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Analysis of the Ruin Probability of a Hawkes Process with Variable Memory under Partial Payments to Shareholders and Dependent on Claims via the Spearman Copula

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Abstract

Previous research has mainly focused on risk models constructed from Markov processes. This article is an extension of the risk model based on Hawkes' variable memory process with a partial dividend payment strategy to shareholders, a constant threshold *b*, and a dependence between the amounts of claims and the inter-claim times via the Spearman copula. We study the probability of ultimate ruin associated with this risk model and conduct simulations to observe the behavior of this probability.

Keywords

Gerber-Shiu Functions, Hawkes Process, Spearman Copula, Dividends, Integral-Differential Equation

1. Introduction

In this article, we present the results obtained, in particular, a simplified expression of the ultimate ruin probability from a Hawkes variable memory process [1] via the Spearman copula. We assume in this risk model that the inter-arrivals of claims follow a Hawkes process and the amounts of claims follow an exponential distribution. Risk management is a major issue for financial companies. Mathematical models are constantly being developed to provide a better understanding of risks and their evolution, with the simplifying assumption of independence between the random variables involved in risk modeling (see, for example, references [2] and [3]). However, in certain practical contexts, this assumption is inappropriate and too restrictive. In flood insurance, for example, the occurrence of several floods in a short period can cause significant damage, and therefore large claims, due to the accumulation of water. In earthquake insurance, it is the opposite: in a high-risk area, the longer the time between two earthquakes, the greater the impact of the second earthquake, due to the accumulation of energy.

To address this shortcoming, numerous studies incorporate into the risk model the dependence between certain random variables, particularly the claim amounts and the inter-claim times, using the Farlie Gumbel Morgenstern copula [4]-[9]. Although this copula is widely used in the literature, it exhibits certain limitations. Notably, it fails to capture tail dependencies [10]-[14].

To overcome the limitations of the Farlie-Gumbel-Morgenstern copula while reflecting the operational realities of insurance companies, this article considers a risk model based on the Hawkes process with variable memory. In this framework, we integrate not only the dependence between claim amounts and inter-claim times via the Spearman copula, but also a partial dividend payment strategy to shareholders, governed by a constant threshold b.

In this model, when the surplus process reaches the fixed constant threshold barrier b, bonuses are partially granted to shareholders at a constant rate θ such that $0 < \theta < 1$. Denoting by $R_b(t)$, the surplus process in the presence of the threshold dividend barrier b (with $R_b(t) = u$), the model follows the following dynamics:

$$\mathrm{d}R_b(t) = \begin{cases} c\mathrm{d}t - \mathrm{d}S(t) & \text{if } R_b(t) < b \\ (1-\theta)c\mathrm{d}t - \mathrm{d}S(t) & \text{if } R_b(t) = b \end{cases} \tag{1.1}$$

- $R_b(t)$ is the surplus process in the presence of a dividend barrier threshold b (with $R_b(0) = u$ the initial surplus and $0 < u \le b$);
- c is the constant rate of premium received by the insurer per unit of time;
- t_b is the first moment when the surplus reaches the horizontal barrier b then $t_b = \dfrac{b-u}{c}$;
  $lllc$
- $S(t) = \sum_{i=1}^{N(t)} X_i$ is the process of loss:
  - $\{N(t), t \ge 0\}$ is the total number of claims recorded up to time t, which follows a Hawkes process; (Note that $S(t) = 0$ if $N(t) = 0$);
  - $\{X_i, i \ge 1\}$ is a sequence of random variables representing the individual amounts of claims with a common density function f_X and a distribution function F_X, and assumed to have an exponential distribution with the parameter γ.

The counter-claim times $\{V_i, i \ge 1\}$ form a sequence of random variables governed by a Hawkes process with a spectral density function. The objective of this work is to determine the ultimate ruin probability within the risk model defined by relations (1.1). The remainder of the article is organized as follows: Section 2 discusses the preliminaries of the risk model defined by relation (1.1). Section 3 examines the integro-differential equation satisfied by the Gerber-Shiu function in the context of the model defined by relation (1.1). Section 4 focuses on the Laplace transforms of the Gerber-Shiu functions and the ultimate ruin probability in the same risk framework. Finally, Section 5 provides further discussion on the ultimate ruin probability within the model defined by relations (1.1).

2. Preliminaries

2.1. Hawkes Process

The arrival time laws of a Hawkes process can be established using the intensity function $\lambda(t)$ of the Hawkes process. The intensity function $\lambda(t)$ is a function that describes the infinitesimal probability of an arrival given the history of previous events.

$$\lambda(t) = \lambda + \sum_{i=1}^{k} \mu(t - t_i).$$

When the event of interest occurs, the intensity of the process is modified by the function μ. In a way, this function can be interpreted as a response to the jump in the process. Its introduction in the intensity expression allows for the extension of the modeling capability of point processes to a large number of random phenomena. The function μ can be increasing or decreasing. In this article, we consider the exponentially decreasing function μ defined $\forall t > 0$:

$$\mu(t) = \alpha \mathrm{e}^{-\beta t}$$

We would like to remind that Hawkes processes do not admit a probability density like continuous processes, but admit a spectral density [1] defined by:

$$f(t) = \frac{\lambda}{2\pi}\left[1 + \frac{\alpha\lambda(2\beta - \alpha)}{(\beta - \alpha)^2 + t^2}\right]$$

with α, β, and λ as the parameters of the process.

Proposition 2.1 In order for the process to be well defined, it is necessary that:

$$\beta > \alpha$$

Proof [1].

