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A Linear Microdilation Microcontinuum Theory for Thermoelastic Solids

Karan S. Surana*, **Michael Poskin**

Department of Mechanical Engineering, University of Kansas, Lawrence, KS, USA
Email: *kssurana@ku.edu

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Abstract

This paper presents a linear microdilation microcontinuum theory in which the microconstituents have only one unknown degree of freedom, volumetric strain or a quantity proportional to volumetric strain, and have three known rigid rotational degrees of freedom defined by the classical rotations. In this microdilation theory, the microconstituents, the medium as well as the interaction of the microconstituents all have thermoelastic deformation physics. Additionally, the microconstituents can experience rigid rotations. Due to deformable microconstituents, we begin the derivation with the microconstituent conservation and balance laws using classical continuum mechanics, followed by integral-average definitions that facilitate derivation of macro conservation and balance laws. This microdilation theory is completely different than Eringen's microstretch theory; the differences are discussed in the paper. It is shown that the approach of using smoothing weight functions in deriving macro balance of linear momenta, macro balance of angular momenta and the new balance law proposed for closure of the mathematical model in Eringen's work is neither needed nor used in the present work and is not supported by thermodynamics. All constitutive theories are derived using representation theorem and integrity, hence mathematically consistent and complete. The linear microdilation theory presented in this paper for thermoelastic solids is shown to be thermodynamically and mathematically consistent.

Keywords

Microstretch Microvolumetric Micro, Macro Integral-Average Representation Theorem Balance of Moment of Moments Thermolelastic Conservation and Balance Laws Constitutive Theories

1. Introduction

A thorough review of existing work published by Eringen and others on 3M theories has been presented by Surana *et al.* in a recent paper [1]. The references discussed by the authors [2]-[56], are listed here for the convenience of the readers. Detailed discussions of the works in these references can be found in [1] and are not repeated here for the sake of brevity. The linear microdilation theory presented here is motivated by Eringen's microstretch theory, in which he proposed the microconstituents to have an unknown stretch deformational degree of freedom and three unknown rigid rotations, a total of four degrees of freedom, all unknown. Our view is that microconstituents are not oriented objects in the theory, hence direction of stretch can not be established. Secondly, the use of unknown rigid rotations of the microconstituents always leads to thermodynamically inconsistent nonclassical theory [31]-[54], thus such microstretch theories are not valid theories.

In this paper, we propose a linear microcontinuum theory in which the microelements can have three rigid rotations and in addition have only one deformational degree of freedom. We allow microconstituents to dilate, *i.e.*, the microelement volume can change (increase or decrease), but without distortion of the shape. This deformation physics can be described by a single deformation measure, the volumetric strain, or a quantity proportional to the volumetric strain, hence constitutes the unknown deformational degree of freedom. The Cauchy stress tensor for this deformation is naturally a diagonal tensor, with all three diagonal components being the same. Thus, the Cauchy stress tensor is a pressure field that ensures pure volumetric deformation. This microcontinuum theory is obviously not microstretch theory but is rather microdilation theory. This microdilation theory is the only possible microcontinuum theory if we only allow microconstituents to have only one deformational degree of freedom, the other three being rigid rotations of the microconstituents.

We consider the derivation of the conservation and balance laws and the constitutive theories in which: 1) The deformation/strain measures derived by Surana *et al.* [35] serve as basic measures of deformation for the microdilation theory. The microconstituents are deformable, hence there is a microdeformation gradient tensor associated with them. 2) All conservation and balance laws are initiated for the microdeformation of the microconstituents using laws of thermodynamics of classical continuum mechanics, yielding micro conservation and balance laws. From the micro conservation and balance laws, "integral-average" definitions are introduced that permit the derivation of macro conservation and balance laws and constitutive theories using principles of thermodynamics and well-established concepts in applied mathematics. 3) In deriving conservation and balance laws and constitutive theories for microdilation continua, we maintain and adhere to the concepts of classical rotations, Cauchy moment tensor, theory of isotropic tensors, etc., introduced and used successfully by Surana *et al.* [31]-[54] in conjunction with linear and nonlinear micropolar theories for solid and fluent continua. This is necessary because the physics of rigid rotations of microconstituents exists in all three 3M theories, as it is due to the skew symmetric part of the microdeformation gradient tensor. Thus, we must have exactly same mathematical treatment of the rigid rotations physics in all 3M theories, requiring that we maintain the micropolar theory as a subset of micromorphic theory as well as subset of microdilation theory.

We first proceed with the conservation and balance laws for micro as well as macro deformation physics, with clarity of valid "integral-average" definitions that are essential for deriving macro conservation and balance laws. This is followed by constitutive theories for macro Cauchy stress tensor, microconstituent Cauchy stress tensor, macro Cauchy moment tensor and the heat vector. Constitutive theories are initialized using conjugate pairs in the entropy inequality, establishing constitutive tensors and their argument tensors. Constitutive tensors and argument tensors are adjusted or augmented as required by the desired physics that may not have been considered while deriving the entropy inequality. All four constitutive theories (microconstituent Cauchy stress tensor, macro Cauchy stress tensor, macro Cauchy moment tensor and heat vector) are derived using representation theorem [57]-[69], hence are always thermodynamically and mathematically consistent. Constitutive theories and material coefficients are first derived for the constitutive theories based on integrity (complete basis), and then their simplified forms are presented that are linear in the components of the argument tensors.

The microdilation theory derived here is shown to be thermodynamically and mathematically consistent; hence, it is a valid and physical linear microdilation microcontinuum theory. The microdilation theory presented here is compared with Eringen's microstretch microcontinuum theory.

2. Micro Mechanics in Macro Description

We consider the volume of matter to be composed of material points, same as in classical mechanics. A material point has volume $V+\partial V$ and $\bar{V}+\partial\bar{V}$ in the reference and deformed configurations with center of mass at P and $\bar{P}$. The material point contains microconstituents with their own volumes. We assume that the center of mass of the material point only sees statistically averaged response of the microconstituents in its volume. We further assume that there exists a surrogate configuration of microconstituents in the material point volume such that each microconstituent in this surrogate configuration has identical response at the center of mass of the material point and this response is exactly the same as the statistically averaged response of the original configuration of the microconstituents. With these assumptions and simplifications, we only need to consider micromechanics of a single microconstituent for a material point. For homogeneous and isotropic matter the same treatment holds for all material points, hence for the entire volume of matter.

Figure 1 shows the undeformed and deformed volume of a material point containing microconstituent "α" with its volume $V^{(\alpha)}+\partial V^{(\alpha)}$ and $\bar{V}+\partial\bar{V}^{(\alpha)}$ in the reference and current configuration. $\boldsymbol{x}$ and $\bar{\boldsymbol{x}}$ are locations of the center

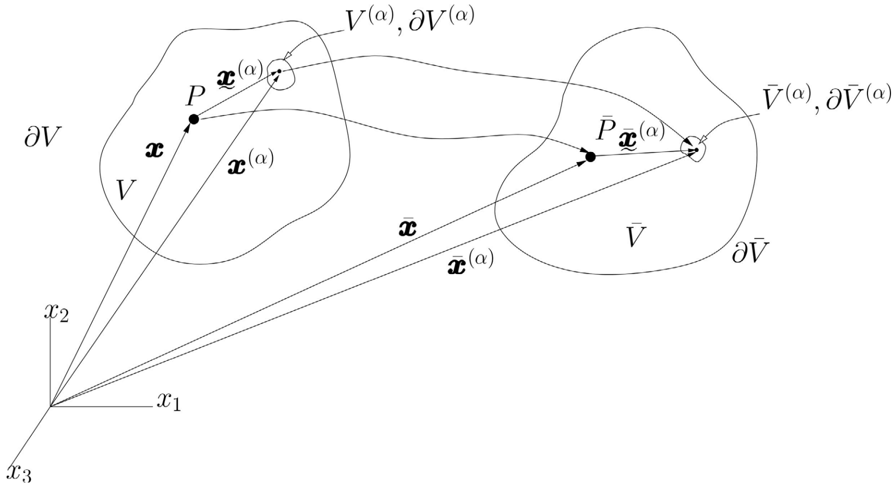

Figure 1. Undeformed and deformed configurations of material point volume.

of mass of the material point volume $V + \partial V$ and $\bar{V} + \partial \bar{V}$ in fixed x-frame. $\boldsymbol{x}^{(\alpha)}$ and $\underset{\sim}{\boldsymbol{x}}^{(\alpha)}$ are locations of the microconstituent with respect to the x-frame and with respect to the center of mass of the material point volume $V + \partial V$. Similarly $\bar{\boldsymbol{x}}^{(\alpha)}$ and $\underset{\sim}{\bar{\boldsymbol{x}}}^{(\alpha)}$ hold for the volume $\bar{V} + \partial \bar{V}$. $\underset{\sim}{\boldsymbol{x}}^{(\alpha)}$ is called the director in the undeformed configuration $V + \partial V$. Likewise $\underset{\sim}{\bar{\boldsymbol{x}}}^{(\alpha)}$ is the deformed director in the current configuration $\bar{V} + \partial \bar{V}$. Deformation of $\underset{\sim}{\boldsymbol{x}}^{(\alpha)}$ characterizes the microdeformation of the microconstituent α, hence the deformation of the deformable material point. Choice of specific deformation physics of $\underset{\sim}{\boldsymbol{x}}^{(\alpha)}$ yields specific microcontinuum theory. In the linear microdilation theory the microconstituents can experience volumetric change without distortion of the volume and can have rigid rotations defined by ${}_c\boldsymbol{\Theta}$, the classical rotations.

The linear and nonlinear deformation measures for 3M theories have been derived by Surana *et al.* [35]. The linear deformation measures are utilized in the present work.

3. Microconstituent Stress Tensor S Due to Micro Cauchy Stress Tensor $\sigma^{(\alpha)}$

In the derivation of the conservation and the balance laws, we use the following integral-average definition.

$$\int_{\bar{V}^{(\alpha)}(t)} \bar{\sigma}_{mk}^{(\alpha)} d\bar{V}^{(\alpha)} \overset{\text{def}}{=} \bar{S}_{mk} d\bar{V} \tag{1}$$

In which $\bar{\boldsymbol{\sigma}}^{(\alpha)}$ is the total Cauchy stress tensor for the microconstituents, thus $\bar{\boldsymbol{S}}$ is also the total stress tensor. In this process there is no concept of additive decomposition of $\bar{\boldsymbol{\sigma}}^{(\alpha)}$ into equilibrium and deviatoric stress tensors, hence volumetric and distortional physics are not considered explicitly. Secondly, microconstituent density is eliminated through integral-average definitions. But $\rho^{(\alpha)}$ is needed if we were to consider constitutive theory for equilibrium stress for the microconstituents. Both of these consideration help us in concluding that the stress tensor $\bar{\boldsymbol{S}}$ or $\boldsymbol{S}$ is due to mechanical loading, hence is a function of work conjugate strain tensor and elastic properties of the microconstituents. Henceforth, we do not consider any additive decomposition of $\boldsymbol{S}$, but consider work conjugate strain tensor and temperature as its argument tensors of $\boldsymbol{S}$ for simple thermoelastic case in deriving the constitutive theory for it.

4. Degrees of Freedom for Microconstituents

The kinematics of the microconstituents can be described using classical continuum mechanics, *i.e.* by the micro displacement gradient tensor ${}^{d}\boldsymbol{J}^{(\alpha)}$. Additive decomposition of ${}^{d}\boldsymbol{J}^{(\alpha)}$ into symmetric (${}_{s}^{d}\boldsymbol{J}^{(\alpha)}$) and skew symmetric (${}_{a}^{d}\boldsymbol{J}^{(\alpha)}$) tensor allows us to separates rigid rotation of microconstituents in ${}_{a}^{d}\boldsymbol{J}^{(\alpha)}$ and the deformation in ${}_{s}^{d}\boldsymbol{J}^{(\alpha)}$. Both tensors are completely defined by the gradients of micro displacements $\boldsymbol{u}^{(\alpha)}$. The skew symmetric tensor defines rotations ${}_{\alpha}\boldsymbol{\Theta}$ of the microconstituents that are the same as classical rotations ${}_{c}\boldsymbol{\Theta}$ and are three known degrees of freedom for the microconstituents. Due to consideration of purely volumetric deformation physics of the microconstituents, we must place some restriction on ${}_{s}^{d}\boldsymbol{J}^{(\alpha)}$ so that resulting tensor only describes volumetric deformation physics. Thus, we must consider the following.

$$\left.\begin{aligned} &{}_{s}^{d}J_{ij}^{(\alpha)} = \varepsilon_v \delta_{ij} \\ &{}_{s}^{d}J_{11}^{(\alpha)} = {}_{s}^{d}J_{22}^{(\alpha)} = {}_{s}^{d}J_{33}^{(\alpha)} = \varepsilon_v = \frac{1}{3}\,\text{volumetric strain} \end{aligned}\right\} \tag{2}$$

Hence, in the microdilation theory the symmetric part of the microdeformation gradient tensor has the form defined by (2). Volumetric deformation (2) defines volumetric strain, thus in the microdilation theory the fourth degree of freedom is the volumetric strain or is proportional to the volumetric strain. Thus, in the microdilation theory presented here there are four degrees of freedom, ${}_{\alpha}\boldsymbol{\Theta}$ and ε_v. ${}_{\alpha}\boldsymbol{\Theta}$ are the same as classical rotations ${}_{c}\boldsymbol{\Theta}$, hence are known, therefore in this microcontinuum theory there is only one unknown degree of freedom, ε_v, associated with the microelements describing pure volumetric deformation of the microconstituents.

5. Conservation and the Balance Laws

In this section we present the derivation of the conservation and balance laws: conservation of mass, balance of linear momenta, balance of angular momenta, balance of moment of moments, and first and second laws of thermodynamics in Eulerian as well as Lagrangian description. We always begin the derivation with microconstituents and show that valid thermodynamic laws are possible to apply for micro deformation using classical continuum mechanics. This is followed by the introduction of "*integral-average*" definitions that hold at the macro level and are used to derive valid conservation and balance laws at the macro level. Even though conservation and balance laws for micro deformation use classical continuum mechanics, due to the use of integral-average definitions the resulting conservation and balance laws at the macro level are in fact modified conservation and balance laws of classical continuum mechanics. Introduction of a new kinematic conjugate pair, rotations and moments in addition to already existing displacements and forces requires an additional balance law, balance of moment of moments [40] [50] [70]. This balance is essential for all 3M theories and is considered in the present work.

5.1. Conservation of Mass

In the following we consider conservation of mass for the microconstituents as well as at the macro level. For the microconstituents in the reference and deformed configurations, conservation of mass can be expressed as:

$$\int_{V^{(\alpha)}} \rho_0^{(\alpha)} dV^{(\alpha)} = \int_{\bar{V}^{(\alpha)}(t)} \bar{\rho}^{(\alpha)} d\bar{V}^{(\alpha)} \tag{3}$$

If microconstituent mass is conserved, then

$$\frac{D}{Dt} \int_{\bar{V}^{(\alpha)}(t)} \bar{\rho}^{(\alpha)} d\bar{V}^{(\alpha)} = 0 \tag{4}$$

Using transport theorem [68] [69], we can write the following from (4)

$$\int_{\bar{V}^{(\alpha)}(t)} \left(\frac{D}{Dt} \bar{\rho}^{(\alpha)} \left(\bar{\boldsymbol{x}}^{(\alpha)}, t \right) + \bar{\rho}^{(\alpha)} \left(\bar{\boldsymbol{x}}^{(\alpha)}, t \right) \frac{\partial \bar{v}_l^{(\alpha)} \left(\bar{\boldsymbol{x}}^{(\alpha)}, t \right)}{\partial \bar{x}_l^{(\alpha)}} \right) d\bar{V}^{(\alpha)} = 0 \tag{5}$$

Using localization theorem, we obtain the following from (5)

$$\frac{D\bar{\rho}^{(\alpha)}}{Dt} + \bar{\rho}^{(\alpha)} \frac{\partial \bar{v}_l^{(\alpha)}}{\partial \bar{x}_l^{(\alpha)}} = 0 \tag{6}$$

Equation (6) is the differential form of the continuity equation in Eulerian description for the microconstituent based on classical continuum mechanics. In Lagrangian description, using (3)

$$\int_{V^{(\alpha)}} \rho_0^{(\alpha)} dV^{(\alpha)} = \int_{V^{(\alpha)}} \rho^{(\alpha)} \left| \boldsymbol{J}^{(\alpha)} \right| dV^{(\alpha)} \tag{7}$$

Equation (7) implies that the two integrands must be equal, hence we have

$$\rho_0^{(\alpha)} = \rho^{(\alpha)} \left| \boldsymbol{J}^{(\alpha)} \right| \tag{8}$$

Equation (8) is the continuity equation for the microconstituents in Lagrangian description based on classical continuum mechanics. In the following we consider macro conservation of mass.

Consider Eulerian as well as Lagrangian descriptions in (3) and integrate over $\bar{V}$ to obtain

$$\int_{\bar{V}(t)} \int_{\bar{V}^{(\alpha)}(t)} \bar{\rho}^{(\alpha)} d\bar{V}^{(\alpha)} = \int_{V} \int_{V^{(\alpha)}} \rho_0^{(\alpha)} dV^{(\alpha)} \tag{9}$$

Define

$$\int_{\bar{V}^{(\alpha)}(t)} \bar{\rho}^{(\alpha)} d\bar{V}^{(\alpha)} \stackrel{\text{def}}{=} \bar{\rho} d\bar{V} \tag{10}$$

Substituting (10) in (9) and setting its material derivative to zero (as mass in conserved in volume $\bar{V}$)

$$\frac{D}{Dt}\int_{\bar{V}} \bar{\rho} d\bar{V} = 0 \tag{11}$$

Using transport theorem [68] [69], we obtain the following from (11)

$$\int_{\bar{V}} \left(\frac{D\bar{\rho}(\bar{\boldsymbol{x}},t)}{Dt} + \bar{\rho}(\bar{\boldsymbol{x}},t)\bar{\boldsymbol{\nabla}}\cdot\bar{\boldsymbol{v}}(\bar{\boldsymbol{x}},t) \right) d\bar{V} = 0 \tag{12}$$

Using localization theorem

$$\frac{D\bar{\rho}}{Dt} + \bar{\rho}\bar{\boldsymbol{\nabla}}\cdot\bar{\boldsymbol{v}} = 0 \tag{13}$$

Equation (13) is the macro continuity equation resulting from macro conservation of mass.

