int_0^t\left(M_{\tilde{R}}+\left(\hat{R}\cdot L_R+1\right)\right)\|u_2-u_1\|_{C([0,T];H)}\,\mathrm{d}s\\&\le T\cdot\left(M_{\tilde{R}}+\left(\hat{R}\cdot L_R+1\right)\right)\|u_2-u_1\|_{C([0,T];H)},\end{aligned}$$

choose a sufficiently small T such that $T\cdot\left(M_{\tilde{R}}+\left(\hat{R}\cdot L_R+1\right)\right)<1$.

(II) For any $t\in[0,T]$ and $z\in B_\omega$, combining (3.10)-(3.13), we obtain

$$\begin{aligned}&\left\|[Ku](t)-\bar{u}(t)\right\|_{C([0,T];H)}\\&\le\int_0^t\left\|\mathrm{e}^{-\mathcal{L}(t-s)}(I-\Delta)^{-1}\left(-f(u(s))-u\left(s-\eta\left[u^s\right]\right)\right)\right\|_{C([0,T];H)}\mathrm{d}s\\&\le\int_0^t\left(\left\|f(u(s))\right\|_{C([0,T];H)}+\left\|u\left(s-\eta\left[u^s\right]\right)\right\|_{C([0,T];H)}\right)\mathrm{d}s\\&\le\int_0^t\left(M_{\tilde{R}}+\left(\hat{R}\cdot L_R+1\right)\right)\|u\|_{C([0,T];H)}\,\mathrm{d}s\\&\le T\cdot\left(M_{\tilde{R}}+\left(\hat{R}\cdot L_R+1\right)\right)\cdot\tilde{R},\end{aligned}\tag{3.14}$$

choose $T<\min\left\{\frac{1}{M_{\tilde{R}}+\left(\hat{R}\cdot L_R+1\right)},\frac{\omega}{\hat{R}\cdot\left(M_{\tilde{R}}+\left(\hat{R}\cdot L_R+1\right)\right)}\right\}$, such that $T\cdot\left(M_{\tilde{R}}+\left(\hat{R}\cdot L_R+1\right)\right)\cdot\tilde{R}<\omega$. Therefore, from (I) (II), we conclude that

$$K:B_\omega\to B_\omega$$

is a contraction mapping, and thus there exists a unique fixed point $u\in C([-h,T];H)$.

Let

$$\bar{u}=\begin{cases}u(t), & t\in[0,T],\\ \varphi(t), & t\in[-h,0],\end{cases}$$

and suppose $\bar{u}\in C((-h,T];H)$, then $\bar{u}$ is a mild solution to Equation (1.1) on the interval $[-h,T]$.

Theorem 3.6. Suppose that conditions (H1)-(H3) hold. Then, for any initial values $\varphi_i\in\mathcal{C}$ with $\|\varphi_i\|_{\mathcal{C}}\le\varpi$, $i=1,2$, there exists a unique global solution $u(t)$ to Equation (1.1) on the interval $[-h,+\infty)$. Moreover, for any $\varpi>0$ and $T>0$, there exists a positive constant $C_{\varpi,T}$ such that

$$\left\|A^{\frac{1}{2}}\left(u_1(t)-u_2(t)\right)\right\|^2+\left\|u_1(t)-u_2(t)\right\|^2\le C_{\varpi,T}\left\|\varphi_1-\varphi_2\right\|_c^2,\ t\in[0,T].$$

Proof. Multiply u by (1.1) and do the inner product in H, we have

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(A^{\frac{1}{2}}u^2+\|u\|^2\right)+A^{\frac{1}{2}}u^2+\mu\|u\|^2\\&=-(f(u),u)-\left(u\left(t-\eta\left[u^t\right]\right),u\right)+(\sigma,u).\end{aligned}\tag{3.15}$$

using (1.9), Hölder's and Young's inequality, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(A^{\frac{1}{2}}u^2+\|u\|^2\right)+\frac{7}{8}A^{\frac{1}{2}}u^2+\frac{\mu}{2}\|u\|^2+C_0\int_\Omega F(u)\mathrm{d}x\\&\le\frac{1}{\mu}\left\|u\left(t-\eta\left[u^t\right]\right)\right\|^2+\frac{1}{\mu}\|\sigma\|^2+K_2.\end{aligned}\tag{3.16}$$