2.2. Measures of Ruin

The probability of insurer ruin is the probability that ruin occurs either over a limited time horizon or over an infinite time horizon. In the latter case, we speak of the ultimate risk probability. Let τ be the time of the insurance company's ruin τ is defined by:

$$\tau = \inf\{t \geq 0; R(t) < 0\} \tag{2.1}$$

When the probability of ruin is always zero, by convention, we denote $\tau = \infty$ in this case.

$$R(t) \geq 0, \forall t \geq 0.$$

The probability of ruin at a finite horizon is defined by:

$$\Psi(u,t) = \mathbb{P}(\tau \in [0;t], R(t) < 0) \tag{2.2}$$

Similarly, the ultimate probability of ruin is defined by:

$$\Psi(u)=\Psi(u,\infty)=\mathbb{P}\left(\tau<\infty,R(t)<0\right) \tag{2.3}$$

2.3. Reduced Penalty Function of Gerber-Shiu

The expected penalty function of Gerber-Shiu or Gerber-Shiu function appeared in 1998 in the works of Gerber and Shiu (see [2]). Today, this function is of great interest to research. Its analysis remains a central issue in both the insurance and finance sectors, as it is a valuable tool not only in the study of ruin probability but also in the calculation of pension and reinsurance premiums, and option pricing. It is defined by:

$$\varphi(u)=E\left[\mathrm{e}^{-\delta\tau}w\left(R\left(\tau^-\right),\left|R(\tau)\right|\right)1_{\tau<\infty}\mid R(0)=u\right] \tag{2.4}$$

where:

- τ is the moment of failure defined by the relationship (2.1);
- τ^- is the moment just before ruin;
- δ is a force of interest;
- The penalty function $w(x,y)$ is a positive function of the surplus just before ruin $R\left(\tau^-\right)$ and the deficit of ruin $\left|R(\tau)\right|,\forall x,y\geq 0$;
- $1_{\tau<\infty}$ is the indicator function that is 1 if event A occurs and 0 otherwise.

2.4. Dependency Structure

In 1959, Abe Sklar introduced the copula function, which did not gain widespread recognition among financial experts until the 1990s [15]. As a method for studying the dependence structures of random variables, the copula possesses unique properties, such as the ability to describe a multivariate distribution function using univariate marginal functions and multivariate correlation structure functions. Copulas are mathematical tools used to model the dependence structure between multiple random variables, independently of their marginal distributions [16]-[19].

2.4.1. Dependency Structure

The concept of tail dependence is essential for analyzing the asymptotic dependence between two random variables. It allows us to describe the level of dependence in the extremes of the distribution, making it a suitable tool for studying dependence between high values (higher tail dependence) and low values (lower tail dependence). This measure is of great importance for extreme value copulas. There are two tail dependence coefficients that are defined as follows:

Definition 2.1 Let X and Y be two continuous random variables with respective distribution functions F and G. The lower tail dependency coefficient λ_L is defined by:

$$\lambda_L(X,Y)=\lim_{u\to 0^+}\mathbb{P}\left(X\leq F^{-1}(\alpha)\mid Y\leq G^{-1}(\alpha)\right)$$

and the higher dependency coefficient λ_U is defined by:

$$\lambda_U(X,Y)=\lim_{u\to 1^-}\mathbb{P}\left(X\le F^{-1}(\alpha)\mid Y\le G^{-1}(\alpha)\right)$$

These measures can be defined in terms of a copula C.

Definition 2.1 Let X and Y be two continuous random variables with copula C, then we have:

$$\lambda_L(X,Y)=\lim_{u\to 0^+}\frac{C(u,u)}{u};$$

and

$$\lambda_U(X,Y)=\lim_{u\to 1^-}\frac{1-2u+C(u,u)}{1-u}$$

Remark

- if $\lambda_L\in\left]0,1\right]$; so it has a lower tail dependency.
- if $\lambda_L=0$; so C has no lower tail dependency.
- if $\lambda_U\in\left]0,1\right]$; so it has a higher tail dependency.
- if $\lambda_U=0$; so it has no higher-order tail dependency.

Many authors [20]-[24], to name a few, have used the Farlie-Gumbel-Morgenstern (FGM) copula to define the dependency structure between the size of demand and the time between requests. The FGM copula is given by:

$$C_\alpha(u,v)=uv+\alpha uv(1-u)(1-v);0\le u,v\le 1.$$

It is not suitable for modeling dependencies on extreme values because $\lambda_L=0$ and $\lambda_U=0$.

2.4.2. Dependence Model Based on the Spearman Copula

In this work, the dependency structure is provided by the Spearman copula defined by: $\forall(u,v)\in[0,1]^2$ and $\gamma\in[0,1]$ by:

$$C_\gamma(u,v)=(1-\gamma)C_I(u,v)+\gamma C_M(u,v)\tag{2.5}$$

where: $C_I(u,v)=uv$; $C_M(u,v)=\min(u,v)$; α is the dependence parameter. It is suitable for modeling dependence on extreme values because $\lambda_L=\gamma$ and $\lambda_U=\gamma$. The Spearman copula can be used to express positive dependencies as well as tail dependencies in many situations. Using Formula (3.1), the random vector of claim amounts and times between claims (X, V) has the joint distribution function given by:

$$\begin{aligned}F_{X,V}(x,t)&=C_\gamma\left(F_X(x),F_V(t)\right)\\&=(1-\gamma)C_I\left(F_X(x),F_V(t)\right)+\gamma C_M\left(F_X(x),F_V(t)\right)\\&=(1-\gamma)F_I(x,t)+\gamma F_M(x,t)\end{aligned}\tag{2.6}$$

where: F_X and F_V are the respective marginal distributions of the random variables X and V.

2.5. Condition of the Net Profit for the Hawkes Model

The condition of net profit is written:

$$c>\frac{\lambda E[X]}{1-\theta}$$

with:

- c is the rate of premium (income per unit of time);
- $E[X]$ is the expected amount of claims;
- $\theta \in [0,1)$ is the feedback parameter of the Hawkes process (measures self-excitation, that is to say, the memory effect of past events);
- The term $\frac{\lambda}{1-\theta}$ represents the effective average intensity of the Hawkes process, taking into account self-excitation;
- The product $\frac{\lambda E[X]}{1-\theta}$ is therefore the average cost of claims per unit of time;
- In order for the insurer's surplus to increase on average, the premium rate c must exceed this average cost.

