

Kinetics in the Weyl and Lagrange Spaces

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Abstract

This article discusses a method for deriving of Liouville equations (as a basis for equations with a self-consistent field) for a system of particles on a Weyl manifold, a generalization of a Riemannian manifold, taking into account an additional scalar potential. A connection is established between the constructed kinetics in Weyl space and the kinetics in Lagrange space.

Keywords

Lagrange Geometry, Scalar Potential, Weyl Manifold

1. Introduction

The possibility of using non-Riemannian geometries for the analysis of dynamics statistical systems in space with (generalized) curvature is new and a very promising approach to an extremely broad class of problems such as classical (topological theory of turbulence, description behavior of nonlocal objects near singularities in theory collapse into black holes and cosmology), and quantum (interactions, described Wigner equations, non-commutative geometry in applications to particle physics and astrophysics, quantum theory of gravity) character. In fact, it already exists at present a significant theoretical basis for these investigations, starting with the works of E. Cartan [1] [2] and P. Finsler [3] [4], up to the creation in recent decades of the theory of Lagrange and generalized Finsler spaces [5] [6]. Moreover, it is quite remarkable that that, starting from the 1970s, a stream of publications devoted to experimental and evaluative aspects of applied non-Riemannian geometry in cosmic ray physics, black hole astrophysics and various generalizations of the special theory of relativity, has grown significantly and is perceived by the scientific community as an independent branch of research in these areas. Enough mention the works: [7] [8] (devoted to the issue cutting off the energy spectrum of primary cosmic protons); [9] [10] (in which the so-called “Double Special Relativity”, that is, a variant of the Special Theory of Relativity, where

as invariants involve both the speed of light c and a certain second fundamental quantity—for example, the Planck length ℓ_p or Planck energy E_p); [11] (concerning the angular distribution temperature fluctuations of the microwave background radiation). A remarkable paper [12] has recently been published that attempts to unify gravitational interactions and electromagnetism using Weyl geometry.

However, the actual behavior of the statistical system (ensemble) particles) in space with Finsler (or more general Lagrange or Cartan metric function) has been covered very extensively until now poorly; the main reason here, apparently, should not be considered only the complexity of the mathematical apparatus of “physical geometry” (or lack of interest in the problems of system dynamics particles in it), how much is the need for its special adaptation with the introduction of non-standard—significantly different in form and the content from the usual “classical”—assumptions in the resulting version of statistical mechanics, thermodynamics and hydrodynamics. However, a careful study of this issue leads to the conclusion that the “correspondence principle” is not applicable here is violated—and moreover, some observed effects may get a simpler and clearer justification.

In this paper, an attempt is made to study the dynamics many-particle system on “Weyl manifolds” characterized by the absence in the general case of invariance of the interval ds , from the point of view of Lagrange and Hamiltonian geometry. The validity of this approach is determined by the possibility quite simply and without introducing any additional assumptions into consideration to connect classical mechanics in phase space with the statistical mechanics of ensembles in General Theory of Relativity and then move on to construction corresponding quantum-statistical theory, which can be considered as a kind of basis for the quantum theory of gravity, which is currently being actively developed.

2. Weyl Geometry and Particle Kinetics on the Weyl Manifolds

The basis for constructing the General Theory of Relativity (GTR), as is known, Riemannian geometry served as the basis. Unlike Newton’s classical mechanics, basically “content” with Euclidean geometry, space-time in GTR has a very wide range of changes in its internal structure, defined by the Riemannian metric $^{[R]}g_{ij}$ (non-degenerate and in general sign-indefinite), that is, in fact, the coefficients in the expression for the (square) interval $ds^2 = ^{[R]}g_{ij}dx^i dx^j$ ($i, j = \overline{0,3}$). Mathematical apparatus of the general theory of relativity makes significant use of affine varieties on which the statement about the constancy of the length of the vector $\xi^i(\tau)$ under parallel translation is true from a point $x^i(\tau)$ to a point $x^i(\tau) + dx^i(\tau)$ on some curve $\varpi(\tau)$ (along which—in tangent affine space—a vector field $\xi^i(\tau)$ is given): $d\left(^{[R]}g^{ij}\xi_i\xi_j\right) = 0$, and this parallel translation is characterized by the Christoffel connection object Γ_{ij}^k (that is, for an infinitesimal displacement along the curve, the coordinates of the vector $\xi^i(s)$ change

according to the law $d\xi^k = -\Gamma_{ij}^k \xi^j dx^i$). The introduction of a connection on a Riemannian space allows us to construct a meaningful theory developed by A. Einstein [13] [14] and D. Hilbert [15].

