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Geometric Aspects of the Liouville and Vlasov Equations Theory in the Phase Space

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Abstract

The article discusses the method of obtaining from general relativistic actions for systems of interacting massive charged particles the corresponding Vlasov-type equations. It is shown that the type of action depends on both the impulse/speed representation in the components of the integral core actions, as well as from the full metric of the phase space of the system.

Keywords

Asaki Metric, Einstein-Hilbert Action, Vlasov-Einstein Equation

1. Introduction

Currently, the key methodology for constructing cosmological models and studying general relativistic properties of multi-particle ensembles and electromagnetic and scalar fields as subsystems can be recognized as the Lagrangian formalism. Essentially, the study of each of the mentioned models comes down to constructing an action of a sufficiently general form, containing terms that can be further interpreted as a kind of generating terms in the resulting "equations of motion" of the system under consideration. However, there are certain subtleties associated with the correct consideration of the structure of the terms in the action, associated with the nonequilibrium nature of the evolution of interacting multiparticle systems in a gravitational field. In the overwhelming majority of methodological manuals on dynamics in the theory of relativity, the consideration is limited to the hydrodynamic level of motion (continuous medium approximation). In this case, it is essentially ignored complex structure (corpuscular, star clusters, etc.) of the general relativistic system under study; as examples, one can refer to fundamental works [1]-[5], content and methodology of presentation, which are the most typical for the literature describing the approaches and applications of

General Theory of Relativity. The classical representation of the energy-momentum tensor/EMT of the type $\mathrm{diag}\left(\rho c^2, -p, -p, -p\right)$ is fundamentally incompatible with the local nature of the metric transformation and the influence on EMT of the motion invariant $g_{\alpha\beta} v^\alpha v^\beta$, since, unlike the metric on the phase space, the EMT in this case does not reduce to a trace form. If one attempts to use the hydrodynamic description in a curved space, it is completely unclear how to formally project the 8-dimensional phase space onto the space of macrovariables: the projection of the phase space onto 4-dimensional spacetime is possible and even obvious in some cases, for example, when the space is homogeneous and isotropic (the projection will provide an additional factor for the metric coefficients on the 4-dimensional manifold).

Further, the structure of individual terms in the summary action must be examined separately in order to exclude from consideration mathematically correct, but physically unfounded consequences; here, naturally, the rule must be observed to maintain a certain balance of the situation, since the "non-physicality" of the resulting equations may indicate about the presence of fundamentally new effects that contradict existing paradigms. And finally, it is necessary to take into account the possibility of optimizing the formalism used in relation to specific problems—here it is implied that the mathematical apparatus for analyzing dynamics of the multiparticle system should be adapted within the a priori physical framework of the problem being solved.

The aim of this work is to develop a method for obtaining an explicit form of equations of the Liouville-Einstein type and Vlasov-Einstein for a system of gravitationally interacting particles of different types (including stars, galaxies, etc. in cosmology) in curved space-time, taking into account the general topological structure of the phase space, which examines the motions of a many-particle general relativistic system. It should be noted that a fairly large number of papers have been devoted to solving this problem (in various formulations). Publications dating back to the 1950s. The most significant works should be noted: A.A. Vlasov [6] [7], N.A. Chernikov [8] and Yu. L. Klimontovich [9]. The current state of the problem is covered in great detail in the books [10]-[12], including, among other things, analysis approaches used by various authors over the past decades. The method of obtaining equations of the Vlasov-Einstein type is currently accepted by the scientific community default is enough worked out, although in reality requires either the use of a very labor-intensive mathematical apparatus based on general relativistic chains of BBGKY [13], or introduction of non-canonical Poisson-Morrison brackets of generalized Hamiltonian dynamics [14], etc. Therefore, it seems appropriate to draw attention to the possibilities of the universal and simple Lagrangian formalism to obtain a Vlasov-type equation for massive particles, taking into account the possible presence of electric charges and an electromagnetic field through which they interact Poisson, Vlasov-Maxwell-Einstein equations [15]); at the same time this formalism without fundamental changes also applicable for deriving the Vlasov-Yang-Mills, Vlasov-Brans-Dicke, etc. equations. (derived in the simplest possible way, *i.e.* using the apparatus of Lagrangian dynamics), in particular, taking into account the fact that evolution particle system must be described sequentially on the 8-dimensional (co)tangent bundle over the space-time manifold; in this case, the dynamic properties of the system, defining the form of the equations of motion, acquire substantially new aspects.