Let

$$E_1(t)=\|u\|^2+\left\|A^{\frac{1}{2}}u\right\|^2\ge 0,$$

Integrating (3.16) over the interval $[0,t]$, we obtain

$$\begin{aligned}&E_1(t)+\int_0^t\left(\mu\|u\|^2+\frac{7}{4}\left\|A^{\frac{1}{2}}u\right\|^2\right)\mathrm{d}s+2C_0\int_0^t\int_\Omega F(u)\mathrm{d}x\\&\le E_1(0)+\frac{2}{\mu}\|\sigma\|^2+\frac{2}{\mu}\int_0^t\|u\|^2\,\mathrm{d}s,\end{aligned}\tag{3.17}$$

Moreover, since for any $s\in[0,T_\varphi)$,

$$\begin{aligned}\left\|u^s\right\|_{\mathcal{C}}&=\max_{\theta\in[-h,0]}\left\|A^{\frac{1}{2}}u^s(\theta)\right\|+\max_{\theta\in[-h,0]}\left\|u^s(\theta)\right\|\\&\le\max_{r\in[s-h,0]}\left\|A^{\frac{1}{2}}u(r)\right\|+\max_{r\in[0,s]}\left\|A^{\frac{1}{2}}u(r)\right\|+\max_{r\in[s-h,0]}\left\|u(r)\right\|+\max_{r\in[0,s]}\left\|u(r)\right\|\\&\le\|\varphi\|_{\mathcal{C}_1}+\sqrt{2}\sqrt{\max_{r\in[0,s]}\left\|A^{\frac{1}{2}}u(r)\right\|^2+\max_{r\in[0,s]}\left\|u(r)\right\|^2}\\&\le\|\varphi\|_{\mathcal{C}_1}+2\sqrt{\max_{r\in[0,s]}\left(\left\|A^{\frac{1}{2}}u(r)\right\|^2+\left\|u(r)\right\|^2\right)}.\end{aligned}\tag{3.18}$$

Substituting (3.18) into (3.17), we obtain

$$\begin{aligned}&\max_{r\in[0,t]}\left(\left\|A^{\frac{1}{2}}u(r)\right\|^2+\left\|u(r)\right\|^2\right)\\&\le C\left(E_1(0)+t\|\varphi\|_{\mathcal{C}_1}^2+\frac{2t}{\mu}\|\sigma\|^2+\int_0^t\max_{r\in[0,s]}\left(\left\|A^{\frac{1}{2}}u(r)\right\|^2+\left\|u(r)\right\|^2\right)\mathrm{d}s\right),\end{aligned}$$

where $C>0$. Applying the integral form of Gronwall's lemma to the above inequality, for $t<T_\varphi$, we have

$$\max_{r\in[0,t]}\left(\left\|A^{\frac{1}{2}}u(r)\right\|^2+\left\|u(r)\right\|^2\right)\le C\left(1+E_1(0)+\|\varphi\|_{\mathcal{C}_1}^2\right)\mathrm{e}^{\alpha_1 t},$$

where $\alpha_1>0$. For any $T>0$, since the above inequality holds on $[0,T_\varphi)\subset[0,T]$, the solution to Equation (1.1) can be extended to the interval $[0,+\infty)$. Consequently, we obtain the continuous dependence on initial values and the uniqueness of the solution.

4. Global Attractor

According to *Theorem* 3.6, we define the following semigroup: $S(t):\mathcal{C}_1\to\mathcal{C}_1$, where for any $t\ge 0$, $S(t)\varphi=u^t$. Here, $u(t)$ is the mild solution to Equation (1.1) satisfying $u^0=\varphi$. We denote $(S(t),\mathcal{C}_1)$ as the dynamical system generated by the solution semigroup corresponding to Equation (1.1).

First, we prove that the dynamical system $(S(t),\mathcal{C}_1)$ is dissipative.

