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Deriving Some Results of Quantum Mechanics from Classical Physics

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Abstract

The idea is that inside the phase of a wave, there exists a variational structure. This allows a correspondence with Hamiltonian Mechanics. This allows a correspondence between waves and particles. Combined with Dispersive Wave Theory, results from Quantum Mechanics follow.

Keywords

Classical Mechanics, Lagrangian Mechanics, Hamiltonian Mechanics, Variational Methods, Hamilton's Principle, Stationary Phase Principle, Extended Canonical Transformation, Whitham's Methods, Eikonal Representation, Dispersive Waves, Quantum Mechanics, Postulates of Quantum Mechanics, Wave-Particle Duality, Born's Postulate, de Broglie Relation, Planck Relation, Schrodinger Equation, Schrodinger's Cat, Group Velocity

1. Introduction: Explaining Wave-Particle Duality

What is a wave? From the point of view of PDEs, it is a disturbance that moves along characteristics(rays). According to Whitham [1], there are two kinds of waves. Waves where the characteristics are inside the amplitude and those where the characteristics are inside the phase. The latter kind is called dispersive waves (which are also described by variational methods). This work is based on the fact that a quantum wavefunction is a dispersive wave. (Briefly, a Dispersive Wave is a wave of the form $\psi = A\mathrm{e}^{i\theta}$, where A is the amplitude and θ is the phase, which in turn is related to the angular frequency ω and wave number k. Once plugged into the governing equation, we are left with a Dispersion relation relating ω and k: $\omega = \Omega(k)$. A more detailed definition is given below.)

Particles also travel along characteristics, as is seen from variational methods. But here, the characteristics are trajectories (described by, say, Hamiltonian mechanics) rather than rays. Thus, wave-particle duality may be seen as ray-trajectory equivalence. The ray-trajectory equivalence is well known from mechanics, see, for example, Section 10-8 in [2]. However, less known is the ray-wave equivalence. That is, given a ray, we may reconstruct the associated wavefronts using Hamilton's Stationary Wave Principle. The fact that waves and particles are described by variational methods explains why these methods are so fundamental in physics, both for classical physics as well as quantum physics (as well as fields). Even the space $\left(p,q\right)$ in classical mechanics is called the phase space. Hamilton realized that his treatment applied equally to particles as well as waves, as is evident from the title of his research paper [3]. Had he seen the experiments demonstrating the wave-particle duality, he might have come up with quantum mechanics himself. It also explains why quantum mechanics may be applied to classical waves, as is the case for plasma physics [4].

Moreover, as there exists an underlying variational structure for both particles and waves explains why both are related using a variational structure… that is classical particle time development may be described by Hamiltonian mechanics and quantum mechanics evolution is described by a Hamiltonian Operator. (Though Quantum Mechanics may also be described by Lagrangian mechanics, as is the case of Feynman Treatment.)

It is worth noting that, as there exists an analogy of Einstein's General Theory of Relativity in terms of geometric optics and rays… then there also exists a reformulation in terms of waves through the wave-ray analogy. This raises hope of quantizing general relativity. Quantum physics is more than a theory, it's a process, every classical theory has its quantum counterpart (through quantization). This is because every Algebraic Expression of energy conservation may be converted to a differential wave equation… through identifying the energy/frequency to the temporal derivative and the momentum/wave-vector to the spatial derivative.

In this work, we attempt to derive some of the postulates of Quantum Mechanics from Classical physics. For future reference, we state the postulates of Quantum Mechanics, see [5]. Here, with abuse of notation, we designate both the observable and the corresponding eigenvalue with the same mathematical designation.

- Postulate I: Observables and Operators

To any observable in physics, say A, there corresponds an operator $\hat{A}$, such that the measurement of A yields eigenvalues A of the operator $\hat{A}$

$$\hat{A}\varphi = A\varphi \tag{1.1}$$

where φ is the eigenfunction of the operator $\hat{A}$ corresponding to the eigenvalue A.

- Postulate II: Measurement in Quantum Mechanics

Measurement of the observable A that yields the value A leaves the system in the state φ_A. Where φ_A is the eigenfunction of the operator $\hat{A}$ corresponding to the eigenvalue A.

- Postulate III: The State Function and the Properties of the System

For a system in the state $\psi(\boldsymbol{r},t)$, the average of physical observable A is

$$\langle A\rangle = \int \psi^{*}\hat{A}\psi \mathrm{d}\boldsymbol{r} \tag{1.2}$$

where, $\mathrm{d}\boldsymbol{r}$ is the element of volume and the average $\langle A\rangle$ is called the expectation value of the observable A.

- Postulate IV: Time Evolution of the State Function

  The state $\psi(\boldsymbol{r},t)$ evolves in time according to:

$$i\hbar\frac{\partial}{\partial t}\psi(x,t) = \hat{H}\psi(x,t) \tag{1.3}$$

This is the time-dependent Schrodinger Equation and $\hat{H}$ is the Hamiltonian Operator.

- Born Postulate:

  For a given wavefunction ψ, the probability density P is:

$$|\psi|^{2} = P \tag{1.4}$$

where $P\mathrm{d}x\mathrm{d}y\mathrm{d}z$ is the probability that the measurement of the particle's position at the time t finds it in the volume $\mathrm{d}x\mathrm{d}y\mathrm{d}z$ about the point (x,y,z).

2. A Definition of Dispersive Waves

According to Whitham [1], there are two main classes of waves. The first class is defined by hyperbolic partial differential equation and these are known as Hyperbolic Waves. The second class is called Dispersive Waves and they are defined by the solution rather than by an equation—unlike Hyperbolic Waves. In 1D, a linear dispersive system is any system that admits solutions of the form

$$\psi = A\mathrm{e}^{i(kx-\omega t)} \tag{1.5}$$

where A is the Amplitude, k is the wave number, ω is the angular frequency. For a linear system, they are all constants. When (1.5) is substituted into the governing equation a relation between ω and k is reached satisfying

$$G(\omega,k) = 0 \tag{1.6}$$

This is the (implicit) Dispersion Relation and contains all the physics. There exists a correspondence between the PDE and the dispersion relation through the relation:

$$-i\partial_{x} \leftrightarrow k, i\partial_{t} \leftrightarrow \omega \tag{1.7}$$

We assume that this correspondence persists for more general definitions of Dispersive Waves for non-constant PDEs and non-constant A,ω,k.

We may write ω in terms of k to get an explicit Dispersion Relation:

$$\omega = \Omega(k) \tag{1.8}$$

The quantity

$$\theta = kx - \omega t \tag{1.9}$$

is called the phase. The phase velocity is

$$v_{p} = \frac{\omega}{k} \tag{1.10}$$

And in general, the phase velocity depends on the wave number. That is, different phases with different $k's$ will travel at different phase velocities resulting in dispersion. For 1D cases, we require that

$$\Omega''(k) \neq 0 \tag{1.11}$$

This condition ensures that we have a dispersion and is stronger than the requirement that v_p is not constant… as it eliminates the case $\Omega = v_p k + const$ as well. As will be seen for example when considering Asymptotic Case later on, we note that the group velocity $v_g = \Omega'(k)$ is the more important propagation velocity and the condition (1.11) ensures that it is not constant.

Whitham [1] introduces a basic definition of Linear Dispersive Waves: They are the ones with solutions (1.5), (1.8) and satisfy the condition (1.11). We make the addition, that was only implicit in Whitham's work, that the correspondence (1.7) holds.

For the Extended Linear Case, *i.e.* non-uniform media or PDEs with non-constant coefficients the correspondence persists but the trio A, ω, k are no longer constant and are instead functions of x and t. Moreover, the dispersion relation will then have an additional dependence on x and t. Here, Fourier Analysis will be of limited use and the problem may be handled by Whitham's Variational Methods.