However, almost simultaneously with the “classical” GTR, alternative versions were proposed, based, in particular, on generalizations of the concept of connectivity introduced in the geometry of G. Weyl [16] and in differentiation according to E. Cartan [17]. Although Weyl’s approach (aimed at unifying electromagnetic and gravitational interactions) was pushed off the “mainstream” of gravity theory development in the decade following its introduction by Einstein’s physically more transparent theory and was not subsequently considered promising, its mathematical formalism has significant value and can be applied (with appropriate modification) to create a theory of particle dynamics on manifolds more general than Riemannian ones.

Let’s first look at some preliminary “basic” questions. As it is known [18], the change in $\delta_{\xi_i}^{\xi}$ components of a vector field ξ_i under parallel translation along a small closed contour σ^{ij} is equal to

$$\delta_{\xi_i}^{\xi} = \frac{1}{2} R_{ijk\ell} \xi^{\ell} d\sigma^{jk}, \tag{1}$$

where

$$R_{ijk\ell} = \frac{1}{2} \left[\frac{\partial^2 ([R] g_{\ell k})}{\partial x_j \partial x_i} + \frac{\partial^2 ([R] g_{ij})}{\partial x_{\ell} \partial x_k} - \frac{\partial^2 ([R] g_{ik})}{\partial x_{\ell} \partial x_j} - \frac{\partial^2 ([R] g_{j\ell})}{\partial x_i \partial x_k} \right] - \Gamma_{ik}^m \Gamma_{\ell j, m} + \Gamma_{ij}^m \Gamma_{\ell k, m}$$

is a Riemann-Christoffel tensor, $[R] g_{\ell k}$ —metric tensor on the Riemannian manifold, which considers the traversal of the contour σ^{ij} by the vector ξ_i . It follows that

$$\xi^i \delta_{\xi_i}^{\xi} = \frac{1}{2} R_{ijk\ell} \xi^i \xi^{\ell} d\sigma^{jk} = 0, \tag{2}$$

since the tensor $R_{ijk\ell}$ is antisymmetric with respect to the indices i and ℓ . Thus, the vector $\delta_{\xi_i}^{\xi}$ is orthogonal to the vector ξ_i ; further, since $|\xi|^2 = \xi^i \xi_i$, then $\delta |\xi|^2 = 2 \delta_{\xi_i}^{\xi} \xi^i$, from which it is clear that the length of the vector $|\xi|$ does not change when transfer along a closed contour.

Is it possible to construct a consistent theory that does not use the last statement (preservation of the vector norm: $|\xi| = const$)? Let us consider instead of relation (1) the following, understood as a definition:

$$\delta_{\xi_i}^{\xi} = \frac{1}{2} \mathcal{R}_{ijk\ell} \xi^{\ell} d\sigma^{jk}, \tag{3}$$

Where the tensor $\mathcal{R}_{ijk\ell}$ has more general kind than ordinary tensor curvature: antisymmetry with respect to i and ℓ is no longer assumed (but is preserved with respect to indices j and k). If we introduce antisymmetric

$^{(as)} \mathcal{R}_{ijk\ell} = (1/2)(\mathcal{R}_{ijk\ell} - \mathcal{R}_{\ell jki})$ and symmetric $^{(s)} \mathcal{R}_{ijk\ell} = (1/2)(\mathcal{R}_{ijk\ell} + \mathcal{R}_{\ell jki})$ with respect to indices i and ℓ tensors, then

$$\delta \xi_i = (1/2) \left({}^{(s)}\mathcal{R}_{ijk\ell} + {}^{(as)}\mathcal{R}_{ijk\ell} \right) \xi^\ell d\sigma^{jk}$$

and, accordingly, $\xi^i \delta \xi_i = (1/2) {}^{(s)}\mathcal{R}_{ijk\ell} \xi^i \xi^\ell d\sigma^{jk} \neq 0$ in general. We impose the following constraint on ${}^{(s)}\mathcal{R}_{ijk\ell}$: ${}^{(s)}\mathcal{R}_{ijk\ell} = [R]g_{i\ell} {}^{(s)}\mathcal{R}_{jk}$, ${}^{(s)}\mathcal{R}_{jk} = rot \vec{T}$, where \vec{T} is some 4-vector. Thus, we obtain that when going around a closed loop the change in the length of the vector $\delta |\vec{\xi}|$ is proportional to the initial length $|\vec{\xi}|$ (and does not depend on the direction of the vector):

$$\delta |\vec{\xi}| = \frac{{}^{(s)}\mathcal{R}_{jk} [R]g_{i\ell} \xi^i \xi^\ell d\sigma^{jk}}{2|\vec{\xi}|} = \frac{1}{2} {}^{(s)}\mathcal{R}_{jk} |\vec{\xi}| d\sigma^{jk}. \tag{4}$$