2. The Structure of the Action of a System of Particles in a Gravitational Field

It is quite obvious that the directly obtained values depend on the structure of the kinetic potential (Lagrangian) of the system. or indirectly its dynamic properties, such as evolution as a whole and changes in the state of partial properties (changes distribution of particles, geometric dimensions of the system and its isolated parts, taking into account the need for introduction or exclusion types of interparticle interactions at the corresponding scales, etc.). However, constructing the form of the Lagrangian, the dynamic consequences of which are physically justified and correspond to reality, is not an entirely trivial task even in the simplest case of a system of point massive neutral particles in an external gravitational field (this will be shown below in sufficient detail). In fact, in almost all textbooks on general relativity, this question is not raised, and the authors a priori accept some specific form of the Lagrangian density as a single-valued "generating function" to establish the form of equations of motion when studying the dynamics of a system. In this case, it is established a priori that the main criterion for the validity of the internal structure of the analyzed action is the possibility of obtaining Einstein's equations on its basis, with the right-hand side containing the energy-momentum tensor of matter distributed in a certain way in the region of space-time under study. However, modification of Lagrangian density structure (for example, if we consider a system of massive charged particles) can lead to a fundamentally incorrect form of kinetic equation (see [16] [17] for details).

Einstein's equations for the gravitational field due to some distribution of material objects (particles with masses $m_i \equiv 1$), as is known, can be obtained from the principle of stationarity of action by variation of the sum of the Einstein-Hilbert actions S_{EH} and the system of particles S_p: $\delta\left(S_{EH} + S_p\right) = 0$ (in the more general case, including the presence of vector and scalar fields, instead of S_p one should consider the full "material" action $S_m = S_p + S_{pf} + S_f$, taking into account these fields S_f and the interaction of fields with particles S_{pf}). In this case, $S_{EH} = \int L_{EH} |g|^{1/2} \mathrm{d}^4 x$, where $g \equiv \det\left(g_{\alpha\beta}\right)$, $g_{\alpha\beta}$—covariant metric tensor of the manifold (space-time) under consideration M_4 (here and further for indices denoted by Greek letters, it is assumed that $\alpha, \beta, \cdots = \overline{0,3}$). The quantity $|g|^{1/2} \mathrm{d}^4 x \equiv \mathrm{d}\mathcal{V}_x^{(4)}$ in this case is an element of a 4-volume in curvilinear coordinates $\left\{x^\alpha\right\}_{\alpha=0,\cdots,3}$. The variation δS_{EH} can be obtained as a result of standard calculations:

$$\delta S_{EH} = -\frac{c^3}{16\pi G}\int\left(R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R - \Lambda g_{\alpha\beta}\right)\delta g^{\alpha\beta}\mathrm{d}\mathcal{V}_x^{(4)},$$

where: G —gravitational constant, $R_{\alpha\beta}$ —tensor Ricci, R —scalar curvature, Λ —the cosmological lambda term. For empty spacetime ($L_p \equiv 0$) from $\delta S_{EH} = 0$ due to the arbitrariness of the variation of the metric tensor from here we have Einstein's classical field equations.

The material action of a system of N discrete particles, numbered with index a ($a = 1,\cdots,N$), can be written as follows (the so-called "natural" representation):

$$S_{p,1} = \sum_a \int_{(s_1)_1}^{(s_a)_2} \left(L_{p,1}\right)_a \mathrm{d}\lambda,\ \left(L_{p,1}\right)_a = -m_a c\left(g_{\alpha\beta}v^\alpha v^\beta\right)_a^{1/2},\ v_a^\alpha \equiv \frac{\mathrm{d}x_a^\beta}{\mathrm{d}\lambda},$$

where: $\mathrm{d}s_a\left(\equiv c\mathrm{d}\tau_a\right)$—element of general relativistic interval $\mathrm{d}s_a^2 = g_{\alpha\beta}\mathrm{d}x_a^\alpha \mathrm{d}x_a^\beta$, $\left(L_{p,1}\right)_a$—local density of kinetic potential (Lagrangian). Summation over individual particles can be replaced by integration over phase space along geodesics using identity transition involving the local distribution function $f_a\left(x^\alpha, v^\mu\right)$ (or $f_a\left(x^\alpha, p_\mu\right)$) particles:

$$\sum_a \int(\ldots)\mathrm{d}s_a \to \int \sum_a f_a\left(x^\alpha, \xi(\mu)\right)(\ldots)\mathrm{d}\mathcal{V}^{(8)},$$

where $\mathrm{d}\mathcal{V}^{(8)}$ is an invariant volume element in 8-dimensional (co)tangent bundle $T^*M_4\left(TM_4\right)$ over 4-dimensional configuration manifold $M_4 = \left\{x^\alpha\right\}_{\alpha=0,\cdots,3}$ respectively, if we assume that $\xi(\mu) = p_\mu$ or $\xi(\mu) = v^\mu$ ($v^\mu \in TM_4$, $p_\mu \in T^*M_4$).

Here, there is an ambiguity in the choice of the structure of the action $S_{p,1}$, associated with the possibility of choosing the variables of the function distributions (*i.e.* $f\left(x^\alpha, p_\beta\right)$ or $f\left(x^\alpha, v^\beta\right)$) and element of the 8-dimensional phase space. Indeed, if for a 4-dimensional space-time manifold M_4

$$\mathrm{d}\mathcal{V}_x^{(4)} = |g|^{1/2}\mathrm{d}^4x$$

is geometrically the only acceptable choice (providing a standard form of field equations Einstein in the analysis of the Einstein-Hilbert action), then for the tangent bundle the element 8-volume, generally speaking, depends on the type of the complete "natural induced" metric [18], defined by the following quadratic forms:

$$\text{I)}\ \ g_{\alpha\beta}\mathrm{d}x^\alpha \mathrm{d}x^\beta,\ \text{II)}\ \ 2g_{\alpha\beta}\mathrm{d}x^\alpha Dv^\beta,\ \text{III)}\ \ g_{\alpha\beta}Dx^\alpha Dv^\beta,$$

where $Dv^\beta \equiv \mathrm{d}v^\beta + \Gamma^\beta_{\mu\nu}v^\mu \mathrm{d}x^\nu$ —covariant differentials of the coordinates of the tangent vector. The use of the "full" (pseudo)metric variant on TM_4 (e.g. I + II, II, II + III, I + III) should be determined additional conditions to a specific physical problem, since the dynamics of a system of particles depends significantly on the choice of the type of metrization TM_4 (for some problems, the optimal choice may be considered metric types that are not combinations of the above quadratic forms [19]-[21]). In the literature on phase space dynamics, special attention is paid to metric of type I + III (apparently due to the fact that it was introduced into consideration earlier than the others), otherwise, Sasaki diagonal lift metric [22]:

$$d\sigma^2_{\mathrm{I+III}} = g_{\alpha\beta} dx^\alpha dx^\beta + g_{\alpha\beta}\left(dv^\alpha + \Gamma^\alpha_{\zeta\mu} v^\zeta dx^\mu\right)\left(dv^\beta + \Gamma^\beta_{\sigma\kappa} v^\sigma dx^\kappa\right) \quad (1)$$

(we will assume that in the right-hand side of the above relation the dimensions of the terms are matched, for example, by redefining differentials dx^α, dv^β).

It is usually implicitly assumed that

$$d\mathcal{V}^{(8)}\left(TM_4\right) = d\mathcal{V}_x^{(4)} d\mathcal{V}_v^{(4)} = \left|g_x\right|^{1/2} d^4x \left|g_v\right|^{1/2} d^4v \quad (2)$$

(and similarly for T^*M_4: $d\mathcal{V}^{(8)}\left(T^*M_4\right) = d^4x d^4p$). However, the situation here is actually much more complex and therefore requires a detailed commentary. Firstly, the presence of the factor $\left|g_x\right|^{1/2}$ for the hypereuclidean volume element d^4x is due to the following: the 4-volume introduced above

$$d\mathcal{V}_x^{(4)} = \left|g_x\right|^{1/2} d^4x = \frac{1}{4!}\left|g_x\right|^{1/2} \varepsilon_{\mu\nu\rho\sigma} dx^\mu \wedge dx^\nu \wedge dx^\rho \wedge dx^\sigma$$

($\varepsilon_{\mu\nu\rho\sigma}$—Levi-Civitta tensor) is invariant under coordinate transformations, while the element d^4x is not invariant.

