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Some Estimates for Commutators of θ-Type Calderón-Zygmund Operator in Variable Exponent Lebesgue Spaces

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Abstract

In this paper, by applying the extension of Rubio de Francia's extrapolation theorem in Lebesgue spaces with variable exponent, the boundedness of the θ-type Calderón-Zygmund operator T_θ is bounded in variable exponent Lebesgue spaces. In addition, the commutators generated by T_θ and Besov, BMO and Lipschitz functions are respectively obtained in $L^{p(\cdot)}(\mathbb{R}^n)$.

Keywords

θ-Type Calderón-Zygmund Operator, Commutator, Lebesgue Space with Variable Exponent

1. Introduction and Main Results

Given a measurable function $p:\mathbb{R}^n \to [1,\infty)$, $L^{p(\cdot)}(\mathbb{R}^n)$ denotes the set of measurable functions f on $\mathbb{R}^n$ such that for some $\lambda > 0$,

$$\int_{\mathbb{R}^n} \left(\frac{|f(x)|}{\lambda} \right)^{p(x)} \mathrm{d}x < \infty.$$

Equipped with the Luxemburg-Nakano norm

$$\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} = \inf \left\{ \lambda > 0 : \int_{\mathbb{R}^n} \left(\frac{|f(x)|}{\lambda} \right)^{p(x)} \mathrm{d}x \le 1 \right\},$$

then $L^{p(\cdot)}(\mathbb{R}^n)$ becomes a Banach function space. If $p(x) = p_0$ is constant, then $L^{p(\cdot)}(\mathbb{R}^n)$ equals the usual Lebesgue spaces $L^{p_0}(\mathbb{R}^n)$.

Lebesgue spaces with variable exponent $L^{p(\cdot)}(\mathbb{R}^n)$ were defined originally by Orlicz (see [1]). As the theory of the spaces with variable exponent was applied in some fields such as fluid dynamics, elasticity dynamics, Calculus of variations and differential equations with non-standard growth conditions (see [2]-[5]), the boundedness of some typical operators is being studied with keen interest (see [6]-[8]). The aim of this paper is to study the boundedness of the θ-type Calderón-Zygmund operators and their commutators in $L^{p(\cdot)}(\mathbb{R}^n)$.

In 1985, Yabuta introduced certain θ-type Calderón-Zygmund operators to facilitate his study of certain classes of pseudodifferential operators (see [9]). Since then, such type of operators is extensively applied in PDE with non-smooth area. Singular integral operators and variable exponent spaces play a fundamental role in the modern theory of partial differential equations (PDEs). These operators provide essential tools for establishing regularity and well-posedness results in non-standard function spaces, while variable exponent spaces offer a natural framework for problems with non-uniform ellipticity or growth conditions. Their interplay has led to significant advances in the analysis of nonlinear PDEs, especially in materials science and fluid dynamics, where physical properties exhibit sharp variations. Further applications have been found for such type of operators (see [10]-[13]).

With the further research, in [10], Quek and Yang introduced certain Calderón-Zygmund operator boundedness on space such as weighted Lebesgue spaces, weighted weak Lebesgue spaces, weighted Hardy spaces and weighted weak Hardy spaces. After that, Ri and Zhang obtained the boundedness of θ-type Calderón-Zygmund operator on Hardy spaces with non-doubling measures and non-homogeneous metric measure spaces (see [11] [12]), and Wang proved the boundedness of θ-type Calderón-Zygmund and commutators in the generalized weighted Morrey spaces (see [13]).

Definition 1.1 [9] Let θ be a non-negative, non-decreasing function on $\mathbb{R}^+=(0,\infty)$ satisfying

$$\int_0^1 \frac{\theta(t)}{t}\mathrm{d}t<\infty. \tag{1.1}$$

A measurable function $K(\cdot,\cdot)$ on $\left(\mathbb{R}^n\times\mathbb{R}^n\left\{(x,x):x\in\mathbb{R}^n\right\}\right)$ is said to be a θ-type Calderón-Zygmund kernel if it satisfies

$$|K(x,y)|\le C|x-y|^{-n} \tag{1.2}$$

and

$$|K(x,y)-K(x',y)|+|K(y,x)-K(y,x')|\le C\theta\left(\frac{|x-x'|}{|x-y|}\right)|x-y|^{-n}, \tag{1.3}$$

when $|x-y|\ge 2|x-x'|$.

Definition 1.2 [9] Let T_θ be a linear operator from $\mathcal{S}$ into its dual $\mathcal{S}'$. One can say that T_θ is a θ-type Calderón-Zygmund operator if:

1) T_θ can be extended to be a bounded linear operator on $L^2(\mathbb{R}^n)$;

2) There is a T_θ-type kernel $K(x,y)$ such that

$$T_\theta f(x) := \int_{\mathbb{R}^n} K(x,y) f(y)\,\mathrm{d}y \tag{1.4}$$

for all $f \in C_c^\infty(\mathbb{R}^n)$ and for all $x \notin \operatorname{supp} f$, where $C_c^\infty(\mathbb{R}^n)$ is the space consisting of all infinitely differentiable functions on $\mathbb{R}^n$ with compact supports.