While for nonlinear PDEs, and the Nonlinear Case, the correspondence still persists and the trio still depends on x and t but this time the Dispersion relation will depend further on the Amplitude. And again, the problem may be treated via Whitham's Variational Methods.

Intuitively speaking, as Whitham [1] mentions, a dispersive wave is an oscillation in time coupled to an oscillation in space.

3. Hamilton's Stationary Phase Principle

There is a Hamiltonian Structure inside the phase … for fast as well as slow changing waves. The following is based on [4].

Hamilton's *phase integral* shows how a *ray* theory could be used to recover the *wave*.

For the Extended Linear Case, in Cartesian Co-ordinates, for a dispersive wave (also known as the eikonal ansatz),

$$\psi(x,t) = A(x,t)\mathrm{e}^{i\theta(x,t)} \tag{1.12}$$

moving along the phase $\theta(x,t)$ and keeping constant, we have:

$$\mathrm{d}\theta = \theta_x \mathrm{d}x + \theta_t = k(x,t)\mathrm{d}x - \omega(x,t)\mathrm{d}t \tag{1.13}$$

where we defined

$$k(x,t) = \theta_x, \omega(x,t) = -\theta_t \tag{1.14}$$

which is a mere generalization of $\theta = kx - \omega t$ … an oscillation that looks locally like a plane wave

But on θ is a constant, $\mathrm{d}\theta = 0$, whence,

$$\text{phase velocity} = v_p = \frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\omega}{k} \tag{1.15}$$

Consider how the phase $\theta(x,t)$ changes along a path $x(\sigma)$ for a fixed time t_0. So given $\theta(x(\sigma),t_0)$ find $\frac{\mathrm{d}\theta}{\mathrm{d}\sigma}$:

$$\begin{aligned}\mathrm{d}\theta(x(\sigma),t_0) &= \frac{\mathrm{d}\theta}{\mathrm{d}x}\mathrm{d}x + \frac{\mathrm{d}\theta}{\mathrm{d}t}\mathrm{d}t_0 \\ &= \frac{\mathrm{d}\theta}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}\sigma}\mathrm{d}\sigma + 0 \\ &= k(x(\sigma))\frac{\mathrm{d}x}{\mathrm{d}\sigma}\mathrm{d}\sigma\end{aligned} \tag{1.16}$$

Thus,

$$\frac{\mathrm{d}\theta}{\mathrm{d}\sigma} = k(x(\sigma)) \tag{1.17}$$

This can be inverted to give:

$$\theta(x(\sigma),t_0) = \int_0^\sigma k(x(\sigma'))\frac{\mathrm{d}x(\sigma')}{\mathrm{d}\sigma}\mathrm{d}\sigma' \tag{1.18}$$

This is just the integral version of (1.17)… we are recovering the phase by integrating over a path σ. Observe that this path is independent of parametrization. This path is not yet a ray.

How do we generalize this expression to include the time t and the frequency ω? Assuming that the path does not change in time and that the phase evolves in time with the wavefronts moving along a fixed path.

$$\theta(x(t),t) = \int_0^t \left\{k(x(t'))\frac{\mathrm{d}x(t')}{\mathrm{d}\sigma} - \omega(x(t'),t')\right\}\mathrm{d}t' \tag{1.19}$$

Here, we have used the fact that the integral (1.18) is independent of parameterization to change from the arbitrary orbit parameter to the physical time. Now (1.19) is simply an integral form of the phase function that satisfies (1.14). Now let us carry out the following steps:

1) *Let's Double the dimensionality of our system* by treating k as *independent* of x (just as we treat p as independent of q in Hamiltonian mechanics).

And now the new space (x,k) is called *ray phase space*—analogous to (p,q) phase space in Hamiltonian Mechanics.

2) Introduce $\Omega(x,k,t)$, a smooth function *on ray phase space and time*.

3) Introduce a smooth path $(x(t),k(t))$ *in ray phase space*, parameterized by the physical time t.

4) Define the phase integral, which is *a functional of the phase space path*

$$\Theta(x(t),k(t)) = \int_0^t k(t')\mathrm{d}x(t') - \Omega(k(t'),x(t'),t')\mathrm{d}t' \tag{1.20}$$

5) Now, invoke a stationarity principle for the phase integral (1.20), allowing variation of $x(t)$ and $k(t)$ separately, holding the endpoints of the path fixed. First, stationarity of Θ with respect to $x(t) \to x(t) + \varepsilon y(t)$, holding $k(t)$

fixed, implies (after integration by parts, and suppressing the arguments for clarity)

$$\Theta\left(x+\varepsilon y,k\right)=\int_0^t k\mathrm{d}\left(x+\varepsilon y\right)-\Omega\left(k,x+\varepsilon y,t'\right)\mathrm{d}t' \tag{1.21}$$

$$\begin{aligned}\frac{\mathrm{d}\Theta\left(x+\varepsilon y,k\right)}{\mathrm{d}\varepsilon}&=\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\int_0^t k\mathrm{d}\left(x+\varepsilon y\right)-\Omega\left(k,x+\varepsilon y,t'\right)\mathrm{d}t'\\&=\int_0^t k\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\left[\mathrm{d}x+\varepsilon\mathrm{d}y\right]-\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Omega\left(k,x+\varepsilon y,t'\right)\mathrm{d}t'\\&=\int_0^t k\mathrm{d}y-\frac{\partial}{\partial\left(x+\varepsilon y\right)}\Omega\left(k,x+\varepsilon y,t'\right)\frac{\partial\left(x+\varepsilon y\right)}{\partial\varepsilon}\mathrm{d}t'\end{aligned} \tag{1.22}$$

Now,

$$\frac{\partial}{\partial\left(x+\varepsilon y\right)}\Omega\left(k,x+\varepsilon y,t'\right)=\frac{\partial}{\partial x}\Omega\left(k,x,t'\right) \tag{1.23}$$

As in both cases, we are differentiating w.r.t. the second argument

While,

$$\frac{\partial\left(x+\varepsilon y\right)}{\partial\varepsilon}=y \tag{1.24}$$

Substituting (1.23) and (1.24) into (1.22), we now have:

$$\frac{\mathrm{d}\Theta\left(x+\varepsilon y,k\right)}{\mathrm{d}\varepsilon}=\int_0^t k\mathrm{d}y-y\frac{\partial\Omega}{\partial x}\mathrm{d}t' \tag{1.25}$$

However, (and here comes the integration by parts)

$$\frac{\mathrm{d}}{\mathrm{d}t'}\left(ky\right)=k\frac{\mathrm{d}y}{\mathrm{d}t'}+y\frac{\mathrm{d}k}{\mathrm{d}t'} \tag{1.26}$$

Integrating w.r.t. $\mathrm{d}t'$ from t to 0, we get:

$$\left[ky\right]_0^t=\int_0^t k\mathrm{d}y+\int_0^t y\mathrm{d}k \tag{1.27}$$

Setting boundary term to zero we get:

$$\int_0^t k\mathrm{d}y=-\int_0^t y\mathrm{d}k \tag{1.28}$$

Substituting the above into (1.25), we have:

$$\frac{\mathrm{d}\Theta\left(x+\varepsilon y,k\right)}{\mathrm{d}\varepsilon}=-\int_0^t y\left\{\mathrm{d}k+\frac{\partial\Omega}{\partial x}\mathrm{d}t\right\} \tag{1.29}$$

Finally,

$$0=\left.\frac{\mathrm{d}\Theta\left(x+\varepsilon y,k\right)}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0}=-\int_0^t y\left\{\mathrm{d}k+\frac{\partial\Omega}{\partial x}\mathrm{d}t\right\} \tag{1.30}$$

Secondly, stationarity of Θ with respect to $k\left(t\right)\to k\left(t\right)+\varepsilon\kappa\left(t\right)$, holding $x\left(t\right)$ fixed, implies

$$0=\left.\frac{\mathrm{d}\Theta\left(x,k+\varepsilon\kappa\right)}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0}=-\int_0^t \kappa\left\{\mathrm{d}x-\frac{\partial\Omega}{\partial k}\mathrm{d}t\right\} \tag{1.31}$$

And now equating the integrands to zero, we end up with the pair:

$$\dot{k}=-\frac{\partial\Omega}{\partial x},\dot{x}=\frac{\partial\Omega}{\partial k} \tag{1.32}$$

Equations (1.32) are called *Hamilton's equations*, and Ω is the *ray Hamiltonian*.