Thus, conservation of mass holds at the micro as well as macro level. In Lagrangian description, consider (10) and introduce the following integral-average definition in the reference configuration.

$$\int_{V^{(\alpha)}} \rho_0^{(\alpha)} dV^{(\alpha)} \stackrel{\text{def}}{=} \rho_0 dV \tag{14}$$

Using (10) and (14) in (9)

$$\int_{\bar{V}} \bar{\rho} d\bar{V} = \int_{V} \rho_0 dV \tag{15}$$

or

$$\int_{V} \rho |\boldsymbol{J}| dV = \int_{V} \rho_0 dV \tag{16}$$

From (16), we obtain

$$\rho_0(\boldsymbol{x}) = \rho(\boldsymbol{x},t)|\boldsymbol{J}| \tag{17}$$

Equation (16) is the macro continuity equation in Lagrangian description resulting from macro conservation of mass.

5.2. Balance of Linear Momenta

Let $\bar{a}_k^{(\alpha)}$, ${}^b\bar{F}_k^{(\alpha)}$, and $\bar{\sigma}_{lk}^{(\alpha)}$ be microconstituent acceleration, body force per unit mass, and Cauchy stress tensor. Due to pure volumetric deformation of the microconstituents, hence pure volumetric strain, the Cauchy stress tensor $\boldsymbol{\sigma}^{(\alpha)}$ is a diagonal tensor with all three diagonal components being the same, *i.e.* $\boldsymbol{\sigma}^{(\alpha)}$ (or $\bar{\boldsymbol{\sigma}}^{(\alpha)}$) is similar to constant pressure or hydrostatic pressure. If we consider a tetrahedron with its oblique plane being part of $\partial\bar{V}^{(\alpha)}$ and its other three orthogonal planes being parallel to the *x*-frame, then Cauchy principle applies to $\bar{\boldsymbol{P}}^{(\alpha)}$, average stress on the oblique; $\bar{\boldsymbol{n}}^{(\alpha)}$, unit exterior normal to the oblique plane of the tetrahedron. $\bar{\boldsymbol{\sigma}}^{(\alpha)}$ acts on the orthogonal planes of the tetrahedron, keeping in mind that $\bar{\sigma}^{(\alpha)}$ is a diagonal tensor with all three diagonal components being the same.

$$\bar{\boldsymbol{P}}^{(\alpha)} = \left(\bar{\boldsymbol{\sigma}}^{(\alpha)}\right)^T \cdot \bar{\boldsymbol{n}}^{(\alpha)} \tag{18}$$

Balance of linear momenta of classical continuum mechanics applies to the microconstituents, hence we can write the following integral form for the volume $\bar{V}^{(\alpha)}$ of the microconstituent [71] [72]

$$\int_{\bar{V}^{(\alpha)}(t)} \left(\bar{\rho}^{(\alpha)}\bar{a}_k^{(\alpha)} - \bar{\rho}^{(\alpha)}\left({}^b\bar{F}_k^{(\alpha)}\right)\right) d\bar{V}^{(\alpha)} - \int_{\partial\bar{V}^{(\alpha)}(t)} \bar{\sigma}_{lk}^{(\alpha)}\bar{n}_l^{(\alpha)} d\bar{A}^{(\alpha)} = 0 \tag{19}$$

or

$$\int_{\bar{V}^{(\alpha)}(t)} \left(\bar{\rho}^{(\alpha)}\bar{a}_k^{(\alpha)} - \bar{\rho}^{(\alpha)}\left({}^b F_k^{(\alpha)}\right) - \sigma_{lk,l}^{(\alpha)}\right) d\bar{V}^{(\alpha)} = 0 \tag{20}$$

Using localization theorem [71] [72], we obtain the following from (20).

$$\bar{\rho}^{(\alpha)}\bar{a}_k^{(\alpha)} - \bar{\rho}^{(\alpha)}\left({}^b\bar{F}_k^{(\alpha)}\right) - \bar{\sigma}_{lk,l}^{(\alpha)} = 0 \tag{21}$$

Equation (21) represents the balance of micro linear momenta in Eulerian description for the microconstituents based on classical contniuum mechanics. Balance of micro linear momenta in Lagrangian description can be directly written using (21).

$$\rho_0^{(\alpha)}a_k^{(\alpha)} - \rho_0^{(\alpha)}\left({}^b F_k^{(\alpha)}\right) - \sigma_{lk,l}^{(\alpha)} = 0 \tag{22}$$

To derive balance of macro linear momenta we introduce the following integral-average definitions.

$$\int_{\bar{V}^{(\alpha)}(t)} \bar{\rho}^{(\alpha)}\bar{a}_k^{(\alpha)} d\bar{V}^{(\alpha)} \overset{\text{def}}{=} \bar{\rho}\bar{a}_k d\bar{V} \tag{23}$$

$$\int_{\bar{V}^{(\alpha)}(t)} \bar{\rho}^{(\alpha)}\left({}^b\bar{F}_k^{(\alpha)}\right) d\bar{V}^{(\alpha)} \overset{\text{def}}{=} \bar{\rho}\,{}^b\bar{F}_k d\bar{V} \tag{24}$$

$$\int_{\partial\bar{V}^{(\alpha)}(t)} \bar{\sigma}_{lk}^{(\alpha)}\bar{n}_l^{(\alpha)} d\bar{A}^{(\alpha)} \overset{\text{def}}{=} \bar{\sigma}_{lk}\bar{n}_l d\bar{A} \tag{25}$$

Using (23)-(25) in (19) and integrating over $\bar{V}$ and $\partial\bar{V}$

$$\int_{\bar{V}(t)} \left(\bar{\rho}\bar{a}_k - \bar{\rho}\left({}^b\bar{F}_k\right)\right) d\bar{V} - \int_{\partial\bar{V}(t)} \bar{\sigma}_{lk}^{(\alpha)}\bar{n}_l d\bar{A} = 0 \tag{26}$$

or

$$\int_{\bar{V}(t)} \left(\bar{\rho}\bar{a}_k - \bar{\rho}\left({}^b\bar{F}_k\right) - \bar{\sigma}_{lk,l}\right) d\bar{V} = 0 \tag{27}$$

Using localization theorem, we obtain the following from (27).

$$\bar{\rho}\left(\bar{a}_k\right) - \bar{\rho}\left({}^b\bar{F}_k\right) - \bar{\sigma}_{lk,l} = 0 \tag{28}$$

Equation (28) in the balance of macro linear momenta in Eulerian description based on classical continuum mechanics. Balance of macro linear moments in Lagrangian description can be written directly using (28).

$$\rho_0 a_k - \rho_0 \left({}^{b}F_k \right) - \sigma_{lk,l} = 0 \tag{29}$$

5.3. Balance of Macro Angular Momenta

Based on this balance law the sum of the moment of the forces and the moments acting on a volume of matter must be zero so that the volume of matter does not experience pure rigid body rotations. In the derivation of the macro balance of angular momenta, we must begin with the balance of angular momenta balance law for the microconstituents. Since the balance of linear momenta for the microconstituents is a statement of the sum of the inertial forces, the body forces and those due to traction on ∂V (converted to stresses using Cauchy principle), balance of angular momenta can be directly written by multiplying balance of linear momenta by $\epsilon_{mkn}\bar{x}_m^{(\alpha)}$ and integrating over $\partial\bar{V}^{(\alpha)}$ and $\partial\bar{V}$. Additionally we must include the moment $\bar{M}_n^{(\alpha)}$ acting on $d\bar{A}^{(\alpha)}$, the moment per unit area over the oblique plane of the tetrahedron with elemental area $d\bar{A}^{(\alpha)}$.

Form **1**

$$\begin{aligned} &\int\limits_{\bar{V}(t)}\int\limits_{\bar{V}^{(\alpha)}(t)} \epsilon_{mkn}\bar{x}_m^{(\alpha)} \left(\bar{\rho}^{(\alpha)}\bar{a}_k^{(\alpha)} - \bar{\rho}^{(\alpha)} \left({}^{b}\bar{F}_k^{(\alpha)} \right) \right) d\bar{V}^{(\alpha)} \\ &- \int\limits_{\partial\bar{V}(t)}\int\limits_{\partial\bar{V}^{(\alpha)}(t)} \epsilon_{mkn}\bar{x}_m^{(\alpha)}\bar{\sigma}_{lk}^{(\alpha)}\bar{n}_l^{(\alpha)} d\bar{A}^{(\alpha)} - \int\limits_{\partial\bar{V}(t)}\int\limits_{\partial\bar{V}^{(\alpha)}(t)} \bar{M}_n^{(\alpha)} d\bar{A}^{(\alpha)} = 0 \end{aligned} \tag{30}$$

We refer to (30) as "Form 1".

Surana *et al.* [1] have shown that the second integral term in (30) can be expressed in the following two alternate forms (Form 2 and Form 3).

Form **2**

In "Form 2", we replace the second integral term in (30) by its equivalent volume integral to obtain

$$\begin{aligned} &\int\limits_{\bar{V}(t)}\int\limits_{\bar{V}^{(\alpha)}(t)} \epsilon_{mkn}\bar{x}_m^{(\alpha)} \left(\bar{\rho}^{(\alpha)}\bar{a}_k^{(\alpha)} - \bar{\rho}^{(\alpha)} \left({}^{b}\bar{F}_k^{(\alpha)} \right) \right) d\bar{V}^{(\alpha)} \\ &- \int\limits_{\bar{V}(t)}\int\limits_{\bar{V}^{(\alpha)}(t)} \epsilon_{mkn}\bar{x}_m^{(\alpha)}\bar{\sigma}_{lk,l}^{(\alpha)} d\bar{V}^{(\alpha)} - \int\limits_{\partial\bar{V}(t)}\int\limits_{\partial\bar{V}^{(\alpha)}(t)} \bar{M}_n^{(\alpha)} d\bar{A}^{(\alpha)} = 0 \end{aligned} \tag{31}$$

The first and third terms in (31) remain the same as in (30).

Form **3**

We consider the following identity.

$$\begin{aligned} \left(\bar{x}_m^{(\alpha)}\bar{\sigma}_{lk}^{(\alpha)} \right)_{,l} &= \bar{x}_{m,l}^{(\alpha)}\bar{\sigma}_{lk}^{(\alpha)} + \bar{x}_m^{(\alpha)}\bar{\sigma}_{lk,l}^{(\alpha)} \\ \therefore\ \bar{x}_m^{(\alpha)}\bar{\sigma}_{lk,l}^{(\alpha)} &= \left(\bar{x}_m^{(\alpha)}\bar{\sigma}_{lk}^{(\alpha)} \right)_{,l} - \bar{x}_{m,l}^{(\alpha)}\bar{\sigma}_{lk}^{(\alpha)} \end{aligned} \tag{32}$$

We substitute from (32) in the second term of (31) to obtain

$$\begin{aligned} &\int\limits_{\bar{V}(t)}\int\limits_{\bar{V}^{(\alpha)}(t)} \epsilon_{mkn}\bar{x}_m^{(\alpha)} \left(\bar{\rho}^{(\alpha)}\bar{a}_k^{(\alpha)} - \bar{\rho}^{(\alpha)} \left({}^{b}\bar{F}_k^{(\alpha)} \right) \right) d\bar{V}^{(\alpha)} \\ &- \int\limits_{\bar{V}(t)}\int\limits_{\bar{V}^{(\alpha)}(t)} \left(\epsilon_{mkn} \left(\bar{x}_m^{(\alpha)}\bar{\sigma}_{lk}^{(\alpha)} \right)_{,l} - \epsilon_{mkn}\bar{x}_m^{(\alpha)}\bar{\sigma}_{lk}^{(\alpha)} \right) d\bar{V}^{(\alpha)} \\ &- \int\limits_{\partial\bar{V}(t)}\int\limits_{\partial\bar{V}^{(\alpha)}(t)} \bar{M}_n^{(\alpha)} d\bar{A}^{(\alpha)} = 0 \end{aligned} \tag{33}$$

This Equation (33) is the third possible form that can be used to derive macro balance of angular momenta.

Balance of angular momenta has been derived in reference [1] for linear micropolar theory. The basic derivations and the final form of this balance law for linear microdilation theory remains the same in appearance as in the case of linear micromorphic theory, but the definition of the stress and strain tensor and some other details are naturally different in the case of microdilation theory. In the following we present important steps of the derivations for all three forms.

***Balance of angular momenta*: *Form* 1**

Consider Form 1 given by (30). We consider each term in Equation (30). Using

$$\bar{x}_m^{(\alpha)} = \bar{x}_m + \underset{\sim}{\bar{x}}_m^{(\alpha)} \tag{34}$$

and defining

$$\int_{\bar{V}^{(\alpha)}(t)} \left(\bar{\rho}^{(\alpha)}\bar{a}_k^{(\alpha)} - \bar{\rho}^{(\alpha)}\left({}^b\bar{F}_k^{(\alpha)}\right)\right)d\bar{V}^{(\alpha)} \overset{\text{def}}{=} \left(\bar{\rho}\bar{a}_k - \bar{\rho}\left({}^b\bar{F}_k\right)\right)d\bar{V} \tag{35}$$

We can obtain the following for the first term in (30) in Lagrangian description.

$$\begin{aligned}&\int_{\bar{V}^{(\alpha)}(t)}\int_{\partial\bar{V}^{(\alpha)}(t)} \epsilon_{mkn}\bar{x}_m^{(\alpha)}\left(\bar{\rho}^{(\alpha)}\bar{a}_k^{(\alpha)} - \bar{\rho}^{(\alpha)}\left({}^b\bar{F}_k^{(\alpha)}\right)\right)d\bar{V}^{(\alpha)} \\ &= \int_V \epsilon_{mkn}x_m\left(\rho_0 a_k - \rho_0\left({}^bF_k\right)\right)dV + \int_V\int_{V^{(\alpha)}} \epsilon_{mkn}\underset{\sim}{x}_m^{(\alpha)}\left(\rho_0^{(\alpha)}a_k^{(\alpha)} - \rho_0^{(\alpha)}\left({}^bF_k^{(\alpha)}\right)\right)dV^{(\alpha)}\end{aligned} \tag{36}$$

Consider the second term in (30), substitute $\bar{x}_m^{(\alpha)}$ from (34) and defining

$$\int_{\partial\bar{V}^{(\alpha)}(t)} \bar{\sigma}_{lk}^{(\alpha)}\bar{n}_l^{(\alpha)}d\bar{A}^{(\alpha)} \overset{\text{def}}{=} \bar{\sigma}_{lk}\bar{n}_l d\bar{A} \tag{37}$$

$$\int_{\bar{V}^{(\alpha)}(t)} \bar{\sigma}_{mk}^{(\alpha)}d\bar{V}^{(\alpha)} \overset{\text{def}}{=} \bar{S}_{mk}d\bar{V} \tag{38}$$

We can obtain the following.

$$\begin{aligned}&\int_{\partial\bar{V}(t)}\int_{\partial\bar{V}^{(\alpha)}(t)} \epsilon_{mkn}\bar{x}_m^{(\alpha)}\bar{\sigma}_{lk}^{(\alpha)}\bar{n}_l^{(\alpha)}d\bar{A}^{(\alpha)} \\ &= \int_V \epsilon_{mkn}\left(\sigma_{mk} + x_m\sigma_{lk,l} + S_{mk}\right)dV + \int_V\int_{V^{(\alpha)}} \epsilon_{mkn}\underset{\sim}{x}_m^{(\alpha)}\sigma_{lk,l}^{(\alpha)}dV^{(\alpha)}\end{aligned} \tag{39}$$

Consider the third term in (30). Defining

$$\int_{\partial\bar{V}^{(\alpha)}(t)} \bar{m}_{ln}^{(\alpha)}\bar{n}_l^{(\alpha)}d\bar{A}^{(\alpha)} \overset{\text{def}}{=} \bar{m}_{ln}\bar{n}_l d\bar{A} \tag{40}$$

using (40), we can obtain the following for the third term in (30).

$$\int_{\partial\bar{V}(t)}\int_{\partial\bar{V}^{(\alpha)}(t)} \bar{M}_n^{(\alpha)}d\bar{A}^{(\alpha)} = \int_{\bar{V}(t)} \bar{m}_{ln,l}d\bar{V} \tag{41}$$

Substituting (36), (39), and (41) in (30), we obtain

$$\begin{aligned}&\int_{\bar{V}(t)} \epsilon_{mkn}\bar{x}_m\left(\bar{\rho}\bar{a}_k - \bar{\rho}\left({}^b\bar{F}_k\right)\right)d\bar{V} + \int_{\bar{V}(t)}\int_{\bar{V}^{(\alpha)}(t)} \epsilon_{mkn}\underset{\sim}{\bar{x}}_m^{(\alpha)}\left(\bar{\rho}^{(\alpha)}\bar{a}_k^{(\alpha)} - \bar{\rho}^{(\alpha)}\left({}^b\bar{F}_k^{(\alpha)}\right)\right)d\bar{V}^{(\alpha)} \\ &- \int_{\bar{V}(t)} \epsilon_{mkn}\left(\bar{\sigma}_{mk} + \bar{x}_m\bar{\sigma}_{lk,l} + \bar{S}_{mk}\right)d\bar{V} - \int_{\bar{V}(t)}\int_{\bar{V}^{(\alpha)}(t)} \epsilon_{mkn}\underset{\sim}{\bar{x}}_m^{(\alpha)}\bar{\sigma}_{lk,l}^{(\alpha)}d\bar{V}^{(\alpha)} - \int_{\bar{V}(t)} \bar{m}_{ln,l}d\bar{V} = 0\end{aligned} \tag{42}$$