3. Integral-Differential Equation Satisfied by the Gerber-Shiu Function

The objective of this section is to determine the differential equation satisfied by the function $\varphi_b(u)$ in a risk model with constant dividend payment threshold b and dependence between the random claim amounts and the time between claims via the Spearman copula. In this risk model [10] [23] [24], the Gerber-Shiu function $\varphi_b(u)$ is given by:

$$\varphi_b(u) = (1-\gamma)\left[I_{b,1}(u) + I_{b,2}(u)\right] + \gamma\left[I_{b,3}(u) + I_{b,4}(u)\right] \tag{3.1}$$

where:

$$I_{b,1}(u) = \int_0^\infty \int_0^{u+ct} \mathrm{e}^{-\delta t} \varphi_b(u+ct-x)\,\mathrm{d}F_I(x,t);$$

$$I_{b,2}(u) = \int_0^\infty \int_{u+ct}^\infty \mathrm{e}^{-\delta t} w(u+ct, x-u-ct)\,\mathrm{d}F_I(x,t);$$

$$I_{b,3}(u) = \int_0^\infty \int_0^{u+ct} \mathrm{e}^{-\delta t} \varphi_b(u+ct-x)\,\mathrm{d}F_M(x,t);$$

$$I_{b,4}(u) = \int_0^\infty \int_{u+ct}^\infty \mathrm{e}^{-\delta t} w(u+ct, x-u-ct)\,\mathrm{d}F_M(x,t).$$

To determine the integro-differential equation satisfied by the Gerber-Shiu function in the risk model defined by relation (1.1), we adopt the following approach:

- The first loss occurs at a time t before the surplus process reaches the barrier b $\left(t < \frac{b-u}{c}\right)$. The amount x is such that $x < u + ct$.
- The first loss occurs at a time t before the surplus process reaches the barrier b $\left(t < \frac{b-u}{c}\right)$. The amount x is such that $x > u + ct$.
- The first loss occurs at a time t after the surplus process has crossed the barrier b $\left(t > \frac{b-u}{c}\right)$. The amount x is such that $x < b + (1-\theta)c(t-t_b)$.
- The first loss occurs at a time t after the surplus process has crossed the barrier b $\left(t > \frac{b-u}{c}\right)$. The amount x is such that $x > b + (1-\theta)c(t-t_b)$.

By conditioning on the time and the amount of the first claim, and taking into account the various scenarios above, we have:

$$\begin{aligned} I_{b,1}(u) = {} & \int_0^{t_b}\int_0^{u+ct} \mathrm{e}^{-\delta t}\varphi_b(u+ct-x)\,\mathrm{d}F_I(x,t) \\ & + \int_{t_b}^{\infty}\int_0^{b+ac(t-t_b)} \mathrm{e}^{-\delta t}\varphi_b\left(b+ac(t-t_b)-x\right)\mathrm{d}F_I(x,t) \end{aligned} \tag{3.2}$$

with $t_b = \dfrac{b-u}{c}$ and $a = 1-\theta$

$$\begin{aligned} I_{b,2}(u) = {} & \int_0^{t_b}\int_{u+ct}^{\infty} \mathrm{e}^{-\delta t} w(u+ct, x-u-ct)\,\mathrm{d}F_I(x,t) \\ & + \int_{t_b}^{\infty}\int_{b+ac(t-t_b)}^{\infty} \mathrm{e}^{-\delta t} w\left(b+ac(t-t_b), x-b-ac(t-t_b)\right)\mathrm{d}F_I(x,t) \end{aligned} \tag{3.3}$$

The C_I copula being the independent part of the Spearman copula, we have:

$$\mathrm{d}F_I(x,t) = \frac{\lambda}{2\pi}\left[1+\frac{\alpha\lambda(2\beta-\alpha)}{(\beta-\alpha)^2+t^2}\right] f_X(x)\,\mathrm{d}x\mathrm{d}t \tag{3.4}$$

By placing $I_b(u) = I_{b,1}(u) + I_{b,2}(u)$, and by using the relationships (3.2), (3.3), and (3.4), we have:

$$\begin{aligned} I_b(u) = {} & \frac{\lambda}{2\pi}\int_0^{t_b}\int_0^{u+ct} \mathrm{e}^{-\delta t}\varphi_b(u+ct-x)\left[1+\frac{\alpha\lambda(2\beta-\alpha)}{(\beta-\alpha)^2+t^2}\right] f_X(x)\,\mathrm{d}x\mathrm{d}t \\ & + \frac{\lambda}{2\pi}\int_{t_b}^{\infty}\int_0^{b+ac(t-t_b)} \mathrm{e}^{-\delta t}\varphi_b\left(b+ac(t-t_b)-x\right)\left[1+\frac{\alpha\lambda(2\beta-\alpha)}{(\beta-\alpha)^2+t^2}\right] f_X(x)\,\mathrm{d}x\mathrm{d}t \\ & + \frac{\lambda}{2\pi}\int_0^{t_b}\int_{u+ct}^{\infty} \mathrm{e}^{-\delta t} w(u+ct, x-u-ct)\left[1+\frac{\alpha\lambda(2\beta-\alpha)}{(\beta-\alpha)^2+t^2}\right] f_X(x)\,\mathrm{d}x\mathrm{d}t \\ & + \frac{\lambda}{2\pi}\int_{t_b}^{\infty}\int_{b+ac(t-t_b)}^{\infty} \mathrm{e}^{-\delta t} w\left(b+ac(t-t_b), x-b-ac(t-t_b)\right)\left[1+\frac{\alpha\lambda(2\beta-\alpha)}{(\beta-\alpha)^2+t^2}\right] f_X(x)\,\mathrm{d}x\mathrm{d}t \end{aligned} \tag{3.5}$$

To simplify the notation of relation (3.5), we set:

$$\begin{aligned} & \omega(u) = \int_u^{\infty} w(u, x-u) f_X(x)\,\mathrm{d}x \quad \text{and} \\ & \sigma_b(u) = \int_0^{u}\varphi_b(u-x) f_X(x)\,\mathrm{d}x + \omega(u) \end{aligned} \tag{3.6}$$