Given the smooth function $\Omega(x,k,t)$ and the initial conditions $\left(x(t_0),k(t_0)\right)$, Hamilton's equations determine a *unique* path in ray phase space. Paths that satisfy Hamilton's equations we will call *rays* once more, in order to distinguish them from arbitrary paths in ray phase space. But note that the dimensionality of the space the ray inhabits has doubled from Fermat's formulation.

Constructing the eikonal phase using rays that satisfy (1.32) in the phase integral (1.20) requires that we follow a *family* of rays. For that family of rays, if at each point x there is only one ray, then k is a well-defined *function of* x, and we can write $k(x,t)$ once more. The phase space integral (1.20) then reduces to the previous form of the phase integral (1.19), thereby showing that once more $k = \partial\theta/\partial x$. For this family of rays, we also have the identity:

$$\frac{\partial\theta}{\partial t} = -\omega(x,t) = -\Omega(x,k,t) \tag{1.33}$$

Thus, along rays we may rewrite (1.32) as:

$$\dot{k} = -\frac{\partial\omega}{\partial x}, \dot{x} = \frac{\partial\omega}{\partial k} \tag{1.34}$$

Observe that now we may reconstruct the wave (or phase) from the ray equations above. Given k, ω, x, we may use (1.20) to reconstruct the phase Θ, suppressing functional dependence we have:

$$\Theta = \int_0^t \left\{ k\frac{\mathrm{d}x}{\mathrm{d}t'} - \Omega \right\} \mathrm{d}t' = \int_0^t \{k\dot{x} - \Omega\}\,\mathrm{d}t' \tag{1.35}$$

And we have a correspondence between rays and waves. Moreover, as Ω is also a function of k, then it is the dispersion relation rather than just simply the angular frequency ω.

Another key point is that Hamilton's "ray" equations look very much like Hamilton's equations for particle mechanics:

$$\dot{p} = -\frac{\partial H}{\partial q}, \dot{q} = \frac{\partial H}{\partial q} \tag{1.36}$$

That is we have a time derivative of the independent variables equal to a (plus or minus) partial derivative of a dependent variable, with the correspondence:

$$\begin{aligned} &q = x \\ &p \leftrightarrow k \\ &H \leftrightarrow \Omega = \omega \end{aligned} \tag{1.37}$$

Why does (1.34) look like (1.36)? It is because (1.35), the phase integral, looks like a modified Hamilton's Principle under the same correspondence. Whence the variation of the phase integral is like the variation of the Hamiltonian yielding the same characteristic equations. We have just shown the equivalence of the rays to particle characteristics whence establishing the relation between waves and particles… proving the wave-particle duality. Under the correspondence (1.35) becomes:

$$\begin{aligned}\Theta &= \int_0^t \{k\dot{x} - \Omega\}\,\mathrm{d}t' \\ S &= \int_0^t \{p\dot{q} - H\}\,\mathrm{d}t' = \int_0^t L\mathrm{d}t'\end{aligned} \tag{1.38}$$

Whence we have another correspondence:

$$\Theta \leftrightarrow S \tag{1.39}$$

That is, the phase and action are related and the Principle of Least Action coincides with the Stationarity of the Phase. This is a variational, and so a fundamental, explanation of the wave-particle duality. Both principles play a role in Feynman's Path-Integral.

Remarks on Functional Dependence

The fact that here k is only a function of t and is independent of x, unlike the requirement by the Extended Linear Case, does not affect the correspondence between waves and particles. This is because in the Hamiltonian formulation p (the counterpart of k) is independent of q (the counterpart of x). Later on, we shall give a derivation of (1.32) where k is a function of both x and t.

Moreover, the Hamiltonian $H(p,q,t)$ more directly corresponds to the Dispersion Relation $\Omega(k,x,t)$ rather than the angular frequency $\omega(x,t)$. This is because, the Hamiltonian and the Dispersion relation have a corresponding functional dependence on p and k respectively:

$$\begin{aligned}&H = H(p,q,t) \\ &\updownarrow \qquad\ \ \updownarrow\updownarrow\updownarrow \\ &\Omega = \Omega(k,x,t)\end{aligned} \tag{1.40}$$

While $\omega(x,t)$ only depends on x and t… not k even though ω is numerically equal to Ω.

4. Planck's and de Broglie Relations

Could we make the correspondence (1.37) more precise? The answer turns out to be yes. Observe that Equations (1.34) and (1.36) both possess a Hamiltonian structure. Equations (1.38) satisfy the same variational principle:

$$\begin{aligned}\delta\Theta &= \delta\int_0^t \{k\dot{x} - \Omega\}\,\mathrm{d}t' = 0 \\ \delta S &= \delta\int_0^t \{p\dot{q} - H\}\,\mathrm{d}t' = 0\end{aligned} \tag{1.41}$$

Whence they are related by an Extended Canonical Transformation, see [6]. For simplicity, we set the generating function to zero and we are left with the scale transformations, using $\hbar$ as the scaling constant:

$$\begin{aligned}p &= \hbar k \\ q &= 1\cdot x\end{aligned} \tag{1.42}$$

Leading to:

$$H = (1\cdot\hbar)\Omega = \hbar\Omega = \hbar\omega \tag{1.43}$$

Finally, combining the two Equations (1.42), (1.43) and using (1.41)… we finally have:

$$S = \hbar\Theta \tag{1.44}$$

And the first of (1.42) is the de-Broglie relation with the scaling constant $\hbar = \frac{h}{2\pi}$, h being Planck's constant. Observe that (1.43) is more general than the Einstein-Planck relation $E = \hbar\omega$, as in general the Hamiltonian H is not always equal to the total Energy E.

Furthermore, observe that the direct relation without the scaling transformation is meaningless as the units on either side of the equation do not match. It would be interesting to see the kind of physics that emerges had the generating function not been set to zero … but that's the subject of further research.

5. Another Way of Seeing the Correspondence between Waves and Particles

Another way of seeing the correspondence between momentum and wavevector and that between energy and frequency is presented in this section. The constant phase of a plane wave and the Action are expressed as follows:

$$\begin{aligned} \Theta &= kx - \omega t \\ S &= \int \{ p\dot{q} - H \} \mathrm{d}t \end{aligned} \tag{1.45}$$

where k and ω are constants while p and H are not. If we try to make the second of (1.45) look like the first then we may specialize to the case where p and H are constants. In which case the second of equation of (1.45) becomes:

$$\begin{aligned} S &= \int \{ p\dot{q} - H \} \mathrm{d}t \\ &= \int p\dot{q}dt - \int H\mathrm{d}t, \text{ for constant } H, p \text{ we have} \\ &= pq - Ht \end{aligned} \tag{1.46}$$

And now we have a direct correspondence with the first of (1.45) (remember that $q = x$):

$$\begin{aligned} &\Theta = kx - \omega t \\ &\updownarrow \quad \updownarrow \quad \updownarrow \\ &S = pq - Ht \end{aligned} \tag{1.47}$$

And we have:

$$\begin{aligned} &\Theta \leftrightarrow S \\ &k \leftrightarrow p \\ &\omega \leftrightarrow H \end{aligned} \tag{1.48}$$

If instead we try the other way around that is, we make the second of (1.45) look like the first. Through generalizing k, ω to the non-constant case and introduce an integration sign… then we end up with the approach of the previous sections.