Grouping terms in (42)

$$\begin{aligned}&\int_{\bar{V}(t)} \epsilon_{mkn}\bar{x}_m\left(\bar{\rho}\bar{a}_k-\bar{\rho}\left({}^b\bar{F}_k\right)-\bar{\sigma}_{lk,l}\right)d\bar{V}\\&+\int_{\bar{V}(t)}\int_{\bar{V}^{(\alpha)}(t)} \epsilon_{mkn}\underset{\sim}{\bar{x}}_m^{(\alpha)}\left(\bar{\rho}^{(\alpha)}\bar{a}_k^{(\alpha)}-\bar{\rho}^{(\alpha)}\left({}^b\bar{F}_k^{(\alpha)}\right)-\bar{\sigma}_{lk,l}^{(\alpha)}\right)d\bar{V}^{(\alpha)}\\&-\int_{\bar{V}(t)}\left(\epsilon_{mkn}\left(\bar{\sigma}_{mk}+\bar{S}_{mk}\right)+\bar{m}_{ln,l}\right)d\bar{V}=0\end{aligned} \tag{43}$$

The first and second terms in (43) are zero due to macro and micro balance of linear momenta, thus (43) reduces to

$$\int_{\bar{V}(t)}\left(\epsilon_{mkn}\left(\bar{\sigma}_{mk}+\bar{S}_{mk}\right)+\bar{m}_{ln,l}\right)d\bar{V}=0 \tag{44}$$

Using localization theorem (44) yields

$$\epsilon_{mkn}\left(\bar{\sigma}_{mk}+\bar{S}_{mk}\right)+\bar{m}_{ln,l}=0 \tag{45}$$

Equation (45) is the final form of balance of macro angular momenta for Form 1 in Eulerian description. Equation (45) in Lagrangian description can be written as

$$\epsilon_{mkn}\left(\sigma_{mk}+S_{mk}\right)+m_{ln,l}=0 \tag{46}$$

***Balance of angular momenta*: *Form* 2**

In this case we consider the following.

$$\begin{aligned}&\int_{\bar{V}(t)}\int_{\bar{V}^{(\alpha)}(t)} \epsilon_{mkn}\bar{x}_m^{(\alpha)}\left(\bar{\rho}^{(\alpha)}\bar{a}_k^{(\alpha)}-\bar{\rho}^{(\alpha)}\left({}^b\bar{F}_k^{(\alpha)}\right)\right)d\bar{V}^{(\alpha)}\\&-\int_{\bar{V}(t)}\int_{\bar{V}^{(\alpha)}(t)} \epsilon_{mkn}\bar{x}_m^{(\alpha)}\bar{\sigma}_{lk,l}^{(\alpha)}d\bar{V}^{(\alpha)}-\int_{\partial\bar{V}(t)}\int_{\partial\bar{V}^{(\alpha)}(t)}\bar{M}_n^{(\alpha)}d\bar{A}^{(\alpha)}=0\end{aligned} \tag{47}$$

For the first and third terms we already have details in (36) and (41), thus we need to consider only the second term in (47). Substituting for $\bar{x}_m^{(\alpha)}$ from (34) and defining

$$\int_{\partial\bar{V}^{(\alpha)}(t)}\bar{\sigma}_{lk}^{(\alpha)}\bar{n}_l^{(\alpha)}d\bar{A}^{(\alpha)}\overset{\text{def}}{=}\bar{\sigma}_{lk}\bar{n}_l d\bar{A} \tag{48}$$

$$\int_{\bar{V}^{(\alpha)}(t)}\bar{\sigma}_{mk}^{(\alpha)}d\bar{V}^{(\alpha)}\overset{\text{def}}{=}\bar{S}_{mk}d\bar{V} \tag{49}$$

and following reference [1] we can obtain the following

$$\begin{aligned}&\int_{\bar{V}(t)}\int_{\bar{V}^{(\alpha)}(t)} \epsilon_{mkn}\bar{x}_m^{(\alpha)}\bar{\sigma}_{lk,l}^{(\alpha)}d\bar{V}^{(\alpha)}\\&=\int_V \epsilon_{mkn}\left(\sigma_{mk}+x_m\sigma_{lk,l}+S_{mk}\right)dV+\int_V\int_{V^{(\alpha)}}\epsilon_{mkn}\underset{\sim}{x}_m^{(\alpha)}\sigma_{lk,l}^{(\alpha)}dV^{(\alpha)}\end{aligned} \tag{50}$$

This is exactly the same as what we had obtained for the second term in Form 1 (Equation (39)). Substituting from (36), (41), and (50) in (47), we obtain exactly the same balance of angular momenta as in Form 1. In Lagrangian description we can write

$$\epsilon_{mkn}\left(\sigma_{mk}+S_{mk}\right)+m_{ln,l}=0 \tag{51}$$

Which is the same as the one derived in Form 1 (Equation (46)).

***Balance of angular momenta*: *Form* 3**

In this case we consider the following.

$$\begin{aligned}&\int_{\bar{V}(t)}\int_{\bar{V}^{(\alpha)}(t)}\epsilon_{mkn}\bar{x}_m^{(\alpha)}\left(\bar{\rho}^{(\alpha)}\bar{a}_k^{(\alpha)}-\bar{\rho}^{(\alpha)}\left({}^b\bar{F}_k^{(\alpha)}\right)\right)d\bar{V}^{(\alpha)}\\&-\int_{\bar{V}(t)}\int_{\bar{V}^{(\alpha)}(t)}\left(\epsilon_{mkn}\left(\bar{x}_m^{(\alpha)}\bar{\sigma}_{lk}^{(\alpha)}\right)_{,l}-\epsilon_{mkn}\bar{x}_m^{(\alpha)}\bar{\sigma}_{lk}^{(\alpha)}\right)d\bar{V}^{(\alpha)}\\&-\int_{\partial\bar{V}(t)}\int_{\partial\bar{V}^{(\alpha)}(t)}\bar{M}_n^{(\alpha)}d\bar{A}^{(\alpha)}=0\end{aligned} \tag{52}$$

In this case also the first and last terms are the same as in Form 1 and Form 2, hence (36), (41) hold for these terms. We consider the second term in (52). Substituting for $\bar{x}_m^{(\alpha)}$ from (34) and defining

$$\int_{\bar{V}^{(\alpha)}(t)}\bar{\sigma}_{mk}^{(\alpha)}d\bar{V}^{(\alpha)}\overset{\text{def}}{=}\bar{S}_{mk}d\bar{V} \tag{53}$$

$$\int_{\bar{V}^{(\alpha)}(t)}\bar{\sigma}_{lk}^{(\alpha)}\bar{n}_l^{(\alpha)}d\bar{A}^{(\alpha)}\overset{\text{def}}{=}\bar{\sigma}_{lk}\bar{n}_l d\bar{A} \tag{54}$$

and following reference [1], we can derive the following in Lagrangian description.

$$\begin{aligned}&\int_{\bar{V}^{(\alpha)}(t)}\epsilon_{mkn}\left(\bar{x}_m^{(\alpha)}\bar{\sigma}_{lk}^{(\alpha)}\right)_{,l}d\bar{V}^{(\alpha)}-\int_{\bar{V}(t)}\int_{\bar{V}^{(\alpha)}(t)}\epsilon_{mkn}\bar{\underset{\sim}{x}}_{m,l}^{(\alpha)}\bar{\sigma}_{lk}^{(\alpha)}d\bar{V}^{(\alpha)}\\&=\int_V\epsilon_{mkn}\left(\sigma_{mk}+x_m\sigma_{lk,l}-S_{mk}\right)dV+\int_V\int_{V^{(\alpha)}}\epsilon_{mkn}\underset{\sim}{x}_m^{(\alpha)}\sigma_{lk,l}^{(\alpha)}dV^{(\alpha)}\end{aligned} \tag{55}$$

Substituting from (36), (41), and (55) in (52) we can obtain the following.

$$\begin{aligned}&\int_{\bar{V}(t)}\epsilon_{mkn}\bar{x}_m\left(\bar{\rho}\bar{a}_k-\bar{\rho}\left({}^b\bar{F}_k\right)\right)d\bar{V}+\int_{\bar{V}(t)}\int_{\bar{V}^{(\alpha)}(t)}\epsilon_{mkn}\bar{\underset{\sim}{x}}_m^{(\alpha)}\left(\bar{\rho}^{(\alpha)}\bar{a}_k^{(\alpha)}-\bar{\rho}^{(\alpha)}\left({}^b\bar{F}_k^{(\alpha)}\right)\right)d\bar{V}^{(\alpha)}\\&-\int_{\bar{V}(t)}\epsilon_{mkn}\left(\bar{\sigma}_{mk}+\bar{x}_m\bar{\sigma}_{lk,l}-\bar{S}_{mk}\right)d\bar{V}-\int_{\bar{V}(t)}\int_{\bar{V}^{(\alpha)}(t)}\epsilon_{mkn}\bar{\underset{\sim}{x}}_m^{(\alpha)}\bar{\sigma}_{lk,l}^{(\alpha)}d\bar{V}^{(\alpha)}-\int_{\bar{V}(t)}\bar{m}_{ln,l}d\bar{V}=0\end{aligned} \tag{56}$$

Collecting terms in (56)

$$\begin{aligned}&\int_{\bar{V}(t)}\epsilon_{mkn}\bar{x}_m\left(\bar{\rho}\bar{a}_k-\bar{\rho}\left({}^b\bar{F}_k\right)-\bar{\sigma}_{lk,l}\right)d\bar{V}\\&+\int_{\bar{V}(t)}\int_{\bar{V}^{(\alpha)}(t)}\epsilon_{mkn}\bar{\underset{\sim}{x}}_m^{(\alpha)}\left(\bar{\rho}^{(\alpha)}\bar{a}_k^{(\alpha)}-\bar{\rho}^{(\alpha)}\left({}^b\bar{F}_k^{(\alpha)}\right)-\bar{\sigma}_{lk,l}\right)d\bar{V}^{(\alpha)}\\&-\int_{\bar{V}(t)}\left(\epsilon_{mkn}\left(\bar{\sigma}_{mk}-\bar{S}_{mk}\right)+\bar{m}_{ln,l}\right)d\bar{V}=0\end{aligned} \tag{57}$$

The first two terms in (57) are zero due to balance of macro and micro linear momenta. Thus, (57) reduces to

$$\int_{\bar{V}(t)}\left(\epsilon_{mkn}\left(\bar{\sigma}_{mk}-\bar{S}_{mk}\right)+\bar{m}_{ln,l}\right)d\bar{V}=0 \tag{58}$$

Using localization theorem in (58) we can obtain

$$\epsilon_{mkn}\left(\bar{\sigma}_{mk}-\bar{S}_{mk}\right)+\bar{m}_{ln,l}=0 \tag{59}$$

This is the final form of balance of macro angular momenta for Form 3 in Eulerian description. In Lagrangian description (59) can be written as

$$\epsilon_{mkn}\left(\sigma_{mk}-S_{mk}\right)+m_{ln,l}=0 \tag{60}$$

Remarks

1) We note that Forms 1 and 2 yield exactly the same balance of angular momenta. This is no surprise because the second term in the two forms, one is over the boundary $\partial\bar{V}^{(\alpha)}$ and the other is over the volume $\bar{V}^{(\alpha)}$, both can be derived from each other. In Form 3, use of identity is not part of the standard derivation of balance of angular momenta. Its use may or may not result in changes in the balance of angular momenta. However, in our derivation Form 3 results in a negative sign for the $\boldsymbol{S}$ term. At this stage we do not discard Form 3, but keep it till all balance laws have been derived to determine which form of balance of angular momenta is supported by the physics. Hence, for now we maintain both positive and negative signs for $\boldsymbol{S}$ tensor.

$$\epsilon_{mkn}\left(\bar{\sigma}_{mk}\pm\bar{S}_{mk}\right)+\bar{m}_{ln,l}=0 \tag{61}$$

$$\epsilon_{mkn}\left(\sigma_{mk}\pm S_{mk}\right)+m_{ln,l}=0 \tag{62}$$

Since $\bar{S}_{mk}=\bar{S}_{km}$ and $S_{mk}=S_{km}$, $\epsilon_{mkn}S_{mk}=0$ and $\epsilon_{mkn}\bar{S}_{mk}=0$, (61) and (62) reduce to the following.

$$\epsilon_{mkn}\left(\bar{\sigma}_{mk}\right)+\bar{m}_{ln,l}=0 \tag{63}$$

and

$$\epsilon_{mkn}\left(\sigma_{mk}\right)+m_{ln,l}=0 \tag{64}$$

Equations (63) and (64) are the balance of angular momenta in Eulerian and Lagrangian descriptions.

We note that balance of angular momenta (64) in microdilation microcontinuum theory is the same as in micropolar and micromorphic theories. This is no surprise because the rigid rotation physics of the microconstituents is identical for all 3M theories. Equation (64) gives three equations for balance of angular momenta about the *x*-axes. From balance of angular momenta we note that the skew symmetric components of the nonsymmetric Cauchy stress tensor ($\bar{\boldsymbol{\sigma}}$ or $\boldsymbol{\sigma}$) are balanced by the gradients of the Cauchy moment tensor. This is a rather significant observation, as it precludes the skew symmetric part of $\bar{\boldsymbol{\sigma}}$ or $\boldsymbol{\sigma}$ from being a constitutive tensor or part of a constitutive tensor, *i.e.* if we consider additive decomposition of $\boldsymbol{\sigma}$

$$\boldsymbol{\sigma}={}_s\boldsymbol{\sigma}+{}_a\boldsymbol{\sigma} \tag{65}$$

Then $\boldsymbol{\sigma}$ and ${}_a\boldsymbol{\sigma}$ cannot be constitutive tensors. Only ${}_s\boldsymbol{\sigma}$ can be considered a valid choice of constitutive tensor.

2) The first and the second law of thermodynamics for linear micromorphic microcontinuum were also derived in reference [1]. The derivation of these bal-

ance laws follows the same procedure as presented in [1], however since the microdeformation is completely different in this microdilation theory than that of linear micromorphic theory, the stress and the strain measures are completely different compared to linear micromorphic theory and there are other differences in the derivation, thus the derivation details are needed for understanding the intricacies in the derivation. For these reasons we present the derivations of both the first and second laws of thermodynamics in the following.

5.4. First Law of Thermodynamics

Since the conservation and balance laws of classical continuum mechanics hold for microconstituents we can begin with the integral form of the energy equation resulting from the first law of thermodynamics for the microconstituents in Eulerian description.

$$\int_{\bar{V}^{(\alpha)}(t)} \left(\bar{\rho}^{(\alpha)} \dot{\bar{e}}^{(\alpha)} - \bar{\sigma}_{kl}^{(\alpha)} \bar{v}_{l,k}^{(\alpha)} - \bar{q}_{k,k}^{(\alpha)} \right) d\bar{V}^{(\alpha)} - \int_{\partial \bar{V}^{(\alpha)}(t)} \bar{M}_k^{(\alpha)} \left({}_c^r \bar{\Theta} \right)_k d\bar{A}^{(\alpha)} = 0 \tag{66}$$

In which $\bar{e}^{(\alpha)}$ is the specific internal energy, $\bar{\boldsymbol{q}}^{(\alpha)}$ is the heat vector, and ${}_c^r \bar{\boldsymbol{\Theta}}$ are classical rotation rates (due to $\boldsymbol{\nabla}^{(\alpha)} \times \bar{\boldsymbol{v}}^{(\alpha)}$). We integrate (66) over $\bar{V}$ and $\partial \bar{V}$.

$$\int_{\bar{V}(t)} \int_{\bar{V}^{(\alpha)}(t)} \left(\bar{\rho}^{(\alpha)} \dot{\bar{e}}^{(\alpha)} - \bar{\sigma}_{kl}^{(\alpha)} \bar{v}_{l,k}^{(\alpha)} - \bar{q}_{k,k}^{(\alpha)} \right) d\bar{V}^{(\alpha)} - \int_{\partial \bar{V}(t)} \int_{\partial \bar{V}^{(\alpha)}(t)} \bar{M}_k^{(\alpha)} \left({}_c^r \bar{\Theta} \right)_k d\bar{A}^{(\alpha)} = 0 \tag{67}$$

We consider each term within the integrals of (67). Consider the first term (say t1)

$$\mathrm{t1} = \int_{\bar{V}(t)} \int_{\bar{V}^{(\alpha)}(t)} \bar{\rho}^{(\alpha)} \dot{\bar{e}}^{(\alpha)} d\bar{V}^{(\alpha)} = \int_V \int_{V^{(\alpha)}} \rho_0^{(\alpha)} \dot{e}^{(\alpha)} dV^{(\alpha)} = \int_V \frac{D}{Dt} \int_{V^{(\alpha)}} \rho_0^{(\alpha)} e^{(\alpha)} dV^{(\alpha)} \tag{68}$$

Define

$$\int_{V^{(\alpha)}} \rho_0^{(\alpha)} e^{(\alpha)} dV^{(\alpha)} \stackrel{\text{def}}{=} \rho_0 e dV \tag{69}$$

Substituting (69) in (68)

$$\mathrm{t1} = \int_V \frac{D}{Dt} \left(\rho_0 e \right) dV = \int_V \rho_0 \dot{e} dV = \int_{\bar{V}(t)} \bar{\rho} \dot{\bar{e}} d\bar{V} \tag{70}$$

Consider the second term of (67) (say t2)

$$\begin{aligned} \mathrm{t2} &= \int_{\bar{V}(t)} \int_{\bar{V}^{(\alpha)}(t)} \bar{\sigma}_{kl}^{(\alpha)} \bar{v}_{l,k}^{(\alpha)} d\bar{V}^{(\alpha)} = \int_{\bar{V}(t)} \int_{\bar{V}^{(\alpha)}(t)} \left(\left(\bar{\sigma}_{kl}^{(\alpha)} \bar{v}_l^{(\alpha)} \right)_{,k} - \bar{\sigma}_{kl,k}^{(\alpha)} \bar{v}_l^{(\alpha)} \right) d\bar{V}^{(\alpha)} \\ &= \int_{\partial \bar{V}(t)} \int_{\partial \bar{V}^{(\alpha)}(t)} \bar{\sigma}_{kl}^{(\alpha)} \bar{v}_l^{(\alpha)} \bar{n}_k^{(\alpha)} d\bar{A}^{(\alpha)} - \int_{\bar{V}(t)} \int_{\bar{V}^{(\alpha)}(t)} \bar{\sigma}_{kl,k}^{(\alpha)} \bar{v}_l^{(\alpha)} d\bar{V}^{(\alpha)} \end{aligned} \tag{71}$$