Note that the classical Calderón-Zygmund operator with standard kernel (see [4] [14]) is a special case of θ-type operator T_θ when $\theta(t) = t^\delta$ with $0 < \delta \le 1$.

Definition 1.3 [15] Let $0 < \alpha < 1$, $1 < r, s < \infty$. Then, the Besov spaces $\dot{\Lambda}_\alpha^{r,s}(\mathbb{R}^n)$ consists of all functions f in $L^r(\mathbb{R}^n)$ for which the norm

$$\|f\|_{\dot{\Lambda}_\alpha^{r,s}(\mathbb{R}^n)} = \|f\|_r + \left(\int_{\mathbb{R}^n} \frac{\left(\|f(x+t) - f(x)\|_r \right)^s}{|t|^{n+\alpha s}} \mathrm{d}t \right)^{\frac{1}{s}}$$

is finite.

Definition 1.4 [16] Let $p(\cdot) \in \mathcal{P}(\cdot)$. If there exist $C > 0$ such that for any $x, y \in \mathbb{R}^n$,

$$|p(x) - p(y)| \le \frac{C}{-\log(|x-y|)},\quad |x-y| < 1/2;$$

$$|p(x) - p(0)| \le \frac{C}{\log(e + 1/|x|)},\quad |y| \ge |x|.$$

then $p(\cdot)$ is said to satisfy the log-Hölder condition.

We denote

$$p_- = \operatorname{essinf}\{p(x) : x \in \mathbb{R}^n\},\quad p_+ = \operatorname{esssup}\{p(x) : x \in \mathbb{R}^n\}.$$

Then, $\mathcal{P}(\mathbb{R}^n)$ consists all $p(\cdot)$ satisfying $p_- > 1$ and $p_+ < \infty$.

Let M be the Hardy-Littlewood maximal operator. We denote $\mathcal{B}(\mathbb{R}^n)$ to be the set of all functions $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ satisfying the condition that M is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$.

Given a measurable function $f \in L_{\mathrm{loc}}(\mathbb{R}^n)$, the sharp maximal operators $M_{r,\rho}$ and $M^\#$ are respectively defined by

$$M_{r,\rho} f(x) = \sup_{x \in Q} \left(\frac{1}{|Q|^{1-\frac{r\rho}{n}}} \int_Q |f(y)|^r \,\mathrm{d}y \right)^{\frac{1}{r}},$$

and

$$M^\# f(x) = \sup_{x \in Q} \int_Q |f(y) - f_Q| \,\mathrm{d}y,$$

where $f_Q = \frac{1}{|Q|} \int_Q f(y)\,\mathrm{d}y$, $r \ge 1$ and $0 < \rho < \frac{n}{r}$.

When $0<\beta\le 1$, the Homogeneous Lipschitz spaces $\mathrm{Lip}_\beta\left(\mathbb{R}^n\right)$ is the space of functions such that

$$\|f\|_{\mathrm{Lip}_\beta}=\sup_{x,h\in\mathbb{R}^n;h\neq 0}\frac{\left|f\left(x+h\right)-f\left(x\right)\right|}{|h|^\beta}<\infty. \tag{1.5}$$

Theorem 1.5 Suppose that $p(\cdot)\in\mathcal{B}\left(\mathbb{R}^n\right)$, θ satisfies (1.1). Then, there exists a constant C independent of f such that

$$\left\|T_\theta\left(f\right)\right\|_{L^{p(\cdot)}\left(\mathbb{R}^n\right)}\le C\|f\|_{L^{p(\cdot)}\left(\mathbb{R}^n\right)}.$$

Theorem 1.6 Suppose that $q(x)$ satisfies the log-Hölder's inequality and θ satisfies

$$\int_0^1\frac{\theta(t)\cdot|\log t|}{t}\mathrm{d}t<\infty. \tag{1.6}$$

If $b\in\dot{\wedge}_\alpha^{r,s}\left(\mathbb{R}^n\right),0<\alpha<1$, then for any $f(x)\in L^{q(\cdot)}\left(\mathbb{R}^n\right)$, there exists a constant C independent of f such that

$$\left\|\left[b,T_\theta\right]f\right\|_{L^{q(\cdot)}\left(\mathbb{R}^n\right)}\le C\|b\|_{\dot{\wedge}_\alpha^{r,s}}\|f\|_{L^{q(\cdot)}\left(\mathbb{R}^n\right)}.$$