Thus, in the constructed geometry, the transfer of vectors of non-zero length along a closed contour requires specification transport paths and, as a consequence, comparison of lengths related to different points manifolds in the geometry under consideration (*Weyl geometry*) impossible (since the result of the comparison depends on the paths along which the vectors are transferred). This is the fundamental difference with the ideology of Riemannian geometry, which assumes the presence of a reference scale that is invariant with respect to movements.

Let us formulate more strictly the basic concepts of Weyl geometry as applied to the general N -dimensional case. We introduce an N -dimensional smooth manifold \mathcal{M} and define Riemannian metric on it $[R]g_{ik}$ ($i, k = \overline{0, N-1}$). Two Riemannian metrics $[R]g_{ik}$ and $[R]\tilde{g}_{ik}$ will be called *equivalent* if $[R]\tilde{g}_{ik} = \exp(\zeta) [R]g_{ik}$, $\zeta(\bar{x})$ is a smooth function on the manifold \mathcal{M} . A *Weyl structure* on \mathcal{M} —mapping $\hat{\mathcal{W}}: \mathcal{R}_{\mathcal{M}} \rightarrow F^{(1)}(\mathcal{M})$, satisfying the condition $\hat{\mathcal{W}}(\exp(\zeta) [R]g_{ik}) = \phi^{(1)} - d\zeta$, where $\mathcal{R}_{\mathcal{M}}$ is the equivalence class of Riemannian metrics on \mathcal{M} , $\Phi^{(1)}(\mathcal{M})$ is the space of 1-forms on \mathcal{M} , $\phi^{(1)} \equiv \hat{\mathcal{W}}([R]g_{ik})$ —1-form called a *metric potential*; the manifold $\mathcal{M}|_{\hat{\mathcal{W}}}$ with the Weyl structure we will call a *Weyl manifold*. Path connectivity on $\mathcal{M}|_{\hat{\mathcal{W}}}$ is *compatible* with its structure $\hat{\mathcal{W}}$ provided $[R]g_{ik,m} + [R]g_{ik}\phi_m^{(1)} = 0$; such a (structure-compatible) connectivity $[W]\Gamma_{jk}^i$ exists on every Weyl manifold (see, for example, [19]), and its components have the form:

$$[W]\Gamma_{jk}^i = \frac{[R]g^{i\ell}}{2} \left([R]g_{j,k} + [R]g_{k\ell,j} - [R]g_{jk,\ell} \right) - \frac{1}{2} \left([R]g_j^i \phi_k^{(1)} + [R]g_k^i \phi_j^{(1)} - [R]g_{jk} (\phi^{(1)})^i \right). \tag{5}$$

The condition of equivalence of metrics in terms of intervals is equivalent to the requirement $d\tilde{s} = \exp(\zeta/2) ds$, so we introduce a change in the scale system; in this case, the metric potential is transformed as follows way:

$$\phi_k^{(1)} \rightarrow \tilde{\phi}_k^{(1)} = \phi_k^{(1)} + \frac{\partial \zeta}{\partial x^k}, \tag{6}$$

where $\{x^k\}$ is the local coordinate system on $\mathcal{M}|_{\hat{\mathcal{W}}}$. The equation of motion is along the geodesic (for a massive particle) on the Weyl manifold, can be obtained by following, for example, the method [20]; it has the form

$$\frac{dv^i}{d\tau} + [{}^W]\Gamma_{jk}^i v^j v^k + \frac{1}{2}(\phi_j^{(1)} v^j) v^i = 0, \quad v^i = \frac{dx^i}{d\tau}, \tag{7}$$

where τ is the affine parameter (the proper time of the particle).