We note that

$$\bar{v}_l^{(\alpha)} = \bar{v}_l + \bar{L}_{lm}^{(\alpha)} \underset{\sim}{\bar{x}}_m^{(\alpha)} \tag{72}$$

$$\bar{\sigma}_{kl,k}^{(\alpha)} = \bar{\rho}^{(\alpha)} \bar{a}_l^{(\alpha)} - \bar{\rho}^{(\alpha)} \left({}^b \bar{F}_l^{(\alpha)} \right) \tag{73}$$

Substituting (72) and (73) in (71)

$$\begin{aligned} t2 = &\int_{\partial\bar{V}(t)} \int_{\partial\bar{V}^{(\alpha)}(t)} \bar{\sigma}_{kl}^{(\alpha)} \left(\bar{v}_l^{(\alpha)} + \bar{L}_{lm}^{(\alpha)} \underset{\sim}{\bar{x}}_m^{(\alpha)}\right) \bar{n}_k^{(\alpha)} d\bar{A}^{(\alpha)} \\ &- \int_{\bar{V}(t)} \int_{\bar{V}^{(\alpha)}(t)} \left(\bar{\rho}^{(\alpha)} \bar{a}_l^{(\alpha)} - \bar{\rho}^{(\alpha)} \left({}^b\bar{F}_l^{(\alpha)}\right)\right) \left(\bar{v}_l + \bar{L}_{lm}^{(\alpha)} \underset{\sim}{\bar{x}}_m^{(\alpha)}\right) d\bar{V}^{(\alpha)} \end{aligned} \tag{74}$$

$$\begin{aligned} t2 = &\int_{\partial\bar{V}(t)} \bar{v}_l \int_{\partial\bar{V}^{(\alpha)}(t)} \bar{\sigma}_{kl}^{(\alpha)} \bar{n}_k^{(\alpha)} d\bar{A}^{(\alpha)} + \int_{\partial\bar{V}(t)} \int_{\partial\bar{V}^{(\alpha)}(t)} \bar{\sigma}_{kl}^{(\alpha)} \bar{L}_{lm}^{(\alpha)} \underset{\sim}{\bar{x}}_m^{(\alpha)} \bar{n}_k^{(\alpha)} d\bar{A}^{(\alpha)} \\ &= -\int_{\bar{V}(t)} \bar{v}_l \int_{\bar{V}^{(\alpha)}(t)} \left(\bar{\rho}^{(\alpha)} \bar{a}_l^{(\alpha)} - \bar{\rho}^{(\alpha)} \left({}^b\bar{F}_l^{(\alpha)}\right)\right) d\bar{V}^{(\alpha)} \\ &- \int_{\bar{V}(t)} \int_{\bar{V}^{(\alpha)}(t)} \left(\bar{\rho}^{(\alpha)} \bar{a}_l^{(\alpha)} - \bar{\rho}^{(\alpha)} \left({}^b\bar{F}_l^{(\alpha)}\right)\right) \bar{L}_{lm}^{(\alpha)} \underset{\sim}{\bar{x}}_m^{(\alpha)} d\bar{V}^{(\alpha)} \end{aligned} \tag{75}$$

Define

$$\int_{\partial\bar{V}^{(\alpha)}(t)} \bar{\sigma}_{kl}^{(\alpha)} \bar{n}_k^{(\alpha)} d\bar{A}^{(\alpha)} \stackrel{\text{def}}{=} \bar{\sigma}_{kl} \bar{n}_k d\bar{A} \tag{76}$$

and

$$\int_{\bar{V}^{(\alpha)}(t)} \left(\bar{\rho}^{(\alpha)} \bar{a}_l^{(\alpha)} - \bar{\rho}^{(\alpha)} \left({}^b\bar{F}_l^{(\alpha)}\right)\right) d\bar{V}^{(\alpha)} \stackrel{\text{def}}{=} \left(\bar{\rho}\bar{a}_l - \bar{\rho}\left({}^b\bar{F}_l\right)\right) d\bar{V} \tag{77}$$

Substituting (76) and (77) in (75)

$$\begin{aligned} t2 = &\int_{\bar{V}(t)} \bar{v}_l \bar{\sigma}_{kl} \bar{n}_k d\bar{A} + \int_{\partial\bar{V}(t)} \int_{\partial\bar{V}^{(\alpha)}(t)} \bar{\sigma}_{kl}^{(\alpha)} \bar{L}_{lm}^{(\alpha)} \underset{\sim}{\bar{x}}_m^{(\alpha)} \bar{n}_k^{(\alpha)} d\bar{A}^{(\alpha)} \\ &- \int_{\bar{V}(t)} \bar{v}_l \left(\bar{\rho}\bar{a}_l - \bar{\rho}\left({}^b\bar{F}_l\right)\right) d\bar{V} \\ &- \int_{\bar{V}(t)} \int_{\bar{V}^{(\alpha)}(t)} \left(\bar{\rho}^{(\alpha)} \bar{a}_l^{(\alpha)} - \bar{\rho}^{(\alpha)} \left({}^b\bar{F}_l^{(\alpha)}\right)\right) \bar{L}_{lm}^{(\alpha)} \underset{\sim}{\bar{x}}_m^{(\alpha)} d\bar{V}^{(\alpha)} \end{aligned} \tag{78}$$

We note that

$$\int_{\partial\bar{V}(t)} \bar{\sigma}_{kl}^{(\alpha)} \bar{v}_l \bar{n}_k d\bar{A} = \int_{\bar{V}(t)} \left(\bar{\sigma}_{kl}\bar{n}_l\right)_{,k} d\bar{V} = \int_{\bar{V}(t)} \left(\bar{\sigma}_{kl}\bar{v}_{l,k} + \bar{v}_l \bar{\sigma}_{kl,k}\right) d\bar{V} \tag{79}$$

and

$$\begin{aligned} &\int_{\partial\bar{V}(t)} \bar{L}_{lm}^{(\alpha)} \int_{\partial\bar{V}^{(\alpha)}(t)} \bar{\sigma}_{kl}^{(\alpha)} \underset{\sim}{\bar{x}}_m^{(\alpha)} \bar{n}_k^{(\alpha)} d\bar{A}^{(\alpha)} \\ &= \int_V \int_{V^{(\alpha)}} L_{lm}^{(\alpha)} \left(\bar{\sigma}_{kl}^{(\alpha)} \underset{\sim}{\bar{x}}_m^{(\alpha)}\right)_{,k} d\bar{V} \\ &= \int_{\bar{V}(t)} \int_{\bar{V}^{(\alpha)}(t)} \left(\bar{L}_{lm}^{(\alpha)} \bar{\sigma}_{kl,k}^{(\alpha)} \underset{\sim}{\bar{x}}_m^{(\alpha)} + \bar{L}_{lm}^{(\alpha)} \bar{\sigma}_{kl}^{(\alpha)} \underset{\sim}{\bar{x}}_{m,k}^{(\alpha)}\right) d\bar{V}^{(\alpha)} \\ &= \int_{\bar{V}(t)} \int_{\bar{V}^{(\alpha)}(t)} \bar{\sigma}_{kl,k}^{(\alpha)} \bar{L}_{lm}^{(\alpha)} \underset{\sim}{\bar{x}}_m^{(\alpha)} d\bar{V}^{(\alpha)} + \int_{\bar{V}(t)} \int_{\bar{V}^{(\alpha)}(t)} \bar{\sigma}_{ml}^{(\alpha)} \bar{L}_{lm}^{(\alpha)} \underset{\sim}{\bar{x}}_{m,k}^{(\alpha)} d\bar{V}^{(\alpha)} \end{aligned} \tag{80}$$

Define

$$\int_{\bar{V}^{(\alpha)}(t)} \bar{L}_{lm}^{(\alpha)} \bar{\sigma}_{ml}^{(\alpha)} dV^{(\alpha)} \stackrel{\text{def}}{=} \bar{S}_{ml} \bar{L}_{lm}^{(\alpha)} d\bar{V} \tag{81}$$

Using (81) in (80)

$$\int_{\partial\bar{V}(t)}\int_{\partial\bar{V}^{(\alpha)}(t)}\bar{\sigma}_{kl}^{(\alpha)}\bar{L}_{lm}^{(\alpha)}\bar{\underset{\sim}{x}}_{m}^{(\alpha)}\bar{n}^{(\alpha)}d\bar{A}^{(\alpha)}=\int_{\bar{V}(t)}\int_{\bar{V}^{(\alpha)}(t)}\bar{\sigma}_{kl,k}^{(\alpha)}\bar{L}_{lm}^{(\alpha)}\bar{\underset{\sim}{x}}_{m}^{(\alpha)}d\bar{V}^{(\alpha)}+\int_{\bar{V}(t)}\bar{S}_{ml}\bar{L}_{lm}^{(\alpha)}d\bar{V} \quad (82)$$

Substituting (79) and (82) in (78)

$$\begin{aligned}\text{t2}=&\int_{\bar{V}(t)}\left(\bar{\sigma}_{kl}\bar{v}_{l,k}+\bar{v}_{l}\bar{\sigma}_{kl,k}\right)d\bar{V}+\int_{\bar{V}(t)}\int_{\bar{V}^{(\alpha)}(t)}\bar{\sigma}_{kl,k}^{(\alpha)}\bar{L}_{lm}^{(\alpha)}\bar{\underset{\sim}{x}}_{m}^{(\alpha)}d\bar{V}^{(\alpha)}\\&+\int_{\bar{V}(t)}\bar{S}_{ml}\bar{L}_{lm}^{(\alpha)}d\bar{V}-\int_{\bar{V}(t)}\bar{v}_{l}\left(\bar{\rho}\bar{a}_{l}-\bar{\rho}\left({}^{b}\bar{F}_{l}\right)\right)d\bar{V}\\&-\int_{\bar{V}(t)}\int_{\bar{V}^{(\alpha)}(t)}\left(\bar{\rho}^{(\alpha)}\bar{a}_{l}^{(\alpha)}-\bar{\rho}^{(\alpha)}\left({}^{b}\bar{F}_{l}^{(\alpha)}\right)\right)\bar{L}_{lm}^{(\alpha)}\bar{\underset{\sim}{x}}_{m}^{(\alpha)}d\bar{V}^{(\alpha)}\end{aligned} \quad (83)$$

Collecting coefficients in (83), we can write (83) as

$$\begin{aligned}\text{t2}=&\int_{\bar{V}(t)}\bar{\sigma}_{kl}\bar{v}_{l,k}d\bar{V}-\int_{\bar{V}(t)}\bar{v}_{l}\left(\bar{\rho}\bar{a}_{l}-\bar{\rho}\left({}^{b}\bar{F}_{l}\right)-\bar{\sigma}_{kl,k}\right)d\bar{V}\\&+\int_{\bar{V}(t)}\int_{\bar{V}^{(\alpha)}(t)}\bar{L}_{lm}^{(\alpha)}\bar{\underset{\sim}{x}}_{m}^{(\alpha)}\left(\bar{\rho}^{(\alpha)}\bar{a}_{l}^{(\alpha)}-\bar{\rho}^{(\alpha)}\left({}^{b}\bar{F}_{l}^{(\alpha)}\right)-\bar{\sigma}_{kl,k}^{(\alpha)}\right)d\bar{V}^{(\alpha)}\\&+\int_{\bar{V}(t)}\bar{S}_{ml}\bar{L}_{lm}^{(\alpha)}d\bar{V}\end{aligned} \quad (84)$$

The second and third terms in (84) are zero due to balance of macro and micro linear momenta, hence we obtain the following from (84)

$$\text{t2}=\int_{\bar{V}(t)}\left(\bar{\sigma}_{kl}\bar{v}_{l,k}+\bar{S}_{ml}\bar{L}_{lm}^{(\alpha)}\right)d\bar{V} \quad (85)$$

Consider the third term in (67) (say t3)

$$\text{t3}=\int_{\bar{V}(t)}\int_{\bar{V}^{(\alpha)}(t)}\bar{q}_{k,k}^{(\alpha)}d\bar{V}^{(\alpha)}=\int_{\partial\bar{V}(t)}\int_{\partial\bar{V}^{(\alpha)}(t)}\bar{q}_{k}^{(\alpha)}\bar{n}_{k}^{(\alpha)}d\bar{A}^{(\alpha)} \quad (86)$$

Define

$$\int_{\partial\bar{V}^{(\alpha)}(t)}\bar{q}_{k}^{(\alpha)}\bar{n}_{k}^{(\alpha)}d\bar{A}^{(\alpha)}\overset{\text{def}}{=}\bar{q}_{k}\bar{n}_{k}d\bar{A} \quad (87)$$

Substituting (82) in (86)

$$\text{t3}=\int_{\partial\bar{V}(t)}\bar{q}_{k}\bar{n}_{k}d\bar{A}=\int_{\bar{V}(t)}\bar{q}_{k,k}d\bar{V} \quad (88)$$

Consider the fourth term in (67) (say t4)

$$\text{t4}=\int_{\partial\bar{V}(t)}\int_{\bar{V}^{(\alpha)}(t)}\bar{M}_{k}^{(\alpha)}\left({}_{c}^{r}\Theta\right)_{k}d\bar{A}^{(\alpha)}=\int_{\partial\bar{V}(t)}\left({}_{c}^{r}\Theta\right)_{k}\int_{\partial\bar{V}^{(\alpha)}(t)}\bar{m}_{lk}\bar{n}_{l}^{(\alpha)}d\bar{A}^{(\alpha)} \quad (89)$$

Define

$$\int_{\partial\bar{V}^{(\alpha)}(t)}\bar{m}_{lk}^{(\alpha)}\bar{n}_{l}^{(\alpha)}d\bar{A}^{(\alpha)}\overset{\text{def}}{=}\bar{m}_{lk}\bar{n}_{l}d\bar{A} \quad (90)$$

Substituting (90) in (89)

$$\text{t4}=\int_{\partial\bar{V}(t)}\left({}_{c}^{r}\bar{\Theta}\right)_{k}\bar{m}_{lk}\bar{n}_{l}d\bar{A}=\int_{\partial\bar{V}(t)}{}_{c}^{r}\bar{\boldsymbol{\Theta}}\cdot\left(\bar{\boldsymbol{m}}\right)^{T}\cdot d\bar{\boldsymbol{A}}=\int_{\bar{V}(t)}\bar{\boldsymbol{\nabla}}\cdot\left({}_{c}^{r}\bar{\boldsymbol{\Theta}}\cdot\left(\bar{\boldsymbol{m}}\right)^{T}\right)d\bar{V} \quad (91)$$

We can show that [47] [50] [71] [72]

$$\bar{\nabla}\cdot\left({}_c^r\bar{\boldsymbol{\Theta}}\cdot\bar{\boldsymbol{m}}^T\right)={}_c^r\bar{\boldsymbol{\Theta}}\cdot\left(\bar{\nabla}\cdot\bar{\boldsymbol{m}}\right)+\bar{\boldsymbol{m}}:\left({}^{r}_{c}\bar{\Theta}\bar{\boldsymbol{J}}\right) \tag{92}$$

Using (92) in (91)

$$t4=\int_{\bar{V}(t)}\left({}_c^r\bar{\boldsymbol{\Theta}}\cdot\left(\bar{\nabla}\cdot\bar{\boldsymbol{m}}\right)+\bar{\boldsymbol{m}}:\left({}^{r}_{c}\bar{\Theta}\bar{\boldsymbol{J}}\right)\right)d\bar{V} \tag{93}$$

Substituting (70), (85), (88) and (93) in (67), we can write (67) as follows

$$\begin{aligned}&\int_{\bar{V}(t)}\bar{\rho}\dot{\bar{e}}d\bar{V}-\int_{\bar{V}(t)}\bar{\sigma}_{kl}\bar{v}_{l,k}d\bar{V}-\int_{\bar{V}(t)}\bar{S}_{ml}\bar{L}_{lm}^{(\alpha)}d\bar{V}+\int_{\bar{V}(t)}\bar{q}_{k,k}d\bar{V}\\&-\int_{\bar{V}(t)}\left({}_c^r\bar{\boldsymbol{\Theta}}\cdot\left(\bar{\nabla}\cdot\bar{\boldsymbol{m}}\right)+\bar{\boldsymbol{m}}:\left({}^{r}_{c}\bar{\Theta}\bar{\boldsymbol{J}}\right)\right)d\bar{V}=0\end{aligned} \tag{94}$$

or

$$\int_{\bar{V}(t)}\left(\bar{\rho}\dot{\bar{e}}-\bar{\sigma}_{kl}\bar{v}_{l,k}-\bar{S}_{ml}\bar{L}_{lm}^{(\alpha)}+\bar{q}_{k,k}-\left({}_c^r\bar{\boldsymbol{\Theta}}\cdot\left(\bar{\nabla}\cdot\bar{\boldsymbol{m}}\right)+\bar{\boldsymbol{m}}:\left({}^{r}_{c}\bar{\Theta}\bar{\boldsymbol{J}}\right)\right)\right)d\bar{V}=0 \tag{95}$$

Using localization theorem, we obtain the following from (95)

$$\bar{\rho}\dot{\bar{e}}-\bar{\sigma}_{kl}\bar{v}_{l,k}-\bar{S}_{ml}\bar{L}_{lm}^{(\alpha)}+\bar{q}_{k,k}-\left({}_c^r\bar{\boldsymbol{\Theta}}\cdot\left(\bar{\nabla}\cdot\bar{\boldsymbol{m}}\right)+\bar{\boldsymbol{m}}:\left({}^{r}_{c}\bar{\Theta}\bar{\boldsymbol{J}}\right)\right)=0 \tag{96}$$

This is the final form of the macro energy equation in Eulerian description resulting from the first law of thermodynamics. The energy equation in Lagrangian description can be written directly from (96).

$$\rho_0\dot{e}-\sigma_{kl}v_{l,k}-S_{ml}L_{lm}^{(\alpha)}+q_{k,k}-\left({}_c\dot{\boldsymbol{\Theta}}\cdot\left(\nabla\cdot\boldsymbol{m}\right)+\boldsymbol{m}:{}^{c\theta}\dot{\boldsymbol{J}}\right)=0 \tag{97}$$