Theorem 1.7 Let $b\in\mathrm{Lip}_\beta\left(\mathbb{R}^n\right)$. Suppose that $0<\beta<1$, θ satisfies (1.6) and $p(\cdot)\in\mathcal{P}\left(\mathbb{R}^n\right)$ is such that $p_+<\frac{n}{\beta}$. Define $q(\cdot)$ by

$$\frac{1}{p(x)}-\frac{1}{q(x)}=\frac{\beta}{n}.$$

If $q(\cdot)\left(1-\frac{\beta}{n}\right)\in\mathcal{B}\left(\mathbb{R}^n\right)$, then there exists a constant C independent of f such that

$$\left\|\left[b,T_\theta\right](f)\right\|_{L^{q(\cdot)}\left(\mathbb{R}^n\right)}\le C\|b\|_{\mathrm{Lip}_\beta\left(\mathbb{R}^n\right)}\|f\|_{L^{p(\cdot)}\left(\mathbb{R}^n\right)}.$$

Theorem 1.8 Let $b\in\mathrm{BMO}\left(\mathbb{R}^n\right)$. Suppose that $p(\cdot)\in\mathcal{B}\left(\mathbb{R}^n\right)$ and θ satisfies (1.6). Then, there exists a constant C independent of f such that

$$\left\|\left[b,T_\theta\right](f)\right\|_{L^{p(\cdot)}\left(\mathbb{R}^n\right)}\le C\|b\|_*\|f\|_{L^{p(\cdot)}\left(\mathbb{R}^n\right)}.$$

2. Preliminary Lemmas

Lemmas 2.1 [17] Let $p(\cdot)\in\mathcal{B}\left(\mathbb{R}^n\right)$, then for all $f(x)\in L^{p(\cdot)}\left(\mathbb{R}^n\right)$, we have

$$\|f\|_{L^{p(\cdot)}\left(\mathbb{R}^n\right)}\lesssim\left\|M^\# f\right\|_{L^{p(\cdot)}\left(\mathbb{R}^n\right)}.$$

Lemmas 2.2 [15] Let $b\in\dot{\wedge}_\alpha^{r,s}\left(\mathbb{R}^n\right)$, then for $0<\alpha<1$, $1<r<s<\infty$, we have

$$\sup_{y\in B}\frac{1}{|B|^{1+\frac{r}{n}-\frac{1}{r}}}\int_B\left|b-b_B\right|\mathrm{d}y\le\sup_{y\in B}\frac{1}{|B|^{\frac{\alpha}{n}+\frac{1}{s}-\frac{1}{r}}}\left(\int_B\left|b-b_B\right|^s\mathrm{d}y\right)^{\frac{1}{s}}\le C\|b\|_{\dot{\wedge}_\alpha^{r,s}\left(\mathbb{R}^n\right)}.$$

Lemmas 2.3 [18] Let $p(\cdot)\in p(\cdot)\in\mathcal{P}(\mathbb{R}^n)$. Then, $C_c^\infty(\mathbb{R}^n)$ is dense in $L^{p(\cdot)}(\mathbb{R}^n)$, where $C_c^\infty(\mathbb{R}^n)$ denotes the infinity times differentiable functions on $\mathbb{R}^n$ with compact support set.

Lemmas 2.4 [14] Let $p(\cdot)\in p(\cdot)\in\mathcal{P}(\mathbb{R}^n)$. If endowing the spaces $L^{p(\cdot)}(\mathbb{R}^n)$ with the following Orlicz-type norm:

$$\|f\|_{L^{p(\cdot)}}^0=\sup\left\{\int_{\mathbb{R}^n}\left|f(x)g(x)\mathrm{d}x:\|g\|_{L^{p(\cdot)}}\le 1\right|\right\},$$

then the norm $\|\cdot\|_{L^{p(\cdot)}}^0$ above is equivalent to the Luxemburg-Nakano norm $\|\cdot\|_{L^{p(\cdot)}}$.

Lemmas 2.5 [19] Let $p(\cdot)\in\mathcal{P}(\mathbb{R}^n)$. Then, the following conditions are equivalent:

1) $p(\cdot)\in\mathcal{B}(\mathbb{R}^n)$.
2) $p'(\cdot)\in\mathcal{B}(\mathbb{R}^n)$.
3) $p(\cdot)/q\in\mathcal{B}(\mathbb{R}^n)$ for some $1<q<p_-$.
4) $(p(\cdot)/q)'\in\mathcal{B}(\mathbb{R}^n)$ for some $1<q<p_-$.

Lemmas 2.6 [20] Let $p(\cdot)\in p(\cdot)\in\mathcal{P}(\mathbb{R}^n)$. Then, for $f\in L^{p(\cdot)}(\mathbb{R}^n)$ and $g\in L^{p'(\cdot)}(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n}\left|f(x)g(x)\right|\mathrm{d}x\le C_p\|f\|_{L^{p(\cdot)}}\|g\|_{L^{p(\cdot)}},$$

where $C_p=1+1/p^- -1/p^+$.