6. An Alternative Derivation of the Hamiltonian Structure inside the Phase along with a Discussion of Group Velocity

The following derivation and discussion this time is based on Whitham [1]. Consider again a phase θ with a local wavenumber k and local angular frequency ω

$$\begin{aligned} &\Theta = \Theta(x,t) \\ &k = \frac{\partial \Theta}{\partial x} \\ &\omega = -\frac{\partial \Theta}{\partial t} \end{aligned} \tag{1.49}$$

Taking further derivatives with respect to x and t gives,

$$\begin{aligned} &\frac{\partial k}{\partial t} = \frac{\partial^2 \Theta}{\partial x \partial t} \\ &\frac{\partial \omega}{\partial x} = -\frac{\partial^2 \Theta}{\partial t \partial x} \end{aligned} \tag{1.50}$$

Assuming continuous 2nd derivatives we have:

$$\frac{\partial k}{\partial t} + \frac{\partial \omega}{\partial x} = 0 \tag{1.51}$$

which is the continuity equation for the wave number. This is seen if we let k denote wave density and ω denote the wave flux. What we have is the conservation of waves!

Observing that

$$\frac{\partial \omega}{\partial x} = \frac{\partial \omega}{\partial k}\frac{\partial k}{\partial x} = v_g \frac{\partial k}{\partial x} \tag{1.52}$$

We see that (1.51) becomes

$$\frac{\partial k}{\partial t} + v_g \frac{\partial k}{\partial x} = 0 \tag{1.53}$$

What this states is that the group velocity v_g is the propagation velocity of the wave number k (This is the first role of the group velocity. The second role—to be seen later—is the propagation of the amplitude squared, A^2). To see this, consider

$$k = k(x,t) = const \tag{1.54}$$

Then

$$\mathrm{d}k = \frac{\partial k}{\partial x}\mathrm{d}x + \frac{\partial k}{\partial t}\mathrm{d}t = 0 \tag{1.55}$$

Or

$$\frac{\mathrm{d}k}{\mathrm{d}t} = \frac{\partial k}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial k}{\partial t} = 0 \tag{1.56}$$

Compared with (1.53), we find that

$$\frac{\partial \omega}{\partial k} = v_g = \frac{\mathrm{d}x}{\mathrm{d}t} \tag{1.57}$$

The general solution of (1.53) for an initial distribution $k = f(x)$ at $t = 0$ is

$$\begin{aligned} &k = f(\xi) \\ &x = \xi + C(\xi)t \end{aligned} \tag{1.58}$$

where,

$$C(\xi)=v_g\left(f(\xi)\right) \tag{1.59}$$

If k is initially centered at the origin then it is determined from

$$x=v_g t \tag{1.60}$$

This is just the solution obtained by asymptotic methods, where x and t are so large that the distribution appears centered at the origin.

If now we extend the dispersion relation to the case of nonuniform media, then it can be written as

$$\omega=\Omega(k,x,t) \tag{1.61}$$

Then (the dependence directly on x and the dependence through k),

$$\frac{\partial\omega}{\partial x}=\frac{\partial\Omega}{\partial k}\frac{\partial k}{\partial x}+\frac{\partial\Omega}{\partial x} \tag{1.62}$$

Substituting into (1.51), we find that:

$$\frac{\partial k}{\partial t}+\frac{\partial\Omega}{\partial k}\frac{\partial k}{\partial x}=-\frac{\partial\Omega}{\partial x} \tag{1.63}$$

But from (1.61), this is just

$$\frac{\partial k}{\partial t}+\frac{\partial\omega}{\partial k}\frac{\partial k}{\partial x}=-\frac{\partial\Omega}{\partial x} \tag{1.64}$$

From (1.57) and (1.56), the LHS can be written as:

$$\frac{\mathrm{d}k}{\mathrm{d}t}=-\frac{\partial\Omega}{\partial x}\text{ on }\frac{\mathrm{d}x}{\mathrm{d}t}=\frac{\partial\Omega}{\partial k} \tag{1.65}$$

And we have recovered the Hamiltonian structure in the phase. (Compare to (1.34) and (1.36).)

Note that from (1.61), we have

$$\begin{gathered}
\mathrm{d}\omega(x,t)=\mathrm{d}\Omega(k,x,t)\\
\frac{\partial\omega}{\partial x}\mathrm{d}x+\frac{\partial\omega}{\partial t}\mathrm{d}t=\frac{\partial\Omega}{\partial k}\mathrm{d}k+\frac{\partial\Omega}{\partial x}\mathrm{d}x+\frac{\partial\Omega}{\partial t}\mathrm{d}t\\
\frac{\partial\omega}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t}+\frac{\partial\omega}{\partial t}=\frac{\partial\Omega}{\partial k}\frac{\mathrm{d}k}{\mathrm{d}t}+\frac{\partial\Omega}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t}+\frac{\partial\Omega}{\partial t}
\end{gathered} \tag{1.66}$$

But from (1.65), we have

$$\frac{\partial\omega}{\partial x}v_g+\frac{\partial\omega}{\partial t}=-\frac{\partial\Omega}{\partial k}\frac{\partial\Omega}{\partial x}+\frac{\partial\Omega}{\partial x}\frac{\partial\Omega}{\partial k}+\frac{\partial\Omega}{\partial t} \tag{1.67}$$

Or

$$\frac{\mathrm{d}\omega}{\mathrm{d}t}=\frac{\partial\omega}{\partial x}v_g+\frac{\partial\omega}{\partial t}=\frac{\partial\Omega}{\partial t} \tag{1.68}$$

That is, the frequency is constant on each characteristic when the medium is time independent as remarked by Whitham. Furthermore, it may be deduced from (1.68) that

$$\frac{\mathrm{d}\Omega}{\mathrm{d}t}=\frac{\partial\Omega}{\partial t} \tag{1.69}$$

Mirroring the relation from Classical Mechanics that

$$\frac{\mathrm{d}H}{\mathrm{d}t}=\frac{\partial H}{\partial t} \tag{1.70}$$

which says that the Hamiltonian is constant in time if it does not explicitly depend on time. This Boosts confidence that the Hamiltonian $H(p,q,t)$ does indeed correspond to the Dispersion Relation $\Omega(k,x,t)$

As a bonus, from the definition of ω and k in (1.49), we may rewrite (1.61) as

$$-\frac{\partial\Theta}{\partial t}=\Omega\left(\frac{\partial\Theta}{\partial x},x,t\right) \tag{1.71}$$

which is the Hamilton-Jacobi Equation using the correspondence in (1.48).

7. Schrodinger's Equation and Unitary Evolution

For a dispersive wave (and this is how the eigenfunction formulation now kicks in), we have the correspondence:

$$-i\partial_x \leftrightarrow k, i\partial_t \leftrightarrow \omega \tag{1.72}$$

$$\begin{aligned} i\partial_t\psi &= \omega\psi \\ &= \Omega\psi \\ &= \frac{H}{h}\psi \end{aligned} \tag{1.73}$$

However, we may define that H is the eigenfunction of some operator $\hat{H}$. as it is proportional to the frequency ω and is virtually identical up to a constant. So, we now have,

$$\hat{H}\psi = H\psi \tag{1.74}$$

Thus,

$$\begin{aligned} i\partial_t\psi &= \frac{1}{\hbar}H\psi \\ &= \frac{1}{\hbar}\hat{H}\psi \end{aligned} \tag{1.75}$$

And so, we have,

$$i\hbar\partial_t\psi = \hat{H}\psi \tag{1.76}$$

And we have recovered unitary evolution.