Secondly, by analogy with the invariant volume element on the configuration 4-manifold M_4 in the literature the corresponding one is introduced into consideration in a standard manner element in the space VM_4 tangent to it: $d\mathcal{V}_v^{(4)} = \left|g_v\right|^{1/2} d^4v$ (see, for example, [11]). However, relation (2) should be considered as correct only in the adapted ("angolonomic") system coordinates [23] (for more details see below). In addition, metrics of types I + III, II, I + II are only special (degenerate) cases characterized by the condition $\left|g_v\right| = \left|g_x\right| \equiv \left|g\right|$, in the class "$g$-natural metrics" [18] on TM_4 (this class also contains a wide subclass of metrics for which of incompressibility of the geodesic flow on TM_4 are violated).

3. Introduction of the Distribution Function into the Lagrangian Density and Angolonomic Basis of 8-Dimensional Space

For a multiparticle system, the action $S_{p,1}$ can be represented through introducing a set of partial distribution functions in the 8-dimensional space $f_a(x,v)$:

$$S_p[f] = -\sum_a m_a c^2 \int \sqrt{g^{\alpha\beta}(\boldsymbol{x}) v^\alpha v^\beta}\, f_a(\boldsymbol{x},\boldsymbol{v})\, d\mathcal{V}^{(8)}\left(TM_4\right),$$

and the reverse transition from the action $S_p[f]$ to the action $S_{p,1}$ is performed by the substitution $f_a(\boldsymbol{x},\boldsymbol{v}) = \delta\left(\boldsymbol{x} - \boldsymbol{x}_a(\lambda)\right)\delta\left(\boldsymbol{v} - \boldsymbol{v}_a(\lambda)\right)$. If we take into account that on the 8-dimensional manifold TM_4 a metric of type I + III is defined a priori, then structure of the (8 × 8) matrix $G^{\mathrm{I+III}}_{\mathfrak{AB}}\Big|_{\mathfrak{A}=\overline{0,7};\mathfrak{B}=\overline{0,7}}$ the coefficients of the full metric are as follows:

$$G^{\mathrm{I+III}}_{\mathfrak{AB}}(\boldsymbol{x},\boldsymbol{v}) = \begin{pmatrix} g_{\alpha\beta}(\boldsymbol{x}) + g_{\mu\nu}(\boldsymbol{x}) N^\mu_\alpha(\boldsymbol{x},\boldsymbol{v}) N^\nu_\beta(\boldsymbol{x},\boldsymbol{v}) & N_{\zeta'\alpha}(\boldsymbol{x},\boldsymbol{v}) \\ N_{\gamma'\beta}(\boldsymbol{x},\boldsymbol{v}) & g_{\gamma'\zeta'}(\boldsymbol{x}) \end{pmatrix},$$

$$N_{\alpha}^{\zeta}(\boldsymbol{x},\boldsymbol{v}) = v^{\beta}\Gamma_{\beta\alpha}^{\zeta}(\boldsymbol{x}),\ \gamma',\zeta' = \overline{4,7},$$

where N_{α}^{ζ} are the so-called gauge potentials [24]. By transforming the coordinates, this matrix can be brought to a "block-diagonal" form, that is, in other words, go to adapted coordinate basis on the tangent bundle TM_4, in which the 8-metric can be represented in the following form (its coefficients now depend only on one variable $\tilde{\boldsymbol{x}}$, being a function old independent arguments $(\boldsymbol{x},\boldsymbol{v})$):

$$\tilde{G}_{\mathfrak{A}\mathfrak{B}}^{\mathrm{I+III}}(\tilde{\boldsymbol{x}}) = \begin{pmatrix} g_{\alpha\beta}(\tilde{\boldsymbol{x}}(\boldsymbol{x},\boldsymbol{v})) & 0 \\ 0 & g_{\mu'\nu'}(\tilde{\boldsymbol{x}}(\boldsymbol{x},\boldsymbol{v})) \end{pmatrix},\ g_{\mu'\nu'} = g_{\mu\nu}.$$