The relation (3.5) becomes:

$$\begin{aligned} I_b(u) = {} & \frac{\lambda}{2\pi}\int_0^{t_b}\int_0^{u+ct} \mathrm{e}^{-\delta t}\varphi_b(u+ct-x)\left[1+\frac{\alpha\lambda(2\beta-\alpha)}{(\beta-\alpha)^2+t^2}\right] f_X(x)\,\mathrm{d}x\mathrm{d}t \\ & + \frac{\lambda}{2\pi}\int_0^{t_b}\int_{u+ct}^{\infty} \mathrm{e}^{-\delta t} w(u+ct, x-u-ct)\left[1+\frac{\alpha\lambda(2\beta-\alpha)}{(\beta-\alpha)^2+t^2}\right] f_X(x)\,\mathrm{d}x\mathrm{d}t \\ & + \frac{\lambda}{2\pi}\int_{t_b}^{\infty}\int_0^{b+ac(t-t_b)} \mathrm{e}^{-\delta t}\varphi_b\left(b+ac(t-t_b)-x\right)\left[1+\frac{\alpha\lambda(2\beta-\alpha)}{(\beta-\alpha)^2+t^2}\right] f_X(x)\,\mathrm{d}x\mathrm{d}t \\ & + \frac{\lambda}{2\pi}\int_{t_b}^{\infty}\int_{b+ac(t-t_b)}^{\infty} \mathrm{e}^{-\delta t} w\left(b+ac(t-t_b), x-b-ac(t-t_b)\right)\left[1+\frac{\alpha\lambda(2\beta-\alpha)}{(\beta-\alpha)^2+t^2}\right] f_X(x)\,\mathrm{d}x\mathrm{d}t \end{aligned}$$

$$\begin{aligned} I_b(u) = {} & \frac{\lambda}{2\pi}\int_0^{t_b}\left[1+\frac{\alpha\lambda(2\beta-\alpha)}{(\beta-\alpha)^2+t^2}\right]\mathrm{e}^{-\delta t}\bigg(\int_0^{u+ct}\varphi_b(u+ct-x)f_X(x)\mathrm{d}x \\ & +\int_{u+ct}^{\infty}w(u+ct,x-u-ct)f_X(x)\mathrm{d}x\bigg)\mathrm{d}t \\ & +\frac{\lambda}{2\pi}\int_{t_b}^{\infty}\left[1+\frac{\alpha\lambda(2\beta-\alpha)}{(\beta-\alpha)^2+t^2}\right]\mathrm{e}^{-\delta t}\bigg(\int_0^{b+ac(t-t_b)}\varphi_b(b+ac(t-t_b)-x)f_X(x)\mathrm{d}x \\ & +\int_{b+ac(t-t_b)}^{\infty}w(b+ac(t-t_b),x-b-ac(t-t_b))f_X(x)\mathrm{d}x\bigg)\mathrm{d}t \end{aligned}$$

$$\begin{aligned} I_b(u) = {} & \frac{\lambda}{2\pi}\int_0^{t_b}\left[1+\frac{\alpha\lambda(2\beta-\alpha)}{(\beta-\alpha)^2+t^2}\right]\mathrm{e}^{-\delta t}\left(\sigma_b(u+ct)\right)\mathrm{d}t \\ & +\frac{\lambda}{2\pi}\int_{t_b}^{\infty}\left[1+\frac{\alpha\lambda(2\beta-\alpha)}{(\beta-\alpha)^2+t^2}\right]\mathrm{e}^{-\delta t}\bigg(\int_0^{u+ct}\varphi_b(u+ct-x)f_X(x)\mathrm{d}x \\ & +\int_{u+ct}^{\infty}w(u+ct,x-ct)f_X(x)\mathrm{d}x\bigg)\mathrm{d}t \end{aligned}$$

$$\begin{aligned} I_b(u) = {} & \frac{\lambda}{2\pi}\times\Bigg(\int_0^{\frac{b-u}{c}}\left[1+\frac{\alpha\lambda(2\beta-\alpha)}{(\beta-\alpha)^2+t^2}\right]\mathrm{e}^{-\delta t}\sigma_b(u+ct)\mathrm{d}t \\ & +\int_{\frac{b-u}{c}}^{\infty}\left[1+\frac{\alpha\lambda(2\beta-\alpha)}{(\beta-\alpha)^2+t^2}\right]\mathrm{e}^{-\delta t}\sigma_b(u+ct)\mathrm{d}t\Bigg) \end{aligned}$$

So

$$I_b(u)=\frac{\lambda}{2\pi}\int_0^{\infty}\left[1+\frac{\alpha\lambda(2\beta-\alpha)}{(\beta-\alpha)^2+t^2}\right]\mathrm{e}^{-\delta t}\sigma_b(u+ct)\mathrm{d}t \tag{3.7}$$

Let's move on to the calculation of integrals $I_{b,3}(u)$ and $I_{b,4}(u)$ in the relation (3.1). The copular structure of the support C_M is $D=\left\{(u;v)\in[0;1]^2 : u=v\right\}$.

On the estate $[0;1]^2 \setminus D$, $\frac{\partial^2 C_M}{\partial u \partial v}=0$; and on *D*, C_M is uniformly distributed.

Since the dependence structure is described by the copula *M C*, they are monotonic and there almost certainly exists an increasing function *l*, such that *X* = *l*(*V*) (see Nelsen, 2006 [7]: p. 27). The distribution function of *X* is:

$$f_X(x)=f_V\left(l^{-1}(x)\right)$$

$$\frac{\lambda}{2\pi}\left[1+\frac{\alpha\lambda(2\beta-\alpha)}{(\beta-\alpha)^2+x^2}\right]=\frac{\lambda}{2\pi}\left[1+\frac{\alpha\lambda(2\beta-\alpha)}{(\beta-\alpha)^2+\left[l^{-1}(x)\right]^2}\right]$$

$$\frac{\alpha\lambda(2\beta-\alpha)}{(\beta-\alpha)^2+x^2}=\frac{\alpha\lambda(2\beta-\alpha)}{(\beta-\alpha)^2+\left[l^{-1}(x)\right]^2}$$

$$(\beta-\alpha)^2+x^2=(\beta-\alpha)^2+\left[l^{-1}(x)\right]^2$$

$$l^{-1}(x)=t$$

$$x=l(t) \tag{3.8}$$

The joint distribution $F(x,t)$ of the random vector (*X*, *W*) has support on the set:

$$D'=\{(x,t):x=l(t)\}$$

The joint distribution of the random vector (*X*, *V*) is singular. Its distribution over the domain D' is:

$$G(t)=F_M(l(t);t)$$

Thus, we obtain in general:

$$I_{b,3}(u)=\int_L \varphi_b(u+ct-l(t))\mathrm{e}^{-\delta t}\mathrm{d}G(t)$$

$$\begin{aligned}
I_{b,3}(u)&=\frac{\lambda}{2\pi}\int_L\left[1+\frac{\alpha\lambda(2\beta-\alpha)}{(\beta-\alpha)^2+\left[l^{-1}(x)\right]^2}\right]\varphi_b(u+ct-l(t))\mathrm{e}^{-\delta t}\mathrm{d}t\\
&=\frac{\lambda}{2\pi}\int_0^\infty\left[1+\frac{\alpha\lambda(2\beta-\alpha)}{(\beta-\alpha)^2+[t]^2}\right]\varphi_b(u+ct-t)\mathrm{e}^{-\delta t}\mathrm{d}t\\
&=\frac{\lambda}{2\pi}\int_0^\infty\left[1+\frac{\alpha\lambda(2\beta-\alpha)}{(\beta-\alpha)^2+[t]^2}\right]\varphi_b(u+(c-1)t)\mathrm{e}^{-\delta t}\mathrm{d}t
\end{aligned}$$

Let's ask $c_1=c-1$ and $z=u+at$ that means $t=\frac{z-u}{c_1}$ and $\mathrm{d}t=\frac{1}{c_1}\mathrm{d}z$.

$$I_{b,3}(u)=\frac{\lambda}{2\pi c_1}\int_u^\infty\left[1+\frac{\alpha\lambda(2\beta-\alpha)}{(\beta-\alpha)^2+\left[\frac{z-u}{c_1}\right]^2}\right]\varphi_b(z)\mathrm{e}^{-\delta\left(\frac{z-u}{c_1}\right)}\mathrm{d}z$$

$$I_{b,3}(u)=\frac{\lambda}{2\pi c_1}\mathrm{e}^{\frac{\delta}{c_1}u}\int_u^\infty\left[1+\frac{\alpha\lambda(2\beta-\alpha)}{(\beta-\alpha)^2+\left[\frac{z-u}{c_1}\right]^2}\right]\varphi_b(z)e^{-\frac{\delta}{c_1}z}\mathrm{d}z$$

$$I_{b,4}(u)=\frac{\lambda}{2\pi}\int_{\mathbb{R}\setminus L}\left[1+\frac{\alpha\lambda(2\beta-\alpha)}{(\beta-\alpha)^2+\left[l^{-1}(x)\right]^2}\right]w(u+ct-x,x-u-ct)\mathrm{e}^{-\delta t}\mathrm{d}t$$

With $L=\{tu+ct\geq l(t)\}=\mathbb{R}^+$, we have: $I_{b,4}(u)=0$.

So, the Gerber-Shiu function of ruin is determined by:

$$\varphi_b(u)=(1-\gamma)I_b(u)+\gamma I_{b,3}(u).$$

By using the derivative of Equation (3.7), we arrive at the following Lemma 3.1.

Lemma 3.1 The Gerber-Shiu function in the risk model defined by relation (1.1) satisfies the following integro-differential equation:

$$\frac{\partial}{\partial u}\varphi_b(u)=\frac{\delta}{c}\varphi_b(u)+\frac{\lambda(1-\gamma)}{2\pi c}\mathrm{e}^{\frac{\delta}{c}u}\int_u^{\infty}\left[\frac{2\alpha\lambda(2\beta-\alpha)\left(\frac{y-u}{c^2}\right)}{\left((\beta-\alpha)^2+\left(\frac{y-u}{c}\right)^2\right)^2}\right]\mathrm{e}^{-\frac{\delta}{c}y}\sigma_b(y)\,\mathrm{d}y$$

$$+\frac{\gamma\delta}{c_1}I_{b,3}(u)+\frac{\gamma\lambda}{2\pi c_1}\mathrm{e}^{\frac{\delta}{c_1}u}\int_u^{\infty}\left[\frac{2\alpha\lambda(2\beta-\alpha)\left(\frac{z-u}{c_1^2}\right)}{\left((\beta-\alpha)^2+\left(\frac{z-u}{c_1}\right)^2\right)^2}\right]\mathrm{e}^{-\frac{\delta}{c_1}z}\varphi_b(z)\,\mathrm{d}z$$

Proof

The derivative of φ_b gives us:

$$\frac{\partial}{\partial u}\varphi_b(u)=(1-\gamma)\frac{\partial}{\partial u}I_b(u)+\gamma\frac{\partial}{\partial u}I_{b,3}(u). \tag{3.9}$$

Equation (3.7) with $y=u+ct$ gives us:

$$\frac{\partial}{\partial u}I_b(u)=\frac{\partial}{\partial u}\left(\frac{\lambda}{2\pi}\int_u^{\infty}\left[1+\frac{\alpha\lambda(2\beta-\alpha)}{(\beta-\alpha)^2+t^2}\right]\mathrm{e}^{-\delta t}\sigma_b(u+ct)\,\mathrm{d}t\right)$$

$$=\frac{\lambda}{2\pi c}\times\left(\int_u^{\infty}\frac{\delta}{c}\mathrm{e}^{\frac{\delta}{c}u}\left[1+\frac{\alpha\lambda(2\beta-\alpha)}{(\beta-\alpha)^2+\left(\frac{y-u}{c}\right)^2}\right]\mathrm{e}^{-\frac{\delta}{c}y}\sigma_b(y)\,\mathrm{d}y\right.$$

$$\left.+\mathrm{e}^{\frac{\delta}{c}u}\int_u^{\infty}\left[\frac{2\alpha\lambda(2\beta-\alpha)\left(\frac{y-u}{c^2}\right)}{\left((\beta-\alpha)^2+\left(\frac{y-u}{c}\right)^2\right)^2}\right]\mathrm{e}^{-\frac{\delta}{c}y}\sigma_b(y)\,\mathrm{d}y\right)$$

$$\frac{\partial}{\partial u}I_b(u)=\frac{\delta}{c}I_b(u)+\frac{\lambda}{2\pi c}\mathrm{e}^{\frac{\delta}{c}u}\int_u^{\infty}\left[\frac{2\alpha\lambda(2\beta-\alpha)\left(\frac{y-u}{c^2}\right)}{\left((\beta-\alpha)^2+\left(\frac{y-u}{c}\right)^2\right)^2}\right]\mathrm{e}^{-\frac{\delta}{c}y}\sigma_b(y)\,\mathrm{d}y \tag{3.10}$$