In which

$$\begin{aligned}v_{l,k}&=\dot{J}_{l,k}\\\bar{L}_{lm}^{(\alpha)}&=\dot{J}_{lm}^{(\alpha)}\end{aligned} \tag{98}$$

5.5. Second Law of Thermodynamics

Let $\bar{\eta}^{(\alpha)}$ be the entropy density in the microconstituent volume $\bar{V}^{(\alpha)}$, $\bar{h}^{(\alpha)}$ be the entropy flux imparted to the volume $\bar{V}^{(\alpha)}$ by the surrounding medium and $\bar{s}^{(\alpha)}$ be the source of entropy in $\bar{V}^{(\alpha)}$ due to the non contacting sources (bodies). Then the rate of increase of entropy in volume $\bar{V}^{(\alpha)}$ of the microconstituent from all contacting and noncontacting sources is given by

$$\frac{D}{Dt}\int_{\bar{V}^{(\alpha)}(t)}\bar{\eta}^{(\alpha)}\bar{\rho}^{(\alpha)}dV^{(\alpha)}\geq\int_{\partial\bar{V}^{(\alpha)}(t)}\bar{h}^{(\alpha)}d\bar{A}^{(\alpha)}+\int_{\bar{V}^{(\alpha)}(t)}\bar{s}^{(\alpha)}\bar{\rho}^{(\alpha)}d\bar{V}^{(\alpha)} \tag{99}$$

Integrating (99) over $\bar{V}$ and $d\bar{V}$

$$\int_{\bar{V}(t)}\frac{D}{Dt}\int_{\bar{V}^{(\alpha)}(t)}\bar{\eta}^{(\alpha)}\bar{\rho}^{(\alpha)}d\bar{V}^{(\alpha)}\geq\int_{\partial\bar{V}(t)}\int_{\partial\bar{V}^{(\alpha)}(t)}\bar{h}^{(\alpha)}d\bar{A}^{(\alpha)}+\int_{\bar{V}(t)}\int_{\bar{V}^{(\alpha)}(t)}\bar{s}^{(\alpha)}\bar{\rho}^{(\alpha)}d\bar{V}^{(\alpha)} \tag{100}$$

Using

$$\bar{h}^{(\alpha)}=-\bar{\Psi}_k^{(\alpha)}\bar{n}_k^{(\alpha)} \tag{101}$$

$$\overline{\Psi}_k^{(\alpha)} = \frac{\overline{q}_k^{(\alpha)}}{\overline{\theta}} \tag{102}$$

$$\therefore\ \overline{h}^{(\alpha)} = -\frac{\overline{q}_k^{(\alpha)}\overline{n}_k^{(\alpha)}}{\overline{\theta}} \tag{103}$$

$$\text{and}\quad \overline{s}^{(\alpha)} = \frac{\overline{r}^{(\alpha)}}{\overline{\theta}} \tag{104}$$

Using (103) and (104) in (100)

$$\int_{\overline{V}(t)} \frac{D}{Dt} \int_{\overline{V}^{(\alpha)}(t)} \overline{\eta}^{(\alpha)}\overline{\rho}^{(\alpha)} d\overline{V}^{(\alpha)} \geq \int_{\partial\overline{V}(t)} - \int_{\partial\overline{V}^{(\alpha)}(t)} \frac{\overline{q}_k^{(\alpha)}\overline{n}_k^{(\alpha)}}{\overline{\theta}} d\overline{A}^{(\alpha)} + \int_{\overline{V}(t)} \int_{\overline{V}^{(\alpha)}(t)} \frac{\overline{r}^{(\alpha)}\overline{\rho}^{(\alpha)}}{\overline{\theta}} d\overline{V}^{(\alpha)} \tag{105}$$

Define

$$\frac{D}{Dt} \int_{\overline{V}(t)} \overline{\eta}^{(\alpha)}\overline{\rho}^{(\alpha)} d\overline{V}^{(\alpha)} \overset{\text{def}}{=} \dot{\overline{\eta}}\,\overline{\rho}\, d\overline{V} \tag{106}$$

$$\int_{\partial\overline{V}^{(\alpha)}(t)} \frac{\overline{q}_k^{(\alpha)}\overline{n}_k^{(\alpha)}}{\overline{\theta}} d\overline{A}^{(\alpha)} \overset{\text{def}}{=} \frac{\overline{q}_k\overline{n}_k}{\overline{\theta}} d\overline{A} \tag{107}$$

$$\int_{\overline{V}^{(\alpha)}(t)} \frac{\overline{r}^{(\alpha)}\overline{\rho}^{(\alpha)}}{\overline{\theta}} d\overline{V}^{(\alpha)} \overset{\text{def}}{=} \frac{\overline{r}\,\overline{\rho}}{\overline{\theta}} d\overline{V} \tag{108}$$

Using (106), (107) and (108) in (105), we obtain

$$\int_{\overline{V}(t)} \overline{\rho}\dot{\overline{\eta}}\, d\overline{V} \geq \int_{\partial\overline{V}(t)} \frac{\overline{q}_k}{\overline{\theta}} \overline{n}_k d\overline{A} + \int_{\overline{V}(t)} \frac{\overline{r}\,\overline{\rho}}{\overline{\theta}} d\overline{V} \tag{109}$$

or

$$\int_{\overline{V}(t)} \overline{\rho}\dot{\overline{\eta}}\, d\overline{V} \geq \int_{\overline{V}(t)} \left(\frac{\overline{q}_k}{\overline{\theta}}\right)_{,k} d\overline{V} + \int_{\overline{V}(t)} \frac{\overline{r}\,\overline{\rho}}{\overline{\theta}} d\overline{V} \tag{110}$$

or

$$\int_{\overline{V}(t)} \left(\overline{\rho}\dot{\overline{\eta}} + \frac{\overline{q}_{k,k}}{\overline{\theta}} - \frac{\overline{q}_k}{\overline{\theta}^2}\overline{\theta}_{,k} + \frac{\overline{r}\,\overline{\rho}}{\overline{\theta}} \right) d\overline{V} \geq 0 \tag{111}$$

Using localization theorem

$$\overline{\rho}\dot{\overline{\eta}} + \frac{\overline{q}_{k,k}}{\overline{\theta}} - \frac{\overline{q}_k\overline{\theta}_{,k}}{\overline{\theta}^2} + \frac{\overline{r}\,\overline{\rho}}{\overline{\theta}} \geq 0 \tag{112}$$

Multiply (112) throughout by $\overline{\theta}$

$$\overline{\rho}\dot{\overline{\eta}}\overline{\theta} + \overline{q}_{k,k} - \frac{\overline{q}_k\overline{\theta}_{,k}}{\overline{\theta}} + \overline{r}\,\overline{\rho} \geq 0 \tag{113}$$

Let

$$\overline{\Phi} = \overline{e} - \overline{\eta}\overline{\theta} \tag{114}$$

$$\therefore\ \dot{\overline{\Phi}} = \dot{\overline{e}} - \dot{\overline{\eta}}\overline{\theta} - \overline{\eta}\dot{\overline{\theta}} \tag{115}$$

$$\therefore\ \overline{\rho}\overline{\theta}\dot{\overline{\eta}} = \overline{\rho}\dot{\overline{e}} - \overline{\rho}\dot{\overline{\Phi}} - \overline{\rho}\overline{\eta}\dot{\overline{\theta}} \tag{116}$$

Substituting from (116) into (113)

$$-\bar{\rho}\left(\dot{\bar{\Phi}}+\bar{\eta}\dot{\bar{\theta}}\right)+\overline{\rho\dot{e}}+\bar{q}_{k,k}-\frac{\bar{q}_k\bar{\theta}_{,k}}{\bar{\theta}}-\bar{r}\bar{\rho}\geq 0 \tag{117}$$

Substituting $\overline{\rho\dot{e}}$ from energy Equation (96) in (117)

$$-\bar{\rho}\left(\dot{\bar{\Phi}}+\bar{\eta}\dot{\bar{\theta}}\right)+\bar{\boldsymbol{\sigma}}:\bar{\boldsymbol{L}}+\bar{\boldsymbol{S}}:\bar{\boldsymbol{L}}^{(\alpha)}-\bar{\nabla}\cdot\bar{\boldsymbol{q}}+\left({}_c^r\bar{\boldsymbol{\Theta}}\cdot\left(\bar{\nabla}\cdot\bar{\boldsymbol{m}}\right)+\bar{\boldsymbol{m}}:{}_c^r\bar{\boldsymbol{\Theta}}\bar{\boldsymbol{J}}\right)$$
$$+\bar{r}\bar{\rho}+\bar{\nabla}\cdot\bar{\boldsymbol{q}}-\frac{\bar{q}_k\bar{\theta}_{,k}}{\bar{\theta}}-\bar{r}\bar{\rho}\geq 0 \tag{118}$$

$\bar{\nabla}\cdot\bar{\boldsymbol{q}}$ term and $\bar{r}\bar{\rho}$ terms cancel and we can write the following after changing the sign.

$$\bar{\rho}\left(\dot{\bar{\Phi}}+\bar{\eta}\dot{\bar{\theta}}\right)-\bar{\boldsymbol{\sigma}}:\bar{\boldsymbol{L}}-\bar{\boldsymbol{S}}:\bar{\boldsymbol{L}}^{(\alpha)}+\frac{\bar{q}_k\bar{\theta}_{,k}}{\bar{\theta}}-\left({}_c^r\bar{\boldsymbol{\Theta}}\cdot\left(\bar{\nabla}\cdot\bar{\boldsymbol{m}}\right)+\bar{\boldsymbol{m}}:{}_c^r\bar{\boldsymbol{\Theta}}\bar{\boldsymbol{J}}\right)\leq 0 \tag{119}$$

This is the final form of the macro entropy inequality in Eulerian description. In Lagrangian description we can write the following directly from (119)

$$\rho_0\left(\dot{\Phi}+\eta\dot{\theta}\right)-\boldsymbol{\sigma}:\dot{\boldsymbol{J}}-\boldsymbol{S}:\dot{\boldsymbol{J}}^{(\alpha)}-\frac{q_k\theta_{,k}}{\theta}-\left({}_c\boldsymbol{\Theta}\cdot\left(\nabla\cdot\boldsymbol{m}\right)+\boldsymbol{m}:{}^c\boldsymbol{\Theta}\dot{\boldsymbol{J}}\right)\leq 0 \tag{120}$$

5.6. Balance of Moment of Moments Balance Law: Macro

Since in all 3M microcontinuum theories rotations and moments is an additional kinematically conjugate pair in addition to displacements and forces (or stresses), hence based on Yang *et al.* [70] and Surana *et al.* [40] [51], the balance of moment of moments balance law is essential in all 3M theories. Using the macro Cauchy stress tensor, the Cauchy moment tensor and following references [40] [51], we can derive the following for this balance law.

$$\epsilon_{ijk}\bar{m}_{ij}=0 \text{ or } \epsilon_{ijk}m_{ij}=0 \tag{121}$$

That is, the Cauchy moment tensor is symmetric in all three 3M microcontinuum theories.

5.7. Summary of Macro Conservation and Balance Laws in Eulerian Description

The differential form of the continuity equation, balance of linear momenta, balance of angular momenta, energy equation, entropy inequality and balance of moment of moments in Eulerian description are given in the following.

$$\dot{\bar{\rho}}+\bar{\rho}\left(\bar{\nabla}\cdot\bar{\boldsymbol{v}}\right)=0 \tag{122}$$

$$\bar{\rho}\bar{a}_k-\bar{\rho}\left({}^b\bar{F}_k\right)-\bar{\sigma}_{lk,l}=0 \tag{123}$$

$$\epsilon_{mkn}\left(\bar{\sigma}_{mk}\pm\bar{S}_{mk}\right)+\bar{m}_{ln,l}=0 \tag{124}$$

$$\overline{\rho\dot{e}}-\bar{\boldsymbol{\sigma}}:\bar{\boldsymbol{L}}-\bar{\boldsymbol{S}}:\bar{\boldsymbol{L}}^{(\alpha)}+\bar{\nabla}\cdot\bar{\boldsymbol{q}}-\left({}_c^r\bar{\boldsymbol{\Theta}}\cdot\left(\bar{\nabla}\cdot\bar{\boldsymbol{m}}\right)+\bar{\boldsymbol{m}}:{}_c^r\bar{\boldsymbol{\Theta}}\bar{\boldsymbol{J}}\right)=0 \tag{125}$$

$$\bar{\rho}\left(\dot{\bar{\Phi}}+\bar{\eta}\dot{\bar{\theta}}\right)-\bar{\boldsymbol{\sigma}}:\bar{\boldsymbol{L}}-\bar{\boldsymbol{S}}:\bar{\boldsymbol{L}}^{(\alpha)}+\frac{\bar{\boldsymbol{q}}\cdot\bar{\boldsymbol{g}}}{\bar{\theta}}-\left({}_c^r\bar{\boldsymbol{\Theta}}\cdot\left(\bar{\nabla}\cdot\bar{\boldsymbol{m}}\right)+\bar{\boldsymbol{m}}:{}_c^r\bar{\boldsymbol{\Theta}}\bar{\boldsymbol{J}}\right)\leq 0 \tag{126}$$

$$\epsilon_{ijk}\bar{m}_{ij}=0 \tag{127}$$

This mathematical model consists of eight equations: Continuity (1), balance of linear momenta (3), balance of angular momenta (3), and energy equation (1) in twenty five dependent variables: $\bar{\rho}(1), \bar{\boldsymbol{v}}(3), \bar{\boldsymbol{\sigma}}(9), \bar{\boldsymbol{m}}(6), \bar{\boldsymbol{S}}(1), \bar{\theta}(1), \bar{\boldsymbol{q}}(3)$ and ${}_s\bar{\boldsymbol{L}}^{(\alpha)}(1)$. Thus, we need 17 more equations for closure of the mathematical model. Constitutive theories yield sixteen equations: $\boldsymbol{\sigma}(6), \bar{\boldsymbol{m}}(6), \bar{\boldsymbol{q}}(3)$ and $\bar{\boldsymbol{S}}(1)$. Thus, we need one additional equation for closure. This is discussed in Section 5.9.

5.8. Summary of Macro Conservation and Balance Laws in Lagrangian Description

The differential form of the continuity equation, balance of linear momenta, balance of angular momenta, energy equation, entropy inequality and balance of moment of moments in Lagrangian description are given in the following (readily obtained from (122)-(127)).

$$\rho_0(\boldsymbol{x}) = |\boldsymbol{J}|\rho(\boldsymbol{x},t) \tag{128}$$

$$\rho_0 a_k - \rho_0\left({}^bF_k\right) - \sigma_{lk,l} = 0 \tag{129}$$

$$\epsilon_{mkn}\left(\sigma_{mk} \pm S_{mk}\right) + m_{ln,l} = 0 \tag{130}$$

$$\rho_0\dot{e} - \boldsymbol{\sigma} : \dot{\boldsymbol{J}} - \boldsymbol{S} : \dot{\boldsymbol{J}}^{(\alpha)} - \nabla\cdot\boldsymbol{q} - \left({}_c\dot{\Theta}\cdot(\nabla\cdot\boldsymbol{m}) + \boldsymbol{m} : {}^{c\Theta}\dot{\boldsymbol{J}}\right) = 0 \tag{131}$$

$$\rho_0\left(\dot{\Phi} + \eta\dot{\theta}\right) - \boldsymbol{\sigma} : \dot{\boldsymbol{J}} - \boldsymbol{S} : \dot{\boldsymbol{J}}^{(\alpha)} + \frac{\boldsymbol{q}\cdot\boldsymbol{g}}{\theta} - \left({}_c\dot{\Theta}\cdot(\nabla\cdot\boldsymbol{m}) + \boldsymbol{m} : {}^{c\Theta}\dot{\boldsymbol{J}}\right) \le 0 \tag{132}$$

$$\epsilon_{ijk} m_{ij} = 0 \tag{133}$$

This mathematical model consists of seven partial differential equations: balance of linear momenta (3), balance of angular momenta (3) energy equation (1) in twenty four dependent variables: $\boldsymbol{u}(3), \boldsymbol{\sigma}(9), \boldsymbol{S}(1), \boldsymbol{m}(6), \boldsymbol{q}(3), \theta(1)$ and ${}^d_s\bar{\boldsymbol{J}}^{(\alpha)}(1)$. Thus, additional seventeen equations are needed for closure of the mathematical model. Constitutive theories provide sixteen equations: $\boldsymbol{\sigma}(6), \boldsymbol{S}(1), \boldsymbol{m}(6), \boldsymbol{q}(3)$. Thus, an additional equation is needed for closure. This additional equation is discussed in Section 5.9.

5.9. Additional Equation in the Mathematical Model

From the conservation and balance laws, we note that the microconstituent stress tensor $\boldsymbol{S}$ only appears in the energy equation and the entropy inequality. This of course implies that if we were to solve a boundary value problem for isothermal physics, in which case the energy equation is not part of the mathematical model, then the microconstituent stress $\boldsymbol{S}$ is completely absent from the mathematical model. This is certainly not physical, as the microconstituent deformation contributes to the macro deformation for boundary value problems, as well as initial value problems. Thus, we must have another relation that considers $\boldsymbol{S}$ and the symmetric part of $\boldsymbol{\sigma}$.

Eringen [7]-[25] and those following his work suggest that in the derivation of the balance of angular momenta the permutation tensor must be dropped to obtain another balance law, moments of $\boldsymbol{S}$ and symmetric part of $\boldsymbol{\sigma}$ that must balance with the gradients of the symmetric part of the Cauchy moment tensor. In Eringen's work the nonsymmetric moment tensor also contains the permutation tensor in balance of angular momenta (this is not valid), hence yields three equations containing gradients of the skew symmetric part of the Cauchy moment tensor and the skew symmetric part of stress tensor $\boldsymbol{\sigma}$. One additional equation is obtained by balancing moments of the symmetric part of $\boldsymbol{\sigma}$ and $\boldsymbol{S}$ with the gradients of the symmetric part of a third rank moment tensor. It is suggested that these four equations are sufficient to address four degrees of freedom for the microconstituents; three rigid rotations and a pure stretch.