Lemma 4.1 (Dissipativity). Suppose that conditions (H1)-(H3) hold and $\sigma\in L^2(\Omega)$. Then, for any μ_0, there exists $h_0=h(\mu_0)$ such that for every $(\mu,h)\in[\mu_0,+\infty)\times(0,h_0]$, the system $(S(t),\mathcal{C}_1)$ is dissipative. That is, there exists $R>0$, such that for any $\rho>0$,

$$\|S(t)\varphi\|_{\mathcal{C}_1}\le R,\ \forall\varphi\in\mathcal{C}_1,\ \|\varphi\|_{\mathcal{C}_1}\le\rho,\ t\ge t_\rho,$$

where $t_\rho>0$ depends on ρ, and for any fixed $\mu_0>0$, the dissipative radius R is independent of both the damping coefficient $\mu\ge\mu_0$ and the time delay $h\in(0,h_0]$.

Proof. Similar to the a priori estimates in Section 3.1, we have

$$\|\nabla u\|^2+\|u\|^2\le 4\mathrm{e}^{-\gamma t}\left(E(0)+\alpha h\|\varphi\|_{\mathcal{C}_1}^2\right)+\frac{4C}{\gamma}. \tag{4.1}$$

Replacing t with $t+\theta$ in the above inequality (where $\theta\in[-h,0]$), the following holds

$$\begin{aligned}\|\nabla u(t+\theta)\|^2+\|u(t+\theta)\|^2&\le 4\mathrm{e}^{-\gamma(t+\theta)}\left(E(0)+\alpha h\|\varphi\|_{\mathcal{C}_1}^2\right)+C\\&\le 4\mathrm{e}^{-\gamma(t-h)}\left(E(0)+\alpha h\|\varphi\|_{\mathcal{C}_1}^2\right)+\frac{4C}{\gamma}.\end{aligned} \tag{4.2}$$

Therefore, from (4.2), we obtain

$$\begin{aligned}\|u^t\|_{\mathcal{C}_1}^2&=\max_{\theta\in[-h,0]}\|\nabla u(t+\theta)\|^2+\max_{\theta\in[-h,0]}\|u(t+\theta)\|^2\\&\le 2\max_{\theta\in[-h,0]}\left(\|\nabla u(t+\theta)\|^2+\|u(t+\theta)\|^2\right)\\&\le 8\mathrm{e}^{-\gamma(t-h)}\left(E(0)+\alpha h\|\varphi\|_{\mathcal{C}_1}^2\right)+\frac{8C}{\gamma},\end{aligned} \tag{4.3}$$

The above inequality implies that there exists $t\ge t_\rho$ such that the ball $B_0=\overline{B}(0,R)$ is a bounded absorbing set for the dynamical system $(S(t),\mathcal{C}_1)$, where $R>\frac{2\sqrt{2C}}{\gamma}$.

Lemma 4.2 (Quasi-Stability). Suppose that conditions (H1)-(H3) hold and $\sigma\in L^2(\Omega)$. Then, there exist constants $C_1(R)$, $C_2(R)>0$ and $\overline{\lambda}$ such that for any two solutions u_1,u_2 to Equation (1.1) with initial values $\varphi_1,\varphi_2\in\mathcal{C}_1$, the following properties hold

$$\|u_i(t)\|^2+\|\nabla u_i(t)\|^2\le R^2,\ t\ge -h,\ i=1,2 \tag{4.4}$$

and the quasi-stability estimate

$$\begin{aligned}&\left\|u_1(t)-u_2(t)\right\|^2+\left\|\nabla u_1(t)-\nabla u_2(t)\right\|^2\\&\leq C_1(R)\mathrm{e}^{-\bar{\lambda}t}\left\|\varphi_1-\varphi_2\right\|_{\mathcal{C}_1}^2+C_2(R)\max_{r\in[0,t]}\left\|A^{\frac{1}{2}-\epsilon}\left(u_1(r)-u_2(r)\right)\right\|^2,\end{aligned}\tag{4.5}$$

where $\epsilon>0$ is a small positive constant.