Let consider the Liouville equation on the Weyl manifolds. For simplicity we turn again to the case of 4-dimensional space-time. Denoting δN is the rate of change of the total number of particles in a small region \mathcal{V} of a spherical stratification $S\mathcal{M}$ (otherwise—mass hypersurface obtained from the tangent bundle $T\mathcal{M}$ by imposing constraints $[{}^R]g_{ij} v^i v^j = 0$ or 1), following the method [20], we have $\delta N = \int_{\mathcal{V}} (1 - \Pi(g)) \phi_i^{(1)} v^i f \Omega$, where $\Pi(g)$ is determined from ratios $n^i_{,i} = (1 - \Pi(g)) \phi_i^{(1)} n$, $n^i = \int v^i f d\bar{v}$ —density of particles, f is the distribution function, Ω is the volume element on $S\mathcal{M}$. On the other hand,

$$\delta N = \int_{\mathcal{V}} (\mathcal{J}(f) + (\nabla \mathcal{J})f) \Omega, \quad \mathcal{J} \equiv v^i \frac{\partial}{\partial x^i} + \frac{dv^i}{d\tau} \frac{\partial}{\partial v^i}, \tag{8}$$

where $dv^i/d\tau$ is determined from (7). Therefore, the phase volume transport equation (Liouville) acquires view

$$v^i \left(\frac{\partial}{\partial x^i} - \left([{}^W]\Gamma_{ik}^j v^k + \frac{1}{2} \phi_i^{(1)} [{}^R]g_k^j v^k \right) \frac{\partial}{\partial v^j} \right) f + \left(1 + \frac{\Pi(g)}{2} \right) \phi_k^{(1)} v^k f = 0, \tag{9}$$

where we used the ratio $(\nabla \mathcal{J})_{S\mathcal{M}} = (3/2) \phi_k^{(1)} v^k$. In terms of 4-impulse (for massive particles $p^i = mv^i$ at $v^i v_i = 1$) we get

$$p^i \left(\frac{\partial}{\partial x^i} - \left([{}^W]\Gamma_{ik}^j + \frac{1}{2} \phi_i^{(1)} [{}^R]g_k^j \right) p^k \frac{\partial}{\partial v^j} \right) f + \left(1 + \frac{\Pi(g)}{2} \right) \phi_k^{(1)} p^k f = 0. \tag{10}$$

3. Dynamics of Particles in Lagrange Geometry

The smooth *Lagrangian on bundle* $T\mathcal{M}$ over differentiable real manifold \mathcal{M} (of dimension n) there is a mapping $L : (x, y) \in T\mathcal{M} \rightarrow L(x, y) \in \mathbb{R}^1$ class C^∞ on the manifold $T\mathcal{M} \setminus \{0\}$ and continuous on the kernel $\{0\} : \mathcal{M} \rightarrow T\mathcal{M}$ of the projection endomorphism $\pi : \mathcal{M} \rightarrow T\mathcal{M}$. Hessian (with respect to y^i) of the Lagrangian L (on $T\mathcal{M} \setminus \{0\}$) $[{}^L]g_{ij}(x, y) = \frac{1}{2} \frac{\partial^2 L(x, y)}{\partial y^i \partial y^j}$ is a d -tensor field [5], covariant of rank 2, and symmetric (here i, j, k, \dots can also refer to the variable y , so so that $y^i = \dot{x}^i = y^{i\dot{}}$). Lagrangian is *regular* if for a given Hessian the following holds: $rank \left\| [{}^L]g_{ij}(x, y) \right\| = n$ on $T\mathcal{M} \setminus \{0\}$.

Lagrange space is a pair $L^n = (\mathcal{M}, L(x, y))$, formed with n -dimensional smooth manifold \mathcal{M} and regular Lagrangian $L(x, y)$, for which a d -tensor $[{}^L]g_{ij}(x, y)$ has over the bundle $T\mathcal{M} \setminus \{0\}$ constant signature.