Again, using (1.42) and (1.72), we have for a dispersive wave:

$$\begin{aligned} -i\partial_x\psi &= k\psi \\ &= \frac{p}{\hbar}\psi \end{aligned} \tag{1.77}$$

However, we may define that p is the eigenfunction of some operator $\hat{p}$. as it is proportional to the wavevector k and is virtually identical up to a constant. So, we now have,

$$-i\partial_x\psi = \frac{p}{\hbar}\psi \tag{1.78}$$

And as

$$\hat{p}\psi = p\psi \tag{1.79}$$

We have,

$$-i\partial_x\psi = \frac{\hat{p}}{\hbar}\psi \tag{1.80}$$

Or,

$$\hat{p}\psi = -i\hbar\partial_x\psi \tag{1.81}$$

For a time-independent Hamiltonian, we have,

$$H = \frac{p^2}{2m} + V(x) \tag{1.82}$$

Starting from (1.73), we have

$$\begin{aligned} i\hbar\partial_t\psi &= H\psi \\ &= \left(\frac{p^2}{2m} + V(x)\right)\psi \\ &= \frac{p^2}{2m}\psi + V(x)\psi \end{aligned} \tag{1.83}$$

Using (1.81), we have:

$$\begin{aligned} i\hbar\partial_t\psi &= \frac{p^2}{2m}\psi + V(x)\psi \\ &= \frac{1}{2m}\hat{p}^2\psi + V(x)\psi \end{aligned} \tag{1.84}$$

And we have reproduced Schrodinger's Equation. The point is that the relation to dispersive waves explains why only the momentum p has an equivalent differential operator unlike $V(x)$, or x for that matter.

8. Discussion of the Quantum Mechanics Postulates

Equation (1.76) matches Equation (1.3) and so we have reproduced Postulate IV the Time Evolution Postulate. We have also derived the momentum operator equation. Moreover, the form of Equations (1.74) and (1.78) matches the form found in Postulate I on Observables and Operators… in that we have an operator acting on an eigenfunction to yield an eigenvalue times the eigenfunction. Also, the eigenvalue turns out to be the physical quantity we seek (which in the case of Equations (1.74) and (1.78) is the Hamiltonian and the momentum).

Regarding the troublesome Measurement Postulate (Postulate II above), observe the form of the momentum operator Equation (1.79) that determines the momentum p. It is a differential equation and so it determines (along with the Boundary Conditions) the eigenfunction (which is the state function) ψ along with the eigenvalue (the quantity to be measured). Thus, the very act of measurement of p induces a new differential equation—different from the time evolution equation—and so comes up with a new state function (which may be different from that determined by the Schrodinger Equation) and the old state function is destroyed… Moreover, a new observable is reached. The very Eigen-formulation of formulation of Quantum Mechanics where measurement of a new quantity is induced by a new operator—as in Postulate I—is what leads to the destruction of the old state function and along with it the destruction of the old state. This Eigen-formulation (along with the rule for the Momentum and Hamiltonian Operators) is inherited from Dispersive Waves. Dispersive Waves possess an eigen-structure as for example is seen in Photonics where the dispersion relation of Electromagnetic waves is of prime importance see for example [7]. The Schrodinger Equation then describes how this new state-function/eigenfunction evolves in time with the measurement as the initial condition.

Regarding Postulate III on the Expectation Value … Averaging is quite natural when dealing with waves or oscillations. For example, Root Mean Square of a sine wave is the relevant quantity when dealing with AC circuits or adiabatic invariants in two-timing mechanics (see [1]). Sandwiching an operator between two state functions in order to extract the quantity of interest is quite natural and may happen in for example photonics in order to extract the Dispersion Relation from the Master Equation (see Appendix A of [7] for the similarities of Photonics and Quantum Mechanics both consisting of an Eigen-formulation). In fact, Whitham also extracts the Dispersion Relation via Averaging in his Averaged Lagrangian Formulation.

The correspondence between the differential operators and the wave quantities k and ω maps the PDE governing wave motion to the Dispersion Relation (an algebraic expression which contains all the physics of the wave motion). This Dispersion Relation is in turn mapped to the Hamiltonian (which is another algebraic expression containing all the physics of the particle motion) via the de Broglie and Einstein-Planck relations. Thus overall, we have a mapping from the PDE wave equation (which in the elementary linear case amounts to “pulling down” information from the phase) to the Particle’s (Algebraic) Hamiltonian.

The fact that we used the Extended Linear Case of the Definition of Dispersive Waves means that the solutions display wave behavior… Such as: Interference and Linear Superposition. In fact, this was one of the motivations for Schrodinger to write down a Linear PDE for the time evolution of the wave function.

Finally, it may be noted that the Born’s Postulate regarding probabilistic behavior of matter will be discussed towards the end.

9. Quantization

From Hamiltonian Mechanics, we have:

$$\begin{aligned}\{x,p\} &= \{q,p\} \\ &= \frac{\partial q}{\partial q}\frac{\partial p}{\partial p} - \frac{\partial q}{\partial p}\frac{\partial p}{\partial q} \\ &= 1\end{aligned} \tag{1.85}$$

While from dispersion waves theory, (we expect that the operator $\hat{x}$ corresponds simply to multiplication of ψ by x just like the operator equivalent of $V(x)$ is simply multiplication of ψ by x), we have:

$$\begin{aligned}[\hat{x},\hat{p}]\psi &= [x,-i\hbar\partial_x]\psi \\ &= [x(-i\hbar\partial_x)-(-i\hbar\partial_x)x]\psi \\ &= i\hbar[-x\partial_x\psi+\partial_x(x\psi)] \\ &= i\hbar[-x\partial_x\psi+x\partial_x\psi+1\cdot\psi] \\ &= i\hbar\psi\end{aligned} \tag{1.86}$$

Or

$$[x,p]=i\hbar \tag{1.87}$$

10. Uncertainty Principle

And the Uncertainty Principle follows from the Fourier Transform, related to Dispersion Wave Theory,