Proof. Let u_1 and u_2 be two solutions to Equation (1.1). Then $w=u_1(t)-u_2(t)$ is a solution to the following equation

$$\begin{aligned}&\partial_t w-\Delta\partial_t w-\Delta w+\mu w\\&=-\left(u_1\left(t-\eta\left[u_1^t\right]\right)-u_2\left(t-\eta\left[u_2^t\right]\right)\right)-\left(f(u_1)-f(u_2)\right).\end{aligned}\tag{4.6}$$

According to *Lemma* 4.1, the system is dissipative, so it is obvious that (4.4) holds.

Define the energy functional as

$$E_w(t)=\frac{1}{2}\left(\|\nabla w\|^2+\|w\|^2\right).\tag{4.7}$$

By multiplying $w(t)$ by (4.6) and integrating over the interval $[t,T]$, we have

$$\begin{aligned}&E_w(T)-E_w(t)+\int_t^T\|\nabla w(s)\|^2\mathrm{d}s+\mu\int_t^T\|w(s)\|^2\mathrm{d}s\\&\leq\int_t^T\left(f(u_2(s))-f(u_1(s)),w(s)\right)\mathrm{d}s\\&\quad+\int_t^T\left(u_2\left(s-\eta\left[u_2^s\right]\right)-u_1\left(s-\eta\left[u_1^s\right]\right),w(s)\right)\mathrm{d}s.\end{aligned}\tag{4.8}$$

By condition (1.4), we have

$$\begin{aligned}&\left|\int_\Omega\left(f(u_2(t))-f(u_1(t))\right)w(t)\mathrm{d}x\right|\\&\leq\int_\Omega\left|f'\left(u_1+\xi(u_2-u_1)\right)\right|\left\|u_2(t)-u_1(t)\right\|\left\|w(t)\right\|\mathrm{d}x\\&\leq C\left\|\nabla\left(u_2(t)-u_1(t)\right)\right\|\|w\|\\&\leq\frac{\varepsilon}{2}\|\nabla w\|^2+\frac{C_R}{2\varepsilon}\|w\|^2,\ \forall\varepsilon>0,\end{aligned}\tag{4.9}$$

where $0<\xi<1$ and $\varepsilon>0$. Using assumption (1.7), we obtain

$$\begin{aligned}&\left|\int_\Omega\left(u_2\left(t-\eta\left[u_2^t\right]\right)-u_1\left(t-\eta\left[u_1^t\right]\right)\right)w(t)\mathrm{d}x\right|\\&\leq u_2\left(t-\eta\left[u_2^t\right]\right)-u_1\left(t-\eta\left[u_1^t\right]\right)\left\|w(t)\right\|\\&\leq\max_{\theta\in[-h,0]}A^{\frac{1}{2}-\epsilon}w(t+\theta)^2+C_R\left\|w(t)\right\|^2.\end{aligned}\tag{4.10}$$

Combining (4.9)-(4.10) with (4.8), we derive the following inequality

$$\begin{aligned}&\left|E_w(T)-E_w(t)+\mu\int_t^T\|w(s)\|^2\mathrm{d}s\right|\\&\leq\frac{\varepsilon}{2}\int_t^T\|\nabla w(s)\|^2\mathrm{d}s+\int_t^T\max_{\theta\in[-h,0]}A^{\frac{1}{2}-\epsilon}w(t+\theta)^2\mathrm{d}s\\&\quad+C_R\left(1+\frac{1}{2\varepsilon}\right)\int_t^T\|w(s)\|^2\mathrm{d}s.\end{aligned}\tag{4.11}$$

For any $\varepsilon>0$, choose a sufficiently large μ such that the following inequality holds

$$C_R\left(1+\frac{1}{2\varepsilon}\right)<\frac{\mu}{2},\ \frac{\varepsilon}{2}<1. \tag{4.12}$$