In the same way, we have:

$$\frac{\partial}{\partial u}I_{b,3}(u)=\frac{\delta}{c_1}I_{b,3}(u)+\frac{\lambda}{2\pi c_1}\mathrm{e}^{\frac{\delta}{c_1}u}\int_u^{\infty}\left[\frac{2\alpha\lambda(2\beta-\alpha)\left(\frac{z-u}{c_1^2}\right)}{\left((\beta-\alpha)^2+\left(\frac{z-u}{c_1}\right)^2\right)^2}\right]\mathrm{e}^{-\frac{\delta}{c_1}z}\varphi_b(z)\,\mathrm{d}z \tag{3.11}$$

Using Equations (3.9), (3.10), and (3.11), we deduce:

$$\frac{\partial}{\partial u}\varphi_b(u)=\frac{\delta}{c}\varphi_b(u)+\frac{\lambda(1-\gamma)}{2\pi c}\mathrm{e}^{\frac{\delta}{c}u}\int_u^{\infty}\left[\frac{2\alpha\lambda(2\beta-\alpha)\left(\frac{y-u}{c^2}\right)}{\left((\beta-\alpha)^2+\left(\frac{y-u}{c}\right)^2\right)^2}\right]\mathrm{e}^{-\frac{\delta}{c}y}\sigma_b(y)\mathrm{d}y$$
$$+\frac{\gamma\delta}{c_1}I_{b,3}(u)+\frac{\gamma\lambda}{2\pi c_1}\mathrm{e}^{\frac{\delta}{c_1}u}\int_u^{\infty}\left[\frac{2\alpha\lambda(2\beta-\alpha)\left(\frac{z-u}{c_1^2}\right)}{\left((\beta-\alpha)^2+\left(\frac{z-u}{c_1}\right)^2\right)^2}\right]\mathrm{e}^{-\frac{\delta}{c_1}z}\varphi_b(z)\mathrm{d}z. \tag{3.12}$$

4. Laplace Transforms of Gerber-Shiu Functions $\varphi_b(u)$ and the Probability of Ultimate Ruin

The goal of this section is to determine the Laplace transform of the Gerber-Shiu functions $\varphi_b(u)$ and the ultimate ruin probability in the risk model defined by relation (1.1).

Lemma 4.1 The Gerber-Shiu function $\varphi_b(u)$ in the risk model defined by relation (1.1) has a Laplace transform $\hat{\varphi}_b(s)$ given by:

$$\hat{\varphi}_b(s)=\frac{N_1(s)+\gamma N_2(s)}{N_3(s)+\gamma N_4(s)} \tag{4.1}$$

with:

$$N_1(s)=\varphi_b(0)+\left(\frac{\alpha\lambda^2(2\beta-\alpha)cs}{2\pi(s-p)(\beta-\alpha)}\right)\left(\frac{\rho}{p+\rho}\hat{\varphi}_b(p)+\hat{w}_b(p)+\hat{w}_b(s)\right)$$

$$N_2(s)=\frac{\lambda(q-p)}{2\pi c_1(s-q)}\left(\frac{\alpha\lambda(2\beta-\alpha)cs}{\beta-\alpha}\hat{\varphi}_b(q)-\frac{\partial}{\partial q}\hat{\varphi}_b(q)\right)-\left(\frac{\alpha\lambda^2(2\beta-\alpha)cs}{2\pi(s-p)(\beta-\alpha)}\right)$$
$$\times\left(\frac{\rho}{p+\rho}\hat{\varphi}_b(p)+\hat{w}_b(p)+\hat{w}_b(s)\right)+\frac{\alpha\lambda^2(2\beta-\alpha)s}{2\pi(s-q)(\beta-\alpha)}\hat{\varphi}_b(q)$$

$$N_3(s)=s-q-\frac{\alpha\lambda^2(2\beta-\alpha)s}{2\pi(s-p)(\beta-\alpha)(s+\rho)}$$

$$N_4(s)=\frac{\lambda(q-p)\left[\beta-\alpha-\alpha\lambda(2\beta-\alpha)cs\right]}{2\pi a(s+q)(\beta-\alpha)}+\frac{\alpha\lambda^2(2\beta-\alpha)s}{2\pi(s-q)(\beta-\alpha)}$$
$$+\frac{\alpha\lambda^2(2\beta-\alpha)s}{2\pi(s-p)(\beta-\alpha)(s+\rho)}$$

Proof

$$I_{b,3}(u)=\frac{\lambda}{2\pi c_1}\mathrm{e}^{\frac{\delta}{c_1}u}\int_u^{\infty}\left[1+\frac{\alpha\lambda(2\beta-\alpha)}{(\beta-\alpha)^2+\left[\frac{z-u}{c_1}\right]^2}\right]\varphi_b(z)\mathrm{e}^{-\frac{\delta}{c_1}z}\mathrm{d}z$$

$$
=\frac{\lambda}{2\pi c_1}\int_0^{\infty}\mathrm{e}^{\frac{\delta}{c_1}u}\mathrm{e}^{-\frac{\delta}{c_1}z}\varphi_b(z)\mathrm{d}z+\mathrm{e}^{\frac{\delta}{c_1}u}\int_u^{\infty}\left[\frac{\alpha\lambda(2\beta-\alpha)}{(\beta-\alpha)^2+\left[\frac{z-u}{c_1}\right]^2}\right]\varphi_b(z)\mathrm{e}^{-\frac{\delta}{c_1}z}\mathrm{d}z
$$

$$
-\frac{\lambda}{2\pi c_1}\mathrm{e}^{\frac{\delta}{c_1}u}\int_0^{u}\left[1+\frac{\alpha\lambda(2\beta-\alpha)}{(\beta-\alpha)^2+\left[\frac{z-u}{c_1}\right]^2}\right]\varphi_b(z)\mathrm{e}^{-\frac{\delta}{c_1}z}\mathrm{d}z
$$