Given $0<\alpha<n$, define the fractional integral operator I_α by

$$I_\alpha(f)(x)=\int_{\mathbb{R}^n}\frac{f(y)}{|x-y|^{n-\alpha}}\mathrm{d}y.$$

Lemmas 2.7 [20] Let $p(\cdot),q(\cdot)\in\mathcal{P}(\mathbb{R}^n)$ be such that $p_+<n/\alpha$ and

$$\frac{1}{p(x)}-\frac{1}{q(x)}=\frac{\alpha}{n}.$$

If $q(\cdot)(n-\alpha)/n\in\mathcal{B}(\mathbb{R}^n)$, then

$$\|I_\alpha\|_{L^{q(\cdot)}(\mathbb{R}^n)}\le C\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}.$$

For $\delta>0$, $f\in L_{\mathrm{loc}}^\delta(\mathbb{R}^n)$, let

$$M_\delta(f)(x)=\sup_{x\in Q}\left(|Q|\int_Q|f(y)|^\delta\,\mathrm{d}y\right)^{\frac{1}{\delta}},$$

and

$$f_\delta^{\#}=\sup_{x\in Q}\inf_{c\in\mathbb{R}}\left(|Q|\int_Q|f(y)-c|^\delta\,\mathrm{d}y\right)^{\frac{1}{\delta}},$$

The non-increasing rearrangement of a measurable function f on $\mathbb{R}^n$ is defined by

$$f^*(t)=\inf\{\lambda>0:|x\in\mathbb{R}^n:|f(x)>\lambda\}|\le t\},\ (0<t<\infty).$$

Furthermore, for $\tau\in(0,1)$ and a measurable function f on $\mathbb{R}^n$, the local sharp maximal operator $M_\tau^{\#}$ is defined by

$$M_\tau^{\#}(f)(x)=\sup_{x\in Q}\inf_{c\in\mathbb{R}}\left((f-c)\chi_Q\right)^*(\tau|Q|).$$

Lemmas 2.8 [6] Let $\delta>0$, $\tau\in(0,1)$ and $f\in L^1_{\text{loc}}(\mathbb{R}^n)$. Then, for any $x\in\mathbb{R}^n$

$$M_\tau^{\#}(f)(x)\le(1/\tau)^{1/\delta}f_\delta^{\#}(x).$$

Lemmas 2.9 [2] Let $g\in L^1_{\text{loc}}$, $\tau\in(0,1)$ and a measurable function f satisfying

$$\left|\{x:|f(x)>\alpha|\}\right|<\infty \text{ for all } \alpha>0. \tag{2.1}$$

Then

$$\int_{\mathbb{R}^n}|f(x)g(x)|\mathrm{d}x\le C\int_{\mathbb{R}^n}M_\tau^{\#}(f)(x)M(g)(x)\mathrm{d}x.$$

Lemmas 2.10 [21] Suppose that θ satisfies (1.1) and $\int_0^1\frac{\theta(t)\cdot|\log t|}{t}\mathrm{d}t<\infty$ respectively in 1) and 2). Then, for all $f\in C_c^\infty$ and $x\in\mathbb{R}^n$, there exists a constant $C=C_{\delta>0}$ such that:

1) If $0<\delta<1$, then

$$M_\delta^{\#}(T_\theta f)(x)\le CM(f)(x).$$

2) If $0<\delta<l<1$, $m\in\mathbb{N}$ and $b\in\mathrm{BMO}(\mathbb{R}^n)$, then

$$\left(([b,T_\theta]f)(x)\right)_\delta^{\#}\le C\left(\|b\|_*M_l([b,T_\theta](f))(x)+\|b\|_*M^2(f)(x)\right).$$

Lemmas 2.11 Let $1<r,s<\infty$, $b\in\dot\wedge_\alpha^{r,s}(\mathbb{R}^n)$, $\frac{1}{t}+\frac{1}{s}=1$. Then, for any $f\in C_c^\infty(\mathbb{R}^n)$, there is a constant C, we have

$$M^{\#}([b,T_\theta]f)(x)\lesssim m^{\frac{n}{2}}\|b\|_{\dot\wedge_\alpha^{r,s}(\mathbb{R}^n)}\left[M_{t,\frac{\alpha r-n}{r}}(T_\theta f)(x)+M_{t,\frac{\alpha r-n}{r}}f(x)+M_{\frac{s}{\sqrt{s}-1}}f(x)\right],\ x\in\mathbb{R}^n.$$