Now, taking the inner product of Equation (4.6), with $w(t)$ and integrating over the interval $[0,T]$, we obtain

$$\begin{aligned}&\left(w(T),w(T)\right)-\left(w(0),w(0)\right)+\left(\nabla w(T),\nabla w(T)\right)-\left(\nabla w(0),\nabla w(0)\right)\\&+\int_0^T\|\nabla w(s)\|^2\,\mathrm{d}s+\mu\int_0^T\left(w(s),w(s)\right)\mathrm{d}s\\&\le\tilde{C}_R\int_0^T\max_{\theta\in[-h,0]}A^{\frac{1}{2}-\epsilon}w(t+\theta)^2\,\mathrm{d}s+\tilde{C}_R\int_0^T\|w(s)\|^2\,\mathrm{d}s.\end{aligned} \tag{4.13}$$

Furthermore, by Hölder's inequality and Young's inequality, we have

$$\mu\int_0^T\left(w(s),w(s)\right)\mathrm{d}s\le\frac{1}{2}\int_0^T\|w(s)\|^2\,\mathrm{d}s+\frac{\mu^2}{2}\int_0^T\|w(s)\|^2\,\mathrm{d}s.$$

According to the definition of the energy functional $E_w(t)$, we get

$$\begin{aligned}\frac{1}{2}\int_0^T\|\nabla w(s)\|^2\,\mathrm{d}s&\le\frac{3}{2}\int_0^T\|w(s)\|^2\,\mathrm{d}s+C\left(E_w(0)+E_w(T)\right)\\&\quad+\tilde{C}_R(\mu)\int_0^T\max_{\theta\in[-h,0]}A^{\frac{1}{2}-\epsilon}w(t+\theta)^2\,\mathrm{d}s.\end{aligned}$$

Setting $t=0$ in (4.11) and combining it with (4.12), we obtain

$$\begin{aligned}E_w(0)+E_w(T)&\le 2E_w(T)+\frac{3\mu}{2}\int_0^T\|w(s)\|^2\,\mathrm{d}s+\varepsilon\int_0^T\|\nabla w(s)\|^2\,\mathrm{d}s\\&\quad+\int_0^T\max_{\theta\in[-h,0]}A^{\frac{1}{2}-\epsilon}w(t+\theta)^2\,\mathrm{d}s.\end{aligned} \tag{4.14}$$

Integrating (4.11) over the interval $[0,T]$ and combining it with (4.12), we have

$$\begin{aligned}TE_w(T)&\le\int_0^T E_w(s)\,\mathrm{d}s+\varepsilon T\int_0^T\|\nabla w(s)\|^2\,\mathrm{d}s\\&\quad+T\int_0^T\max_{\theta\in[-h,0]}A^{\frac{1}{2}-\epsilon}w(t+\theta)^2\,\mathrm{d}s.\end{aligned} \tag{4.15}$$

Setting $t=0$ in (4.11) and combining with (4.12), we can obtain

$$\frac{\mu}{2}\int_0^T\|w(s)\|^2\,\mathrm{d}s\le E_w(0)+\varepsilon\int_0^T\|\nabla w(s)\|^2\,\mathrm{d}s+\int_0^T\max_{\theta\in[-h,0]}A^{\frac{1}{2}-\epsilon}w(t+\theta)^2\,\mathrm{d}s. \tag{4.16}$$

Adding (4.15) and (4.16), assuming $\mu>8$, and substituting the result into (4.15), we get

$$\begin{aligned}&\left(\frac{\mu}{2}-2\right)\int_0^T\|w(s)\|^2\,\mathrm{d}s+\frac{1}{2}\int_0^T E_w(s)\,\mathrm{d}s+\frac{1}{2}TE_w(T)\\&\le\varepsilon(T+1)\int_0^T\|\nabla w(s)\|^2\,\mathrm{d}s+C\left(E_w(0)+E_w(T)\right)\\&\quad+\tilde{C}_R'(\mu)\left(1+\frac{T}{2}\right)\int_0^T\max_{\theta\in[-h,0]}A^{\frac{1}{2}-\epsilon}w(t+\theta)^2\,\mathrm{d}s.\end{aligned} \tag{4.17}$$