Let’s ask : $q=\frac{\delta}{c_1}$, $h_1(u)=\mathrm{e}^{qu}$, $h_2(z-u)=\frac{\alpha\lambda(2\beta-\alpha)}{(\beta-\alpha)^2+\left[\frac{z-u}{c_1}\right]^2}$,

$f(u)=h_2(u)\mathrm{e}^{-qu}$ and $h_3(z-u)=\mathrm{e}^{-q(z-u)}$, we obtain:

$$
I_{b,3}(u)=\frac{\lambda}{2\pi c_1}\left[h_1(u)\left(h_2(u)-\frac{\partial}{\partial q}\hat{\varphi}_b(q)\right)-h_3(u)*\varphi_b(u)-f(u)*\varphi_b(u)\right] \quad (4.2)
$$

Using Equation (4.2), we deduce the Laplace transform of $I_{b,3}(u)$

$$
\hat{I}_{b,3}(s)=\frac{\lambda}{2\pi c_1}\left(\hat{h}_1(s)\left[\hat{h}_2(s)\hat{\varphi}_b(q)-\frac{\partial}{\partial q}\hat{\varphi}_b(q)\right]-\hat{h}_3(s)\hat{\varphi}_b(s)-\hat{f}(s)\hat{\varphi}_b(s)\right)
$$

$$
\begin{aligned}
\hat{I}_{b,3}(s)=&\frac{\lambda}{2\pi c_1(s-q)}\left(\frac{\alpha\lambda(2\beta-\alpha)cs}{\beta-\alpha}\hat{\varphi}_b(q)-\frac{\partial}{\partial q}\hat{\varphi}_b(q)\right)\\
&-\frac{\lambda}{2\pi c_1(s+q)}\hat{\varphi}_b(s)\left(1-\frac{\alpha\lambda(2\beta-\alpha)cs}{\beta-\alpha}\right)
\end{aligned} \quad (4.3)
$$

Let’s ask $p=\frac{\delta}{c}$ and

$$
K_1(u)=\mathrm{e}^{\frac{\delta}{c}u}\int_u^{\infty}\left[\frac{2\alpha\lambda(2\beta-\alpha)\left(\frac{y-u}{c^2}\right)}{\left((\beta-\alpha)^2+\left(\frac{y-u}{c}\right)^2\right)^2}\right]\mathrm{e}^{-\frac{\delta}{c}y}\sigma_b(y)\mathrm{d}y
$$

$$
K_1(u)=g_1(u)\hat{\sigma}_b(p)-g_1(u)*\sigma_b(u) \quad (4.4)
$$

with $\sigma(u)=\varphi_b(u)*f_X(u)+w(u)$, that gives us:

$$
\hat{\sigma}(s)=\frac{\rho}{s+\rho}\hat{\varphi}_b(s)+\hat{w}(s) \quad (4.5)
$$

By using Equations (4.4) and (4.5), we obtain:

$$
\hat{K}_1(u)=\frac{\alpha\lambda(2\beta-\alpha)cs}{(\beta-\alpha)(s-p)}\left(\frac{\rho}{p+\rho}\hat{\varphi}_b(p)+\hat{w}(p)-\frac{\rho}{s+\rho}\hat{\varphi}_b(s)-\hat{w}(s)\right) \quad (4.6)
$$

Let’s ask $q=\frac{\delta}{c_1}$ and

$$K_2(u)=\mathrm{e}^{\frac{\delta}{c_1}u}\int_u^{\infty}\left[\frac{2\alpha\lambda(2\beta-\alpha)\left(\frac{y-u}{c_1^2}\right)}{\left((\beta-\alpha)^2+\left(\frac{y-u}{c_1^2}\right)^2\right)^2}\right]\mathrm{e}^{-\frac{\delta}{c_1}y}\sigma_b(z)\,\mathrm{d}z$$

$$K_2(u)=g_2(u)\hat{\sigma}_b(q)-g_2(u)*\varphi_b(u)$$

$$\hat{K}_2(s)=\frac{\alpha\lambda(2\beta-\alpha)c_1s}{(\beta-\alpha)(s-q)}\left(\hat{\varphi}_b(q)-\hat{\varphi}_b(s)\right) \tag{4.7}$$

Equations (3.12), (4.3), (4.6), and (4.7) give:

$$\frac{\partial}{\partial u}\hat{\varphi}_b(s)=p\hat{\varphi}_b(s)+\gamma(q-p)\hat{I}_{b,3}(s)+\frac{\lambda(1-\gamma)}{2\pi c}\hat{K}_1(s)+\frac{\lambda\gamma}{2\pi c_1}\hat{K}_2(s)$$

$$s\hat{\varphi}_b(s)-\varphi(0)=p\hat{\varphi}_b(s)+\gamma(q-p)\hat{I}_{b,3}(s)+\frac{\lambda(1-\gamma)}{2\pi c}\hat{K}_1(s)+\frac{\lambda\gamma}{2\pi c_1}\hat{K}_2(s) \tag{4.8}$$

By simplifying Equation (4.8), we obtain (4.1).

Theorem 4.1 The Laplace transform of the ultimate ruin probability in the risk model defined by relation (1.1) is given by:

$$\hat{\Psi}_b(s)=\frac{T_1(s)+\gamma T_2(s)}{T_3(s)+\gamma T_4(s)} \tag{4.9}$$

where

$$T_1(s)=\Psi_b(0)+A\hat{\Psi}_b(p)+A\hat{w}_b(p)-\frac{A}{s+\rho}$$

$$T_2(s)=B\hat{\Psi}_b(q)-B\hat{\Psi}_b(p)-B\hat{w}_b(p)+\frac{B}{s+\rho}$$

$$T_3(s)=s-\frac{D}{s+\rho}$$

$$T_4(s)=B+\frac{D}{s+\rho}$$

with

$$A=\frac{\alpha\lambda^2c(2\beta-\alpha)}{2\pi(\beta-\alpha)},B=\frac{\alpha\lambda^2(2\beta-\alpha)}{2\pi(\beta-\alpha)},D=\frac{\alpha\lambda^2\rho(2\beta-\alpha)}{2\pi(\beta-\alpha)}$$

Proof

By setting $w(x,y)=1$, we have: $\hat{\omega}(s)=\frac{1}{s+\rho}$.