Thus, we can say that the transition to angolonomic variables for the metric I + III on the tangent manifold, on the one hand, allows you to simply write down the simplest functionality in the form invariant element 8-volume $\mathcal{V}^{(8)} = \int_{\tilde{\Omega}_x\times\tilde{\Omega}_v} |g(\tilde{\boldsymbol{x}})|^{1/2}\,\mathrm{d}\tilde{\boldsymbol{x}}\,|g(\tilde{\boldsymbol{x}})|^{1/2}\,\mathrm{d}\tilde{\boldsymbol{v}}$, but, on the other hand, when calculating the action functional S_p (and, in general, S_m) with a kernel depending on the "physical" coordinates $(\boldsymbol{x},\boldsymbol{v})$, reducing the invariant volume element to a simple is accompanied by a complication of the form of the functional core, written in transformed "adapted" coordinates $(\tilde{\boldsymbol{x}},\tilde{\boldsymbol{v}})$.

Accordingly, the action of S_p is representable through the kernel defined on the cotangent bundle T^*M_4 over configuration space M_4:

$$S_p = -\sum_a m_a c^2 \iint \frac{\sqrt{p_{\mu}((\tilde{\boldsymbol{x}},\tilde{\boldsymbol{p}}))\,g^{\xi\mu}(x(\tilde{\boldsymbol{x}},\tilde{\boldsymbol{p}}))\,p_{\xi}(\tilde{\boldsymbol{x}},\tilde{\boldsymbol{p}})}}{g^{\xi 0}(\boldsymbol{x}(\tilde{\boldsymbol{x}},\tilde{\boldsymbol{p}}))\,p_{\xi}(\tilde{\boldsymbol{x}},\tilde{\boldsymbol{p}})} f_a(\boldsymbol{x}(\tilde{\boldsymbol{x}},\tilde{\boldsymbol{p}}),p(\tilde{\boldsymbol{x}},\tilde{\boldsymbol{p}}))\,\mathrm{d}\tilde{\boldsymbol{x}}\mathrm{d}\tilde{\boldsymbol{p}},$$

in this case, instead of the Sasaki metric on the 8-dimensional $(\boldsymbol{x},\boldsymbol{v})$-bundle, it is necessary to consider the Sato metric [21] on phase $(\boldsymbol{x},\boldsymbol{p})$-space.

4. Forms of Action of a Multiparticle System and the Dependence of Its Dynamics on the Choice of Metric on the Tangent Bundle

Let us present the main variants of actions containing various representations of $L_{p,1}[f]$:

$$S_{p,1}^{[1]} \propto \sum_a \int_{\tilde{\Omega}_x^4\times\tilde{\Omega}_v^4} K_{p,a}^{[1]}(x^{\alpha},v^{\beta}\,|\,g_{\mu\nu})\,\mathrm{d}\tilde{\mathcal{V}}^{(8)},$$

$$S_{p,1}^{[2]} \propto \sum_a \int_{\tilde{\Omega}_x^4\times\tilde{\Omega}_p^4} K_{p,a}^{[2]}(x^{\alpha},p_{\beta}\,|\,g_{\mu\nu})\,\mathrm{d}\tilde{\mathcal{V}}^{(8)},$$

$$K_{p,a}^{[1]} = m_a f_a(x^{\alpha},v^{\beta})\sqrt{g_{\mu\nu}v^{\mu}v^{\nu}},\ K_{p,a}^{[2]} = m_a f_a(x^{\alpha},p_{\beta}(v^{\zeta}))\sqrt{g_{\mu\nu}v^{\mu}v^{\nu}},$$

$$K_{p,a}^{[3]} = m_a f_a(x^{\alpha},v^{\beta}(p_{\zeta}))\sqrt{g_{\mu\nu}v^{\mu}(p_{\zeta})v^{\nu}(p_{\zeta})}.$$

In addition to these types of general relativistic action for a multiparticle system, the literature considers a whole class of actions of similar form, the only limitation of which is the possibility of obtaining from each representative of the given class Euler-Lagrange equations (EL) of the form