$$\Delta k\Delta x=\frac{1}{2} \tag{1.88}$$

Multiplying both sides by $\hbar$, we have,

$$\Delta p\Delta x=\frac{\hbar}{2} \tag{1.89}$$

11. Schrodinger's Cat and the Uncertainty Principle… Reproduced via a Classical System

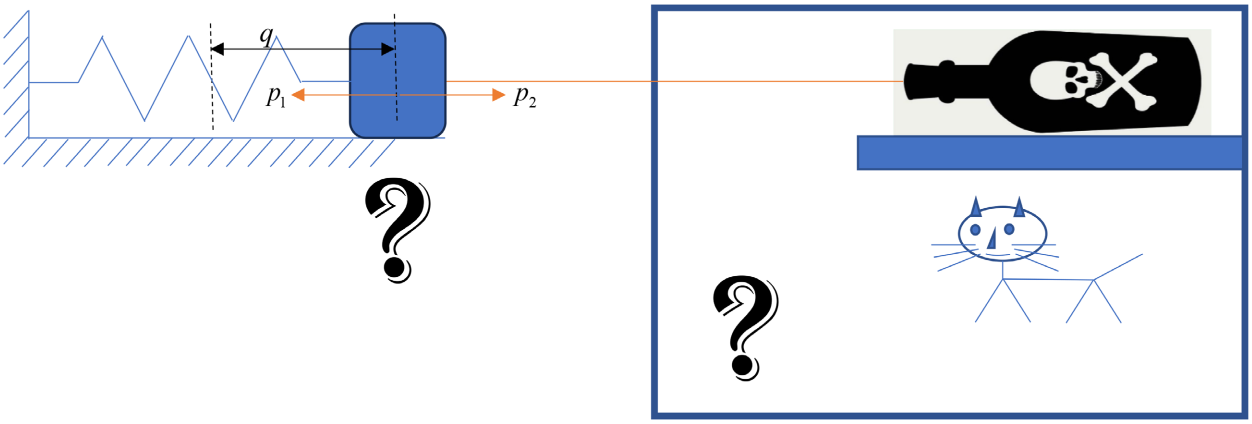

Figure 1. The Simple Harmonic Oscillator on the left is a classical system. Yet for a given position q (displacement off equilibrium position)of the block with mass m, we do not know for sure the linear momentum, as the block might be swinging from left to right or from right to left. That is, even though we are 100% sure about the position of the block we are uncertain of the linear momentum (we do not know whether it is p_1 or p_2). Coupling this to the poison in the box that contains a cat we reproduce the dilemma of Schrodinger's cat. Because we are uncertain about the momentum of the block, we are uncertain about whether the bottle of poison has been opened and the cat has died or not. Thus, we need to open the box to find out! We have thus reproduced uncertainty from a Classical System—the Simple Harmonic Oscillator.

The Simple Harmonic Oscillator, as shown in Figure 1, is a classical system and we have 100% information on the position of the block… but we still do not know which way it is going to swing. Hence, we do not know if it is going to pop open the bottle of poison and kill the cat or not.

The uncertainty in the cat's state may be traced back to the multi-valuedness (and hence uncertainty in the classical sense as say buckling of a symmetric pole) of the momentum of a classical system given a certain position.

11.1. Geometric Picture

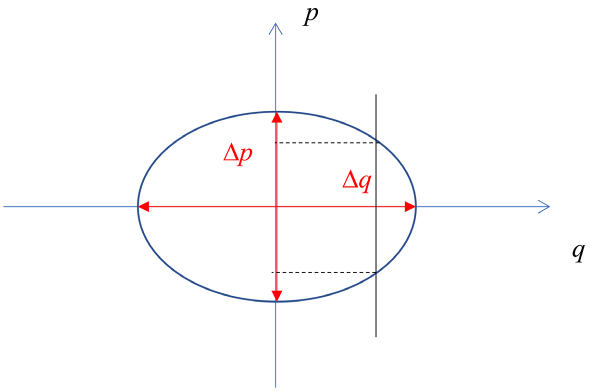

Figure 2. In Phase Space representation of the Simple Harmonic Oscillator, it is clear that a single value of the position q leads to two values of the momentum p. This multi-valuedness in the classical system leads to a natural uncertainty. The extent of the ellipse leads to a rough estimate of the range of values of position Δq and the range of values of the momentum Δp.

Let us write down the Hamiltonian of the Simple Harmonic Oscillator. As q does not depend on time explicitly and the force due to the spring is derivable from a conservative potential, then we have (see [6]),

$$H = T + V \tag{1.90}$$

where, T is the kinetic energy and V is the potential energy,

$$\begin{aligned} T &= \frac{1}{2m} p^2 \\ V &= \frac{1}{2} k q^2 \end{aligned} \tag{1.91}$$

where k is the spring constant. Hence,

$$H = \frac{1}{2m}\left(p^2 + m^2\omega^2 q^2\right) \tag{1.92}$$

where for the natural oscillating frequency ω, we have,

$$\omega^2 = \frac{k}{m} \tag{1.93}$$

For simplicity, we set $m = 1$ and we may rewrite (1.92) as,

$$\frac{p^2}{2H} + \frac{q^2}{2H/\omega^2} = 1 \tag{1.94}$$

which is the equation of the ellipse shown in **Figure 2**. Let us calculate the area of the ellipse.

$$\begin{aligned}\text{Area of an ellipse} &= \pi \times \text{Semi-Major Axis} \times \text{Semi-Minor Axis} \\ &= \pi \frac{\Delta p}{2}\frac{\Delta q}{2} \\ &= \pi (2H)^{1/2}\left(2H/\omega^2\right)^{1/2} \\ &= \pi \frac{2H}{\omega}\end{aligned} \tag{1.95}$$

So,

$$\frac{\Delta p}{2}\frac{\Delta q}{2} = \pi \frac{2H}{\omega} = 2\pi\hbar \tag{1.96}$$

Or,

$$\Delta p \Delta q = 8\pi\hbar \tag{1.97}$$

We have not reproduced the quantum uncertainty principle but we have a classical system with $\Delta p \Delta q \neq 0$.

But we also have,

$$\begin{aligned}\text{Area} &= \oint p\mathrm{d}q \\ &= \iint \mathrm{d}p\mathrm{d}q \\ &= \text{Integral of Poincare Invariant} \\ &= \text{Integral of the Jacobian or 2D volume element}\end{aligned} \tag{1.98}$$

Interestingly enough here (for $m = 1$),

$$\hbar = (\text{Multiple of}) \text{ Integral Poincare Invariant} \tag{1.99}$$

Observe that,

$$\oint p\mathrm{d}q \neq 0 \Leftrightarrow [p, q] \neq 0 \tag{1.100}$$

where the equation on the left is global or integral while the equation on the right is local or differential.

11.2. Statistical Picture

From [6], we have for $m = 1$,

$$q(t) = \frac{\sqrt{2E}}{\omega}\cos\omega t,\ p(t) = \sqrt{2E}\sin\omega t \tag{1.101}$$

For sinusoidal waveforms (with no DC component), we have using Standard Deviations:

$$\sigma_p \sigma_q = q_{rms} p_{rms} = \sqrt{E}\frac{\sqrt{E}}{\omega} = E/\omega = \hbar \tag{1.102}$$

And we are missing a factor of 1/2. This similar to results found by Old Quantum Mechanics, which Einstein endorsed.

This statistical picture also shows a valuable analogy where averaging method is used, *i.e.* rms that relates waves/AC-electrical-theory to particles/DC-electrical-theory.

12. Wave-Particle Duality (on a Small Scale)

The following is based on [4]. Consider a General Linear Wave Equation:

$$D\left(-i\partial_x, i\partial_t\right)\psi\left(x,t\right)=0 \tag{1.103}$$

where $D\left(-i\partial_x, i\partial_t\right)$ is a linear combination of the partial derivatives ∂_x, ∂_t (as is the case with Schrodinger Equation and Dirac Equation). The imaginary number i has been introduced so that we have a direct correspondence to the wave vector and the frequency:

$$-i\partial_x \leftrightarrow k, i\partial_t \leftrightarrow \omega \tag{1.104}$$

Using the Anstanz $\psi\left(x,t\right)=A\left(x,t\right)\mathrm{e}^{i(kx-\omega t)}$, we assume the envelope $A\left(x,t\right)$ is varying at a rate slower than the carrier wave $\mathrm{e}^{i(kx-\omega t)}$… that is slower than the wavelength and period. This is encoded mathematically as follows:

$$k_0 \gg \frac{\left|A_x\right|}{\left|A\right|}, \omega_0 \gg \frac{\left|A_t\right|}{\left|A\right|} \tag{1.105}$$

Introducing,

$$L^{-1} \sim \frac{\left|A_x\right|}{\left|A\right|}, T^{-1} \sim \frac{\left|A_t\right|}{\left|A\right|} \tag{1.106}$$