Incorrect definition of micromoment tensor, presence of permutation tensor with the moment tensor, and nonsymmetry of moment tensor have lead to incorrect balance of angular momenta in Eringen's work. The additional equations needed for closure in Eringen's work are derived by dropping the permutation tensor in balance of angular momenta. In the resulting equations, the symmetric parts of $\boldsymbol{\sigma}$ and $\boldsymbol{S}$ are balanced by the gradients of the symmetric part of the moment tensor. These equations are of concern due to two reasons, wrong definition of moment tensor and there is no such balance law in thermodynamics as suggested and used by Eringen for obtaining additional equations. More discussion and detail on this can be found in reference [1] and is not repeated for the sake of brevity.

Derivation of Additional Equations

It is perhaps advantageous to consider a case in which microconstituents have six deformational degrees of freedom (as in the linear micromorphic theory of reference [1]) in the following discussion and derivation as this is the general case. At the end of the derivation we specialize the results for the linear microdilation theory considered in this paper. We begin with (62) and note that $\boldsymbol{\sigma}$ consists of ${}_a\boldsymbol{\sigma}$ (three independent components) and ${}_s\boldsymbol{\sigma}$ (six independent components), for a total of nine whereas $\boldsymbol{S}$ contains only six independent components. The presence of a permutation tensor on the left hand side of (62) forces us to discard the symmetric components of both $\boldsymbol{\sigma}$ and $\boldsymbol{S}$ and we are only left with ${}_a\boldsymbol{\sigma}$ balanced by the gradients of the symmetric moment tensor (both part of nonclassical physics). We note that the relationship between ${}_s\boldsymbol{\sigma}$ and $\boldsymbol{S}$ is implicitly present in the balance of angular momenta (62), but due to the permutation tensor it is eliminated. This information can be recovered by eliminating the permutation tensor on the left side of (62), we premultiply (62) by $(\epsilon_{mkn})^{-1}$, the inverse of ϵ_{mkn}.

$$(\epsilon_{mkn})^{-1}(\epsilon_{mkn})(\sigma_{mk} \pm S_{mk}) + (\epsilon_{mkn})^{-1} m_{ln,l} = 0 \tag{134}$$

or

$$\sigma_{mk} \pm (\epsilon_{mkn})^{-1} m_{ln,l} = 0 \tag{135}$$

But the inverse of ϵ_{mkn} (for values of 1, 2, 3 for m,k,n) is ϵ_{mkn}, thus we can write Equation (135) as follows.

$$\sigma_{mk} \pm S_{mk} + \epsilon_{mkn}\left(m_{ln,l}\right) = 0 \tag{136}$$

or

$$_a\sigma_{mk} + {}_s\sigma_{mk} \pm S_{mk} + \epsilon_{mkn}\left(m_{ln,l}\right) = 0 \tag{137}$$

Since

$$_a\sigma_{mk} + \epsilon_{mkn}\left(m_{ln,l}\right) = 0 \tag{138}$$

due to balance of angular momenta, (137) reduces to

$$_s\sigma_{mk} \pm S_{mk} = 0 \tag{139}$$

At this stage the choice of negative sign is physical as it would imply $_s\boldsymbol{\sigma} = \boldsymbol{S}$, which is the right physics at the interface between microconstituents and the medium. Thus, we can write (139) as

$$_s\sigma_{mk} - S_{mk} = 0 \tag{140}$$

In the case of microdilation theory $\boldsymbol{S}$ is a diagonal tensor with all diagonal components being the same, thus in this case (140) reduces to

$$tr\left({}_s\boldsymbol{\sigma}\right) - tr\left(\boldsymbol{S}\right) = 0 \tag{141}$$

Equation (141) provides the additional equation needed for closure of the mathematical model.

Also we note that balance of angular momenta in classical continuum mechanics is a statement of balance of moment of forces. Addition of moment tensor due to nonclassical mechanics to this balance law is justified by classical thermodynamics without much explanation. This balance law works perfectly for micropolar case in which the microconstituents can only have rigid rotations that cause the moment tensor. When microconstituents are deformable volume average stress terms result from the Cauchy stress tensor $\bar{\boldsymbol{\sigma}}^{(\alpha)}$ or $\boldsymbol{\sigma}^{(\alpha)}$ of the microconstituents. At the interface between the microconstituent and the medium $\boldsymbol{S} = {}_s\boldsymbol{\sigma}$ must hold (static equilibrium), but this physics is not present in the derivation of the balance of angular momenta. $\boldsymbol{S}$ is treated as another stress tensor like $\boldsymbol{\sigma}$ (as in Form 1 and Form 2) hence will naturally have a positive sign in the balance of angular momenta in Form 1 and Form 2. Use of identity in Form 3 changes the sign from positive to negative for the term that is used to obtain volume average $\boldsymbol{S}$, thus we are able to obtain the desired equation(s) with negative sign for $\boldsymbol{S}$ that is supported by the physics at the interface between the microconstituents and the medium.

6. Constitutive Theories for Linear Microdilation Solid Medium

6.1. Constitutive Tensors and Their Argument Tensors

The initial determination of constitutive tensors and their arguments tensors is facilitated by the conjugate pairs in the entropy inequality and the axiom of causality. Some of these choices can be altered or changed if the physics requires so, and the argument tensors may require augmenting by using additional tensors based on desired physics. In general, we follow the details presented in references [71] [72]. Once the constitutive tensors and their argument tensors are established, we follow the theory of isotropic tensors or representation theorem for deriving the constitutive theories and the standard procedure of the Taylor series expansion of the coefficients in the linear combinations of the basis of the space of the constitutive tensor.

Consider entropy inequality (132)

$$\rho_0\left(\dot{\Phi}+\eta\dot{\theta}\right)-\boldsymbol{\sigma}:\dot{\boldsymbol{J}}-\boldsymbol{S}:\dot{\boldsymbol{J}}^{(\alpha)}+\frac{\boldsymbol{q}\cdot\boldsymbol{g}}{\theta}-\left({}_c\dot{\boldsymbol{\Theta}}\cdot\left(\nabla\cdot\boldsymbol{m}\right)+\boldsymbol{m}:{}^{c\Theta}\dot{\boldsymbol{J}}\right)\le 0 \tag{142}$$

Macro Cauchy stress tensor $\boldsymbol{\sigma}$ is nonsymmetric, hence it can not be a constitutive tensor due to the restriction of representation theorem. We need to consider additive decomposition of $\boldsymbol{\sigma}$ into symmetric (${}_s\boldsymbol{\sigma}$) and skew symmetric (${}_a\boldsymbol{\sigma}$) tensors.

$$\boldsymbol{\sigma}={}_s\boldsymbol{\sigma}+{}_a\boldsymbol{\sigma} \tag{143}$$

Secondly,

$$\dot{\boldsymbol{J}}={}^d\dot{\boldsymbol{J}}={}^d_s\dot{\boldsymbol{J}}+{}^d_a\dot{\boldsymbol{J}}=\dot{\boldsymbol{\varepsilon}}+{}^d_a\dot{\boldsymbol{J}} \tag{144}$$

In which the displacement gradient tensor ${}^d\boldsymbol{J}$ is additively decomposed into symmetric and skew symmetric tensors.

Likewise

$${}^{c\Theta}\boldsymbol{J}={}^{c\Theta}_s\boldsymbol{J}+{}^{c\Theta}_a\boldsymbol{J};\quad {}^{c\Theta}\dot{\boldsymbol{J}}={}^{c\Theta}_s\dot{\boldsymbol{J}}+{}^{c\Theta}_a\dot{\boldsymbol{J}} \tag{145}$$

$$\dot{\boldsymbol{J}}^{(\alpha)}={}^d_s\dot{\boldsymbol{J}}^{(\alpha)}+{}^d_a\dot{\boldsymbol{J}}^{(\alpha)}=\dot{\boldsymbol{\varepsilon}}^{(\alpha)}+{}^d_a\dot{\boldsymbol{J}}^{(\alpha)} \tag{146}$$

Using (144)-(146) in the entropy inequality (142) and noting that

$${}_s\boldsymbol{\sigma}:{}^d_a\dot{\boldsymbol{J}}=0;\ \ {}_a\boldsymbol{\sigma}:{}^d_s\dot{\boldsymbol{J}}^{(\alpha)}=0;\ \ \boldsymbol{S}:{}^d_a\dot{\boldsymbol{J}}^{(\alpha)}=0;\ \ \boldsymbol{m}:{}^{c\Theta}_a\dot{\boldsymbol{J}}=0 \tag{147}$$

We can write (142) as

$$\rho_0\left(\dot{\Phi}+\eta\dot{\theta}\right)-{}_s\boldsymbol{\sigma}:\dot{\boldsymbol{\varepsilon}}-{}_a\boldsymbol{\sigma}:{}^d_a\dot{\boldsymbol{J}}-\boldsymbol{S}:\dot{\boldsymbol{\varepsilon}}^{(\alpha)}-\boldsymbol{m}:{}^{c\Theta}_s\dot{\boldsymbol{J}}+{}_c\dot{\boldsymbol{\Theta}}\cdot\left(\nabla\cdot\boldsymbol{m}\right)\le 0 \tag{148}$$

From balance of angular momenta

$$\nabla\cdot\boldsymbol{m}=-\boldsymbol{\epsilon}:\boldsymbol{\sigma} \tag{149}$$

Substituting (149) in (148)

$$\rho_0\left(\dot{\Phi}+\eta\dot{\theta}\right)-{}_s\boldsymbol{\sigma}:\dot{\boldsymbol{\varepsilon}}-\boldsymbol{S}:\dot{\boldsymbol{\varepsilon}}^{(\alpha)}-\boldsymbol{m}:{}^{c\Theta}_s\dot{\boldsymbol{J}}+\frac{\boldsymbol{q}\cdot\boldsymbol{g}}{\theta}-{}_c\dot{\boldsymbol{\Theta}}\cdot\left(\boldsymbol{\epsilon}:\boldsymbol{\sigma}\right)\le 0 \tag{150}$$

A simple calculation shows that

$${}_a\boldsymbol{\sigma}:{}^{c\Theta}_a\dot{\boldsymbol{J}}={}_c\dot{\boldsymbol{\Theta}}\cdot\left(\boldsymbol{\epsilon}:\boldsymbol{\sigma}\right) \tag{151}$$

Using (151) in (150), (150) reduces to the following.

$$\rho_0\left(\dot{\Phi}+\eta\dot{\theta}\right)-{}_s\boldsymbol{\sigma}:\dot{\boldsymbol{\varepsilon}}-\boldsymbol{S}:\dot{\boldsymbol{\varepsilon}}^{(\alpha)}-\boldsymbol{m}:{}^{c\Theta}_s\dot{\boldsymbol{J}}-\frac{\boldsymbol{q}\cdot\boldsymbol{g}}{\theta}\le 0 \tag{152}$$

Further additive decomposition of ${}_s\boldsymbol{\sigma}$ into equilibrium stress tensor ${}^e_s\boldsymbol{\sigma}$ and deviatoric stress tensor ${}^d_s\boldsymbol{\sigma}$ is necessary to address volumetric and distortional deformation physics that are mutually exclusive.

$$ {}_s\boldsymbol{\sigma} = {}^e_s\boldsymbol{\sigma} + {}^d_s\boldsymbol{\sigma} \tag{153} $$

Using (153) in (152) we get

$$ \rho_0\left(\dot{\Phi}+\eta\dot{\theta}\right) - {}^e_s\boldsymbol{\sigma}:\dot{\boldsymbol{\varepsilon}} - {}^d_s\boldsymbol{\sigma}:\dot{\boldsymbol{\varepsilon}} - \boldsymbol{S}:\dot{\boldsymbol{\varepsilon}}^{(\alpha)} - \boldsymbol{m}:{}^{c}_{s}{}^{\Theta}\dot{\boldsymbol{J}} - \frac{\boldsymbol{q}\cdot\boldsymbol{g}}{\theta} \le 0 \tag{154} $$

The conjugate pairs in the entropy inequality (154) suggest ${}^e_s\boldsymbol{\sigma}, {}^d_s\boldsymbol{\sigma}, \boldsymbol{S}, \boldsymbol{m}$ and $\boldsymbol{q}$ are valid choices of constitutive tensors. Initial choice of argument tensors are (temperature θ is included due to non isothermal physics) given in the following (based on conjugate pairs).

$$ {}^e_s\boldsymbol{\sigma} = {}^e_s\boldsymbol{\sigma}\left(\rho,\theta\right) \tag{155} $$

$$ {}^d_s\boldsymbol{\sigma} = {}^d_s\boldsymbol{\sigma}\left(\boldsymbol{\varepsilon},\theta\right) \tag{156} $$

$$ \boldsymbol{S} = \boldsymbol{S}\left(\boldsymbol{\varepsilon}^{(\alpha)},\theta\right) \tag{157} $$

$$ \boldsymbol{m} = \boldsymbol{m}\left({}^{c}_{s}{}^{\Theta}\boldsymbol{J},\theta\right) \tag{158} $$

$$ \boldsymbol{q} = \boldsymbol{q}\left(\boldsymbol{g},\theta\right) \tag{159} $$

Even though we do not need a constitutive theory for Φ, its argument tensors are essential to establish because it is used to simplify entropy inequality and to derive the constitutive theory for ${}^e_s\boldsymbol{\sigma}$ (using entropy inequality in Eulerian description). Additionally, presence of η in (154) must also be addressed. Initially we assign the same argument tensors to Φ and η. ρ and θ must surely be argument tensors of Φ and η, other argument tensors of Φ and η are chosen based on the principle of equipresence as we have no other means of establishing them. However, the principle of equipresence is not used in (155)-(159) as the conjugate pairs in the entropy inequality clearly dictate the choice of their argument tensors.

$$ \Phi = \Phi\left(\rho,\boldsymbol{\varepsilon},\boldsymbol{\varepsilon}^{(\alpha)},{}^{c}_{s}{}^{\Theta}\boldsymbol{J},\boldsymbol{g},\theta\right) \tag{160} $$

$$ \eta = \eta\left(\rho,\boldsymbol{\varepsilon},\boldsymbol{\varepsilon}^{(\alpha)},{}^{c}_{s}{}^{\Theta}\boldsymbol{J},\boldsymbol{g},\theta\right) \tag{161} $$

6.2. Constitutive Theory for Cauchy Equilibrium Stress ${}^e_s\boldsymbol{\sigma}$

In Lagrangian description, density $\rho(\boldsymbol{x},t)$ is deterministic from the continuity equation, $\rho(\boldsymbol{x},t) = \frac{\rho_0}{|\boldsymbol{J}|}$ once the deformation gradient tensor $\boldsymbol{J}$ is known, hence density $\rho(\boldsymbol{x},t)$ cannot be an argument tensor of the constitutive tensors [71] [72]. However, compressibility and incompressibility physics are related to density and temperature. Thus, the constitutive theory for ${}^e_s\boldsymbol{\sigma}$ cannot be derived using entropy inequality (154) in Lagrangian description. Instead we must consider the entropy inequality in the Eulerian description, similar to (148). The derivation of the constitutive theory for ${}^e_s\boldsymbol{\sigma}$ using entropy inequality in Eulerian description has been presented in references [71] [72] and is not repeated here for the sake of brevity. For compressible and incompressible solid media, the constitutive theory for ${}_s^e\boldsymbol{\sigma}$ and the reduced from of the entropy inequality are given by

$$ {}_s^e\boldsymbol{\sigma} = p(\rho,\theta)\boldsymbol{\delta};\ \ p(\rho,\theta) = -\rho^2 \frac{\partial \Phi}{\partial \rho}\ (\text{compressible}) \tag{162} $$

$$ {}_s^e\boldsymbol{\sigma} = p(\theta)\boldsymbol{\delta}\ \ (\text{incompressible}) \tag{163} $$

$$ -{}_s^d\boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} - \boldsymbol{S} : \dot{\boldsymbol{\varepsilon}}^{(\alpha)} - \boldsymbol{m} : {}_s^{c}{}^{\Theta}\dot{\boldsymbol{J}} - \frac{\boldsymbol{q}\cdot\boldsymbol{g}}{\theta} \le 0 \tag{164} $$

In the following we present the derivation of constitutive theories for ${}_s^d\boldsymbol{\sigma}, \boldsymbol{S}, \boldsymbol{m}$ and $\boldsymbol{q}$ based on (156)-(161) and the representation theorem. ${}_s^d\boldsymbol{\sigma}, \boldsymbol{S}$ and $\boldsymbol{m}$ are symmetric tensors of rank two and $\boldsymbol{q}$ is a tensor of rank one. Based on conjugate pairs in (164), for thermoelastic microdilation theory the following choices are valid (same as in (156)-(161)).

$$ {}_s^d\boldsymbol{\sigma} = {}_s^d\boldsymbol{\sigma}(\boldsymbol{\varepsilon},\theta) \tag{165} $$

$$ \boldsymbol{S} = \boldsymbol{S}\left(\boldsymbol{\varepsilon}^{(\alpha)},\theta\right) \tag{166} $$

$$ \boldsymbol{m} = \boldsymbol{m}\left({}_s^{c}{}^{\Theta}\dot{\boldsymbol{J}},\theta\right) \tag{167} $$

$$ \boldsymbol{q} = \boldsymbol{q}(\boldsymbol{g},\theta) \tag{168} $$

6.3. Constitutive Theory for ${}_s^d\sigma$ Cauchy Stress Tensor

We consider (165) and derive the constitutive theory for ${}_s^d\boldsymbol{\sigma}$ using representation theorem. Let ${}^{\sigma}\underset{\sim}{\boldsymbol{G}}^i; i = 1,2,\cdots,N^{\sigma}$ be the combined generators of the argument tensors of ${}_s^d\boldsymbol{\sigma}$ in (165) that are symmetric tensors of rank two, then $\boldsymbol{I}, {}^{\sigma}\underset{\sim}{\boldsymbol{G}}^i; i = 1,2,\cdots,N^{\sigma}$ constitute the basis of the space of constitutive tensor ${}_s^d\boldsymbol{\sigma}$, hence we can express ${}_s^d\boldsymbol{\sigma}$ as a linear combination of the basis.