Now, substituting the estimate of $E_w(0)+E_w(T)$ from (4.14) into (4.17), we have

$$\frac{1}{2}\int_0^T E_w(s)\,\mathrm{d}s+\left(\frac{1}{2}T-2C\right)E_w(T)$$
$$\le(\mu+2)\int_0^T\|w(s)\|^2\,\mathrm{d}s+2\varepsilon(T+1)\int_0^T\|\nabla w(s)\|^2\,\mathrm{d}s \tag{4.18}$$
$$+\tilde{C}_R(\mu)\left(2+\frac{T}{2}\right)\int_0^T\max_{\theta\in[-h,0]}A^{\frac{1}{2}-\epsilon}w(t+\theta)^2\,\mathrm{d}s.$$

Similarly, setting $t=0$ in (4.11) gives

$$\frac{\mu}{2}\int_0^T\|w(s)\|^2\,\mathrm{d}s\le E_w(0)-E_w(T)+\varepsilon\int_0^T\|\nabla w(s)\|^2\,\mathrm{d}s$$
$$+\int_0^T\max_{\theta\in[-h,0]}A^{\frac{1}{2}-\epsilon}w(t+\theta)^2\,\mathrm{d}s. \tag{4.19}$$

Assume that

$$\frac{1}{2}T-2C>1,$$

Substituting (4.19) into (4.18), we thus obtain

$$E_w(T)+\frac{1}{2}\int_0^T E_w(s)\,\mathrm{d}s$$
$$\le C_\mu\left(E_w(0)-E_w(T)\right)+2C_\mu\varepsilon(T+1)\int_0^T\|\nabla w(s)\|^2\,\mathrm{d}s$$
$$+C_\mu\tilde{C}_R(\mu)\left(2+\frac{T}{2}\right)\int_0^T\max_{\theta\in[-h,0]}A^{\frac{1}{2}-\epsilon}w(t+\theta)^2\,\mathrm{d}s,$$

where $C_\mu>0$ denotes a constant depending on μ.

According to the definition of $E_w(t)$, we have $\|\nabla w(s)\|^2\le 2E_w(s)$. Choose $\varepsilon>0$ sufficiently small such that

$$2C_\mu\varepsilon(T+1)<\frac{1}{4}. \tag{4.20}$$

Therefore, we derive

$$E_w(T)\le\frac{C_\mu}{1+C_\mu}E_w(0)+\tilde{C}''_R(T,\mu)\int_0^T\max_{\theta\in[-h,0]}A^{\frac{1}{2}-\epsilon}w(t+\theta)^2\,\mathrm{d}s.$$

Let $\omega=\dfrac{C_\mu}{1+C_\mu}<1$, then there exists a constant $\beta>0$ such that

$$E_w(T)\le \mathrm{e}^{-\beta T}E_w(0)+\tilde{C}''_R(T,\mu)\int_0^T\max_{\theta\in[-h,0]}A^{\frac{1}{2}-\epsilon}w(t+\theta)^2\,\mathrm{d}s. \tag{4.21}$$

Finally, applying *Remark* 3.30 in Reference [10] and repeating the step $mT\mapsto(m+1)T$, we deduce the conclusion (4.4) from the relation (4.21).

Remark 4.2.1 Taking the maximum of inequality (4.5) over the interval $[t-h,t]$, we obtain

$$\|S(t)\varphi_1-S(t)\varphi_2\|_{\mathcal{C}_1}$$
$$\le C_1(R)h\mathrm{e}^{\bar{\lambda}h}\mathrm{e}^{-\bar{\lambda}t}\|\varphi_1-\varphi_2\|_{\mathcal{C}_1}+C_1(R)h\max_{s\in[0,h]}\mu_{\mathcal{C}_1}\left(u_1^s-u_2^s\right),t\ge h,$$

where $\mu_{\mathcal{C}_1}$ denotes a compact seminorm on $\mathcal{C}_1$. Next, based on the dissipativity of *Lemma* 4.1 and the quasi-stability of *Lemma* 4.2, we discuss the existence and properties of the global attractor. A complete orbit of the time-delay dynamical system $(S(t),\mathcal{C}_1)$ can be described as a function $u\in C\left(R,D\left(A^{\frac{1}{2}}\right)\right)$ satisfying $S(t)u^s=u^{s+t}$ for all $s\in R$ and $t>0$.