We see that:

$$k_0 L \gg 1, \omega_0 T \gg 1 \tag{1.107}$$

We introduce a small parameter to keep track of asymptotic orders,

$$\varepsilon \sim \left(k_0 L\right)^{-1} \sim \left(\omega_0 T\right)^{-1} \tag{1.108}$$

We also introduce ε in front of each derivative:

$$\begin{aligned} -i\partial_x &\to -i\varepsilon\partial_x \\ i\partial_t &\to i\varepsilon\partial_t \end{aligned} \tag{1.109}$$

Moreover, as the envelope $A\left(x,t\right)$ is varying slower than the phase, we introduce the small parameter into the phase:

$$\psi = A\left(x,t\right)\mathrm{e}^{i\varepsilon^{-1}(k_0 x-\omega_0 t)} \tag{1.110}$$

Now,

$$\begin{aligned} -i\varepsilon\partial_x\psi &= -i\varepsilon\partial_x A\left(x,t\right)\mathrm{e}^{i\varepsilon^{-1}(k_0 x-\omega_0 t)} + A\left(x,t\right)\cdot -i\varepsilon\partial_x \mathrm{e}^{i\varepsilon^{-1}(k_0 x-\omega_0 t)} \\ &= \mathrm{e}^{i\varepsilon^{-1}(k_0 x-\omega_0 t)}\left(-i\varepsilon\partial_x A\left(x,t\right) + A\left(x,t\right)\cdot -i^2 k_0\right) \\ &= \mathrm{e}^{i\varepsilon^{-1}(k_0 x-\omega_0 t)}\left(-i\varepsilon\partial_x A\left(x,t\right) + k_0 A\left(x,t\right)\right) \\ &= \mathrm{e}^{i\varepsilon^{-1}(k_0 x-\omega_0 t)}\left(k_0 - i\varepsilon\partial_x\right)A\left(x,t\right) \end{aligned} \tag{1.111}$$

$$\begin{aligned} i\varepsilon\partial_t\psi &= i\varepsilon\partial_t A\left(x,t\right)\mathrm{e}^{i\varepsilon^{-1}(k_0 x-\omega_0 t)} + A\left(x,t\right)\cdot i\varepsilon\partial_t \mathrm{e}^{i\varepsilon^{-1}(k_0 x-\omega_0 t)} \\ &= \mathrm{e}^{i\varepsilon^{-1}(k_0 x-\omega_0 t)}\left(i\varepsilon\partial_t A\left(x,t\right) + A\left(x,t\right)\cdot -i^2 \omega_0\right) \\ &= \mathrm{e}^{i\varepsilon^{-1}(k_0 x-\omega_0 t)}\left(i\varepsilon\partial_t A\left(x,t\right) + \omega_0 A\left(x,t\right)\right) \\ &= \mathrm{e}^{i\varepsilon^{-1}(k_0 x-\omega_0 t)}\left(\omega_0 + i\varepsilon\partial_t\right)A\left(x,t\right) \end{aligned} \tag{1.112}$$

So now, the Differential Equation (1.103) becomes after using (1.109), (1.110), (1.111) and (1.112):

$$D\left(k_0-i\varepsilon\partial_x,\omega_0+i\varepsilon\partial_t\right)A\left(x,t\right)=0 \tag{1.113}$$

Expanding both $D\left(k_0-i\varepsilon\partial_x,\omega_0+i\partial_t\right)$ and $A\left(x,t\right)$ in powers of ε:

$$D\left(k_0+\Delta k,\omega_0+\Delta\omega\right)=D\left(k_0,\omega_0\right)+\varepsilon D_k\Delta k+\varepsilon D_\omega\Delta\omega+\cdots \tag{1.114}$$

$$A\left(x,t\right)=A_0\left(x,t\right)+\varepsilon A_1\left(x,t\right)+\cdots \tag{1.115}$$

where,

$$-i\varepsilon\partial_x\leftrightarrow\Delta k,i\varepsilon\partial_t\leftrightarrow\Delta\omega \tag{1.116}$$

$$D_k=\left.\frac{\partial D\left(k,\omega\right)}{\partial k}\right|_{k_0,\omega_0},D_\omega=\left.\frac{\partial D\left(k,\omega\right)}{\partial\omega}\right|_{k_0,\omega_0} \tag{1.117}$$

So, (1.113) gives

$$\left(D\left(k_0,\omega_0\right)+\varepsilon D_k\Delta k+\varepsilon D_\omega\Delta\omega+\cdots\right)\left(A_0\left(x,t\right)+\varepsilon A_1\left(x,t\right)+\cdots\right)=0 \tag{1.118}$$

Or,

$$\left(D\left(k_0,\omega_0\right)+i\varepsilon\left(-D_k\partial_x+D_\omega\partial_t\right)+\cdots\right)\left(A_0\left(x,t\right)+\varepsilon A_1\left(x,t\right)+\cdots\right)=0 \tag{1.119}$$

As k_0,ω_0 were chosen such that:

$$D\left(k_0,\omega_0\right)=0 \tag{1.120}$$

To first order in ε, we have:

$$\left(-D_k\partial_x+D_\omega\partial_t\right)A_0\left(x,t\right)=0 \tag{1.121}$$

As

$$v_g=\frac{\mathrm{d}\Omega}{\mathrm{d}k}=-\frac{D_w}{D_k} \tag{1.122}$$

We have

$$\left(\partial_t+v_g\partial_x\right)A_0\left(x,t\right)=0 \tag{1.123}$$

Thus,

$$A_0\left(x,t\right)=A_0\left(x-v_gt\right) \tag{1.124}$$

And to first order (1.110) becomes:

$$\psi=A_0\left(x,t\right)\mathrm{e}^{i\varepsilon^{-1}\left(k_0x-\omega_0t\right)}=A_0\left(x,t\right) \tag{1.125}$$

As for small ε the oscillatory part varies rapidly. Now, from (1.124), we have:

$$\begin{gathered}\psi=A_0\left(x,t\right)=A_0\left(x-v_gt\right),\text{or}\\ \psi\left(x,t\right)=\psi\left(x-v_gt\right)\end{gathered} \tag{1.126}$$

That is, on a small scale, in the linear regime, ψ is a waveform moving at the velocity v_g which is the particle's velocity $\dot{x}$ (by (1.34)). This explains why according to the linear Schrodinger equation the particle displays wave behavior such as interference. The associated wave moves at the particles velocity and displays wave behavior thus explaining wave-particle duality.

On a larger scale, there might be interference but not at the particle's speed explaining why wave-particle duality is not observed on larger scales. Interference may still occur as ψ still has the form of a wave but the interference will not coincide with the particle as the wave and the particle are not moving at the same velocity. An exception is the Linear Schrodinger Equation, where linearity allows superposition and interference at larger scales.

13. Born's Postulate and Probabilistic Behavior

Intuitively, for a purely classically wave the energy (which is conserved as is equal to the integration of the intensity w.r.t. x) is moving at speed v_g, the group velocity. Thus the quantity $\int_{-\infty}^{\infty} |\psi|^2 \, \mathrm{d}x$ is conserved.

$$\int_{-\infty}^{\infty} |\psi|^2 \, \mathrm{d}x = const \tag{1.127}$$

However, in the QM context, the energy (or more precisely the Hamiltonian) is equal to the frequency and so the conservation of intensity could mean the conservation of probability (after normalization) and we have recovered unitarity… essential for Quantum Physics. Moreover, the particle is also moving at speed $\dot{x} = v_g$ and so we may identify the particle with the square of the intensity.

In other words, if the entirety of $\int_{-\infty}^{\infty} |\psi|^2 \, \mathrm{d}x$ is associated with a particle (as both are moving at speed v_g). Then, parts of the elongated sum, $|\psi|^2 \, \mathrm{d}x$, maybe associated with parts of the particle. However, as the particle is indivisible, then $|\psi|^2 \, \mathrm{d}x$ is associated with the probability of finding the particle at the interval $(x, x + \mathrm{d}x)$. And we have derived Born's Postulate.