Proof Let $f\in C_c^\infty(\mathbb{R}^n)$. In order to prove our theorem, we only need to show that for any x_0, there is a cube Q center at x_0, and a constant C_Q, such that

$$\begin{aligned}&\frac{1}{|Q|}\int_Q\left|[b,T_\theta]f(y)-C_Q\right|\mathrm{d}y\\&\lesssim m^{\frac{n}{2}}\|b\|_{\dot\wedge_\alpha^{r,s}}\left[M_{t,\frac{\alpha r-n}{r}}(T_\theta f)(x)+M_{t,\frac{\alpha r-n}{r}}f(x)+\sup_{x\in Q^*}Mf(x)\right],\ x\in\mathbb{R}^n\end{aligned}$$

Decompose f as

$$f=f_1+f_2=f\chi_{Q^*}+f\chi_{\mathbb{R}^n\setminus Q^*},$$

where Q^* denotes to be a cube, which center is the same with Q and

$l\left(Q^*\right)=2\sqrt{n}l\left(Q\right)$.

Set $C_Q=T_\theta\left(b-b_{Q^*}f_2\right)\left(x_0\right)$. Noting that $\left[b,T_\theta\right]f=\left[\left(b-b_{Q^*}\right),T_\theta\right]f$, then it follows

$$
\begin{aligned}
&\frac{1}{|Q|}\int_Q\left|\left[b,T_\theta\right]f\left(y\right)-C_Q\right|\mathrm{d}y\\
&\le\frac{1}{|Q|}\int_Q\left|b\left(y\right)-b_{Q^*}\right|\left|T_\theta f\left(y\right)\right|\mathrm{d}y+\frac{1}{|Q|}\int_Q\left|T_\theta\left(\left(b-b_{Q^*}\right)f_1\right)\left(y\right)\right|\mathrm{d}y\\
&\quad+\frac{1}{|Q|}\int_Q\left|T_\theta\left(\left(b-b_{Q^*}\right)f_2\right)\left(y\right)-T_\theta\left(\left(b-b_{Q^*}\right)f_2\right)\left(x_0\right)\right|\mathrm{d}y\\
&=I_1+I_2+I_3.
\end{aligned}
$$

We first estimate the term I_1. By using the Hölder's inequality and Lemma 2.2, we have

$$
\begin{aligned}
I_1&=\frac{1}{|Q|}\int_Q\left|b\left(y\right)-b_{Q^*}\right|\left|T_\theta f\left(y\right)\right|\mathrm{d}y\\
&\le\left(\frac{1}{|Q|}\int_Q\left|b\left(y\right)-b_{Q^*}\right|^s\mathrm{d}y\right)^{\frac{1}{s}}\left(\frac{1}{|Q|}\int_Q\left|T_\theta f\left(y\right)\right|^t\mathrm{d}y\right)^{\frac{1}{t}}\\
&\le\frac{1}{|Q|^{\frac{1}{r}-\frac{\alpha}{n}}}\|b\|_{\dot{\wedge}_\alpha^{r,s}}\left(\frac{1}{|Q|}\int_Q\left|T_\theta f\left(y\right)\right|^t\mathrm{d}y\right)^{\frac{1}{t}}\\
&\le C\|b\|_{\dot{\wedge}_\alpha^{r,s}}M_{t,\frac{\alpha r-n}{r}}\left(T_\theta f\right)\left(x\right).
\end{aligned}
$$

Secondly, we consider I_2. Taking $p:=\sqrt{s}$, by using the Hölder's inequality and the fact that $\|T_\theta f\|_{L^p\left(\mathbb{R}^n\right)}\le C\|f\|_{L^p\left(\mathbb{R}^n\right)}$ [13], then we have

$$
\begin{aligned}
I_2&\le\frac{1}{|Q|}\int_Q\left|T_\theta\left(\left(b-b_{Q^*}\right)f_1\right)\left(y\right)\right|\mathrm{d}y\\
&\le C\frac{1}{|Q|}\left(\int_Q\left|T_\theta\left(\left(b-b_{Q^*}\right)f_1\right)\left(y\right)\right|^p\mathrm{d}y\right)^{\frac{1}{p}}|Q|^{1-\frac{1}{p}}\\
&\le C\frac{1}{|Q|}\left(\int_{Q^*}\left|b\left(y\right)-b_{Q^*}\right|^p\left|f\left(y\right)\right|^p\mathrm{d}x\right)^{\frac{1}{p}}|Q|^{1-\frac{1}{p}}\\
&\le C\left(\int_{Q^*}\left|b\left(y\right)-b_{Q^*}\right|^{pp}\right)^{\frac{1}{pp}}\left(\int_{Q^*}\left|f\left(y\right)\right|^{pp'}\mathrm{d}x\right)^{\frac{1}{pp'}}|Q|^{1-\frac{1}{p}}\\
&\le C\left(\int_{Q^*}\left|b\left(y\right)-b_{Q^*}\right|^s\right)^{\frac{1}{s}}\frac{1}{|Q|^{\frac{1}{p}}}\left(\int_{Q^*}\left|f\left(y\right)\right|^{pp'}\mathrm{d}x\right)^{\frac{1}{pp'}}\\
&\le C\left(\frac{1}{|Q^*|}\int_{Q^*}\left|b\left(y\right)-b_{Q^*}\right|^s\right)^{\frac{1}{s}}\frac{1}{|Q|^{\frac{1}{p}}}\left(\frac{1}{|Q^*|}\int_{Q^*}\left|f\left(y\right)\right|^{pp'}\mathrm{d}x\right)^{\frac{1}{pp'}}\left|Q^*\right|^{\frac{1}{pp'}}\left|Q^*\right|^{\frac{1}{s}}\\
&\le C\|b\|_{\dot{\wedge}_\alpha^{r,s}\left(\mathbb{R}^n\right)}M_{\frac{s}{\sqrt{s}-1}}\left(f\right)\left(x\right).
\end{aligned}
$$