The variational problem for the Lagrangian L leads to the Euler-Lagrange equations:

$$\frac{d}{d\tau} \left(\frac{\partial L}{\partial y^i} \right) - \frac{\partial L}{\partial x^i} = 0, \quad y^i = \frac{dx^i}{d\tau}, \quad \frac{dy^i}{d\tau} = -2 [{}^L]G^i(x, y), \tag{11}$$

where $x^i(\tau)$ depends on the parameter τ , which are equivalent to nonlinear

geodesic equations that determine the dynamics of a particle in the Lagrange space

$$\frac{d^2 x^i}{d\tau^2} + 2^{[L]}G^i \left(x^k, \frac{dx^j}{d\tau} \right) = 0, \tag{12}$$

where $^{[L]}G^i$ there are local coefficients of *canonical quasiflow* (for the space L^n) \mathcal{S} on $T\mathcal{M}$:

$$^{[L]}G^i(x, y) = \frac{1}{4} ^{[L]}g^{ij} \left(\frac{\partial^2 L}{\partial y^j \partial x^k} y^k - \frac{\partial L}{\partial x^j} \right), \quad ^{[L]}g^{ij} = \left(^{[L]}g_{ij} \right)^{-1}. \tag{13}$$

Note that for Riemann space

$$^{[L]}G^i(x, y) = (1/2) \Gamma_{jk}^i(x) (dx^k/d\tau) (dx^j/d\tau).$$

Canonical quasiflow defines accordingly *canonical N-connection* on the tangent bundle $T\mathcal{M}$ (according to Theorem 3.1 of [21]):

$$\begin{aligned} ^{[L]}N_j^i &= \frac{\partial ^{[L]}G^i}{\partial y^j} = \frac{1}{4} \frac{\partial ^{[L]}g^{ip}}{\partial y^j} \left(\frac{\partial^2 L}{\partial y^p \partial x^m} y^m - \frac{\partial L}{\partial x^p} \right) \\ &+ \frac{1}{4} ^{[L]}g^{ip} \left(2 \frac{\partial ^{[L]}g_{jp}}{\partial x^m} y^m - \frac{\partial^2 L}{\partial y^j \partial x^p} \right) + \frac{1}{4} ^{[L]}g^{ip} \frac{\partial^2 L}{\partial y^p \partial x^j}. \end{aligned} \tag{14}$$

The kinetic equation for the distribution function $f(x, p)$ can be obtained from Hamilton's equations

$$m \frac{\delta x^i}{\delta \tau} = p^i, \quad m \frac{\delta p_i}{\delta \tau} - ^{[L]}\Gamma_{ij}^k p_k p^j = 0,$$

(since on a symplectic manifold the Hamiltonian phase flow is conserved): $\{\mathcal{H}, f\} = 0$, where $\{\mathcal{H}, f\}$ is the Poisson bracket on the cotangent bundle:

$$\frac{\partial \mathcal{H}}{\partial p_i} = \frac{\delta x^i}{\delta \tau} = \frac{p^i}{m}, \quad -m \left(\frac{\partial \mathcal{H}}{\partial x^i} \right) = ^{[L]}\Gamma_{ij}^k p_k p^j. \tag{15}$$

Thus, the kinetic equation (for massive neutral particles in the absence of external non-metric fields) can be written as

$$\begin{aligned} \left(\frac{\partial \mathcal{H}}{\partial p_i} \right) \left(\frac{\delta f}{\delta x^i} \right) - \left(\frac{\partial \mathcal{H}}{\partial x^i} \right) \left(\frac{\delta f}{\delta p_i} \right) &= p^i \hat{\mathcal{D}}_i f(x, p) = 0, \\ \hat{\mathcal{D}}_i &\equiv \frac{\delta}{\delta x^i} - ^{[L]}\Gamma_{ij}^m p^j \frac{\partial}{\partial p^m}, \end{aligned} \tag{16}$$

i.e.

$$p^i \frac{\delta f}{\delta x^i} - ^{[L]}\Gamma_{ik}^j p^i p^k \frac{\partial f}{\partial p^j} = 0. \tag{17}$$

Analogs of Christoffel coefficients in Lagrange geometry are obtained trivially if we take into account the definitions horizontal lift of the vector field $\partial/\partial x^i$ ($i = \overline{0, N-1}$):

$$\frac{\delta}{\delta x^i} := \left(\frac{\partial}{\partial x^i} \right)^H = \frac{\partial}{\partial x^i} - N_i^j(x, y) \frac{\partial}{\partial y^j}. \tag{18}$$