Asymptotic Approach

An argument from Whitham [1] based on asymptotic methods frees us from associating the Energy to the square of the Amplitude. Consider the continuous superposition of dispersive plane waves.

$$\psi(x,t) = \int_{-\infty}^{\infty} F(\kappa) \mathrm{e}^{i(\kappa x - \Omega(\kappa))t} \mathrm{d}\kappa \tag{1.128}$$

where κ is the wave number, $F(\kappa)$ is the Amplitude density and $\Omega(\kappa)$ is the dispersion relation (numerically equal to the angular frequency).

For large x, t and with $\frac{x}{t}$ held fixed, we rewrite the above as

$$\psi(x,t) = \int_{-\infty}^{\infty} F(\kappa) \mathrm{e}^{i\chi(\kappa)t} \mathrm{d}\kappa \tag{1.129}$$

With

$$\chi(\kappa) = \Omega(\kappa) - \kappa \frac{x}{t} \tag{1.130}$$

And $\frac{x}{t}$ is a fixed parameter (allowing us to consider waves moving at a selected asymptotic velocity $\frac{x}{t}$ highlighting the relation between the velocity and the contributing asymptotic modes) and only the dependence on κ is shown. The method of Stationary Phase says that the main contribution in (1.129) comes from the neighborhood of stationary points or slowly varying points and not from rapid oscillations.

At stationary point $\kappa = k$ we have,

$$\chi'(k) = \Omega'(k) - \frac{x}{t} = 0 \tag{1.131}$$

Expanding the Fourier Component $F(\kappa)$ and $\chi(\kappa)$ in the neighbourhood of the stationary point $\kappa = k$, we have,

$$\begin{gathered} F(\kappa) \simeq F(k) \\ \chi(\kappa) \simeq \chi(k) + 0 + \frac{1}{2}(\kappa - k)^2 \chi''(k) \\ \chi''(k) \neq 0 \end{gathered} \tag{1.132}$$

Using (1.132), (1.129) becomes,

$$\psi(x,t) \simeq F(k) \mathrm{e}^{i\chi(k)t} \int_{-\infty}^{\infty} \mathrm{e}^{i\frac{1}{2}\left((\kappa-k)^2 \chi''(k)\right)t} \mathrm{d}\kappa \tag{1.133}$$

The remaining integral is an error integral as can be seen by rotating the path of integration by $\pm\frac{\pi}{4}$ the sign is the same as that of $\chi''(k)$. And now the main contribution becomes

$$\psi(x,t) \simeq F(k) \sqrt{\frac{2\pi}{t\left|\chi''(k)\right|}} \mathrm{e}^{-i\left(\chi(k)t + \frac{\pi}{4}\operatorname{sgn}\chi''(k)\right)} \tag{1.134}$$

Differentiating (1.130) w.r.t. κ twice and noting that $\frac{x}{t}$ is a fixed parameter, then we have at the stationary point k

$$\chi''(k) = \Omega''(k) \tag{1.135}$$

And so, in terms of Ω, the main contribution becomes

$$\psi(x,t) \simeq F(k) \sqrt{\frac{2\pi}{t\left|\Omega''(k)\right|}} \mathrm{e}^{i\left(kx - \Omega t - \frac{\pi}{4}\operatorname{sgn}\Omega''(k)\right)} \tag{1.136}$$

And from (1.131) the asymptotic is moving at the group velocity

$$v_g = \Omega'(k) = \frac{x}{t} \tag{1.137}$$

And the amplitude of the contribution is

$$A(x,t) = F(k) \sqrt{\frac{2\pi}{t\left|\Omega''(k)\right|}} \tag{1.138}$$

Consider the integral where we continuously sum over the intensity $|\psi|^2 = |A|^2$

$$\begin{aligned} Q(t) &= \int_{x_1}^{x_2} AA^* \mathrm{d}x \\ &= 8\pi \int_{x_1}^{x_2} \frac{F(k)F^*(k)}{t\left|\Omega''(k)\right|} \mathrm{d}x \end{aligned} \tag{1.139}$$

Now, using (1.137), we change variables from x to k:

$$\begin{aligned} x &= \Omega'(k)t \\ \mathrm{d}x &= t\Omega''(k)\,\mathrm{d}k \end{aligned} \tag{1.140}$$

And so, for $\Omega''(k) > 0$ (1.139) becomes

$$Q(t) = 8\pi\int_{k_1}^{k_2} F(k)F^*(k)\,\mathrm{d}k \tag{1.141}$$

where x_1, x_2 are related to k_1, k_2 via:

$$\begin{aligned} x_1 &= \Omega'(k_1)t \\ x_2 &= \Omega'(k_2)t \end{aligned} \tag{1.142}$$

Observe that now if k_1, k_2 do not change in time then $Q(t)$ is a constant with x_1, x_2 are moving with the corresponding group velocities. Thus, the total amount of $|A|^2$ between any pair of group lines (the end points of the integration are lines characterized by the group velocity) is a constant. It is in this sense that $|A|^2$ travels at the group velocity. And as stated before the integration $Q(t) = \int_{x_1}^{x_2} AA^*\mathrm{d}x$ may be normalized to represent probability, without mentioning conservation of energy.

For the special case of a wave packet initially localized in space (Δx is very small) and with amplitude concentrated around a particular value k (Δk is again very small), then the resulting disturbance is confined to the neighbourhood of the group line k and the wave packet as a whole moves with the group velocity $\Omega'(k)$. In the literature, discussions of group velocity are usually limited to this special case of localized initial disturbance, while Whitham's argument is more general and applies to a range of values of wave numbers.

14. Conclusions

In this work, we stated the assumption that the Quantum Wavefunction is a Dispersive Wave. Based on Whitham's work [1], we have come up with a definition of Dispersive Waves. Then, we uncovered the Variational Structure inside the phase and deduced the Hamiltonian Equations for the Ray Hamiltonian. This permitted an Extended Canonical Transformation between a wave's Stationary Principle and a particle's Hamilton's Principle. This opened the way for further progress. By setting the generating function to zero, the Hamiltonian structures of the waves and particles had been related by a scaling transformation (a subset of Extended Canonical Transformation). The de Broglie and Einstein-Planck relations were derived and the latter slightly generalized… It would be interesting to see the kind of physics that would emerge had we not set the generating function to zero. However, this is the subject of further research.

Later, the Schrodinger Equation was derived and some light was shed upon the Postulates of Quantum Mechanics, including the troublesome Measurement Postulate. The alteration of a state-function/eigenfunction through measurement is due to the fact that measurement is encoded mathematically as an eigenoperator which dictates the eigenfunction/state-function. This Eigen-formulation is inherited from Dispersive Waves. The Schrodinger Equation then describes how the state-function evolves in time with the measurement as the initial condition.

The quantization of the momentum followed from the correspondence between the differential operator and the wave number.

Waves and rays were unified as it was shown that it is possible to go from the phase of the wave to the ray information and back. Particles and rays were unified long ago… using this fact and the unification of Waves and rays; Waves and particles were unified. It was shown on a small scale that the particle and associated wave coincide as on that scale they both travel at the same velocity v_g … thus verifying wave-particle duality observed on a small scale in the lab.

The Uncertainty Principle was discussed and a classical system was constructed, using the Simple Harmonic Oscillator, that displayed uncertainty in momentum for a given known position. Finally, the Born Postulate regarding probabilistic behavior was derived along with a physical argument. It seems that Dispersive Waves do provide a classical framework for understanding Quantum Mechanics.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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