Now, we turn to estimate of I_3. Noting the fact that

$$\sum_{k=0}^{\infty}\theta\left(2^{-k}\right)\le C\sum_{k=0}^{\infty}\int_{2^{-k+1}}^{2^{-k}}\frac{\theta\left(2^{-k}\right)}{2^{1-k}}\mathrm{d}t\le C\int_0^1\frac{\theta(t)}{t}\mathrm{d}t<\infty.$$

For I_3, $y\in Q$, $z\in\left(Q^*\right)$, so we have $2\left|y-x_0\right|\le\left|y-z\right|$, one has

$$\begin{aligned}
I_3&=\frac{1}{|Q|}\int_Q\left|T_\theta\left(\left(b-b_{Q^*}\right)f_2\right)(y)-T_\theta\left(\left(b-b_{Q^*}\right)f_2\right)\left(x_0\right)\right|\mathrm{d}y\\
&\le C\frac{1}{|Q|}\int_Q\int_{\mathbb{R}^n\setminus Q^*}\left|\left(b(z)-b_{Q^*}\right)f(z)\right|\cdot\left|K(y,z)-K\left(x_0,z\right)\right|\mathrm{d}z\mathrm{d}y\\
&\le C\frac{1}{|Q|}\int_Q\sum_{k=0}^{\infty}\int_{2^{k+1}Q^*\setminus 2^kQ^*}\left|\left(b(z)-b_{Q^*}\right)f(z)\right|\cdot\frac{1}{|y-z|^n}\theta\left(\frac{\left|x_0-y\right|}{|y-z|}\right)\mathrm{d}z\mathrm{d}y\\
&\le C\frac{1}{|Q|}\int_Q\sum_{k=0}^{\infty}\theta\left(2^{-k}\right)\int_{2^{k+1}Q^*\setminus 2^kQ^*}\left|\left(b(z)-b_{Q^*}\right)f(z)\right|\frac{1}{\left|2^{k+1}Q^*\right|}\theta\left(2^{-k}\right)\mathrm{d}z\mathrm{d}y\\
&\le C\sum_{k=0}^{\infty}\theta\left(2^{-k}\right)\left(\int_{2^{k+1}Q^*}\frac{1}{\left|2^{k+1}Q^*\right|}\left|b(z)-b_{Q^*}\right|^s\mathrm{d}z\right)^{\frac{1}{s}}\left(\frac{1}{\left|2^{k+1}Q^*\right|}\cdot\int_{2^{k+1}Q^*}\left|f(z)\right|^t\mathrm{d}z\right)^{\frac{1}{t}}\\
&\le C\|b\|_{\dot{\Lambda}_\alpha^{r,s}}M_{t,\frac{\alpha r-n}{r}}f(x).
\end{aligned}$$

Combining the estimates for I_1, I_2 and I_3, the proof of Lemma 2.11 is finished.

Then, we have the following conclusion.

3. Proof of Main Results

Proof of Theorem 1.5. By applying the extension of Rubio de Francia's extrapolation theorem in the scale of the variable Lebesgue spaces [22], together with T_θ is bounded in $L^p\left(\mathbb{R}^n\right)$ [10], and combining with the results of Cruz-Uribe and Wang [23], we can know, let $p(\cdot)\in\mathcal{P}\left(\mathbb{R}^n\right)$ and ω be a weight. If the Hardy-Littlewood maximal operator M is bounded on $L^{p(\cdot)}(\omega)$ and on $L^{p'(\cdot)}\left(\omega^{-\frac{1}{p(\cdot)-1}}\right)$, then holds for all $f\in L^{p(\cdot)}(\omega)$ and all measurable functions g, we have

$$\left\|T_\theta(f)\right\|_{\|f\|_{L^{p(\cdot)}(\omega)}}\le C\|f\|_{\|f\|_{L^{p(\cdot)}(\omega)}}.$$

Due to the boundedness of the T_θ in the aforementioned space $L^{p(\cdot)}(\omega)$, weakening the conditions of $p(x)$, then boundedness of T_θ in the $L^{p(\cdot)}\left(\mathbb{R}^n\right)$ must hold, namely

$$\left\|T_\theta(f)\right\|_{L^{p(\cdot)}\left(\mathbb{R}^n\right)}\le C\|f\|_{L^{p(\cdot)}\left(\mathbb{R}^n\right)}.$$

This finishes the proof of **Theorem 1.5**.