$$ {}_s^d\boldsymbol{\sigma} = {}^{\sigma}\alpha^0 \boldsymbol{I} + \sum_{i=1}^{N^{\sigma}} {}^{\sigma}\alpha^i \left({}^{\sigma}\underset{\sim}{\boldsymbol{G}}^i\right) \tag{169} $$

In which ${}^{\sigma}\alpha^i = {}^{\sigma}\alpha^i\left({}^{\sigma}\underset{\sim}{I}^j,\theta\right); i = 1,2,\cdots,N^{\sigma}; j = 1,2,\cdots,M^{\sigma}$ are the combined invariants of the argument tensor of ${}_s^d\boldsymbol{\sigma}$ in (165). The material coefficients ${}^{\sigma}\alpha^i; i = 1,2,\cdots,N^{\sigma}$ in the linear combination (169) are determined by considering Taylor series expansion of ${}^{\sigma}\alpha^i; i = 1,2,\cdots,N^{\sigma}$ in ${}^{\sigma}\underset{\sim}{I}^j; j = 1,2,\cdots,M^{\sigma}$ and θ about a known configuration $\underline{\Omega}$ and retaining only up to linear terms in ${}^{\sigma}\underset{\sim}{I}^j; j = 1,2,\cdots,M^{\sigma}$ and θ (for simplicity of the resulting constitutive theory).

$$ {}^{\sigma}\alpha^i = {}^{\sigma}\alpha^i\Big|_{\underline{\Omega}} + \sum_{j=1}^{M^{\sigma}} \frac{\partial\, {}^{\sigma}\alpha^i}{\partial\, {}^{\sigma}\underset{\sim}{I}^j}\Bigg|_{\underline{\Omega}} \left({}^{\sigma}\underset{\sim}{I}^j - {}^{\sigma}\underset{\sim}{I}^j\Big|_{\underline{\Omega}}\right) + \frac{\partial\, {}^{\sigma}\alpha^i}{\partial\theta}\Bigg|_{\underline{\Omega}} \left(\theta - \theta|_{\underline{\Omega}}\right);\ i = 0,1,\cdots,N^{\sigma} \tag{170} $$

Substituting ${}^{\sigma}\alpha^0$ and ${}^{\sigma}\alpha^i$ from (170) in (169), we obtain

$$ \begin{aligned} {}_s^d\boldsymbol{\sigma} = &\left({}^{\sigma}\alpha^0\Big|_{\underline{\Omega}} + \sum_{j=1}^{M^{\sigma}} \frac{\partial\left({}^{\sigma}\alpha^0\right)}{\partial\left({}^{\sigma}\underset{\sim}{I}^j\right)}\Bigg|_{\underline{\Omega}} \left({}^{\sigma}\underset{\sim}{I}^j - {}^{\sigma}\underset{\sim}{I}^j\Big|_{\underline{\Omega}}\right) + \frac{\partial\left({}^{\sigma}\alpha^0\right)}{\partial\theta}\Bigg|_{\underline{\Omega}} \left(\theta - \theta|_{\underline{\Omega}}\right)\right)\boldsymbol{I} \\ &+ \sum_{i=1}^{N^{\sigma}} \left(\left({}^{\sigma}\alpha^i\right)\Big|_{\underline{\Omega}} + \sum_{j=1}^{M^{\sigma}} \frac{\partial\left({}^{\sigma}\alpha^i\right)}{\partial\left({}^{\sigma}\underset{\sim}{I}^j\right)}\Bigg|_{\underline{\Omega}} \left({}^{\sigma}\underset{\sim}{I}^j - {}^{\sigma}\underset{\sim}{I}^j\Big|_{\underline{\Omega}}\right) + \frac{\partial\left({}^{\sigma}\alpha^i\right)}{\partial\theta}\Bigg|_{\underline{\Omega}} \left(\theta - \theta|_{\underline{\Omega}}\right)\right) {}^{\sigma}\underset{\sim}{\boldsymbol{G}}^i \end{aligned} \tag{171} $$

Collecting coefficients of $\boldsymbol{I}, {}^{\sigma}\underset{\sim}{I}^{j}\boldsymbol{I}, {}^{\sigma}\underset{\sim}{\boldsymbol{G}}^{i}, {}^{\sigma}\underset{\sim}{I}^{j}\left({}^{\sigma}\underset{\sim}{\boldsymbol{G}}^{i}\right), \left(\theta-\theta|_{\underset{\sim}{\Omega}}\right){}^{\sigma}\underset{\sim}{\boldsymbol{G}}^{i}$ and $\left(\theta-\theta|_{\underset{\sim}{\Omega}}\right)\boldsymbol{I}$, we can write (171) as follows:

$$\begin{aligned}{}_{s}^{d}\boldsymbol{\sigma} = \sigma_0\boldsymbol{I} + \sum_{j=1}^{M^\sigma}{}^{\sigma}\underset{\sim}{a}_j\left({}^{\sigma}\underset{\sim}{I}^{j}\right) + \sum_{i=1}^{N^\sigma}{}^{\sigma}\underset{\sim}{b}_i\left({}^{\sigma}\underset{\sim}{\boldsymbol{G}}^{i}\right) + \sum_{i=1}^{N^\sigma}\sum_{j=1}^{M^\sigma}{}^{\sigma}\underset{\sim}{c}_{ij}\left({}^{\sigma}\underset{\sim}{I}^{j}\right)\left({}^{\sigma}\underset{\sim}{\boldsymbol{G}}^{i}\right) \\ -\sum_{i=1}^{N^\sigma}{}^{\sigma}\underset{\sim}{d}^{i}\left(\theta-\theta|_{\underset{\sim}{\Omega}}\right){}^{\sigma}\underset{\sim}{\boldsymbol{G}}^{i} - {}^{\sigma}\alpha_{tm}\left(\theta-\theta|_{\underset{\sim}{\Omega}}\right)\boldsymbol{I}\end{aligned} \tag{172}$$

The material coefficients ${}^{\sigma}\underset{\sim}{a}_j, {}^{\sigma}\underset{\sim}{b}_i, {}^{\sigma}\underset{\sim}{c}_{ij}, {}^{\sigma}\underset{\sim}{d}_i, {}^{\sigma}\alpha_{tm}$ and σ^0 are defined in the following:

$$\begin{aligned}
&\sigma_0 = {}^{\sigma}\alpha^0\Big|_{\underset{\sim}{\Omega}} - \sum_{j=1}^{M^\sigma}\left.\frac{\partial\left({}^{\sigma}\alpha^0\right)}{\partial\left({}^{\sigma}\underset{\sim}{I}^{j}\right)}\right|_{\underset{\sim}{\Omega}}\left(-{}^{\sigma}\underset{\sim}{I}^{j}\Big|_{\underset{\sim}{\Omega}}\right) \\
&\underset{\sim}{a}_j = \left.\frac{\partial\left({}^{\sigma}\alpha^0\right)}{\partial\left({}^{\sigma}\underset{\sim}{I}^{j}\right)}\right|_{\underset{\sim}{\Omega}};\quad \underset{\sim}{b}_i = {}^{\sigma}\alpha^i\Big|_{\underset{\sim}{\Omega}} + \sum_{j=1}^{M^\sigma}\left.\frac{\partial\left({}^{\sigma}\alpha^i\right)}{\partial\left({}^{\sigma}I^{j}\right)}\right|_{\underset{\sim}{\Omega}}\left(-{}^{\sigma}\underset{\sim}{I}^{j}\Big|_{\underset{\sim}{\Omega}}\right) \\
&\underset{\sim}{c}_{ij} = \left.\frac{\partial\left({}^{\sigma}\alpha^i\right)}{\partial\left({}^{\sigma}\underset{\sim}{I}^{j}\right)}\right|_{\underset{\sim}{\Omega}};\quad {}^{\sigma}\underset{\sim}{d}_i = -\left.\frac{\partial\left({}^{\sigma}\alpha^i\right)}{\partial\theta}\right|_{\underset{\sim}{\Omega}} \\
&{}^{\sigma}\alpha_{tm} = -\left.\frac{\partial\left({}^{\sigma}\alpha^0\right)}{\partial\theta}\right|_{\underset{\sim}{\Omega}} \\
&i = 1,2,\cdots,N^\sigma;\ j = 1,2,\cdots,M^\sigma
\end{aligned} \tag{173}$$

The constitutive theory (172) and the material coefficients (173) are based on integrity, *i.e.*, complete basis of the space of constitutive tensor ${}_{s}^{d}\boldsymbol{\sigma}$. Desired simplified theories can be derived from (172) by retaining desired generators and invariants. A constitutive theory for ${}_{s}^{d}\boldsymbol{\sigma}$ in which ${}_{s}^{d}\boldsymbol{\sigma}$ is a linear function of the components of the argument tensor is given by (after redefining material coefficients).

$${}_{s}^{d}\boldsymbol{\sigma} = \sigma_0\boldsymbol{I} + 2\mu_\sigma\boldsymbol{\varepsilon} + \lambda_\sigma(\text{tr}\boldsymbol{\varepsilon})\boldsymbol{I} - {}^{\sigma}\alpha_{tm}\left(\theta-\theta|_{\underset{\sim}{\Omega}}\right)\boldsymbol{I} \tag{174}$$

σ_0 is the initial stress field, μ_σ and λ_σ are Lamé's constants. These can be expressed in terms of modulus of elasticity E or Poisson's ratio ν. ${}^{\sigma}\alpha_{tm}$ is the thermal modulus.

6.4. Constitutive Theory for Stress Tensor S Due to Micro Cauchy Stress Tensor $\boldsymbol{\sigma}^{(\alpha)}$

We consider (166) and derive the constitutive theory for $\boldsymbol{S}$ using representation theorem. First, we note the that S_{ij} is a diagonal tensor with $S_{11} = S_{22} = S_{33} = S$. From the rate of work conjugate pair $\boldsymbol{S}:\dot{\boldsymbol{\varepsilon}}^{(\alpha)}$, the work conjugate pair is $\boldsymbol{S}:\boldsymbol{\varepsilon}^{(\alpha)}$, in which

$$\boldsymbol{\varepsilon}^{(\alpha)} = \varepsilon_v\boldsymbol{\delta} \tag{175}$$

Thus, we can write

$$\boldsymbol{S} = \boldsymbol{S}\left(\boldsymbol{\varepsilon}^{(\alpha)}, \theta\right) \tag{176}$$

$\boldsymbol{S}$ and $\boldsymbol{\varepsilon}^{(\alpha)}$ both are symmetric tensors of rank two.

Let ${}^s\underset{\sim}{\boldsymbol{G}}^i; i = 1,2,\cdots,N^s$ be the combined generators of the argument tensors of $\boldsymbol{S}$ and let ${}^s\underset{\sim}{I}^j; j = 1,2,\cdots,M^s$ be the combined invariants of the same argument tensors of $\boldsymbol{S}$ in (176), then $\boldsymbol{I}, {}^s\underset{\sim}{\boldsymbol{G}}^i; i = 1,2,\cdots,N^s$ constitutes the basis of the space of tensor $\boldsymbol{S}$, hence we can express $\boldsymbol{S}$ as a linear combination of the basis of its space

$$\boldsymbol{S} = {}^s\alpha^0 \boldsymbol{I} + \sum_{i=1}^{N^s} {}^s\alpha^i \left({}^s\underset{\sim}{\boldsymbol{G}}^i\right) \tag{177}$$

In which the coefficient ${}^s\alpha^i = \left({}^s\underset{\sim}{I}^j, \theta\right); j = 1,2,\cdots,M^s$. The material coefficients are determined by expanding ${}^s\alpha^i; i = 1,2,\cdots,N^s$ with Taylor series in ${}^s\underset{\sim}{I}^j; j = 1,2,\cdots,M^s$ and θ about a known configuration $\underset{\sim}{\Omega}$ and retaining only up to linear terms in ${}^s\underset{\sim}{I}^j; j = 1,2,\cdots,M^s$ and θ (for simplicity).

$${}^s\alpha^i = {}^s\alpha^i\Big|_{\underset{\sim}{\Omega}} + \sum_{j=1}^{M^s} \left.\frac{\partial\left({}^s\alpha^i\right)}{\partial\left({}^s\underset{\sim}{I}^j\right)}\right|_{\underset{\sim}{\Omega}} \left({}^s\underset{\sim}{I}^j - {}^s\underset{\sim}{I}^j\Big|_{\underset{\sim}{\Omega}}\right) + \left.\frac{\partial\left({}^s\alpha^i\right)}{\partial\theta}\right|_{\underset{\sim}{\Omega}} \left(\theta - \theta\big|_{\underset{\sim}{\Omega}}\right); \; i = 1,2,\cdots,N^s \tag{178}$$

Substituting ${}^s\alpha^0$ and ${}^s\alpha^i; i = 1,2,\cdots,N^s$ into (177), we obtain

$$\begin{aligned} \boldsymbol{S} = & \left({}^s\alpha^0\Big|_{\underset{\sim}{\Omega}} + \sum_{j=1}^{M^s} \left.\frac{\partial\left({}^s\alpha^0\right)}{\partial\left({}^s\underset{\sim}{I}^j\right)}\right|_{\underset{\sim}{\Omega}} \left({}^s\underset{\sim}{I}^j - {}^s\underset{\sim}{I}^j\Big|_{\underset{\sim}{\Omega}}\right) + \left.\frac{\partial\left({}^s\alpha^0\right)}{\partial\theta}\right|_{\underset{\sim}{\Omega}} \left(\theta - \theta\big|_{\underset{\sim}{\Omega}}\right) \right) \boldsymbol{I} \\ & + \sum_{i=1}^{N^s} \left({}^s\alpha^i\Big|_{\underset{\sim}{\Omega}} + \sum_{j=1}^{M^s} \left.\frac{\partial\left({}^s\alpha^i\right)}{\partial\left({}^s\underset{\sim}{I}^j\right)}\right|_{\underset{\sim}{\Omega}} \left({}^s\underset{\sim}{I}^j - {}^s\underset{\sim}{I}^j\Big|_{\underset{\sim}{\Omega}}\right) + \left.\frac{\partial\left({}^s\alpha^i\right)}{\partial\theta}\right|_{\underset{\sim}{\Omega}} \left(\theta - \theta\big|_{\underset{\sim}{\Omega}}\right) \right) {}^s\underset{\sim}{\boldsymbol{G}}^i \end{aligned} \tag{179}$$

Substituting and collecting coefficients of $\boldsymbol{I}, {}^s\underset{\sim}{I}^j\boldsymbol{I}, {}^s\underset{\sim}{\boldsymbol{G}}^i, {}^s\underset{\sim}{I}^j\left({}^s\underset{\sim}{\boldsymbol{G}}^i\right), \left(\theta - \theta\big|_{\underset{\sim}{\Omega}}\right){}^s\underset{\sim}{\boldsymbol{G}}^i$ and $\left(\theta - \theta\big|_{\underset{\sim}{\Omega}}\right)\boldsymbol{I}$ in (179), the following can be written.

$$\begin{aligned} \boldsymbol{S} = & S_0 \boldsymbol{I} + \sum_{j=1}^{M^s} {}^s\underset{\sim}{a}_j \left({}^s\underset{\sim}{I}^j\right)\boldsymbol{I} + \sum_{i=1}^{N^s} {}^s\underset{\sim}{b}_i \left({}^s\underset{\sim}{\boldsymbol{G}}^i\right) + \sum_{i=1}^{N^s}\sum_{j=1}^{M^s} {}^s\underset{\sim}{c}_{ij} \left({}^s\underset{\sim}{I}^j\right)\left({}^s\underset{\sim}{\boldsymbol{G}}^i\right) \\ & - \sum_{i=1}^{N^s} {}^s\underset{\sim}{d}_i \left(\theta - \theta\big|_{\underset{\sim}{\Omega}}\right){}^s\underset{\sim}{\boldsymbol{G}}^i - {}^s\alpha_{tm}\left(\theta - \theta\big|_{\underset{\sim}{\Omega}}\right)\boldsymbol{I} \end{aligned} \tag{180}$$

The material coefficients are given by

$$\begin{aligned} & S_0 = {}^s\alpha^0\Big|_{\underset{\sim}{\Omega}} - \sum_{j=1}^{M^s} \left.\frac{\partial\left({}^s\alpha^0\right)}{\partial\left({}^s\underset{\sim}{I}^j\right)}\right|_{\underset{\sim}{\Omega}} \left(-{}^s\underset{\sim}{I}^j\Big|_{\underset{\sim}{\Omega}}\right) \\ & {}^s\underset{\sim}{a}_j = \left.\frac{\partial\left({}^s\alpha^0\right)}{\partial\left({}^s\underset{\sim}{I}^j\right)}\right|_{\underset{\sim}{\Omega}}; \quad {}^s\underset{\sim}{b}_i = {}^s\alpha^i\Big|_{\underset{\sim}{\Omega}} + \sum_{j=1}^{M^s} \left.\frac{\partial\left({}^s\alpha^i\right)}{\partial\left({}^s\underset{\sim}{I}^j\right)}\right|_{\underset{\sim}{\Omega}} \left(-{}^s\underset{\sim}{I}^j\Big|_{\underset{\sim}{\Omega}}\right) \\ & {}^s\underset{\sim}{c}_{ij} = \left.\frac{\partial\left({}^s\alpha^i\right)}{\partial\left({}^s\underset{\sim}{I}^j\right)}\right|_{\underset{\sim}{\Omega}}; \quad {}^s\underset{\sim}{d}_i = -\left.\frac{\partial\left({}^s\alpha^i\right)}{\partial\theta}\right|_{\underset{\sim}{\Omega}} \\ & {}^s\alpha_{tm} = -\left.\frac{\partial\left({}^s\alpha^0\right)}{\partial\theta}\right|_{\underset{\sim}{\Omega}} \end{aligned} \tag{181}$$

The constitutive theory (181) is based on integrity, *i.e.* complete basis of the space of constitutive tensor $\boldsymbol{S}$. Simplified constitutive theories can be obtained from (181) by retaining only the desired generators and invariants. A constitutive theory in which $\boldsymbol{S}$ is a linear function of the components of its argument tensors is given by

$$\boldsymbol{S} = S_0 \boldsymbol{I} + 2\mu_s \left(\boldsymbol{\varepsilon}_v^{(\alpha)}\right) + \lambda_s \left(\mathrm{tr}\left(\boldsymbol{\varepsilon}_v^{(\alpha)}\right)\right)\boldsymbol{I} - {}^s\alpha_{tm}\left(\theta - \theta|_{\underset{\sim}{\Omega}}\right)\boldsymbol{I} \tag{182}$$

Remarks

1) In this particular case

$$\boldsymbol{S} = s\boldsymbol{\delta}; \quad \boldsymbol{\varepsilon}^{(\alpha)} = \varepsilon_v \boldsymbol{\delta} \tag{183}$$

2) Also $N^s = 2$ and $M^s = 3$, and we have

$$\begin{aligned} &{}^s\underset{\sim}{\boldsymbol{G}}^1 = \varepsilon_v \boldsymbol{\delta}; \quad {}^s\underset{\sim}{\boldsymbol{G}}^2 = \left(\varepsilon_v\right)^2 \boldsymbol{\delta} \\ &{}^s\underset{\sim}{I}^1 = 3\varepsilon_v; \quad {}^s\underset{\sim}{I}^2 = 3\left(\varepsilon_v\right)^2; \quad {}^s\underset{\sim}{I}^3 = 3\left(\varepsilon_v\right)^3 \end{aligned} \tag{184}$$

3) Thus, we have the following linear constitutive theory

$$\boldsymbol{S} = S_0 \boldsymbol{I} + \left(2\mu_s + \lambda_s\right)\left(\varepsilon_v + 3\varepsilon_v\right)\boldsymbol{I} - {}^s\alpha_{tm}\left(\theta - \theta|_{\underset{\sim}{\Omega}}\right)\boldsymbol{I} \tag{185}$$

4) ε_v is a dependent variable in the mathematical model.