Thus, we obtain for the connection components in the Lagrange space

$${}^{[L]}\Gamma_{jk}^i = \Gamma_{jk}^i - \frac{\mathbf{g}^{mi}}{2} \left(\frac{\partial^{[L]}\mathbf{g}_{mj}}{\partial y^\ell} N_k^\ell + \frac{\partial^{[L]}\mathbf{g}_{km}}{\partial y^\ell} N_j^\ell - \frac{\partial^{[L]}\mathbf{g}_{jk}}{\partial y^\ell} N_m^\ell \right), \quad (19)$$

$$\Gamma_{jk}^i = \frac{\mathbf{g}^{mi}}{2} \left(\frac{\partial^{[L]}\mathbf{g}_{mj}}{\partial x^k} + \frac{\partial^{[L]}\mathbf{g}_{km}}{\partial x^j} - \frac{\partial^{[L]}\mathbf{g}_{jk}}{\partial x^m} \right). \quad (20)$$

4. Relationship between Dynamics on Weyl Manifolds and Lagrange Geometry

The equations of motion and kinetic equations obtained above the equations for Weyl and Lagrange manifolds look like are quite similar to each other. Therefore, one can ask the question on the interpretation in terms of Lagrangian dynamics of the metric potential $\phi^{(1)}$ in order to identify its physical meaning.

Equations of motion on Weyl manifolds and in the Lagrange space have the form (7) and (12), respectively. To establish a correspondence between these equations it is necessary to consider the ratio

$${}^{[W]}\Gamma_{jk}^i + \frac{1}{2}\phi_j^{(1)[R]}\mathbf{g}_i^k = \frac{1}{2}{}^{[L]}\mathbf{g}^{ij} \left(\frac{\partial^2 L}{\partial y^i \partial x^k} y^k - \frac{\partial L}{\partial x^j} \right), \quad (21)$$

allowing for considerable arbitrariness in the choice of the canonical quasi-flow for the correct definition of the metric potential. For the Riemannian structures flow ($2G_{,j,k}^i(x,y) = {}^{[R]}\Gamma_{jk}^i$) then we have for $\phi^{(1)}$ the system of equations

$${}^{[R]}\mathbf{g}_j^i \phi_k^{(1)} + {}^{[R]}\mathbf{g}_k^i \phi_j^{(1)} - {}^{[R]}\mathbf{g}_{jk}^i (\phi^{(1)})^i = {}^{[R]}\mathbf{g}_i^k \phi_j^{(1)}.$$

For a more general quasi-flow with $2G_{,j,k}^i(x,y) = {}^{[L]}\Gamma_{jk}^i$ we get

$$\Gamma_{jk}^i - \frac{\mathbf{g}^{mi}}{2} \left(\frac{\partial^{[L]}\mathbf{g}_{mj}}{\partial y^\ell} N_k^\ell + \frac{\partial^{[L]}\mathbf{g}_{km}}{\partial y^\ell} N_j^\ell - \frac{\partial^{[L]}\mathbf{g}_{jk}}{\partial y^\ell} N_m^\ell \right) = {}^{[W]}\Gamma_{jk}^i + \frac{1}{2}\phi_j^{(1)[R]}\mathbf{g}_i^k,$$

where the connection coefficients Γ_{jk}^i in the last formula are defined in (20). We again obtained a system of linear equations for metric potential.

Similarly, we can consider the comparison of kinetic Equations (10) and (17). However, here there is a certain subtlety here—the fact is that the equation Liouville for a Weyl manifold is actually self-consistent (due to the definition of the coefficient $\Pi(g)$). However, the structure of the transport equation in both cases is the same, and if we add to (17) the term, taking into account the metric interaction of particles in the system, then the solutions of both mentioned equations will behave identically. In particular, by setting $\phi^{(1)}$ as a parameter, we obtain the branching solutions of the kinetic equation (of the Hammerstein type) in the neighborhood critical value $\phi_0^{(1)} = 0$ (apparently, there are other bifurcation points determined by non-trivial $\phi^{(1)}$ —their should be interpreted as structural large-scale transitions in the system particles).

Thus, the particle dynamics and kinetics of multiparticle systems on Weyl manifolds can be viewed as the corresponding dynamics and kinetics of systems with

corresponding Lagrangians. Therefore, scale covariance of quantities should be considered not an exotic feature of Weyl theory, but a consequence of the applied mathematical formalism.

5. Conclusion

The Lagrangian geometry and the mechanics of systems on the Weyl manifolds are closely related. A consequence of this connection is the unification of the powerful mathematical apparatus of these approaches, which leads to highly nontrivial conclusions, in particular concerning the foundations of special and general relativity (the introduction of the Caratheodory transformation along with the Lorentz transformation, the development of the theory of generalized curvature) and certain astrophysical problems (the transfer of a particle system through the event horizon of a black hole, the collapse, and the formation of singularities).

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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