Remark 4.2.2 $\mu_{\mathcal{C}_1}$ is a compact seminorm. This is because in the space $\mathcal{C}_1=([-h,0];H^1)$, the seminorm induced by the operator $A^{\frac{1}{2}-\varepsilon}$ is compact. This is due to the fact that $A=-\Delta$ has a discrete spectrum, and its inverse is a compact operator. Thus, $A^{-\varepsilon}$ is a compact operator, which makes the norm induced by $A^{\frac{1}{2}-\varepsilon}$ on H^1 compact with respect to the $\mathcal{C}_1$ norm.

Theorem 4.3 (Global Attractor). Suppose that conditions (H1)-(H3) hold. Then the dynamical system $(S(t),\mathcal{C}_1)$ generated by Problem (1.1) possesses a compact global attractor.

Proof. From *Lemma* 4.1, we know that the system $(S(t),\mathcal{C}_1)$ is dissipative. Thus, we only need to verify that $(S(t),\mathcal{C}_1)$ is asymptotically smooth. By *Lemma* 4.2, the dynamical system $(S(t),\mathcal{C}_1)$ is quasi-stable on any positively invariant bounded set B. According to *Theorem* 2.1, the dynamical system $(S(t),\mathcal{C}_1)$ is asymptotically smooth, and therefore the existence of a compact global attractor is established.

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

References

[1] Xiao, Y. (2002) Attractors for a Nonclassical Diffusion Equation. *Acta Mathematicae Applicatae Sinica*, *English Series*, **18**, 273-276. https://doi.org/10.1007/s102550200026

[2] Sun, C.Y., Wang, S.Y. and Zhong, C.K. (2007) Global Attractors for a Nonclassical Diffusion Equation. *Acta Mathematica Sinica*, *English Series*, **23**, 1271-1280. https://doi.org/10.1007/s10114-005-0909-6

[3] Xie, Y.Q., Li, Q.S. and Zhu, K.X. (2016) Attractors for Nonclassical Diffusion Equations with Arbitrary Polynomial Growth Nonlinearity. *Nonlinear Analysis*: *Real World Applications*, **31**, 23-37. https://doi.org/10.1016/j.nonrwa.2016.01.004

[4] Hu, Z. and Wang, Y. (2012) Pullback Attractors for a Nonautonomous Nonclassical Diffusion Equation with Variable Delay. *Journal of Mathematical Physics*, **53**, 1-17. https://doi.org/10.1063/1.4736847

[5] Zhu, K.X. and Sun, C.Y. (2015) Pullback Attractors for Nonclassical Diffusion Equations with Delays. *Journal of Mathematical Physics*, **56**, Article 092703. https://doi.org/10.1063/1.4931480

[6] Ruess, W.M. (1996) Existence of Solutions to Partial Differential Equations with Delay. *Communications in Mathematical Physics*, **178**, 259-288.

[7] Zhang, J.W., Xie, Z. and Xie, Y.Q. (2024) Long-Time Behavior of Nonclassical Diffusion Equations with Memory on Time-Dependent Spaces. *Asymptotic Analysis*, **137**, 267-289. https://doi.org/10.3233/ASY-231887

[8] Brezis, H. (2010) Functional Analysis, Sobolev Spaces and Partial Differential Equations. Springer. https://doi.org/10.1007/978-0-387-70914-7

[9] Du, Z.H. (2019) Pullback Attractors for Non-Autonomous Reaction-Diffusion Equations with State-Dependent Delay. Master's Thesis, Lanzhou University.

[10] Wu, J.H. (1998) Theory and Applications of Partial Differential Equations. *Choice Reviews Online*, **36**, Article 1009. https://doi.org/10.5860/choice.36-1009

[11] Chueshov, I. (1999) Introduction to the Theory of Infinite-Dimensional Dissipative Systems. ACTA, Kharkov.