$\mathrm{d}^2x^\alpha/\mathrm{d}\tau^2+\Gamma^\alpha_{\beta\zeta}\left(\mathrm{d}x^\beta/\mathrm{d}\tau\right)\left(\mathrm{d}x^\zeta/\mathrm{d}\tau\right)=0$ (taking into account possible reparametrization). The most typical representatives of this set of actions are will be $S^{[m]}_{p,2}$ ($m=1,2$), equivalent in form to the above $S^{[m]}_{p,1}$, but with the replacement of cores with the following:

$$\mathcal{K}^{[1]}_{p,a}=m_a f_a\left(x^\alpha,v^\beta\right)\frac{g_{\mu\nu}}{2}v^\mu v^\nu,\ \mathcal{K}^{[2]}_{p,a}=m_a f_a\left(x^\alpha,p_\beta\left(v^\zeta\right)\right)\frac{g_{\mu\nu}}{2}v^\mu v^\nu,$$

$$\mathcal{K}^{[3]}_{p,a}=m_a f_a\left(x^\alpha,v^\beta\left(p_\zeta\right)\right)\frac{g_{\mu\nu}}{2}v^\mu\left(p_\zeta\right)v^\nu\left(p_\zeta\right).$$

However, when moving to a more complex—in particular, additive—structure of the material actions $S_p\to S_p+S_{pf}$ equations of geodesic motion for operations with kernels $K^{[i]}_{p,a}$ and $\mathcal{K}^{[i]}_{p,a}$ will have different forms.

If in the expressions for $S^{[1,2,3]}_{p,1}$ we integrate with the weights $\left|g_x\right|^{1/2}\mathrm{d}^4x\left|g_v\right|^{1/2}$, then *we define the dynamics on the tangent bundle* TM_4. The metric g_v is introduced a priori on the tangent space—this is the most significant difference from the "standard" approach (used in the vast majority of publications on this topic), which assumes that g_x uniquely determines the value of g_v when writing out the explicit form of the volume element $\mathrm{d}\mathcal{V}^{(4)}_v$. But this is not legal without fixing the metric on TM_4.

"*The definition of dynamics*" means, *firstly*, that on TM_4 a differential structure is introduced, induced from the "basic" configuration varieties M_4 in such a way that the natural the mapping $\pi:TM_4\to M_4$ is a differentiable submersion and the triple $\left(TM_4,\pi,M_4\right)$ is smooth vector bundle. If $\left(U,\varphi=\left(x^\alpha\right)\right)$ is a local chart in M_4, then any curve $\omega:I\to M_4$ ($\operatorname{im}\omega\subset U$), passing through m. $\boldsymbol{X}$ at $\chi=0$, is analytically representable as $x^\alpha=x^\alpha\left(\chi\right)$, $\chi\in I$, $\varphi\left(\boldsymbol{X}\right)=\left(x^\alpha\left(0\right)\right)$ $\left(\alpha=\overline{1,4}\right)$. The tangent vector $\left[\omega\right]_X$ is defined by a pair of real quantities $x^\alpha=x^\alpha\left(0\right)$, $v^\alpha=\left(x^\alpha\right)_\chi\left(0\right)$. In this case, the pair $\left(\pi^{-1}\left(U\right),\Phi\right)$, where $\Phi\left(\left[\omega\right]_X\right)=\left(x^\alpha,v^\alpha\right)\in\mathbb{R}^8$, $\forall\left[\omega\right]_X\in\pi^{-1}\left(U\right)$ is a local map on TM_4; the set of these induced local maps $\left(\pi^{-1}\left(U\right),\phi=\left(x^\alpha,v^\alpha\right)\right)$ defines a structure on TM_4 such that TM_4 is an 8-dimensional is a smooth manifold, and $\left(TM_4,\pi,M_4\right)$ is a differentiable vector bundle.

Secondly, the Euler-Lagrange equations of geodesic movement (in which it is assumed that the dependent variables are now are the components of the vector $TM_4\ni\boldsymbol{Y}\equiv\left\{y^{\mathfrak{N}}\right\}_{\mathfrak{N}=\overline{1,8}}=\left\{x^\alpha,v^\alpha\right\}_{\alpha=\overline{1,4}}$, and the independent variable is geodesic parameter $\sigma\in\mathbb{R}^1$ from (1)), according to [22], take the Euler-Lagrange equations form for the Lagrangian $L\left(\sigma,y^{\mathfrak{N}},\left(y^{\mathfrak{N}}\right)_\sigma\right)$; the number of equations will increase from 4 to 8, of which 7 are independent. In the case of a metric of type I + III at this the explicit form of these equations is as follows (for the particle with which is associated 1-partial distribution function $f_a\left(\boldsymbol{x},\boldsymbol{v},t\right)$):