By setting $\delta=0$ and $w(x,y)=1$ in relation (4.1), the equations $N_1(s)$, $N_2(s)$, $N_3(s)$ and $N_4(s)$ give the equations respectively $T_1(s)$, $T_2(s)$, $T_3(s)$ and $T_4(s)$, which allow obtaining (4.9).

5. Probability of Ultimate Ruin

Theorem 5.1 The probability of ultimate ruin in the risk model defined by relation

(1.1) is explicitly expressed as follows:

$$\Psi_b(u)=\frac{\alpha\lambda^2(2\beta-\alpha)(c-\gamma)}{2\pi(\beta-\alpha)(R+\rho)}\mathrm{e}^{Ru} \tag{5.1}$$

where:

$$R=-\frac{(\rho+\gamma B)+\sqrt{(\rho+\gamma B)^2+4\rho D(1-\gamma)}}{2}<0$$

Proof

Equation (4.9) gives us:

$$\begin{aligned}\hat{\Psi}_b(s)=&\frac{(s+\rho)\left(\Psi_b(0)+A\hat{\Psi}_b(p)+A\hat{w}_b(p)\right)-A}{s^2+(\rho+\gamma B)s-\rho D+\gamma\rho D}\\&+\frac{\gamma\left((s+\rho)\left(B\hat{\Psi}_b(q)-B\hat{\Psi}_b(p)-B\hat{w}_b(p)\right)+B\right)}{s^2+(\rho+\gamma B)s-\rho D+\gamma\rho D}\end{aligned} \tag{5.2}$$

The denominator of Equation (5.2) is clearly a second-degree polynomial in the variable s with a discriminant $\Delta=(\rho+\gamma B)^2+4\rho D(1-\gamma)>0$, which means that it admits two roots given by the following relationships:

$$R_1=-\frac{(\rho+\gamma B)+\sqrt{(\rho+\gamma B)^2+4\rho D(1-\gamma)}}{2}<0$$

and

$$R_2=-\frac{(\rho+\gamma B)-\sqrt{(\rho+\gamma B)^2+4\rho D(1-\gamma)}}{2}>0$$

The numerator is a polynomial of degree 1 in variable s, so Equation (5.2) in partial fraction decomposition can be written in the following form:

$$\hat{\Psi}_b(s)=\frac{a}{s-R_1}+\frac{d}{s-R_2}=\frac{(a+d)s-aR_2-dR_1}{(s-R_1)(s-R_2)} \tag{5.3}$$

The injectivity of the Laplace transform of (5.3) gives:

$$\Psi_b(u)=a\mathrm{e}^{R_1u}+d\mathrm{e}^{R_2u} \tag{5.4}$$

We know that $\lim_{u\to+\infty}\Psi_b(u)=0$ and $\Psi_b(0)=a+d$, Furthermore $R_1<0$ and $R_2>0$ then using Equation (5.4), we have $d = 0$. Using this information and Equations (5.2), (5.3), and (5.4), we deduce that $a=\frac{\alpha\lambda^2(2\beta-\alpha)(c-\gamma)}{2\pi(\beta-\alpha)(R+\rho)}$ and $\Psi_b(u)=a\mathrm{e}^{R_1u}$.

Example By setting the parameters $\beta = 0.5$, $\alpha = 0.4$, $\lambda = 0.3$, $\rho = 0.2$, $\pi = 3.14$, $c = 15$, $b = 10$, using MATLAB, we present the curves associated with the ruin probability related to different values of the dependence parameter γ (see Figure 1).

The probability of ruin $\Psi_b(u)$ is a decreasing function of the dependence parameter γ. We vary the value of u by increasing to observe the behavior of the probability of ruin at the infinite horizon. Figure 1 shows a simulation of the probability of ruin as a function of the initial reserve u. The contribution rate c is constant throughout the simulation for $\gamma = 0.4$ and $\gamma = 0.8$. We thus obtain the values of the probability of ruin. It is clearly visible from the figure that an increase in u leads to a decrease in the probability of ruin, regardless of the value of q taken in the interval 0 to 1. When the initial reserve varies and tends towards plus infinity (+∞), then the probability of ruin ($\Psi_b(u)$) tends towards 0, which is quite normal for a risk model.

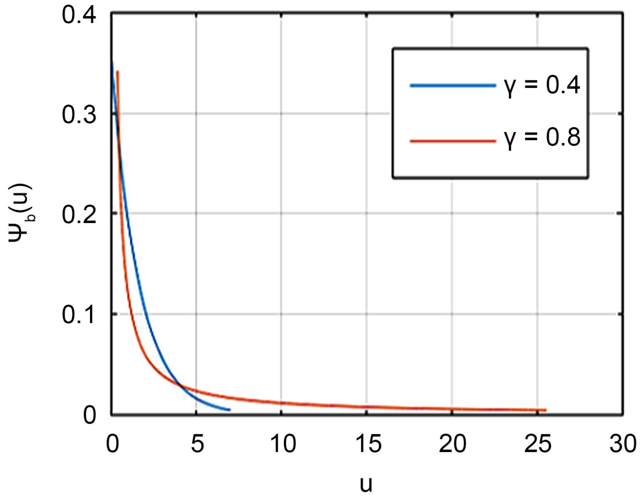

Figure 1. $\Psi_b(u)$ for different values of γ.

6. Conclusion

In this article, we have determined the ultimate ruin probability in a risk model based on the Hawkes process with a partial dividend policy for shareholders and a dependence between the claim amounts and the times between claims via the Spearman copula. This allowed us to obtain a simplified expression of the ultimate ruin probability and to carry out simulations. The self-excitation and memory properties of Hawkes processes provide a more realistic representation of the arrival of claims, which is crucial for effective risk management in insurance. We established an explicit expression of the ultimate probability as a function of a well-known value of the degree of dependence γ, which allowed us to perform simulations. But in the context of our article, we introduce copulas to express the dependence between the intervals of arrival of claims and that of the amounts of claims.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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