Proof of Theorem 1.6. From Lemma 2.1 and Theorem 2.11, we easily see

$$\left\|[b,T_\theta]f\right\|_{L^{q(\cdot)}(\mathbb{R}^n)} \le C\left\|M^{\#}[b,T_\theta]f\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}$$
$$\le C\|b\|_{\dot{\Lambda}_\alpha^{r,s}(\mathbb{R}^n)}\left[\left\|M_{t,\frac{\alpha r-n}{r}}(Tf)(x)\right\|_{L^{q(\cdot)}(\mathbb{R}^n)} + \left\|M_{t,\frac{\alpha r-n}{r}}f(x)\right\|_{L^{q(\cdot)}(\mathbb{R}^n)} + \left\|M_{\frac{s}{\sqrt{s-1}}}f(x)\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}\right].$$

It is easy to verify that $\left\|M_{\frac{s}{\sqrt{s-1}}}f(x)\right\|_{L^{q(\cdot)}(\mathbb{R}^n)} \le C\|f(x)\|_{L^{q(\cdot)}(\mathbb{R}^n)}$, and

$$\left\|M_{t,\frac{\alpha r-n}{r}}(Tf)(x)\right\|_{L^{q(\cdot)}(\mathbb{R}^n)} = \left\|\left(M_{1,\frac{(\alpha r-n)t}{r}}|T_\theta f|^t\right)^{\frac{1}{t}}\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}$$
$$= \left\|M_{1,\frac{(\alpha r-n)t}{r}}|T_\theta f|^t\right\|^{\frac{1}{t}}_{L^{\frac{q(\cdot)}{t}}(\mathbb{R}^n)}$$
$$\le C\left\||T_\theta f|^t\right\|^{\frac{1}{t}}_{L^{\frac{q(\cdot)}{t}}(\mathbb{R}^n)}$$
$$\le C\|f\|_{L^{q(\cdot)}(\mathbb{R}^n)}.$$

Also, we can get

$$\left\|M_{t,\frac{\alpha r-n}{r}}f\right\|_{L^{q(\cdot)}(\mathbb{R}^n)} \le C\left\|f^t\right\|^{\frac{1}{t}}_{\frac{L^{q(\cdot)}(\mathbb{R}^n)}{t}} = \|f\|_{L^{q(\cdot)}(\mathbb{R}^n)}.$$

Thus, we have

$$\left\|[b,T_\theta]f\right\|_{L^{q(\cdot)}(\mathbb{R}^n)} \le Cm^{\frac{n}{2}}\|b\|_{\dot{\Lambda}_\alpha^{r,s}}\|f\|_{L^{q(\cdot)}(\mathbb{R}^n)}.$$

This finishes the proof of **Theorem 1.6**.

Proof of Theorem 1.7. Let $b \in \mathrm{Lip}_\beta(\mathbb{R}^n)$, $0<\beta<1$. Then, by (1.5), we shall get

$$|b(x)-b(y)| \le |x-y|^\beta\|b\|_{\mathrm{Lip}_\beta}.$$

Thus, for any $f \in L^{p(\cdot)}(\mathbb{R}^n)$,

$$[b,T_\theta](f) = \int_{\mathbb{R}^n}[b(x)-b(y)]K(x,y)f(y)\mathrm{d}y$$
$$\left|[b,T_\theta](f)\right| = \left|\int_{\mathbb{R}^n}[b(x)-b(y)]K(x,y)f(y)\mathrm{d}y\right|$$
$$\le C\int_{\mathbb{R}^n}|x-y|^\beta\|b\|_{\mathrm{Lip}_\beta}K(x,y)f(y)\mathrm{d}y$$
$$\le C\int_{\mathbb{R}^n}\|b\|_{\mathrm{Lip}_\beta}\frac{|f(y)|}{|x-y|^{n-\beta}}f(y)\mathrm{d}y$$
$$\le C\|b\|_{\mathrm{Lip}_\beta}I_\beta(|f|).$$

Applying Lemma 2.7, we get

$$\left\|\left[b,T_\theta\right](f)\right\|_{L^{q(\cdot)}(\mathbb{R}^n)} \le C\|b\|_{\mathrm{Lip}_\beta} I_\beta\left(|f|\right) \le C\|b\|_{\mathrm{Lip}_\beta}\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}.$$

Hence, the proof of **Theorem 1.7** is finished.