6.5. Constitutive Theory for Cauchy Moment Tensor *m*

We consider (167) and derive the constitutive theory for $\boldsymbol{m}$ using representation theorem. Let ${}^m\underset{\sim}{\boldsymbol{G}}^i; i = 1, 2, \cdots, N^m$ be the combined generators of the argument tensors of $\boldsymbol{m}$ in (167) that are symmetric tensors of rank two and let ${}^m\underset{\sim}{I}^j; j = 1, 2, \cdots, M^m$ be the combined invariants of the same argument tensors of m in (167). Then $\boldsymbol{I}, {}^m\underset{\sim}{\boldsymbol{G}}^i; i = 1, 2, \cdots, N^m$ constitutes the integrity, *i.e.* the basis of the space of tensor $\boldsymbol{m}$, hence we can express $\boldsymbol{m}$ as a linear combination of the basis of its space.

$$\boldsymbol{m} = {}^m\alpha^0 \boldsymbol{I} + \sum_{i=1}^{N^m} {}^m\alpha^i \left({}^m\underset{\sim}{\boldsymbol{G}}^i\right) \tag{186}$$

In which ${}^m\alpha^i = {}^m\alpha^i\left({}^m\underset{\sim}{I}^j, \theta\right); j = 1, 2, \cdots, M^m$. The material coefficients in (186) are determined by considering Taylor series expansion of ${}^m\alpha^i; i = 1, 2, \cdots, N^m$ in ${}^m\underset{\sim}{I}^j; j = 1, 2, \cdots, M^m$ and θ about a known configuration $\underset{\sim}{\Omega}$ and retaining only up to linear terms in ${}^m\underset{\sim}{I}^j$ and θ.

$${}^m\alpha^i = {}^m\alpha^i\Big|_{\underset{\sim}{\Omega}} + \sum_{j=1}^{M^m} \frac{\partial\left({}^m\alpha^i\right)}{\partial\left({}^m\underset{\sim}{I}^j\right)}\Bigg|_{\underset{\sim}{\Omega}} \left({}^m\underset{\sim}{I}^j - {}^m\underset{\sim}{I}^j\Big|_{\underset{\sim}{\Omega}}\right) + \frac{\partial\left({}^m\alpha^i\right)}{\partial\theta}\Bigg|_{\underset{\sim}{\Omega}} \left(\theta - \theta|_{\underset{\sim}{\Omega}}\right); \; i = 0, 1, \cdots, N^m \tag{187}$$

Substituting ${}^m\alpha^0$ and ${}^m\alpha^i$ from (187) in (186) and collecting the coefficients of $\boldsymbol{I}, {}^m\underset{\sim}{I}^j \boldsymbol{I}, {}^m\underset{\sim}{\boldsymbol{G}}^i, {}^m\underset{\sim}{I}^j\left({}^m\underset{\sim}{\boldsymbol{G}}^i\right), \left(\theta - \theta|_{\underset{\sim}{\Omega}}\right){}^m\underset{\sim}{\boldsymbol{G}}^i$ and $\left(\theta - \theta|_{\underset{\sim}{\Omega}}\right){}^m\underset{\sim}{I}^j$, we can write (186) as follows.

$$\boldsymbol{m} = m_0 \boldsymbol{I} + \sum_{j=1}^{M^m} {}^m\underset{\sim}{a}_j \left({}^m\underset{\sim}{I}^j \right) \boldsymbol{I} + \sum_{i=1}^{N^m} {}^m\underset{\sim}{b}_i \left({}^m\underset{\sim}{\boldsymbol{G}}^i \right) + \sum_{i=1}^{N^m} \sum_{j=1}^{M^m} {}^m\underset{\sim}{c}_{ij} \left({}^m\underset{\sim}{I}^j \right) \left({}^m\underset{\sim}{\boldsymbol{G}}^i \right) - \sum_{i=1}^{N^m} {}^m\underset{\sim}{d}_i \left(\theta - \theta|_\Omega \right) {}^m\underset{\sim}{\boldsymbol{G}}^i - {}^m\alpha_{tm} \left(\theta - \theta|_\Omega \right) \boldsymbol{I} \tag{188}$$

The material coefficients ${}^m\underset{\sim}{a}_j, {}^m\underset{\sim}{b}_i, {}^m\underset{\sim}{c}_{ij}, {}^m\alpha_{tm}$ and m^0 can be functions of ${}^m\underset{\sim}{I}^j\big|_\Omega; j = 1, 2, \cdots, M^m$ and $\theta|_\Omega$ and are given by the following.

$$\begin{aligned} m_0 &= {}^m\alpha^0 - \sum_{j=1}^{M^m} \left. \frac{\partial \left({}^m\alpha^0 \right)}{\partial \left({}^m\underset{\sim}{I}^j \right)} \right|_\Omega \left(-{}^m\underset{\sim}{I}^j\big|_\Omega \right) \\ {}^m\underset{\sim}{a}_j &= \left. \frac{\partial \left({}^m\alpha^0 \right)}{\partial \left({}^m\underset{\sim}{I}^j \right)} \right|_\Omega ; \quad {}^m\underset{\sim}{b}_i = {}^m\alpha^i\big|_\Omega + \sum_{j=1}^{M^m} \left. \frac{\partial \left({}^m\alpha^i \right)}{\partial \left({}^m\underset{\sim}{I}^j \right)} \right|_\Omega \left(-{}^m\underset{\sim}{I}^j\big|_\Omega \right) \\ {}^m\underset{\sim}{c}_{ij} &= \left. \frac{\partial \left({}^m\alpha^i \right)}{\partial \left({}^m\underset{\sim}{I}^j \right)} \right|_\Omega ; \quad {}^m\underset{\sim}{d}_i = -\left. \frac{\partial \left({}^m\alpha^i \right)}{\partial \theta} \right|_\Omega \\ {}^m\alpha_{tm} &= -\left. \frac{\partial \left({}^m\alpha^0 \right)}{\partial \theta} \right|_\Omega \end{aligned} \tag{189}$$

This constitutive theory (188) is based on integrity, *i.e.* complete basis of the space of tensor $\boldsymbol{m}$. Simplified theories can be derived for (188) by retaining only the desired generators and invariants. A constitutive theory in which $\boldsymbol{m}$ is a linear function of the components of its argument tensors is given by

$$\boldsymbol{m} = m_0 \boldsymbol{I} + 2\mu_m \left({}^{c}_{s}\!\overset{\Theta}{\boldsymbol{J}} \right) + \lambda_m \left(\operatorname{tr} \left({}^{c}_{s}\!\overset{\Theta}{\boldsymbol{J}} \right) \right) \boldsymbol{I} - {}^m\alpha_{tm} \left(\theta - \theta|_\Omega \right) \boldsymbol{I} \tag{190}$$

6.6. Constitutive Theory for $\boldsymbol{q}$

Consider Equation (159)

$$\boldsymbol{q} = \boldsymbol{q}(\boldsymbol{g}, \theta) \tag{191}$$

Following references [71] [72], it is straightforward to derive the constitutive theory for $\boldsymbol{q}$ using representation theorem. We have

$$\boldsymbol{q} = -\kappa \boldsymbol{g} - \kappa_1 (\boldsymbol{g} \cdot \boldsymbol{g}) \boldsymbol{g} - \kappa_2 \left(\theta - \theta|_\Omega \right) \boldsymbol{g} \tag{192}$$

κ, κ_1 and κ_2 are material coefficients that can be functions of $(\boldsymbol{g} \cdot \boldsymbol{g})|_\Omega$ and $\theta|_\Omega$. A linear constitutive theory for $\boldsymbol{q}$ is of course the standard Fourier heat conduction law

$$\boldsymbol{q} = -\kappa \boldsymbol{g} \tag{193}$$

7. Thermodynamic and Mathematical Consistency of the Microdilation Theory Presented in This Paper

Surana *et al.* [1] have given details of thermodynamic and mathematical consistency of the linear micromorphic theory presented by them. This material in principle also applies to the linear microdilation theory presented in this paper as the basic laws of thermodynamics and principles of mathematics are the same. We present a summary in the following.

Thermodynamic and mathematical consistency ensures that all aspects of derivation of the theory are supported by thermodynamics as well as established concepts of applied mathematics. Using classical continuum mechanics for microdeformation physics, consistent integral-average definitions from the microdeformation physics that are valid for macro deformation, appropriate decomposition of stress tensor to accommodate deformation physics and to ensure valid constitutive tensors for representation theorem, correct definition of strain measure by removing rigid rotations, and use of classical rotations as rigid rotations of the microconstituents are used in the present work that are completely supported by thermodynamics and by well established concepts in mathematics. Strict use of representation theorem in deriving constitutive theories and use of an additional balance law due to rotations and moments as a new kinematically conjugate pair, thus eliminating spurious constitutive theories and ensuring that violation of thermodynamic principles and well known mathematical concepts does not occur in the derivation of the microdilation theory are significant aspects and strengths of the work presented in the paper.

8. Microstretch Theory of Eringen

The linear microstretch theory of Eringen is a widely accepted and used microcontinuum theory based on the published works. The linear microdilation theory presented here is not a microstretch theory but is the only possible theory if the microconstituents are allowed to have only one deformational degree of freedom. We discuss details in the following.

We discussed in earlier sections that microstretch requires direction of stretch, hence orientation of microconstituents in reference to longitudinal and transverse directions. We obviously do not have this. We have shown that with only a single deformational degree of freedom for the microconstituents, only microdilational microcontinuum theory is possible in which volumetric strain is the degree of freedom. This is a fundamental difference between the microdilation theory presented here and Erengin's microstrech theory. In the present work we always use additive decomposition of $\boldsymbol{J}^{(\alpha)}$ into symmetric and skew symmetric tensors, thus separating deformation and rigid rotations of the microconstituents. This is necessary to have the correct definitions of strain measures. In Eringen's microstretch theory, rigid rotations of the microconstituents are unknown degrees of freedom. Surana *et al.* [31]-[54] have shown that with these unknown rotations of the microconstituents, a thermodynamically consistent and valid theory is not possible. Surana *et al.* have also shown that in the presence of microconstituents the classical rotation field (a free field in the absence of microconctituents) in fact becomes the rigid rotation field of the microconstituents. Use of ${}_{c}\boldsymbol{\Theta}$ as rigid rotational degrees of freedom for the microconstituents results in thermodynamically and mathematically consistent microcontinuum theories. In the theory presented here, as well as in previous works of Surana *et al.* [31]-[54], when the rigid rotations of the microconstituents are resisted by the surrounding solid medium, the moments are created which results in the Cauchy moment tensor through Cauchy principle. Eringen's definition of moment tensor is in error, as it derived using Cauchy stress tensor of the microconstituent, which is due to classical continuum mechanics, hence can not possibly give a moment tensor that is purely due to nonclassical physics.

Use of weight functions in balance of linear momenta for microconstituents has no basis. We have shown that the negative sign for the " $\boldsymbol{S}$ " term in balance of angular momenta also results due to using identity, hence no need for weight functions. In Eringen's work linear microstretch theory has four unknown degrees of freedom, a microstretch and three unknown rigid rotations of the microconstituents. In the present microdilation theory there is only one unknown degree of freedom, volumetric strain or a quantity proportional to volumetric strain, the rigid rotations of the microconstituents are described by classical rotations hence are known. Additional equation(s) needed in Eringen's work for the closure of the mathematical model are obtained by introducing a new balance law that has no thermodynamic basis. In our work it is shown that the additional equations needed are implicitly present in balance of angular momenta and we have presented a procedure to extract them. Choice of nonsymmetric constitutive tensors, use of potentials or functionals in nonsymmetric tensors used to derive constitutive theories can not be supported by representation theorem. Lack of use of balance of moment of moments in Eringen's work is in violation of thermodynamics and leads to spurious conjugate pairs, hence invalid constitutive theories and lack of thermodynamic equilibrium in the deforming matter in the presence of rotations and moments as a kinematically conjugate pair. Use of principle of equipresence for all constitutive tensors introduces nonphysical coupling between classical and nonclassical physics.

9. Summary and Conclusions

A linear microdilation microcontinuum theory has been presented for solid matter in which mechanics of thermoelasticity is considered for microconstituents, solid medium, as well as the interaction of the microconstituents with the solid medium. We summarize the work presented in this paper and draw some conclusions.

1) In the present microdilation microcontinuum theory, a microconstituent has four degrees of freedom: a volumetric strain and three rigid rotations defined by classical rotations ${}_{c}\boldsymbol{\Theta}$. Since ${}_{c}\boldsymbol{\Theta}$ are known, the microconstituent has only one unknown degree of freedom, volumetric strain. In Eringen's work a microconstituent has a microstretch and three unknown rigid rotations ${}_{\alpha}\boldsymbol{\Theta}$, a total of four unknown degrees of freedom.

2) In the work presented here, we always perform additive decomposition of $\boldsymbol{J}^{(\alpha)}$ or ${}^{d}\boldsymbol{J}^{(\alpha)}$ to separate deformation from the rigid rotations. This is essential

in the derivations of constitutive theories. In Eringen's theory, strain measure definitions contain rigid rotations, a violation of basic physics.

3) Our work recognizes that rotations ${}_c\boldsymbol{\Theta}$ and Cauchy moment tensor are a new kinematically conjugate pair in all three 3M theories, hence it requires two balance laws just as the displacements and forces kinematically conjugate pair does in classical continuum mechanics. This necessitates new balance law in all 3M theories [40] [51] [70], balance of moment of moments. Thus, this balance law is supported by classical thermodynamic framework. This balance law is never used in Eringen's work; the consequence of this is spurious conjugate pairs in the entropy inequality and spurious constitutive theories.

4) Varying rotations ${}_c\boldsymbol{\Theta}$ in the deforming solid medium when resisted creates moments. Our derivation shows that the Cauchy moment tensor and the symmetric part of the gradients of ${}_c\boldsymbol{\Theta}$ are kinematically work conjugate. This physics is purely due to nonclassical mechanics, hence has no interaction or any connection to classical continuum theory. Based on this, the "integral-average" definition of moment tensor in Eringen's work is incorrect as it is based on $\bar{\boldsymbol{\sigma}}^{(\alpha)}$ which is purely due to classical continuum mechanics.

5) Use of weight function $\phi^{(\alpha)}\left(\underset{\sim}{\bar{x}}{}_m^{(\alpha)}\right)$ in the derivation of macro balance of linear momenta, balance of angular momenta and moment of momentum has no thermodynamic basis.

6) In our work, the constitutive tensors of rank two are symmetric tensors and their argument tensors of rank two are also symmetric tensors, hence permitting the use of representation theorem in deriving constitutive theory that are naturally mathematically consistent. This is in contrast with published works in which constitutive tensors of rank two are often non symmetric tensors with non symmetric argument tensors. Such constitutive theories derived using assumed potentials or functionals are not justified based on representation theorem, hence leading to mathematical inconsistencies.

7) Conservation of micro inertia advocated by Eringen to be necessary in 3M theories is neither needed nor used in the present work. This law proposed by Eringen has no thermodynamic basis. The need for this balance law is created to obtain three additional equations primarily due to ${}_\alpha\boldsymbol{\Theta}$ being three unknown degrees of freedom. Whereas in our work ${}_\alpha\boldsymbol{\Theta}$ are in fact ${}_c\boldsymbol{\Theta}$, hence known. Other significant differences are that in Eringen's work nine constitutive equations are considered for $\boldsymbol{\sigma}$ as well as $\boldsymbol{m}$. In our work, $\boldsymbol{\sigma} = {}_s\boldsymbol{\sigma} + {}_a\boldsymbol{\sigma}$ decomposition is used and there are only six constitutive equations needed for ${}_s\boldsymbol{\sigma}$. $\boldsymbol{m}$ is symmetric due to balance of moment of moments balance law, hence only six constitutive equations are needed for $\boldsymbol{m}$ as well. We point out again that conservation of micro inertia and moments of symmetric parts of stresses balanced by gradients of the symmetric components the moment tensor are not supported by thermodynamic framework, hence can only be viewed as ad-hoc or phenomenological.

8) Thermodynamic and mathematical consistency of the linear microdilation theory presented in this paper has been established in Section 7. The lack of ther-

modynamic and mathematical consistency in Eringen's linear micromorphic theory has been discussed and illustrated in Section 8.

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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