$$\frac{\mathrm{d}^2x^\alpha}{\mathrm{d}\sigma^2}+\Gamma^\alpha_{\beta\gamma}\frac{\mathrm{d}x^\beta}{\mathrm{d}\sigma}\frac{\mathrm{d}x^\gamma}{\mathrm{d}\sigma}=R^\alpha_{\beta\eta\zeta}\frac{\mathrm{d}x^\beta}{\mathrm{d}\sigma}v^\eta\frac{Dv^\zeta}{\mathrm{d}\sigma}\equiv\hat{\mathcal{F}}\left(x^\beta;v^\eta,v^\zeta\right),\ \frac{D^2v^\alpha}{\mathrm{d}\sigma^2}=0,\qquad(3)$$

where $R^{\alpha}_{\beta\eta\zeta}$ is a tensor curvature diversity space-time M_4; geodesic parameter defined with the help of condition

$$g_{\alpha\beta}\left(\mathrm{d}x^{\alpha}/\mathrm{d}\sigma\right)\left(\mathrm{d}x^{\beta}/\mathrm{d}\sigma\right)+g_{\alpha\beta}\left(Dv^{\alpha}/\mathrm{d}\sigma\right)\left(Dv^{\beta}/\mathrm{d}\sigma\right)=1 \tag{4}$$

(8th independent equation). It should be noted that in the above formulas, generally speaking, $v^{\alpha} \equiv \mathrm{d}x^{\alpha}/\mathrm{d}s \neq \mathrm{d}x^{\alpha}/\mathrm{d}\sigma$. The presence of an effective force $\hat{\mathcal{F}}$ on the right side of the 1st equation of motion is ensured by non-zero curvature of the tangent space; for the Minkowski metric on M_4 system (3) degenerates in Euler-Lagrange equations for one dependent variable, and, accordingly, $\sigma \to s$, where s is the geodesic parameter on M_4. In the case of a type II metric, in the analogue of the first of Equation (3), the force term is identically cancelled out, and the second equation takes the form of the Jacobi equation $D^2v^{\alpha}/\mathrm{d}\sigma^2 + R^{\alpha}_{\beta\eta\zeta}v^{\eta}\left(\mathrm{d}x^{\beta}/\mathrm{d}\sigma\right)\left(\mathrm{d}x^{\zeta}/\mathrm{d}\sigma\right)=0$ (the velocity field coincides with the deviation field along the geodesic M_4), and this demonstrates the fact that the dynamics on TM_4 really depends quite significantly on the method of metrization.

5. Conclusion

In this paper, the methodic of constructing kinetic equations of the Liouville and Vlasov type for the system is considered charged gravitating particles by the method of variation of action taking into account the ambiguity of the Lagrangian densities of kinetic potentials (kernels of action integrals) and a method for topologizing the (co)tangent bundle over a configuration manifold on which the movement of the system of particles occurs. It has been established that during the transition from the Lagrangian representation when writing the generalized Einstein-Hilbert action to the form using distribution functions (respectively, for equations containing moments of distribution functions, *i.e.*, the hydrodynamic type), a certain arbitrariness arises in the choice of variables, requiring a priori assumptions to overcome them. In addition, when deriving equations of the Vlasov type, it is necessary to specifically highlight the used metrization of the (co)tangent bundles, since the Sasaki/Sato diagonal lift metric, used by default in the relevant literature, is not the only possible one: the legitimacy of the appropriate choice must be determined by additional physical considerations. As an example of the dependence of the dynamics on the choice of metrization method, we can consider the first term on the left-hand side of equality (4). On the 4-dimensional Riemannian manifold of general relativity, this term is invariant $I = g_{\alpha\beta}v^{\alpha}v^{\beta}$ (related to the law of conservation of energy). However, introduction of different metrics on the tangent bundle leads to a change in this invariant, which entails a change in the values of the energy-momentum tensor/EMT in Einstein's equations, since when representing the EMT of a system of particles through the integral of their distribution function, the integrand contains the aforementioned invariant I (in accordance with the results previously obtained by the author in [15]-[17]).

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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