Proof of Theorem 1.8. Let $b\in \mathrm{BMO}(\mathbb{R}^n)$, $f\in C_c^\infty(\mathbb{R}^n)$. Then, by Lemma 2.3, we have $f\in L^{p(\cdot)}(\mathbb{R}^n)$. For any $g\in L^{p'(\cdot)}(\mathbb{R}^n)\in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ to be $\|g\|_{L^{p'(\cdot)}(\mathbb{R}^n)}\le 1$, noting the $[b,T_\theta]$ is bounded on usual Lebesgue spaces $L^p(\mathbb{R}^n)$ (see [13]), so it satisfies (2.1) in Lemma 2.9.

Thus, applying Lemma 2.9 and 2.10, we obtain

$$\begin{aligned}\int_{\mathbb{R}^n}[b,T_\theta](f)(x)g(x)\mathrm{d}x &\le C\int_{\mathbb{R}^n} M_\tau^{\#}([b,T_\theta])(f)(x)M(g)(x)\mathrm{d}x\\ &\le C\int_{\mathbb{R}^n}(1/\tau)^{1/\delta}\left([b,T_\theta](f)\right)_\delta^{\#}(x)M(g)(x)\mathrm{d}x,\end{aligned}$$

where $\delta,\tau\in(0,1)$.

By 2) in Lemma 2.10, for $0<\delta<l<1$

$$\begin{aligned}&\int_{\mathbb{R}^n}[b,T_\theta](f)(x)g(x)\mathrm{d}x\\ &\le C\|b\|_*\int_{\mathbb{R}^n}M_l\left([b,T_\theta](f)\right)(x)M(g)(x)\mathrm{d}x\\ &\quad +C\|b\|_*\int_{\mathbb{R}^n}M^2(f)(x)M(g)(x)\mathrm{d}x\\ &=I_1+I_2.\end{aligned}$$

Observing that for $0<l<1$, we have

$$M_l[b,T_\theta](f))(x)\le M\left([b,T_\theta](f)\right)(x)\ \text{ a.e. } x\in\mathbb{R}^n.$$

By Lemma 2.5 and the generalized Hölder's inequality (Lemma 2.6), for $p(\cdot)\in\mathcal{B}(\mathbb{R}^n)$, we have

$$\begin{aligned}I_1&=C\|b\|_*\int_{\mathbb{R}^n}M_l\left([b,T_\theta](f)\right)(x)M(g)(x)\mathrm{d}x\\ &\le C\|b\|_*\int_{\mathbb{R}^n}M\left([b,T_\theta](f)\right)(x)M(g)(x)\mathrm{d}x\\ &\le C\|b\|_*\left\|M\left(T_\theta(f)\right)\right\|_{L^{p(\cdot)}}\|Mg\|_{L^{p'(\cdot)}}\\ &\le C\|b\|_*\left\|T_\theta(f)\right\|_{L^{p(\cdot)}}\|Mg\|_{L^{p'(\cdot)}}\\ &\le C\|b\|_*\|f\|_{L^{p(\cdot)}}\|Mg\|_{L^{p'(\cdot)}}.\end{aligned}$$

On the other hand, also using Lemma 2.5 and the generalized Hölder's inequality (Lemma 2.6), for $p(\cdot)\in\mathcal{B}(\mathbb{R}^n)$,

$$\begin{aligned}I_2&\le C\|b\|_*\left\|M^2(f)\right\|_{L^{p(\cdot)}}\|Mg\|_{L^{p'(\cdot)}}\\ &\le C\|b\|_*\left\|M(f)\right\|_{L^{p(\cdot)}}\|g\|_{L^{p'(\cdot)}}\\ &\le C\|b\|_*\|f\|_{L^{p(\cdot)}}.\end{aligned}$$

According to the estimates of I_1 and I_2 above and Lemma 2.5, we can obtain

$$\int_{\mathbb{R}^n}\left|[b,T_\theta](f)(x)g(x)\right|\mathrm{d}x\le I_1+I_2\le C\|b\|_*\|f\|_{L^{p(\cdot)}},$$

and

$$\left\|[b,T_\theta]\right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le C\left\|[bT_\theta]\right\|^0_{L^{p(\cdot)}(\mathbb{R}^n)} \le C\|b\|_*\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}.$$

Hence, by Lemma 2.3, for any $f \in L^{p(\cdot)}(\mathbb{R}^n)$, we have

$$\left\|[b,T_\theta]\right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le C\|b\|_*\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}.$$

This finishes the proof of **Theorem 1.8**.

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Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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